Mathematical Biosciences
and Engineering

http://www.aimspress.com/journal/MBE

MBE, 18(6): 9430–9473
DOI: 10.3934/mbe.2021464
Received: 10 August 2021
Accepted: 12 October 2021
Published: 28 October 2021

Research article

Analysis of an HTLV/HIV dual infection model with diffusion

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Abstract: In the literature, several HTLV-I and HIV single infections models with spatial dependence have been developed and analyzed. However, modeling HTLV/HIV dual infection with diffusion has not been studied. In this work we derive and investigate a PDE model that describes the dynamics of HTLV/HIV dual infection taking into account the mobility of viruses and cells. The model includes the effect of Cytotoxic T lymphocytes (CTLs) immunity. Although HTLV-I and HIV primarily target the same host, $CD4^{+}T$ cells, via infected-to-cell (ITC) contact, however the HIV can also be transmitted through free-to-cell (FTC) contact. Moreover, HTLV-I has a vertical transmission through mitosis of active HTLV-infected cells. The well-posedness of solutions, including the existence of global solutions and the boundedness, is justified. We derive eight threshold parameters which govern the existence and stability of the eight steady states of the model. We study the global stability of all steady states based on the construction of suitable Lyapunov functions and usage of Lyapunov-LaSalle asymptotic stability theorem. Lastly, numerical simulations are carried out in order to verify the validity of our theoretical results.

Keywords: HTLV/HIV dual infection; diffusion; global stability; mitosis; CTL immune response

1. Introduction

Since past decades humanity has been under attack by many viruses such as hepatitis C virus (HCV), human immunodeficiency virus (HIV), hepatitis B virus (HBV), human T-lymphotropic virus type I (HTLV-I), dengue virus and lastly coronavirus. Both HTLV-I and HIV have similar ways of transmission from infected individual to uninfected one. HTLV-I and HIV primarily target the same host, CD4$^+$T cells. HTLV-I can lead to adult T-cell leukemia (ATL) and HTLV-I-associated myelopathy/tropical spastic paraparesis (HAM/TSP), while HIV causes acquired immunodeficiency syndrome (AIDS). Viral infection models have become an indispensable tool to biological researchers, where they can improve the understanding of a within-host virus dynamics and help in predicting the effect of antiviral drug efficacy on disease's progression [1]. In 1996, Nowak and Bangham [2] have presented an important HIV dynamics model. After that, this model has been extended in many works (see e.g., [3–17]). All of the above mentioned models are given by ordinary or delay differential equations under the assumption that the cells and viruses are well mixed. Wang and Wang [18] have extended the model presented in [2] by incorporating spatial dependence as:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = \rho - \alpha S(x,t) - \varkappa_1 S(x,t)V(x,t), \\ \frac{\partial I(x,t)}{\partial t} = \varkappa_1 S(x,t)V(x,t) - aI(x,t), \\ \frac{\partial V(x,t)}{\partial t} = d_V \Delta V(x,t) + bI(x,t) - \varepsilon V(x,t), \end{cases} \tag{1.1}$$

where $S(x,t)$, $I(x,t)$ and $V(x,t)$ are the concentrations of uninfected cells, active infected cells and free virus particles at position $x = (x_1, x_2, ..., x_m)$ and time t. The parameter ρ represents the creation rate of the uninfected cells. The free virus particles infect the uninfected cells via free-to-cell (FTC) transmission at rate $\varkappa_1 SV$. The infected cells produce viruses at rate bI. The uninfected cells, active infected cells and free virus particles are die with rates αS, aI and εV , respectively. Here, d_V is the diffusion coefficient and Δ is the Laplacian operator. In [19], Kang et al. have studied a four-dimensional diffusive viral infection model with Crowley-Martin infection rate. Model (1.1) has been extended by including different factors; (i) time delay [19, 20], (ii) different forms of the incidence rate [19, 20], (iii) Cytotoxic T lymphocytes (CTLs) immune response [19], and (iv) both CTL and humoral immune responses [21].

In model (1.1), it has been assumed that the virus can only infect the target cell via FTC contact. In case of HIV, the infected cell can infect the target cell via direct infected-to-cell (ITC) contact [22]. Wang et al. [23] have extended model (1.1) by incorporating ITC transmission as:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = \rho - \alpha S(x,t) - \varkappa_1 S(x,t)V(x,t) - \varkappa_2 S(x,t)I(x,t), \\ \frac{\partial I(x,t)}{\partial t} = \varkappa_1 S(x,t)V(x,t) + \varkappa_2 S(x,t)I(x,t) - aI(x,t), \\ \frac{\partial V(x,t)}{\partial t} = d_V \Delta V(x,t) + bI(x,t) - \varepsilon V(x,t), \end{cases} \tag{1.2}$$

where the term $\varkappa_2 SI$ represents the ITC incidence rate. This model has been generalized in [24] by including time delay and general FTC incidence rate function in the form $f(S,V)$. Model (1.2) assumes that the virus particles can move based on Fickian diffusion, while the cells can not. In [25–31], it has been assumed that the viruses, uninfected cells, infected cells and immune cells can diffuse.

Modeling and analysis of HTLV-I single infection have been addressed in several works [32–36]. The effect of CTLs on HTLV-I dynamics has been addressed in many works (see e.g. [37–43]). Lim and Maini [37] have proposed an HTLV-I dynamics model with mitotic division of active HTLV-infected cells and CTL immunity. Both mitotic division of active HTLV-infected cells and CTL immune response have been included in the HTLV-I dynamics in many papers (see e.g., [37,44,45]). All of these HTLV dynamics models did not include the diffusion of the viruses and cells. Wang and Ma [46] have introduced a diffusive HTLV-I infection model with mitotic division of active infected cells and CTL immune response:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_S \Delta S(x,t) + \rho - \alpha S(x,t) - \varkappa_3 S(x,t) Y(x,t), \\ \frac{\partial E(x,t)}{\partial t} = d_E \Delta E(x,t) + \varphi \varkappa_3 S(x,t) Y(x,t) + r^* Y(x,t) - (\psi + \omega) E(x,t), \\ \frac{\partial Y(x,t)}{\partial t} = d_Y \Delta Y(x,t) + \psi E(x,t) - \delta^* Y(x,t) - \mu_2 C^Y(x,t) Y(x,t), \\ \frac{\partial C^Y(x,t)}{\partial t} = d_{C^Y} \Delta C^Y(x,t) + \sigma_2 Y(x,t) - \pi_2 C^Y(x,t), \end{cases}$$

where $E(x,t)$, $Y(x,t)$ and $C^Y(x,t)$ are the concentrations of latent HTLV-infected cells, active HTLV-infected cells and HTLV-specific CTLs at position x and time t, respectively. The uninfected $CD4^+T$ cells become HTLV-infected cells due to ITC contact at rate $\varkappa_3 SY$ (horizontal transmission). The fraction $\varphi \in (0,1)$ represents the probability of new HTLV infections via horizontal transmission could be enter a latent period. The term r^*Y (vertical transmission) represents the rate at which active HTLV-infected cells become latent. The terms ωE and $\delta^* Y$ denote the death rates of the latent and active HTLV-infected cells, respectively. The latent HTLV-infected cells are activated with rate ψE. The active HTLV-infected cells are killed by their specific CTLs at rate $\mu_2 C^Y Y$. The linear term $\sigma_2 Y$ represents the expansion rate of HTLV-specific CTLs. The HTLV-specific CTLs decay at rate $\pi_2 C^Y$.

During the last decades HTLV-I and HIV dual infection has been extensively reported. It has been discovered that the simultaneous infection by the two viruses affects the pathogenic development and influences the outcomes for associated chronic diseases [47]. In fact, concurrent infections with HTLV-I and HIV have occurred frequently in areas where peoples living at high risk activities such as needle injection sharing and unprotected sexual relationships. In addition, HTLV/HIV dual infection have documented in specific geographic regions where both retroviruses become endemic [48], and among those who belonged to a specific ethnic as well. For instance, the dual infection rates in peoples living in some parts of Brazil have reached 16% of HIV-infected patients [49]. In a recent work, it has been estimated that the HIV single infected patients are more exposure to be dually infected with HTLV-I at a higher rate initiating from 100 to 500 times in comparison to the uninfected peoples [50]. Moreover, some seroepidemiologic studies have reported that HTLV-infected patients are at risk to have a concurrent infection with HIV, and vice versa compared to those who are infection-free from the general population [48]. HTLV-I and HIV are mainly attack the $CD4^+T$ cells and lead to immune dysfunctional as well, however, they also conflict no doubt with respect to the etiology of their pathogenic and clinical outcomes [51]. HTLV-I and HIV dual infection appears to have an overlap on the course of associated clinical outcomes with both viruses [48]. Many researchers have reported that HIV infected individuals who are possibly infected with HTLV-I simultaneously can potentially associated with clinical progression to AIDS. In contrast, HIV can modify HTLV-I expression in dual infected patients which leads them to a higher risk of developing HTLV-I related diseases such as TSP/HAM and ATL [48, 50].

While many efforts have been made to investigate mathematical modeling and analysis of both HTLV-I and HIV single infection, almost none have focused on the modeling of HTLV/HIV dual infection dynamics. The only exceptions are the very recent works presented in [52, 53], however, in these papers the diffusion of the viruses and cells has been neglected. Therefore, the contributions of the present paper can be stated as follows: (i) formulate an HTLV/HIV dual infection model taking into account the mobility of cells and viruses, (ii) study the basic properties of the proposed model, (iii) calculate all possible steady states and derive their existence conditions, (iv) study the global stability of all steady states using Lyapunov-LaSalle asymptotic stability theorem, (v) perform some numerical simulations to illustrate the theoretical results.

Since an individual can be infected with two or more viruses in the same time, our model may be helpful to study different dual infections such as Coronavirus/Influenza, HIV/HCV, HIV/HBV and HIV/Malaria.

2. Model formulation

We set up a partial differential equation model that describes the change of concentrations of eight compartments with respect to position x and time t; uninfected CD4$^+$T cells $S(x,t)$, latent HIV-infected cells $L(x,t)$, active HIV-infected cells $I(x,t)$, latent HTLV-infected cells $E(x,t)$, active HTLV-infected cells $Y(x,t)$, free HIV particles $V(x,t)$, HIV-specific CTLs $C^I(x,t)$, and HTLV-specific CTLs $C^Y(x,t)$. We consider the following factors:

(F1) The uninfected CD4$^+$T cells are the main target of each of HTLV-I and HIV;

(F2) There exist latent HIV-infected and HTLV-infected cells;

(F3) Bilinear specific CTL immune response for both HTLV-I and HIV;

(F4) The HIV can spread when an uninfected CD4$^+$T cell is contacted with free HIV particle (FTC infection) or active HIV-infected cell (ITC infection);

(F5) HTLV-I can be transmitted via two routes, (i) horizontal transmission via direct ITC touch via virological synapse, and (ii) vertical transmission by mitotic division of active HTLV-infected cells.

(F6) Spatial diffusion for all compartments.

Taking into account factors (F1)-(F6) we propose the following PDEs model:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_S\Delta S(x,t) + \rho - \alpha S(x,t) - \varkappa_1 S(x,t)V(x,t) - \varkappa_2 S(x,t)I(x,t) - \varkappa_3 S(x,t)Y(x,t), \\ \frac{\partial L(x,t)}{\partial t} = d_L\Delta L(x,t) + (1-\beta)S(x,t)\left[\varkappa_1 V(x,t) + \varkappa_2 I(x,t)\right] - (\lambda+\gamma)L(x,t), \\ \frac{\partial I(x,t)}{\partial t} = d_I\Delta I(x,t) + \beta S(x,t)\left[\varkappa_1 V(x,t) + \varkappa_2 I(x,t)\right] + \lambda L(x,t) - aI(x,t) - \mu_1 C^I(x,t)I(x,t), \\ \frac{\partial E(x,t)}{\partial t} = d_E\Delta E(x,t) + \varphi\varkappa_3 S(x,t)Y(x,t) + \kappa r^* Y(x,t) - (\psi+\omega)E(x,t), \\ \frac{\partial Y(x,t)}{\partial t} = d_Y\Delta Y(x,t) + \psi E(x,t) + (1-\kappa)r^* Y(x,t) - \delta^* Y(x,t) - \mu_2 C^Y(x,t)Y(x,t), \\ \frac{\partial V(x,t)}{\partial t} = d_V\Delta V(x,t) + bI(x,t) - \varepsilon V(x,t), \\ \frac{\partial C^I(x,t)}{\partial t} = d_{C^I}\Delta C^I(x,t) + \sigma_1 C^I(x,t)I(x,t) - \pi_1 C^I(x,t), \\ \frac{\partial C^Y(x,t)}{\partial t} = d_{C^Y}\Delta C^Y(x,t) + \sigma_2 C^Y(x,t)Y(x,t) - \pi_2 C^Y(x,t), \end{cases} \tag{2.1}$$

where $x \in \Gamma$, $t > 0$. A fraction $\beta \in (0,1)$ of new HIV-infected cells will be active, and the remaining part $1-\beta$ will be latent. Latent HIV-infected cells are transmitted to be active at rate λL and die at rate γL. The term $\mu_1 C^I I$ is the killing rate of active HIV-infected cells due to their specific immunity. The term $\kappa r^* Y$, $\kappa \in (0,1)$ represents the rate active HTLV-infected cells that transmit to latent HTLV-infected cells and get-away from the immune system [45]. The expansion rate for HIV-specific CTLs and HTLV-specific CTLs are represented by $\sigma_1 C^I I$ and $\sigma_2 C^Y Y$, respectively. The HIV-specific CTLs decay at rate $\pi_1 C^I$. All remaining parameters have the same biological meaning as explained in Section 1. The spatial domain $\Gamma \subset \mathbb{R}^m$, $m \geq 1$ is connected and bounded with a smooth boundary $\partial\Gamma$, while $d_{\mathcal{U}}$ is the diffusion coefficient where $\mathcal{U} \in \{S, L, I, E, Y, V, C^I, C^Y\}$. The initial conditions are given by

$$\begin{aligned} &S(x,0) = \mathcal{G}_1(x), \quad L(x,0) = \mathcal{G}_2(x), \quad I(x,0) = \mathcal{G}_3(x), \quad E(x,0) = \mathcal{G}_4(x), \quad Y(x,0) = \mathcal{G}_5(x), \\ &V(x,0) = \mathcal{G}_6(x), \quad C^I(x,0) = \mathcal{G}_7(x), \quad C^Y(x,0) = \mathcal{G}_8(x), \quad x \in \bar{\Gamma}, \end{aligned} \tag{2.2}$$

where $\mathcal{G}_i(x)$, $i = 1, ..., 8$, are non-negative and continuous functions. In addition, we take the following homogeneous Neumann boundary conditions:

$$\frac{\partial S}{\partial \vec{V}} = \frac{\partial L}{\partial \vec{V}} = \frac{\partial I}{\partial \vec{V}} = \frac{\partial E}{\partial \vec{V}} = \frac{\partial Y}{\partial \vec{V}} = \frac{\partial V}{\partial \vec{V}} = \frac{\partial C^I}{\partial \vec{V}} = \frac{\partial C^Y}{\partial \vec{V}} = 0, \quad t > 0, \quad x \in \partial\Gamma, \tag{2.3}$$

where $\frac{\partial}{\partial \vec{V}}$ is the outward normal derivative on the boundary $\partial\Gamma$. These boundary conditions indicate that cells and viruses cannot cross the isolated boundary [54].

We assume that $r^* < \min\{\alpha, \omega, \delta^*\}$ [37]. It follows that $(1-\kappa)\, r^* < \delta^*$ and then

$$\delta^* - (1-\kappa)\, r^* > 0.$$

Let $\delta = \delta^* - (1-\kappa)\, r^*$ and $r = \kappa r^*$. Therefore, model (2.1) can be written as:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_S \Delta S(x,t) + \rho - \alpha S(x,t) - \varkappa_1 S(x,t)V(x,t) - \varkappa_2 S(x,t)I(x,t) - \varkappa_3 S(x,t)Y(x,t), \\ \frac{\partial L(x,t)}{\partial t} = d_L \Delta L(x,t) + (1-\beta)\, S(x,t)\left[\varkappa_1 V(x,t) + \varkappa_2 I(x,t)\right] - (\lambda+\gamma)\, L(x,t), \\ \frac{\partial I(x,t)}{\partial t} = d_I \Delta I(x,t) + \beta S(x,t)\left[\varkappa_1 V(x,t) + \varkappa_2 I(x,t)\right] + \lambda L(x,t) - aI(x,t) - \mu_1 C^I(x,t)I(x,t), \\ \frac{\partial E(x,t)}{\partial t} = d_E \Delta E(x,t) + \varphi\varkappa_3 S(x,t)Y(x,t) + rY(x,t) - (\psi+\omega)\, E(x,t), \\ \frac{\partial Y(x,t)}{\partial t} = d_Y \Delta Y(x,t) + \psi E(x,t) - \delta Y(x,t) - \mu_2 C^Y(x,t)Y(x,t), \\ \frac{\partial V(x,t)}{\partial t} = d_V \Delta V(x,t) + bI(x,t) - \varepsilon V(x,t), \\ \frac{\partial C^I(x,t)}{\partial t} = d_{C^I} \Delta C^I(x,t) + \sigma_1 C^I(x,t)I(x,t) - \pi_1 C^I(x,t), \\ \frac{\partial C^Y(x,t)}{\partial t} = d_{C^Y} \Delta C^Y(x,t) + \sigma_2 C^Y(x,t)Y(x,t) - \pi_2 C^Y(x,t). \end{cases} \tag{2.4}$$

3. Well-posedness of solutions

Proposition 1. Assume that $d_S = d_L = d_I = d_E = d_Y = d_V = d_{C^I} = d_{C^Y} = \tilde{d}$. Then, model (2.4) with any initial satisfying (2.2) has a unique, non-negative and bounded solution defined on $\bar{\Gamma} \times [0, +\infty)$.

Proof. We denote $\mathcal{X} = BUC\left(\bar{\Gamma}, \mathbb{R}^8\right)$ the set of all bounded and uniformly continuous functions from $\bar{\Gamma}$ to $\mathbb{R}^8$, with norm $\|\theta\|_{\mathcal{X}} = \sup_{x\in\bar{\Gamma}} |\theta(x)|$. Define the positive cone $\mathcal{X}_+ = BUC\left(\bar{\Gamma}, \mathbb{R}^8_+\right) \subset \mathcal{X}$ which induces a partial order on $\mathcal{X}$. This shows that the space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach lattice [55, 56].

For any initial data $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5, \mathcal{G}_6, \mathcal{G}_7, \mathcal{G}_8)^T \in \mathcal{X}_+$, we define $H = (H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8)^T : \mathcal{X}_+ \to \mathcal{X}$ by

$$\begin{aligned} H_1(\mathcal{G})(x) &= \rho - \alpha\mathcal{G}_1(x) - \varkappa_1\mathcal{G}_1(x)\mathcal{G}_6(x) - \varkappa_2\mathcal{G}_1(x)\mathcal{G}_3(x) - \varkappa_3\mathcal{G}_1(x)\mathcal{G}_5(x), \\ H_2(\mathcal{G})(x) &= (1-\beta)\,\mathcal{G}_1(x)\left[\varkappa_1\mathcal{G}_6(x) + \varkappa_2\mathcal{G}_3(x)\right] - (\lambda+\gamma)\,\mathcal{G}_2(x), \\ H_3(\mathcal{G})(x) &= \beta\mathcal{G}_1(x)\left[\varkappa_1\mathcal{G}_6(x) + \varkappa_2\mathcal{G}_3(x)\right] + \lambda\mathcal{G}_2(x) - a\mathcal{G}_3(x) - \mu_1\mathcal{G}_7(x)\mathcal{G}_3(x), \\ H_4(\mathcal{G})(x) &= \varphi\varkappa_3\mathcal{G}_1(x)\mathcal{G}_5(x) + r\mathcal{G}_5(x) - (\psi+\omega)\,\mathcal{G}_4(x), \\ H_5(\mathcal{G})(x) &= \psi\mathcal{G}_4(x) - \delta\mathcal{G}_5(x) - \mu_2\mathcal{G}_8(x)\mathcal{G}_5(x), \\ H_6(\mathcal{G})(x) &= b\mathcal{G}_3(x) - \varepsilon\mathcal{G}_6(x), \\ H_7(\mathcal{G})(x) &= \sigma_1\mathcal{G}_7(x)\mathcal{G}_3(x) - \pi_1\mathcal{G}_7(x), \\ H_8(\mathcal{G})(x) &= \sigma_2\mathcal{G}_8(x)\mathcal{G}_5(x) - \pi_2\mathcal{G}_8(x). \end{aligned}$$

We note that H is locally Lipschitz on $\mathcal{X}_+$. System (2.4) with initial conditions (2.2) and boundary conditions (2.3) can be rewritten as the following abstract functional differential equation

$$\begin{cases} \frac{d\bar{\mathcal{U}}}{dt} = \Theta\bar{\mathcal{U}} + H(\bar{\mathcal{U}}),\ t > 0, \\ \bar{\mathcal{U}}(0) = \mathcal{G} \in \mathcal{X}_+, \end{cases}$$

where $\bar{\mathcal{U}} = (S, L, I, E, Y, V, C^I, C^Y)^T$ and $\Theta\bar{\mathcal{U}} = \Big(d_S\Delta S, d_L\Delta L, d_I\Delta I, d_E\Delta E, d_Y\Delta Y, d_V\Delta V, d_{C^I}\Delta C^I,$ $d_{C^Y}\Delta C^Y\Big)^T$. It is possible to show that

$$\lim_{h\to 0^+} \frac{1}{h}\text{dist}\,(\mathcal{G}(0) + hH(\mathcal{G}), \mathcal{X}_+) = 0,\ \forall\, \mathcal{G} \in \mathcal{X}_+.$$

It follows from [55–57] that for any $\mathcal{G} \in \mathcal{X}_+$, system (2.4) with (2.2)-(2.3) has a unique non-negative mild solution $(S(x,t), L(x,t), I(x,t), E(x,t), Y(x,t), V(x,t), C^I(x,t), C^Y(x,t))$ defined on $\bar{\Gamma} \times [0, \mathcal{T}_m)$, where $[0, \mathcal{T}_m)$ is the maximal existence time interval on which the solution exists. In addition, this solution also is a classical solution for the given problem.

To prove the boundedness of solutions, we define

$$\Psi(x,t) = S(x,t) + L(x,t) + I(x,t) + \frac{1}{\varphi}[E(x,t) + Y(x,t)] + \frac{a}{2b}V(x,t) + \frac{\mu_1}{\sigma_1}C^I(x,t) + \frac{\mu_2}{\varphi\sigma_2}C^Y(x,t).$$

Since $d_S = d_L = d_I = d_E = d_Y = d_V = d_{C^I} = d_{C^Y} = \tilde{d}$, then using system (2.4) we obtain

$$\begin{aligned}\frac{\partial\Psi(x,t)}{\partial t} - \tilde{d}\Delta\Psi(x,t) = \rho - \alpha S(x,t) - \gamma L(x,t) - \frac{a}{2}I(x,t) - \frac{\omega}{\varphi}E(x,t) \\ - \frac{\delta - r}{\varphi}Y(x,t) - \frac{a\varepsilon}{2b}V(x,t) - \frac{\mu_1\pi_1}{\sigma_1}C^I(x,t) - \frac{\mu_2\pi_2}{\varphi\sigma_2}C^Y(x,t).\end{aligned}$$

We have $\delta - r = \delta^* - r^* > 0$. Hence,

$$\begin{aligned}\frac{\partial\Psi(x,t)}{\partial t} - \tilde{d}\Delta\Psi(x,t) &= \rho - \alpha S(x,t) - \gamma L(x,t) - \frac{a}{2}I(x,t) - \frac{\omega}{\varphi}E(x,t) \\ &\quad - \frac{\delta^* - r^*}{\varphi}Y(x,t) - \frac{a\varepsilon}{2b}V(x,t) - \frac{\mu_1\pi_1}{\sigma_1}C^I(x,t) - \frac{\mu_2\pi_2}{\varphi\sigma_2}C^Y(x,t) \\ &\le \rho - \phi\Bigg[S(x,t) + L(x,t) + I(x,t) + \frac{1}{\varphi}\{E(x,t) + Y(x,t)\} \\ &\quad + \frac{a}{2b}V(x,t) + \frac{\mu_1}{\sigma_1}C^I(x,t) + \frac{\mu_2}{\varphi\sigma_2}C^Y(x,t)\Bigg] = \rho - \phi\Psi(x,t),\end{aligned}$$

where $\phi = \min\{\alpha, \gamma, \frac{a}{2}, \omega, \delta^* - r^*, \varepsilon, \pi_1, \pi_2\}$. Thus, $\Psi(x,t)$ satisfies the following system

$$\begin{cases} \frac{\partial\Psi(x,t)}{\partial t} - \tilde{d}\Delta\Psi(x,t) \le \rho - \phi\Psi(x,t), \\ \Psi(x,0) = \mathcal{G}_1(x) + \mathcal{G}_2(x) + \mathcal{G}_3(x) + \frac{1}{\varphi}[\mathcal{G}_4(x) + \mathcal{G}_5(x)] + \frac{a}{2b}\mathcal{G}_6(x) + \frac{\mu_1}{\sigma_1}\mathcal{G}_7(x) + \frac{\mu_2}{\varphi\sigma_2}\mathcal{G}_8(x) \ge 0, \\ \dfrac{\partial\Psi}{\partial\vec{\mathcal{V}}} = 0. \end{cases}$$

Let $\widetilde{\Psi}(t)$ be a solution of the following ODE

$$\begin{cases} \frac{d\widetilde{\Psi}(t)}{dt} = \rho - \phi\widetilde{\Psi}(t), \\ \widetilde{\Psi}(0) = \max_{x\in\bar{\Gamma}}\Psi(x,0). \end{cases}$$

This gives that $\widetilde{\Psi}(t) \leq \max\left\{\frac{\rho}{\phi}, \max_{x\in\bar{\Gamma}}\Psi(x,0)\right\}$. On the basis of comparison principle (see [58]), we obtain $\Psi(x,t) \leq \widetilde{\Psi}(t)$. Then, we get

$$\Psi(x,t) \leq \max\left\{\frac{\rho}{\phi}, \max_{x\in\bar{\Gamma}}\Psi(x,0)\right\},$$

which implies that $S(x,t), L(x,t), I(x,t), E(x,t), Y(x,t), V(x,t), C^I(x,t)$, and $C^Y(x,t)$ are bounded on $\bar{\Gamma} \times [0, \mathcal{T}_m)$. We deduce from the standard theory for semi-linear parabolic systems that $\mathcal{T}_m = +\infty$ [59]. This shows that solution $\left(S(x,t), L(x,t), I(x,t), E(x,t), Y(x,t), V(x,t), C^I(x,t), C^Y(x,t)\right)$ is defined for all $x \in \Gamma$, $t > 0$ and also is unique and non-negative.

4. Steady state analysis

In this section, we calculate the steady states and derive the threshold parameters which guarantee their existence. The steady states of system (2.4) satisfying the following equations:

$$0 = \rho - \alpha S - \varkappa_1 SV - \varkappa_2 SI - \varkappa_3 SY, \tag{4.1}$$
$$0 = (1-\beta)(\varkappa_1 SV + \varkappa_2 SI) - (\lambda + \gamma) L, \tag{4.2}$$
$$0 = \beta(\varkappa_1 SV + \varkappa_2 SI) + \lambda L - aI - \mu_1 C^I I, \tag{4.3}$$
$$0 = \varphi\varkappa_3 SY + rY - (\psi + \omega) E, \tag{4.4}$$
$$0 = \psi E - \delta Y - \mu_2 C^Y Y, \tag{4.5}$$
$$0 = bI - \varepsilon V, \tag{4.6}$$
$$0 = (\sigma_1 I - \pi_1) C^I, \tag{4.7}$$
$$0 = (\sigma_2 Y - \pi_2) C^Y. \tag{4.8}$$

We find that system (2.4) has eight possible steady states.

(i) Infection-free steady state, $Đ_0 = (S_0, 0, 0, 0, 0, 0, 0, 0)$, where $S_0 = \rho/\alpha$. In this case the body is free from HIV and HTLV.

(ii) Persistent HIV single infection steady state with an ineffective immune response, $Đ_1 = (S_1, L_1, I_1, 0, 0, V_1, 0, 0)$, where

$$S_1 = \frac{S_0}{\Re_1},\ L_1 = \frac{a\varepsilon\alpha(1-\beta)}{(\beta\gamma+\lambda)(\varkappa_1 b + \varkappa_2\varepsilon)}(\Re_1 - 1),\ I_1 = \frac{\varepsilon\alpha}{\varkappa_1 b + \varkappa_2\varepsilon}(\Re_1 - 1),\ V_1 = \frac{\alpha b}{\varkappa_1 b + \varkappa_2\varepsilon}(\Re_1 - 1).$$

The parameter $\Re_1$ represents the basic HIV single infection reproduction number for system (2.4) and is defined as:

$$\Re_1 = \frac{S_0\varkappa_1 b(\beta\gamma+\lambda)}{a\varepsilon(\gamma+\lambda)} + \frac{S_0\varkappa_2(\beta\gamma+\lambda)}{a(\gamma+\lambda)}.$$

The parameter $\Re_1$ decides whether or not a persistent HIV single infection can be established. It is clear that at the steady state $Đ_1$ the HIV single infection persists with ineffective immune response.

(iii) Persistent HTLV single infection steady state with an ineffective immune response, $Đ_2 = (S_2, 0, 0, E_2, Y_2, 0, 0, 0)$, where

$$S_2 = \frac{S_0}{\Re_2}, \quad E_2 = \frac{\alpha\delta}{\varkappa_3\psi}(\Re_2 - 1), \quad Y_2 = \frac{\alpha}{\varkappa_3}(\Re_2 - 1).$$

The parameter $\Re_2$ denotes the basic HTLV single infection reproduction number for system (2.4) and is defined as:

$$\Re_2 = \frac{\varphi\varkappa_3\psi S_0}{(\delta - r)\psi + \delta\omega}.$$

The parameter $\Re_2$ decides whether or not a persistent HTLV single infection can be established. At the steady state $Đ_2$ the HTLV single infection persists with an ineffective immune response.

(iv) Persistent HIV single infection steady state with only effective HIV-specific CTL, $Đ_3 = (S_3, L_3, I_3, 0, 0, V_3, C_3^I, 0)$, where

$$S_3 = \frac{\varepsilon\sigma_1\rho}{\pi_1(\varkappa_1 b + \varkappa_2\varepsilon) + \alpha\varepsilon\sigma_1}, \quad L_3 = \frac{\rho\pi_1(1-\beta)(\varkappa_1 b + \varkappa_2\varepsilon)}{(\gamma+\lambda)[\pi_1(\varkappa_1 b + \varkappa_2\varepsilon) + \alpha\varepsilon\sigma_1]},$$
$$I_3 = \frac{\pi_1}{\sigma_1}, \quad V_3 = \frac{b}{\varepsilon}I_3 = \frac{b\pi_1}{\varepsilon\sigma_1}, \quad C_3^I = \frac{a}{\mu_1}(\Re_3 - 1),$$

and

$$\Re_3 = \frac{\sigma_1\rho(\beta\gamma+\lambda)(\varkappa_1 b + \varkappa_2\varepsilon)}{a(\gamma+\lambda)[\pi_1(\varkappa_1 b + \varkappa_2\varepsilon) + \alpha\varepsilon\sigma_1]},$$

is the HIV-specific CTL immunity reproduction number in case of HIV single infection. The parameter $\Re_3$ determines whether or not the HIV-specific CTL immune response is effective in the absence of HTLV. At the steady state $Đ_3$ the HIV single infection persists with an effective immune response.

(v) Persistent HTLV single infection steady state with only effective HTLV-specific CTL, $Đ_4 = (S_4, 0, 0, E_4, Y_4, 0, 0, C_4^Y)$, where

$$S_4 = \frac{\sigma_2\rho}{\pi_2\varkappa_3 + \alpha\sigma_2}, \quad E_4 = \frac{\pi_2[r(\pi_2\varkappa_3 + \alpha\sigma_2) + \varkappa_3\rho\varphi\sigma_2]}{\sigma_2(\psi+\omega)(\pi_2\varkappa_3 + \alpha\sigma_2)},$$
$$Y_4 = \frac{\pi_2}{\sigma_2}, \quad C_4^Y = \frac{(\delta - r)\psi + \delta\omega}{\mu_2(\psi+\omega)}(\Re_4 - 1).$$

The HTLV-specific CTL immunity reproduction number in case of HTLV single infection is stated as:

$$\Re_4 = \frac{\psi\sigma_2\rho\varphi\varkappa_3}{[(\delta - r)\psi + \delta\omega](\pi_2\varkappa_3 + \alpha\sigma_2)}.$$

The parameter $\Re_4$ determines whether or not the HTLV-specific CTL immune response is effective in the absence of HIV. At the steady state $Đ_4$ the HTLV single infection persists with an effective immune response.

(vi) Persistent HTLV/HIV dual infection steady state with only effective HIV-specific CTL, $Đ_5 = (S_5, L_5, I_5, E_5, Y_5, V_5, C_5^I, 0)$, where

$$S_5 = \frac{(\delta - r)\psi + \delta\omega}{\varphi\varkappa_3\psi} = S_2, \quad I_5 = \frac{\pi_1}{\sigma_1} = I_3, \quad V_5 = \frac{b\pi_1}{\varepsilon\sigma_1} = V_3,$$

$$L_5 = \frac{\pi_1 (1-\beta)(\varkappa_1 b + \varkappa_2 \varepsilon)\left[(\delta - r)\psi + \delta\omega\right]}{\varepsilon \varkappa_3 \sigma_1 \varphi \psi (\gamma + \lambda)}, \quad E_5 = \frac{\delta\left[\pi_1 (\varkappa_1 b + \varkappa_2 \varepsilon) + \alpha \varepsilon \sigma_1\right]}{\varepsilon \varkappa_3 \sigma_1 \psi} (\Re_5 - 1),$$
$$Y_5 = \frac{\pi_1 (\varkappa_1 b + \varkappa_2 \varepsilon) + \alpha \varepsilon \sigma_1}{\varepsilon \varkappa_3 \sigma_1} (\Re_5 - 1), \quad C_5^I = \frac{a}{\mu_1} (\Re_1/\Re_2 - 1).$$

The HTLV infection reproduction number in the presence of HIV infection is stated as:

$$\Re_5 = \frac{\rho \varphi \varepsilon \varkappa_3 \sigma_1 \psi}{\left[(\delta - r)\psi + \delta\omega\right]\left[\pi_1 (\varkappa_1 b + \varkappa_2 \varepsilon) + \alpha \varepsilon \sigma_1\right]}.$$

It is obvious that the parameter $\Re_5$ determines whether or not HIV-infected patients could be dually infected with HTLV. At the steady state $Đ_5$ the HTLV/HIV dual infection persists with only HIV-specific CTL immune response.

(vii) Persistent HTLV/HIV dual infection steady state with only effective HTLV-specific CTL, $Đ_6 = (S_6, L_6, I_6, E_6, Y_6, V_6, 0, C_6^Y)$, where

$$S_6 = \frac{a\varepsilon (\gamma + \lambda)}{(\beta\gamma + \lambda)(\varkappa_1 b + \varkappa_2 \varepsilon)} = S_1, \qquad Y_6 = \frac{\pi_2}{\sigma_2} = Y_4,$$
$$L_6 = \frac{a\varepsilon (1-\beta)(\pi_2 \varkappa_3 + \alpha \sigma_2)}{\sigma_2 (\beta\gamma + \lambda)(\varkappa_1 b + \varkappa_2 \varepsilon)} (\Re_6 - 1), \quad I_6 = \frac{\varepsilon (\pi_2 \varkappa_3 + \alpha \sigma_2)}{\sigma_2 (\varkappa_1 b + \varkappa_2 \varepsilon)} (\Re_6 - 1),$$
$$E_6 = \frac{\pi_2 \left[r (\beta\gamma + \lambda)(\varkappa_1 b + \varkappa_2 \varepsilon) + a\varepsilon\varphi\varkappa_3 (\gamma + \lambda)\right]}{\sigma_2 (\beta\gamma + \lambda)(\psi + \omega)(\varkappa_1 b + \varkappa_2 \varepsilon)},$$
$$V_6 = \frac{b (\pi_2 \varkappa_3 + \alpha \sigma_2)}{\sigma_2 (\varkappa_1 b + \varkappa_2 \varepsilon)} (\Re_6 - 1), \qquad C_6^Y = \frac{(\delta - r)\psi + \delta\omega}{\mu_2 (\psi + \omega)} (\Re_2/\Re_1 - 1).$$

The HIV infection reproduction number in the presence of HTLV infection is stated as:

$$\Re_6 = \frac{\rho \sigma_2 (\beta\gamma + \lambda)(\varkappa_1 b + \varkappa_2 \varepsilon)}{a\varepsilon (\gamma + \lambda)(\pi_2 \varkappa_3 + \alpha \sigma_2)}.$$

It is clear that the parameter $\Re_6$ determines whether or not HTLV-infected patients could be dually infected with HIV. At the steady state $Đ_6$ the HTLV/HIV dual infection persists with only HTLV-specific CTL immune response.

(viii) Persistent HTLV/HIV dual infection steady state with effective HIV-specific CTL and HTLV-specific CTL, $Đ_7 = (S_7, L_7, I_7, E_7, Y_7, V_7, C_7^I, C_7^Y)$, where

$$S_7 = \frac{\varepsilon \sigma_1 \sigma_2 \rho}{\pi_1 \sigma_2 (\varkappa_1 b + \varkappa_2 \varepsilon) + \pi_2 \varkappa_3 \varepsilon \sigma_1 + \alpha \varepsilon \sigma_1 \sigma_2},$$
$$L_7 = \frac{\pi_1 \sigma_2 \rho (1-\beta)(\varkappa_1 b + \varkappa_2 \varepsilon)}{(\gamma + \lambda)\left[\pi_1 \sigma_2 (\varkappa_1 b + \varkappa_2 \varepsilon) + \pi_2 \varkappa_3 \varepsilon \sigma_1 + \alpha \varepsilon \sigma_1 \sigma_2\right]},$$
$$E_7 = \frac{\pi_2 \left[\pi_1 r \sigma_2 (\varkappa_1 b + \varkappa_2 \varepsilon) + r\varepsilon\sigma_1 (\pi_2 \varkappa_3 + \alpha \sigma_2) + \varkappa_3 \varepsilon \sigma_1 \sigma_2 \rho \varphi\right]}{\sigma_2 (\psi + \omega)\left[\pi_1 \sigma_2 (\varkappa_1 b + \varkappa_2 \varepsilon) + \pi_2 \varkappa_3 \varepsilon \sigma_1 + \alpha \varepsilon \sigma_1 \sigma_2\right]},$$
$$I_7 = \frac{\pi_1}{\sigma_1} = I_3 = I_5, \quad Y_7 = \frac{\pi_2}{\sigma_2} = Y_4 = Y_6, \quad V_7 = \frac{b\pi_1}{\varepsilon \sigma_1} = V_3 = V_5,$$
$$C_7^I = \frac{a}{\mu_1} (\Re_7 - 1), \quad C_7^Y = \frac{(\delta - r)\psi + \delta\omega}{\mu_2(\psi + \omega)} (\Re_8 - 1),$$

and

$$\Re_7 = \frac{\sigma_1\sigma_2\rho\,(\beta\gamma+\lambda)\,(\varkappa_1 b+\varkappa_2\varepsilon)}{a\,(\gamma+\lambda)\,[\pi_1\sigma_2\,(\varkappa_1 b+\varkappa_2\varepsilon)+\pi_2\varkappa_3\varepsilon\sigma_1+\alpha\varepsilon\sigma_1\sigma_2]},$$
$$\Re_8 = \frac{\varphi\varkappa_3\varepsilon\sigma_1\sigma_2\rho\psi}{[(\delta-r)\psi+\delta\omega]\,[\pi_1\sigma_2\,(\varkappa_1 b+\varkappa_2\varepsilon)+\pi_2\varkappa_3\varepsilon\sigma_1+\alpha\varepsilon\sigma_1\sigma_2]}.$$

The parameter $\Re_7$ is the competed HIV-specific CTL immunity reproduction number in case of HTLV/HIV dual infection. The parameter $\Re_8$ is the competed HTLV-specific CTL immunity reproduction number in case of HTLV/HIV dual infection. Clearly, $Đ_7$ exists when $\Re_7 > 1$ and $\Re_8 > 1$.

5. Global stability analysis

In this section, we analyze the global asymptotic stability of all steady states by Lyapunov method. For constructing Lyapunov functions we follow the works in [60,61]. To prove Theorems 1–8 we need the arithmetic-geometric mean inequality

$$\frac{1}{n}\sum_{i=1}^{n}\chi_i \geq \sqrt[n]{\prod_{i=1}^{n}\chi_i}, \quad \chi_i \geq 0,\ i = 1,2,...$$

which yields

$$\frac{S_j}{S} + \frac{SIL_j}{S_jI_jL} + \frac{LI_j}{L_jI} \geq 3, \quad j = 1,3,5,6,7, \tag{5.1}$$
$$\frac{S_j}{S} + \frac{SVI_j}{S_jV_jI} + \frac{IV_j}{I_jV} \geq 3, \quad j = 1,3,5,6,7, \tag{5.2}$$
$$\frac{S_j}{S} + \frac{SVL_j}{S_jV_jL} + \frac{LI_j}{L_jI} + \frac{IV_j}{I_jV} \geq 4, \quad j = 1,3,5,6,7, \tag{5.3}$$
$$\frac{S_j}{S} + \frac{SYE_j}{S_jY_jE} + \frac{EY_j}{E_jY} \geq 3, \quad j = 2,4,5,6,7. \tag{5.4}$$

Consider a function $\Phi_j(S, L, I, E, Y, V, C^I, C^Y)$ and define

$$\hat{\Phi}_j(t) = \int_\Gamma \Phi_j(x,t)\,\mathrm{d}x, \quad j = 0,1,...,7.$$

Let Υ_j' be the largest invariant subset of

$$\Upsilon_j = \left\{(S, L, I, E, Y, V, C^I, C^Y) : \frac{d\hat{\Phi}_j}{dt} = 0\right\}, \quad j = 0,1,2,...,7.$$

We define a function

$$F(\upsilon) = \upsilon - 1 - \ln\upsilon.$$

Numann boundary conditions (2.3) and Divergence Theorem imply that

$$0 = \int_{\partial\Gamma} \nabla\mathcal{U}\cdot\vec{\mathcal{V}}\,\mathrm{d}x = \int_\Gamma \mathrm{div}(\nabla\mathcal{U})\,\mathrm{d}x = \int_\Gamma \Delta\mathcal{U}\,\mathrm{d}x,$$

$$0 = \int_{\partial\Gamma} \frac{1}{\mathcal{U}} \nabla\mathcal{U} \cdot \vec{\mathcal{V}}\, \mathrm{d}x = \int_{\Gamma} \mathrm{div}(\frac{1}{\mathcal{U}}\nabla\mathcal{U})\, \mathrm{d}x = \int_{\Gamma} \left(\frac{\Delta\mathcal{U}}{\mathcal{U}} - \frac{\|\nabla\mathcal{U}\|^2}{\mathcal{U}^2} \right) \mathrm{d}x,$$

for $\mathcal{U} \in \{S, L, I, E, Y, V, C^I, C^Y\}$. Thus, we obtain

$$\begin{aligned} &\int_{\Gamma} \Delta\mathcal{U}\, \mathrm{d}x = 0, \\ &\int_{\Gamma} \frac{\Delta\mathcal{U}}{\mathcal{U}}\, \mathrm{d}x = \int_{\Gamma} \frac{\|\nabla\mathcal{U}\|^2}{\mathcal{U}^2}\, \mathrm{d}x. \end{aligned} \tag{5.5}$$

For convenience, we drop the input notation i.e., $(S, L, I, E, Y, V, C^I, C^Y) = (S(x,t), L(x,t), I(x,t), E(x,t), Y(x,t), V(x,t), C^I(x,t), C^Y(x,t))$.

Theorem 1. If $\mathfrak{R}_1 \leq 1$ and $\mathfrak{R}_2 \leq 1$, then $Đ_0$ is globally asymptotically stable (GAS).

Proof. Define $\Phi_0(x,t)$ as:

$$\begin{aligned} \Phi_0(x,t) = {} & S_0 F\left(\frac{S}{S_0}\right) + \frac{\lambda}{\beta\gamma + \lambda} L + \frac{\gamma + \lambda}{\beta\gamma + \lambda} I + \frac{1}{\varphi} E + \frac{\psi + \omega}{\varphi\psi} Y \\ & + \frac{\varkappa_1 S_0}{\varepsilon} V + \frac{\mu_1 (\gamma + \lambda)}{\sigma_1 (\beta\gamma + \lambda)} C^I + \frac{\mu_2 (\psi + \omega)}{\varphi\psi\sigma_2} C^Y. \end{aligned}$$

Clearly, $\hat{\Phi}_0(S, L, I, E, Y, V, C^I, C^Y) > 0$ for all $S, L, I, E, Y, V, C^I, C^Y > 0$, and $\hat{\Phi}_0(S_0, 0, 0, 0, 0, 0, 0, 0) = 0$. We calculate $\frac{\partial\Phi_0}{\partial t}$ along the solutions of model (2.4) as:

$$\begin{aligned} \frac{\partial\Phi_0}{\partial t} = {} & \left(1 - \frac{S_0}{S}\right) [d_S \Delta S + \rho - \alpha S - \varkappa_1 S V - \varkappa_2 S I - \varkappa_3 S Y] \\ & + \frac{\lambda}{\beta\gamma + \lambda} [d_L \Delta L + (1 - \beta)(\varkappa_1 S V + \varkappa_2 S I) - (\lambda + \gamma) L] \\ & + \frac{\gamma + \lambda}{\beta\gamma + \lambda} \left[d_I \Delta I + \beta(\varkappa_1 S V + \varkappa_2 S I) + \lambda L - aI - \mu_1 C^I I\right] \\ & + \frac{1}{\varphi} [d_E \Delta E + \varphi\varkappa_3 S Y + rY - (\psi + \omega) E] \\ & + \frac{\psi + \omega}{\varphi\psi} \left[d_Y \Delta Y + \psi E - \delta Y - \mu_2 C^Y Y\right] + \frac{\varkappa_1 S_0}{\varepsilon} (d_V \Delta V + bI - \varepsilon V) \\ & + \frac{\mu_1 (\gamma + \lambda)}{\sigma_1 (\beta\gamma + \lambda)} \left[d_{C^I} \Delta C^I + \sigma_1 C^I I - \pi_1 C^I\right] + \frac{\mu_2 (\psi + \omega)}{\varphi\psi\sigma_2} \left[d_{C^Y} \Delta C^Y + \sigma_2 C^Y Y - \pi_2 C^Y\right] \\ = {} & \left(1 - \frac{S_0}{S}\right)(\rho - \alpha S) + \varkappa_2 S_0 I + \varkappa_3 S_0 Y - \frac{a(\gamma + \lambda)}{\beta\gamma + \lambda} I + \frac{r}{\varphi} Y - \frac{\delta(\psi + \omega)}{\varphi\psi} Y \\ & + \frac{\varkappa_1 b S_0}{\varepsilon} I - \frac{\mu_1 \pi_1 (\gamma + \lambda)}{\sigma_1 (\beta\gamma + \lambda)} C^I - \frac{\mu_2 \pi_2 (\psi + \omega)}{\varphi\psi\sigma_2} C^Y + d_S \left(1 - \frac{S_0}{S}\right) \Delta S \\ & + \frac{\lambda d_L}{\beta\gamma + \lambda} \Delta L + \frac{d_I (\gamma + \lambda)}{\beta\gamma + \lambda} \Delta I + \frac{d_E}{\varphi} \Delta E + \frac{d_Y (\psi + \omega)}{\varphi\psi} \Delta Y + \frac{\varkappa_1 d_V S_0}{\varepsilon} \Delta V \\ & + \frac{\mu_1 d_{C^I} (\gamma + \lambda)}{\sigma_1 (\beta\gamma + \lambda)} \Delta C^I + \frac{\mu_2 d_{C^Y} (\psi + \omega)}{\varphi\psi\sigma_2} \Delta C^Y. \end{aligned}$$

Using $S_0 = \rho/\alpha$, we obtain

$$\frac{\partial\Phi_0}{\partial t} = -\alpha \frac{(S - S_0)^2}{S} + \frac{a(\gamma + \lambda)}{\beta\gamma + \lambda} (\mathfrak{R}_1 - 1) I + \frac{(\delta - r)\psi + \delta\omega}{\varphi\psi} (\mathfrak{R}_2 - 1) Y$$

$$- \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I - \frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y + d_S\left(1-\frac{S_0}{S}\right)\Delta S$$
$$+ \frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L + \frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I + \frac{d_E}{\varphi}\Delta E + \frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y$$
$$+ \frac{\varkappa_1 d_V S_0}{\varepsilon}\Delta V + \frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I + \frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y.$$

Consequently, we calculate $\frac{d\hat{\Phi}_0}{dt}$ as follows:

$$\begin{aligned}\frac{d\hat{\Phi}_0}{dt} &= -\alpha\int_\Gamma \frac{(S-S_0)^2}{S}\,\mathrm{d}x + \frac{a(\gamma+\lambda)(\Re_1-1)}{\beta\gamma+\lambda}\int_\Gamma I\,\mathrm{d}x + \frac{[(\delta-r)\psi+\delta\omega](\Re_2-1)}{\varphi\psi}\int_\Gamma Y\,\mathrm{d}x\\ &- \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\int_\Gamma C^I\,\mathrm{d}x - \frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}\int_\Gamma C^Y\,\mathrm{d}x + d_S\int_\Gamma\left(1-\frac{S_0}{S}\right)\Delta S\,\mathrm{d}x\\ &+ \frac{\lambda d_L}{\beta\gamma+\lambda}\int_\Gamma \Delta L\,\mathrm{d}x + \frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\int_\Gamma \Delta I\,\mathrm{d}x + \frac{d_E}{\varphi}\int_\Gamma \Delta E + \frac{d_Y(\psi+\omega)}{\varphi\psi}\int_\Gamma \Delta Y\,\mathrm{d}x\\ &+ \frac{\varkappa_1 d_V S_0}{\varepsilon}\int_\Gamma \Delta V\,\mathrm{d}x + \frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\int_\Gamma \Delta C^I\,\mathrm{d}x + \frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\int_\Gamma \Delta C^Y\,\mathrm{d}x.\end{aligned} \tag{5.6}$$

Using equality (5.5), Eq (5.6) is reduced to the following form

$$\begin{aligned}\frac{d\hat{\Phi}_0}{dt} &= -\alpha\int_\Gamma \frac{(S-S_0)^2}{S}\,\mathrm{d}x + \frac{a(\gamma+\lambda)(\Re_1-1)}{\beta\gamma+\lambda}\int_\Gamma I\,\mathrm{d}x + \frac{[(\delta-r)\psi+\delta\omega](\Re_2-1)}{\varphi\psi}\int_\Gamma Y\,\mathrm{d}x\\ &- \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\int_\Gamma C^I\,\mathrm{d}x - \frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}\int_\Gamma C^Y\,\mathrm{d}x - d_S S_0\int_\Gamma \frac{\|\nabla S\|^2}{S^2}\,\mathrm{d}x.\end{aligned}$$

Therefore, $\frac{d\hat{\Phi}_0}{dt} \leq 0$ for all $S, I, Y, C^I, C^Y > 0$ and $\frac{d\hat{\Phi}_0}{dt} = 0$ with equality holding when $(S, I, Y, C^I, C^Y) = (S_0, 0, 0, 0, 0)$. The solutions of system (2.4) converge to Υ_0'. The elements of Υ_0' satisfy $(S, I, Y, C^I, C^Y) = (S_0, 0, 0, 0, 0)$ and then $\frac{\partial S}{\partial t} = \frac{\partial Y}{\partial t} = \Delta S = \Delta Y = 0$. The first and fifth equations of system (2.4) reduce to

$$0 = \frac{\partial S}{\partial t} = \rho - \alpha S_0 - \varkappa_1 S_0 V,$$
$$0 = \frac{\partial Y}{\partial t} = \psi E.$$

This yields $V = E = 0$. Further, we have $\frac{\partial I}{\partial t} = \Delta I = 0$, then the third equation of system (2.4) becomes

$$0 = \frac{\partial I}{\partial t} = \lambda L,$$

which provides that $L = 0$. Hence, $\Upsilon_0' = \{Đ_0\}$ and by applying Lyapunov-LaSalle asymptotic stability theorem [62–64] we get that $Đ_0$ is GAS.

Theorem 2. Let $\Re_1 > 1$, $\Re_2/\Re_1 \leq 1$ and $\Re_3 \leq 1$, then $Đ_1$ is GAS.

Proof. Define a function $\Phi_1(x,t)$ as:

$$\Phi_1(x,t) = S_1 F\left(\frac{S}{S_1}\right) + \frac{\lambda}{\beta\gamma+\lambda}L_1 F\left(\frac{L}{L_1}\right) + \frac{\gamma+\lambda}{\beta\gamma+\lambda}I_1 F\left(\frac{I}{I_1}\right) + \frac{1}{\varphi}E$$

$$+\frac{\psi+\omega}{\varphi\psi}Y+\frac{\varkappa_1 S_1}{\varepsilon}V_1 F\left(\frac{V}{V_1}\right)+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y.$$

Calculating $\frac{\partial\Phi_1}{\partial t}$ as:

$$\begin{aligned}\frac{\partial\Phi_1}{\partial t}&=\left(1-\frac{S_1}{S}\right)[d_S\Delta S+\rho-\alpha S-\varkappa_1 SV-\varkappa_2 SI-\varkappa_3 SY]\\&+\frac{\lambda}{\beta\gamma+\lambda}\left(1-\frac{L_1}{L}\right)[d_L\Delta L+(1-\beta)(\varkappa_1 SV+\varkappa_2 SI)-(\lambda+\gamma)L]\\&+\frac{\gamma+\lambda}{\beta\gamma+\lambda}\left(1-\frac{I_1}{I}\right)\left[d_I\Delta I+\beta(\varkappa_1 SV+\varkappa_2 SI)+\lambda L-aI-\mu_1 C^I I\right]\\&+\frac{1}{\varphi}[d_E\Delta E+\varphi\varkappa_3 SY+rY-(\psi+\omega)E]+\frac{\psi+\omega}{\varphi\psi}\left[d_Y\Delta Y+\psi E-\delta Y-\mu_2 C^Y Y\right]\\&+\frac{\varkappa_1 S_1}{\varepsilon}\left(1-\frac{V_1}{V}\right)[d_V\Delta V+bI-\varepsilon V]+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left[d_{C^I}\Delta C^I+\sigma_1 C^I I-\pi_1 C^I\right]\\&+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}\left[d_{C^Y}\Delta C^Y+\sigma_2 C^Y Y-\pi_2 C^Y\right]\\&=\left(1-\frac{S_1}{S}\right)(\rho-\alpha S)+\varkappa_2 S_1 I+\varkappa_3 S_1 Y-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 SV+\varkappa_2 SI)\frac{L_1}{L}\\&+\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_1-\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}(\varkappa_1 SV+\varkappa_2 SI)\frac{I_1}{I}-\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L\frac{I_1}{I}\\&+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_1+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C^I I_1+\frac{r}{\varphi}Y-\frac{\delta(\psi+\omega)}{\varphi\psi}Y+\varkappa_1 S_1\frac{bI}{\varepsilon}-\varkappa_1 S_1\frac{bI}{\varepsilon}\frac{V_1}{V}\\&+\varkappa_1 S_1 V_1-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y+d_S\left(1-\frac{S_1}{S}\right)\Delta S\\&+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_1}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_1}{I}\right)\Delta I+\frac{d_E}{\varphi}\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y\\&+\frac{d_V\varkappa_1 S_1}{\varepsilon}\left(1-\frac{V_1}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y.\end{aligned}$$

The steady state conditions of Đ$_1$ imply that

$$\begin{aligned}&\rho=\alpha S_1+\varkappa_1 S_1 V_1+\varkappa_2 S_1 I_1,\quad \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 S_1 V_1+\varkappa_2 S_1 I_1)=\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_1,\\&\varkappa_1 S_1 V_1+\varkappa_2 S_1 I_1=\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_1,\quad V_1=\frac{bI_1}{\varepsilon}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\frac{\partial\Phi_1}{\partial t}&=\left(1-\frac{S_1}{S}\right)(\alpha S_1-\alpha S)+(\varkappa_1 S_1 V_1+\varkappa_2 S_1 I_1)\left(1-\frac{S_1}{S}\right)+\varkappa_3 S_1 Y-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_1 V_1\frac{SVL_1}{S_1 V_1 L}\\&-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_1 I_1\frac{SIL_1}{S_1 I_1 L}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 S_1 V_1+\varkappa_2 S_1 I_1)-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_1 V_1\frac{SVI_1}{S_1 V_1 I}\\&-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2 S_1 I_1\frac{S}{S_1}-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 S_1 V_1+\varkappa_2 S_1 I_1)\frac{LI_1}{L_1 I}+\varkappa_1 S_1 V_1+\varkappa_2 S_1 I_1+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C^I I_1\end{aligned}$$

$$
\begin{aligned}
&-\frac{(\delta-r)\psi+\delta\omega}{\varphi\psi}Y-\varkappa_1 S_1 V_1\frac{IV_1}{I_1V}+\varkappa_1 S_1 V_1-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y\\
&+d_S\left(1-\frac{S_1}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_1}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_1}{I}\right)\Delta I+\frac{d_E}{\varphi}\Delta E\\
&+\frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y+\frac{d_V\varkappa_1 S_1}{\varepsilon}\left(1-\frac{V_1}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y\\
&=-\alpha\frac{(S-S_1)^2}{S}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_1 V_1\left(4-\frac{S_1}{S}-\frac{SVL_1}{S_1V_1L}-\frac{LI_1}{L_1I}-\frac{IV_1}{I_1V}\right)\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_1 I_1\left(3-\frac{S_1}{S}-\frac{SIL_1}{S_1I_1L}-\frac{LI_1}{L_1I}\right)+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_1 V_1\left(3-\frac{S_1}{S}-\frac{SVI_1}{S_1V_1I}-\frac{IV_1}{I_1V}\right)\\
&+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2 S_1 I_1\left(2-\frac{S_1}{S}-\frac{S}{S_1}\right)+\frac{(\delta-r)\psi+\delta\omega}{\varphi\psi}\left[\frac{\varphi\varkappa_3\psi S_1}{(\delta-r)\psi+\delta\omega}-1\right]Y\\
&+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}\left(I_1-\frac{\pi_1}{\sigma_1}\right)C^I-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y+d_S\left(1-\frac{S_1}{S}\right)\Delta S\\
&+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_1}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_1}{I}\right)\Delta I+\frac{d_E}{\varphi}\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y\\
&+\frac{d_V\varkappa_1 S_1}{\varepsilon}\left(1-\frac{V_1}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y. \qquad (5.7)
\end{aligned}
$$

Therefore, Eq (5.7) becomes

$$
\begin{aligned}
\frac{\partial\Phi_1}{\partial t}&=-\left[\alpha+\frac{\beta\varkappa_2 I_1(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\frac{(S-S_1)^2}{S}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_1 V_1\left(4-\frac{S_1}{S}-\frac{SVL_1}{S_1V_1L}-\frac{LI_1}{L_1I}-\frac{IV_1}{I_1V}\right)\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_1 I_1\left(3-\frac{S_1}{S}-\frac{SIL_1}{S_1I_1L}-\frac{LI_1}{L_1I}\right)+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_1 V_1\left(3-\frac{S_1}{S}-\frac{SVI_1}{S_1V_1I}-\frac{IV_1}{I_1V}\right)\\
&+\frac{(\delta-r)\psi+\delta\omega}{\varphi\psi}(\Re_2/\Re_1-1)Y+\frac{\mu_1(\gamma+\lambda)[\pi_1(\varkappa_1 b+\varkappa_2\varepsilon)+\alpha\varepsilon\sigma_1]}{\sigma_1(\beta\gamma+\lambda)(\varkappa_1 b+\varkappa_2\varepsilon)}(\Re_3-1)C^I\\
&-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y+d_S\left(1-\frac{S_1}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_1}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_1}{I}\right)\Delta I\\
&+\frac{d_E}{\varphi}\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y+\frac{d_V\varkappa_1 S_1}{\varepsilon}\left(1-\frac{V_1}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y. \qquad (5.8)
\end{aligned}
$$

Calculating the time derivative of $\hat{\Phi}_1(t)$ and using equality (5.5) to get

$$
\begin{aligned}
\frac{d\hat{\Phi}_1}{dt}&=-\left[\alpha+\frac{\beta\varkappa_2 I_1(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\int_\Gamma\frac{(S-S_1)^2}{S}\,\mathrm{d}x\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_1 V_1\int_\Gamma\left(4-\frac{S_1}{S}-\frac{SVL_1}{S_1V_1L}-\frac{LI_1}{L_1I}-\frac{IV_1}{I_1V}\right)\mathrm{d}x\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_1 I_1\int_\Gamma\left(3-\frac{S_1}{S}-\frac{SIL_1}{S_1I_1L}-\frac{LI_1}{L_1I}\right)\mathrm{d}x\\
&+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_1 V_1\int_\Gamma\left(3-\frac{S_1}{S}-\frac{SVI_1}{S_1V_1I}-\frac{IV_1}{I_1V}\right)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{[(\delta - r)\psi + \delta\omega](\Re_2/\Re_1 - 1)}{\varphi\psi}\int_\Gamma Y\,\mathrm{d}x \\
&+ \frac{\mu_1(\gamma+\lambda)[\pi_1(\varkappa_1 b + \varkappa_2\varepsilon) + \alpha\varepsilon\sigma_1](\Re_3 - 1)}{\sigma_1(\beta\gamma+\lambda)(\varkappa_1 b + \varkappa_2\varepsilon)}\int_\Gamma C^I\,\mathrm{d}x \\
&- \frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}\int_\Gamma C^Y\,\mathrm{d}x - d_S S_1\int_\Gamma \frac{\|\nabla S\|^2}{S^2}\,\mathrm{d}x - \frac{\lambda d_L L_1}{\beta\gamma+\lambda}\int_\Gamma \frac{\|\nabla L\|^2}{L^2}\,\mathrm{d}x \\
&- \frac{d_I I_1(\gamma+\lambda)}{\beta\gamma+\lambda}\int_\Gamma \frac{\|\nabla I\|^2}{I^2}\,\mathrm{d}x - \frac{d_V\varkappa_1 S_1 V_1}{\varepsilon}\int_\Gamma \frac{\|\nabla V\|^2}{V^2}\,\mathrm{d}x.
\end{aligned}
$$

Since $\Re_2/\Re_1 \leq 1$ and $\Re_3 \leq 1$, then utilizing inequalities (5.1)–(5.3) we obtain $\frac{d\hat{\Phi}_1}{dt} \leq 0$ for all $S, L, I, Y, V, C^I, C^Y > 0$. Further, $\frac{d\hat{\Phi}_1}{dt} = 0$ at $(S, L, I, V, Y, C^I, C^Y) = (S_1, L_1, I_1, V_1, 0, 0, 0)$. The solutions of system (2.4) tend to Υ_1' which contains elements with $Y = 0$, and hence $\frac{\partial Y}{\partial t} = \Delta Y = 0$. The fifth equation of system (2.4) reduces to

$$
0 = \frac{\partial Y}{\partial t} = \psi E,
$$

which yields $E = 0$. Hence, $\Upsilon_1' = \{\text{Đ}_1\}$ and Đ_1 is GAS by using Lyapunov-LaSalle asymptotic stability theorem.

Theorem 3. If $\Re_2 > 1$, $\Re_1/\Re_2 \leq 1$ and $\Re_4 \leq 1$, then Đ_2 is GAS.

Proof. Let $\Phi_2(x, t)$ be defined as:

$$
\begin{aligned}
\Phi_2(x,t) = {}& S_2 F\left(\frac{S}{S_2}\right) + \frac{\lambda}{\beta\gamma+\lambda}L + \frac{\gamma+\lambda}{\beta\gamma+\lambda}I + \frac{1}{\varphi}E_2 F\left(\frac{E}{E_2}\right) \\
&+ \frac{\psi+\omega}{\varphi\psi}Y_2 F\left(\frac{Y}{Y_2}\right) + \frac{\varkappa_1 S_2}{\varepsilon}V + \frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I + \frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y.
\end{aligned}
$$

We calculate $\frac{\partial\Phi_2}{\partial t}$ as:

$$
\begin{aligned}
\frac{\partial\Phi_2}{\partial t} &= \left(1 - \frac{S_2}{S}\right)[d_S\Delta S + \rho - \alpha S - \varkappa_1 S V - \varkappa_2 S I - \varkappa_3 S Y] \\
&+ \frac{\lambda}{\beta\gamma+\lambda}[d_L\Delta L + (1-\beta)(\varkappa_1 S V + \varkappa_2 S I) - (\lambda+\gamma)L] \\
&+ \frac{\gamma+\lambda}{\beta\gamma+\lambda}\left[d_I\Delta I + \beta(\varkappa_1 S V + \varkappa_2 S I) + \lambda L - aI - \mu_1 C^I I\right] \\
&+ \frac{1}{\varphi}\left(1 - \frac{E_2}{E}\right)[d_E\Delta E + \varphi\varkappa_3 S Y + rY - (\psi+\omega)E] \\
&+ \frac{\psi+\omega}{\varphi\psi}\left(1 - \frac{Y_2}{Y}\right)\left[d_Y\Delta Y + \psi E - \delta Y - \mu_2 C^Y Y\right] + \frac{\varkappa_1 S_2}{\varepsilon}[d_V\Delta V + bI - \varepsilon V] \\
&+ \frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left[d_{C^I}\Delta C^I + \sigma_1 C^I I - \pi_1 C^I\right] + \frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}\left[d_{C^Y}\Delta C^Y + \sigma_2 C^Y Y - \pi_2 C^Y\right] \\
&= \left(1 - \frac{S_2}{S}\right)(\rho - \alpha S) + \varkappa_2 S_2 I + \varkappa_3 S_2 Y - \frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I + \frac{r}{\varphi}Y - \varkappa_3 S Y\frac{E_2}{E} \\
&- \frac{r}{\varphi}Y\frac{E_2}{E} + \frac{\psi+\omega}{\varphi}E_2 - \frac{\delta(\psi+\omega)}{\varphi\psi}Y - \frac{\psi+\omega}{\varphi}E\frac{Y_2}{Y} + \frac{\delta(\psi+\omega)}{\varphi\psi}Y_2
\end{aligned}
$$

$$+\frac{\mu_2(\psi+\omega)}{\varphi\psi}C^Y Y_2+\varkappa_1 S_2\frac{bI}{\varepsilon}-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y$$
$$+d_S\left(1-\frac{S_2}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_2}{E}\right)\Delta E$$
$$+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_2}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_2}{\varepsilon}\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y.$$

Using the steady state conditions for $Đ_2$:

$$\rho=\alpha S_2+\varkappa_3 S_2 Y_2,\quad \varkappa_3 S_2 Y_2+\frac{r}{\varphi}Y_2=\frac{\psi+\omega}{\varphi}E_2=\frac{\delta(\psi+\omega)}{\varphi\psi}Y_2, \tag{5.9}$$

we obtain

$$\begin{aligned}\frac{\partial\Phi_2}{\partial t}&=\left(1-\frac{S_2}{S}\right)(\alpha S_2-\alpha S)+\varkappa_3 S_2 Y_2\left(1-\frac{S_2}{S}\right)+\varkappa_2 S_2 I-\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I-\varkappa_3 S_2 Y_2\frac{SYE_2}{S_2 Y_2 E}\\
&-\frac{r}{\varphi}Y_2\frac{YE_2}{Y_2E}+\varkappa_3 S_2 Y_2+\frac{r}{\varphi}Y_2-\varkappa_3 S_2 Y_2\frac{EY_2}{E_2Y}-\frac{r}{\varphi}Y_2\frac{EY_2}{E_2Y}+\varkappa_3 S_2 Y_2+\frac{r}{\varphi}Y_2\\
&+\frac{\mu_2(\psi+\omega)}{\varphi\psi}C^Y Y_2+\varkappa_1 S_2\frac{bI}{\varepsilon}-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y\\
&+d_S\left(1-\frac{S_2}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_2}{E}\right)\Delta E\\
&+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_2}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_2}{\varepsilon}\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y\\
&=-\alpha\frac{(S-S_2)^2}{S}+\varkappa_3 S_2 Y_2\left(3-\frac{S_2}{S}-\frac{SYE_2}{S_2Y_2E}-\frac{EY_2}{E_2Y}\right)+\frac{r}{\varphi}Y_2\left(2-\frac{YE_2}{Y_2E}-\frac{EY_2}{E_2Y}\right)\\
&+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}\left[\frac{(\varkappa_1 b+\varkappa_2\varepsilon)(\beta\gamma+\lambda)S_2}{a\varepsilon(\gamma+\lambda)}-1\right]I-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I\\
&+\frac{\mu_2(\psi+\omega)}{\varphi\psi}\left(Y_2-\frac{\pi_2}{\sigma_2}\right)C^Y+d_S\left(1-\frac{S_2}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I\\
&+\frac{d_E}{\varphi}\left(1-\frac{E_2}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_2}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_2}{\varepsilon}\Delta V\\
&+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y\\
&=-\alpha\frac{(S-S_2)^2}{S}-\frac{r}{\varphi}\frac{(YE_2-EY_2)^2}{EE_2Y}+\varkappa_3 S_2 Y_2\left(3-\frac{S_2}{S}-\frac{SYE_2}{S_2Y_2E}-\frac{EY_2}{E_2Y}\right)\\
&+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}(\Re_1/\Re_2-1)I-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I+\frac{\mu_2(\psi+\omega)(\alpha\sigma_2+\varkappa_3\pi_2)}{\varphi\psi\varkappa_3\sigma_2}(\Re_4-1)C^Y\\
&+d_S\left(1-\frac{S_2}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_2}{E}\right)\Delta E\\
&+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_2}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_2}{\varepsilon}\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y.\end{aligned} \tag{5.10}$$

Therefore, we take the time derivative of $\hat{\Phi}_2(t)$ along the positive solutions of (2.4) and use equality

(5.5) to find

$$\frac{d\hat{\Phi}_2}{dt} = -\alpha\int_\Gamma \frac{(S-S_2)^2}{S}\,\mathrm{d}x - \frac{r}{\varphi}\int_\Gamma \frac{(YE_2-EY_2)^2}{EE_2Y}\,\mathrm{d}x$$
$$+\varkappa_3 S_2 Y_2\int_\Gamma\left(3-\frac{S_2}{S}-\frac{SYE_2}{S_2Y_2E}-\frac{EY_2}{E_2Y}\right)\mathrm{d}x$$
$$+\frac{a(\gamma+\lambda)(\Re_1/\Re_2-1)}{\beta\gamma+\lambda}\int_\Gamma I\,\mathrm{d}x - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\int_\Gamma C^I\,\mathrm{d}x$$
$$+\frac{\mu_2(\psi+\omega)(\alpha\sigma_2+\varkappa_3\pi_2)(\Re_4-1)}{\varphi\psi\varkappa_3\sigma_2}\int_\Gamma C^Y\,\mathrm{d}x$$
$$-d_S S_2\int_\Gamma\frac{\|\nabla S\|^2}{S^2}\,\mathrm{d}x - \frac{d_E E_2}{\varphi}\int_\Gamma\frac{\|\nabla E\|^2}{E^2}\,\mathrm{d}x - \frac{d_Y Y_2(\psi+\omega)}{\varphi\psi}\int_\Gamma\frac{\|\nabla Y\|^2}{Y^2}\,\mathrm{d}x.$$

Thus, if $\Re_1/\Re_2 \le 1$ and $\Re_4 \le 1$, then from inequality (5.4) we obtain $\frac{d\hat{\Phi}_2}{dt} \le 0$ for all $S, I, E, Y, C^I, C^Y > 0$. In addition, $\frac{d\hat{\Phi}_2}{dt} = 0$ at $(S, E, Y, I, C^I, C^Y) = (S_2, E_2, Y_2, 0, 0, 0)$. The solutions of model (2.4) tend to Υ_2' which satisfy $(S, Y, I) = (S_2, Y_2, 0)$ and hence $\frac{\partial S}{\partial t} = \Delta S = 0$. The first equation of system (2.4) becomes

$$0 = \frac{\partial S}{\partial t} = \rho - \alpha S_2 - \varkappa_1 S_2 V - \varkappa_3 S_2 Y_2.$$

From conditions (5.9) we get $V = 0$. Moreover, we have $\frac{\partial I}{\partial t} = \Delta I = 0$, then the third equation of system (2.4) becomes

$$0 = \frac{\partial I}{\partial t} = \lambda L,$$

which provides $L = 0$. Thus, $\Upsilon_2' = \{Đ_2\}$ and by applying Lyapunov-LaSalle asymptotic stability theorem we get that $Đ_2$ is GAS.

Theorem 4. If $\Re_3 > 1$ and $\Re_5 \le 1$, then $Đ_3$ is GAS.

Proof. Define a function $\Phi_3(x,t)$ as:

$$\Phi_3(x,t) = S_3F\left(\frac{S}{S_3}\right) + \frac{\lambda}{\beta\gamma+\lambda}L_3F\left(\frac{L}{L_3}\right) + \frac{\gamma+\lambda}{\beta\gamma+\lambda}I_3F\left(\frac{I}{I_3}\right) + \frac{1}{\varphi}E + \frac{\psi+\omega}{\varphi\psi}Y$$
$$+\frac{\varkappa_1 S_3}{\varepsilon}V_3F\left(\frac{V}{V_3}\right) + \frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C_3^IF\left(\frac{C^I}{C_3^I}\right) + \frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y.$$

We calculate $\frac{\partial\Phi_3}{\partial t}$ as:

$$\frac{\partial\Phi_3}{\partial t} = \left(1-\frac{S_3}{S}\right)[d_S\Delta S + \rho - \alpha S - \varkappa_1 SV - \varkappa_2 SI - \varkappa_3 SY]$$
$$+\frac{\lambda}{\beta\gamma+\lambda}\left(1-\frac{L_3}{L}\right)[d_L\Delta L + (1-\beta)(\varkappa_1 SV + \varkappa_2 SI) - (\lambda+\gamma)L]$$
$$+\frac{\gamma+\lambda}{\beta\gamma+\lambda}\left(1-\frac{I_3}{I}\right)\left[d_I\Delta I + \beta(\varkappa_1 SV + \varkappa_2 SI) + \lambda L - aI - \mu_1 C^I I\right]$$
$$+\frac{1}{\varphi}[d_E\Delta E + \varphi\varkappa_3 SY + rY - (\psi+\omega)E] + \frac{\psi+\omega}{\varphi\psi}\left[d_Y\Delta Y + \psi E - \delta Y - \mu_2 C^Y Y\right]$$

$$
\begin{aligned}
&+\frac{\varkappa_1 S_3}{\varepsilon}\left(1-\frac{V_3}{V}\right)[d_V\Delta V+bI-\varepsilon V]+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_3^I}{C^I}\right)\\
&\times\left[d_{C^I}\Delta C^I+\sigma_1 C^I I-\pi_1 C^I\right]+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}\left[d_{C^Y}\Delta C^Y+\sigma_2 C^Y Y-\pi_2 C^Y\right]\\
&=\left(1-\frac{S_3}{S}\right)(\rho-\alpha S)+\varkappa_2 S_3 I+\varkappa_3 S_3 Y-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 SV+\varkappa_2 SI)\frac{L_3}{L}+\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_3\\
&-\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}(\varkappa_1 SV+\varkappa_2 SI)\frac{I_3}{I}-\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L\frac{I_3}{I}+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_3\\
&+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C^I I_3+\frac{r}{\varphi}Y-\frac{\delta(\psi+\omega)}{\varphi\psi}Y+\frac{\varkappa_1 S_3}{\varepsilon}bI-\frac{\varkappa_1 S_3}{\varepsilon}bI\frac{V_3}{V}+\varkappa_1 S_3 V_3\\
&-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C_3^I I+\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C_3^I-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y\\
&+d_S\left(1-\frac{S_3}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_3}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_3}{I}\right)\Delta I+\frac{d_E}{\varphi}\Delta E\\
&+\frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y+\frac{d_V\varkappa_1 S_3}{\varepsilon}\left(1-\frac{V_3}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_3^I}{C^I}\right)\Delta C^I\\
&+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y.
\end{aligned}
$$

Using the steady state conditions for Đ$_3$:

$$
\begin{aligned}
&\rho=\alpha S_3+\varkappa_1 S_3 V_3+\varkappa_2 S_3 I_3,\quad \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 S_3 V_3++\varkappa_2 S_3 I_3)=\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_3,\\
&\varkappa_1 S_3 V_3+\varkappa_2 S_3 I_3=\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_3+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C_3^I I_3,\quad I_3=\frac{\pi_1}{\sigma_1},\quad V_3=\frac{b}{\varepsilon}I_3,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{\partial\Phi_3}{\partial t}&=\left(1-\frac{S_3}{S}\right)(\alpha S_3-\alpha S)+(\varkappa_1 S_3 V_3+\varkappa_2 S_3 I_3)\left(1-\frac{S_3}{S}\right)+\left[\varkappa_3 S_3-\frac{(\delta-r)\psi+\delta\omega}{\varphi\psi}\right]Y\\
&-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_3 V_3\frac{SVL_3}{S_3V_3L}-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_3 I_3\frac{SIL_3}{S_3I_3L}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 S_3 V_3+\varkappa_2 S_3 I_3)\\
&-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_3 V_3\frac{SVI_3}{S_3V_3I}-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2 S_3 I_3\frac{S}{S_3}-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 S_3 V_3+\varkappa_2 S_3 I_3)\frac{LI_3}{L_3I}\\
&+\varkappa_1 S_3 V_3+\varkappa_2 S_3 I_3-\varkappa_1 S_3 V_3\frac{IV_3}{I_3V}+\varkappa_1 S_3 V_3-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y\\
&+d_S\left(1-\frac{S_3}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_3}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_3}{I}\right)\Delta I+\frac{d_E}{\varphi}\Delta E\\
&+\frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y+\frac{d_V\varkappa_1 S_3}{\varepsilon}\left(1-\frac{V_3}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_3^I}{C^I}\right)\Delta C^I\\
&+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y\\
&=-\alpha\frac{(S-S_3)^2}{S}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_3 V_3\left(4-\frac{S_3}{S}-\frac{SVL_3}{S_3V_3L}-\frac{LI_3}{L_3I}-\frac{IV_3}{I_3V}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_3 I_3\left(3-\frac{S_3}{S}-\frac{SIL_3}{S_3I_3L}-\frac{LI_3}{L_3I}\right)+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_3 V_3\\
&\times\left(3-\frac{S_3}{S}-\frac{SVI_3}{S_3V_3I}-\frac{IV_3}{I_3V}\right)+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2 S_3 I_3\left(2-\frac{S_3}{S}-\frac{S}{S_3}\right)\\
&+\frac{(\delta-r)\psi+\delta\omega}{\varphi\psi}\left[\frac{\varphi\psi\varkappa_3 S_3}{(\delta-r)\psi+\delta\omega}-1\right]Y-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y\\
&+d_S\left(1-\frac{S_3}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_3}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_3}{I}\right)\Delta I+\frac{d_E}{\varphi}\Delta E\\
&+\frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y+\frac{d_V\varkappa_1 S_3}{\varepsilon}\left(1-\frac{V_3}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_3^I}{C^I}\right)\Delta C^I\\
&+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y\\
&=-\left[\alpha+\frac{\beta\varkappa_2 I_3(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\frac{(S-S_3)^2}{S}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_3 V_3\\
&\times\left(4-\frac{S_3}{S}-\frac{SVL_3}{S_3V_3L}-\frac{LI_3}{L_3I}-\frac{IV_3}{I_3V}\right)+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_3 I_3\left(3-\frac{S_3}{S}-\frac{SIL_3}{S_3I_3L}-\frac{LI_3}{L_3I}\right)\\
&+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_3 V_3\left(3-\frac{S_3}{S}-\frac{SVI_3}{S_3V_3I}-\frac{IV_3}{I_3V}\right)\\
&+\frac{(\delta-r)\psi+\delta\omega}{\varphi\psi}(\Re_5-1)Y-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y+d_S\left(1-\frac{S_3}{S}\right)\Delta S\\
&+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_3}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_3}{I}\right)\Delta I+\frac{d_E}{\varphi}\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\Delta Y\\
&+\frac{d_V\varkappa_1 S_3}{\varepsilon}\left(1-\frac{V_3}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_3^I}{C^I}\right)\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y. \qquad (5.11)
\end{aligned}
$$

After taking the derivative of $\hat{\Phi}_3(t)$ with respect to time t and using equality (5.5), Eq (5.11) will take the form

$$
\begin{aligned}
\frac{d\hat{\Phi}_3}{dt}&=-\left[\alpha+\frac{\beta\varkappa_2 I_3(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\int_\Gamma\frac{(S-S_3)^2}{S}\,\mathrm{d}x\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_3 V_3\int_\Gamma\left(4-\frac{S_3}{S}-\frac{SVL_3}{S_3V_3L}-\frac{LI_3}{L_3I}-\frac{IV_3}{I_3V}\right)\mathrm{d}x\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_3 I_3\int_\Gamma\left(3-\frac{S_3}{S}-\frac{SIL_3}{S_3I_3L}-\frac{LI_3}{L_3I}\right)\mathrm{d}x\\
&+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_3 V_3\int_\Gamma\left(3-\frac{S_3}{S}-\frac{SVI_3}{S_3V_3I}-\frac{IV_3}{I_3V}\right)\mathrm{d}x\\
&+\frac{[(\delta-r)\psi+\delta\omega](\Re_5-1)}{\varphi\psi}\int_\Gamma Y\,\mathrm{d}x-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}\int_\Gamma C^Y\,\mathrm{d}x\\
&-d_S S_3\int_\Gamma\frac{\|\nabla S\|^2}{S^2}\,\mathrm{d}x-\frac{\lambda d_L L_3}{\beta\gamma+\lambda}\int_\Gamma\frac{\|\nabla L\|^2}{L^2}\,\mathrm{d}x-\frac{d_I I_3(\gamma+\lambda)}{\beta\gamma+\lambda}\int_\Gamma\frac{\|\nabla I\|^2}{I^2}\,\mathrm{d}x
\end{aligned}
$$

$$-\frac{d_V\varkappa_1S_3V_3}{\varepsilon}\int_\Gamma\frac{\|\nabla V\|^2}{V^2}\,\mathrm{d}x-\frac{\mu_1d_{C^I}C_3^I(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\int_\Gamma\frac{\|\nabla C^I\|^2}{(C^I)^2}\,\mathrm{d}x.$$

Since $\Re_5\le 1$, then from inequalities (5.1)–(5.3) we obtain $\frac{d\hat{\Phi}_3}{dt}\le 0$ for all $S,L,I,Y,V,C^I,C^Y>0$. In addition $\frac{d\hat{\Phi}_3}{dt}=0$ when $(S,L,I,V,C^I,Y,C^Y)=(S_3,L_3,I_3,V_3,C_3^I,0,0)$. The trajectories of system (2.4) tend to Υ_3' which has elements satisfying $Y=0$. Hence, $\frac{\partial Y}{\partial t}=\Delta Y=0$ and the fifth equation of system (2.4) becomes

$$0=\frac{\partial Y(x,t)}{\partial t}=\psi E,$$

which yields $E=0$ and hence, $\Upsilon_3'=\{Đ_3\}$. Applying Lyapunov-LaSalle asymptotic stability theorem we get $Đ_3$ is GAS.

Theorem 5. If $\Re_4>1$ and $\Re_6\le 1$, then $Đ_4$ is GAS.

Proof. Define $\Phi_4(x,t)$ as:

$$\Phi_4(x,t)=S_4F\left(\frac{S}{S_4}\right)+\frac{\lambda}{\beta\gamma+\lambda}L+\frac{\gamma+\lambda}{\beta\gamma+\lambda}I+\frac{1}{\varphi}E_4F\left(\frac{E}{E_4}\right)+\frac{\psi+\omega}{\varphi\psi}Y_4F\left(\frac{Y}{Y_4}\right)$$
$$+\frac{\varkappa_1S_4}{\varepsilon}V+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}C_4^YF\left(\frac{C^Y}{C_4^Y}\right).$$

Calculating $\frac{\partial\Phi_4}{\partial t}$ as:

$$\frac{\partial\Phi_4}{\partial t}=\left(1-\frac{S_4}{S}\right)[d_S\Delta S+\rho-\alpha S-\varkappa_1SV-\varkappa_2SI-\varkappa_3SY]$$
$$+\frac{\lambda}{\beta\gamma+\lambda}[d_L\Delta L+(1-\beta)(\varkappa_1SV+\varkappa_2SI)-(\lambda+\gamma)L]$$
$$+\frac{\gamma+\lambda}{\beta\gamma+\lambda}\left[d_I\Delta I+\beta(\varkappa_1SV+\varkappa_2SI)+\lambda L-aI-\mu_1C^II\right]$$
$$+\frac{1}{\varphi}\left(1-\frac{E_4}{E}\right)[d_E\Delta E+\varphi\varkappa_3SY+rY-(\psi+\omega)E]$$
$$+\frac{\psi+\omega}{\varphi\psi}\left(1-\frac{Y_4}{Y}\right)\left[d_Y\Delta Y+\psi E-\delta Y-\mu_2C^YY\right]$$
$$+\frac{\varkappa_1S_4}{\varepsilon}[d_V\Delta V+bI-\varepsilon V]+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left[d_{C^I}\Delta C^I+\sigma_1C^II-\pi_1C^I\right]$$
$$+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_4^Y}{C^Y}\right)\left[d_{C^Y}\Delta C^Y+\sigma_2C^YY-\pi_2C^Y\right]$$
$$=\left(1-\frac{S_4}{S}\right)(\rho-\alpha S)+\varkappa_2S_4I+\varkappa_3S_4Y-\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I+\frac{r}{\varphi}Y-\varkappa_3SY\frac{E_4}{E}$$
$$-\frac{r}{\varphi}Y\frac{E_4}{E}+\frac{\psi+\omega}{\varphi}E_4-\frac{\delta(\psi+\omega)}{\varphi\psi}Y-\frac{\psi+\omega}{\varphi}E\frac{Y_4}{Y}+\frac{\delta(\psi+\omega)}{\varphi\psi}Y_4$$
$$+\frac{\mu_2(\psi+\omega)}{\varphi\psi}C^YY_4+\varkappa_1S_4\frac{bI}{\varepsilon}-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y$$
$$-\frac{\mu_2(\psi+\omega)}{\varphi\psi}C_4^YY+\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C_4^Y+d_S\left(1-\frac{S_4}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L$$

$$
\begin{aligned}
&+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_4}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_4}{Y}\right)\Delta Y\\
&+\frac{d_V\varkappa_1 S_4}{\varepsilon}\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_4^Y}{C^Y}\right)\Delta C^Y.
\end{aligned}
$$

Using the steady state conditions for Đ$_4$:

$$
\begin{aligned}
\rho=\alpha S_4+\varkappa_3 S_4 Y_4,\quad Y_4=\frac{\pi_2}{\sigma_2},\\
\varkappa_3 S_4 Y_4+\frac{r}{\varphi}Y_4=\frac{\psi+\omega}{\varphi}E_4=\frac{\delta(\psi+\omega)}{\varphi\psi}Y_4+\frac{\mu_2(\psi+\omega)}{\varphi\psi}C_4^Y Y_4.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\frac{\partial\Phi_4}{\partial t}=&\left(1-\frac{S_4}{S}\right)(\alpha S_4-\alpha S)+\varkappa_3 S_4 Y_4\left(1-\frac{S_4}{S}\right)+\varkappa_2 S_4 I-\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I\\
&-\varkappa_3 S_4 Y_4\frac{SYE_4}{S_4Y_4E}-\frac{r}{\varphi}Y_4\frac{YE_4}{Y_4E}+\varkappa_3 S_4 Y_4+\frac{r}{\varphi}Y_4-\varkappa_3 S_4 Y_4\frac{EY_4}{E_4Y}\\
&-\frac{r}{\varphi}Y_4\frac{EY_4}{E_4Y}+\varkappa_3 S_4 Y_4+\frac{r}{\varphi}Y_4+\varkappa_1 S_4\frac{bI}{\varepsilon}-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I\\
&+d_S\left(1-\frac{S_4}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_4}{E}\right)\Delta E\\
&+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_4}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_4}{\varepsilon}\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I\\
&+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_4^Y}{C^Y}\right)\Delta C^Y\\
=&-\alpha\frac{(S-S_4)^2}{S}+\varkappa_3 S_4 Y_4\left(3-\frac{S_4}{S}-\frac{SYE_4}{S_4Y_4E}-\frac{EY_4}{E_4Y}\right)\\
&+\frac{r}{\varphi}Y_4\left(2-\frac{YE_4}{Y_4E}-\frac{EY_4}{E_4Y}\right)+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}\left[\frac{(\varkappa_1 b+\varkappa_2\varepsilon)(\beta\gamma+\lambda)S_4}{a\varepsilon(\gamma+\lambda)}-1\right]I\\
&-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I+d_S\left(1-\frac{S_4}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I\\
&+\frac{d_E}{\varphi}\left(1-\frac{E_4}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_4}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_4}{\varepsilon}\Delta V\\
&+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_4^Y}{C^Y}\right)\Delta C^Y\\
=&-\alpha\frac{(S-S_4)^2}{S}-\frac{r}{\varphi}\frac{(YE_4-EY_4)^2}{EE_4Y}+\varkappa_3 S_4 Y_4\left(3-\frac{S_4}{S}-\frac{SYE_4}{S_4Y_4E}-\frac{EY_4}{E_4Y}\right)\\
&+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}(\Re_6-1)I-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I+d_S\left(1-\frac{S_4}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\Delta L\\
&+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_4}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_4}{Y}\right)\Delta Y\\
&+\frac{d_V\varkappa_1 S_4}{\varepsilon}\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_4^Y}{C^Y}\right)\Delta C^Y.
\end{aligned}
$$

Calculating $\frac{d\hat{\Phi}_4}{dt}$ and using equality (5.5) we to obtain

$$\begin{aligned}\frac{d\hat{\Phi}_4}{dt} &= -\alpha\int_\Gamma \frac{(S-S_4)^2}{S}\,\mathrm{d}x - \frac{r}{\varphi}\int_\Gamma \frac{(YE_4-EY_4)^2}{EE_4Y}\,\mathrm{d}x\\ &+ \varkappa_3 S_4 Y_4\int_\Gamma\left(3-\frac{S_4}{S}-\frac{SYE_4}{S_4Y_4E}-\frac{EY_4}{E_4Y}\right)\mathrm{d}x\\ &+ \frac{a(\gamma+\lambda)(\Re_6-1)}{\beta\gamma+\lambda}\int_\Gamma I\,\mathrm{d}x - \frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\int_\Gamma C^I\,\mathrm{d}x\\ &- d_S S_4\int_\Gamma \frac{\|\nabla S\|^2}{S^2}\,\mathrm{d}x - \frac{d_E E_4}{\varphi}\int_\Gamma\frac{\|\nabla E\|^2}{E^2}\,\mathrm{d}x\\ &- \frac{d_Y Y_4(\psi+\omega)}{\varphi\psi}\int_\Gamma\frac{\|\nabla Y\|^2}{Y^2}\,\mathrm{d}x\\ &- \frac{\mu_2 d_{C^Y} C_4^Y(\psi+\omega)}{\varphi\psi\sigma_2}\int_\Gamma\frac{\|\nabla C^Y\|^2}{(C^Y)^2}\,\mathrm{d}x.\end{aligned}$$

Hence, if $\Re_6 \le 1$, then from inequality (5.4) we obtain $\frac{d\hat{\Phi}_4}{dt} \le 0$ for all $S, I, E, Y, C^I, C^Y > 0$. Moreover, $\frac{d\hat{\Phi}_4}{dt} = 0$ at $(S, E, Y, C^Y, I, C^I) = (S_4, E_4, Y_4, C_4^Y, 0, 0)$. The solutions of model (2.4) tend to Υ_4' which has elements satisfying $(S, Y, I) = (S_4, Y_4, 0)$, and then $\frac{\partial S}{\partial t} = \Delta S = 0$. Further, the first equation of system (2.4) becomes

$$0 = \frac{\partial S}{\partial t} = \rho - \alpha S_4 - \varkappa_1 S_4 V - \varkappa_3 S_4 Y_4,$$

which yields $V = 0$. Furthermore, we have $\frac{\partial I}{\partial t} = \Delta I = 0$ and the third equation of system (2.4) reduces to

$$0 = \frac{\partial I}{\partial t} = \lambda L,$$

which gives $L = 0$ and hence, $\Upsilon_4' = \{Đ_4\}$. Applying Lyapunov-LaSalle asymptotic stability theorem we get $Đ_4$ is GAS.

Theorem 6. If $\Re_5 > 1$, $\Re_8 \le 1$ and $\Re_1/\Re_2 > 1$, then $Đ_5$ is GAS.

Proof. Define $\Phi_5(x,t)$ as:

$$\begin{aligned}\Phi_5(x,t) &= S_5 F\left(\frac{S}{S_5}\right) + \frac{\lambda}{\beta\gamma+\lambda}L_5 F\left(\frac{L}{L_5}\right) + \frac{\gamma+\lambda}{\beta\gamma+\lambda}I_5 F\left(\frac{I}{I_5}\right) + \frac{1}{\varphi}E_5 F\left(\frac{E}{E_5}\right)\\ &+ \frac{\psi+\omega}{\varphi\psi}Y_5 F\left(\frac{Y}{Y_5}\right) + \frac{\varkappa_1 S_5}{\varepsilon}V_5 F\left(\frac{V}{V_5}\right) + \frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C_5^I F\left(\frac{C^I}{C_5^I}\right) + \frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y.\end{aligned}$$

Calculating $\frac{\partial\Phi_5}{\partial t}$ as:

$$\begin{aligned}\frac{\partial\Phi_5}{\partial t} &= \left(1-\frac{S_5}{S}\right)[d_S\Delta S + \rho - \alpha S - \varkappa_1 SV - \varkappa_2 SI - \varkappa_3 SY]\\ &+ \frac{\lambda}{\beta\gamma+\lambda}\left(1-\frac{L_5}{L}\right)[d_L\Delta L + (1-\beta)(\varkappa_1 SV + \varkappa_2 SI) - (\lambda+\gamma)L]\\ &+ \frac{\gamma+\lambda}{\beta\gamma+\lambda}\left(1-\frac{I_5}{I}\right)\left[d_I\Delta I + \beta(\varkappa_1 SV + \varkappa_2 SI) + \lambda L - aI - \mu_1 C^I I\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\varphi}\left(1-\frac{E_5}{E}\right)\left[d_E\Delta E+\varphi\varkappa_3SY+rY-(\psi+\omega)E\right]\\
&+\frac{\psi+\omega}{\varphi\psi}\left(1-\frac{Y_5}{Y}\right)\left[d_Y\Delta Y+\psi E-\delta Y-\mu_2C^YY\right]\\
&+\frac{\varkappa_1S_5}{\varepsilon}\left(1-\frac{V_5}{V}\right)\left[d_V\Delta V+bI-\varepsilon V\right]+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_5^I}{C^I}\right)\\
&\times\left[d_{C^I}\Delta C^I+\sigma_1C^II-\pi_1C^I\right]+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}\left[d_{C^Y}\Delta C^Y+\sigma_2C^YY-\pi_2C^Y\right]\\
&=\left(1-\frac{S_5}{S}\right)(\rho-\alpha S)+\varkappa_2S_5I+\varkappa_3S_5Y-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\left(\varkappa_1SV+\varkappa_2SI\right)\frac{L_5}{L}\\
&+\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_5-\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\left(\varkappa_1SV+\varkappa_2SI\right)\frac{I_5}{I}-\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L\frac{I_5}{I}\\
&+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_5+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C^II_5+\frac{r}{\varphi}Y-\varkappa_3SY\frac{E_5}{E}-\frac{r}{\varphi}Y\frac{E_5}{E}+\frac{\psi+\omega}{\varphi}E_5\\
&-\frac{\delta(\psi+\omega)}{\varphi\psi}Y-\frac{\psi+\omega}{\varphi}E\frac{Y_5}{Y}+\frac{\delta(\psi+\omega)}{\varphi\psi}Y_5+\frac{\mu_2(\psi+\omega)}{\varphi\psi}C^YY_5+\varkappa_1S_5\frac{bI}{\varepsilon}\\
&-\varkappa_1S_5V_5\frac{bI}{\varepsilon V}+\varkappa_1S_5V_5-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C_5^II+\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C_5^I\\
&-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y+d_S\left(1-\frac{S_5}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_5}{L}\right)\Delta L\\
&+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_5}{I}\right)\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_5}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_5}{Y}\right)\Delta Y\\
&+\frac{d_V\varkappa_1S_5}{\varepsilon}\left(1-\frac{V_5}{V}\right)\Delta V+\frac{\mu_1d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_5^I}{C^I}\right)\Delta C^I+\frac{\mu_2d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y.
\end{aligned}
$$

Using the steady state conditions for Đ$_5$:

$$
\begin{aligned}
&\rho=\alpha S_5+\varkappa_1S_5V_5+\varkappa_2S_5I_5+\varkappa_3S_5Y_5,\\
&\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\left(\varkappa_1S_5V_5+\varkappa_2S_5I_5\right)=\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_5,\\
&\varkappa_1S_5V_5+\varkappa_2S_5I_5=\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_5+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C_5^II_5,\\
&\varkappa_3S_5Y_5+\frac{r}{\varphi}Y_5=\frac{\psi+\omega}{\varphi}E_5=\frac{\delta(\psi+\omega)}{\varphi\psi}Y_5,\quad I_5=\frac{\pi_1}{\sigma_1},\quad V_5=\frac{bI_5}{\varepsilon}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\frac{\partial\Phi_5}{\partial t}&=\left(1-\frac{S_5}{S}\right)(\alpha S_5-\alpha S)+\left(\varkappa_1S_5V_5+\varkappa_2S_5I_5+\varkappa_3S_5Y_5\right)\left(1-\frac{S_5}{S}\right)\\
&-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1S_5V_5\frac{SVL_5}{S_5V_5L}-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2S_5I_5\frac{SIL_5}{S_5I_5L}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\\
&\times\left(\varkappa_1S_5V_5+\varkappa_2S_5I_5\right)-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1S_5V_5\frac{SVI_5}{S_5V_5I}-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2S_5I_5\frac{S}{S_5}\\
&-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\left(\varkappa_1S_5V_5+\varkappa_2S_5I_5\right)\frac{LI_5}{L_5I}+\varkappa_1S_5V_5+\varkappa_2S_5I_5-\varkappa_3S_5Y_5\frac{SYE_5}{S_5Y_5E}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{r}{\varphi}Y_5\frac{YE_5}{Y_5E}+\varkappa_3S_5Y_5+\frac{r}{\varphi}Y_5-\varkappa_3S_5Y_5\frac{EY_5}{E_5Y}-\frac{r}{\varphi}Y_5\frac{EY_5}{E_5Y}+\varkappa_3S_5Y_5\\
&+\frac{r}{\varphi}Y_5-\varkappa_1S_5V_5\frac{IV_5}{I_5V}+\varkappa_1S_5V_5+\frac{\mu_2(\psi+\omega)}{\varphi\psi}\left(Y_5-\frac{\pi_2}{\sigma_2}\right)C^Y\\
&+d_S\left(1-\frac{S_5}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_5}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_5}{I}\right)\Delta I\\
&+\frac{d_E}{\varphi}\left(1-\frac{E_5}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_5}{Y}\right)\Delta Y+\frac{d_V\varkappa_1S_5}{\varepsilon}\left(1-\frac{V_5}{V}\right)\Delta V\\
&+\frac{\mu_1d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_5^I}{C^I}\right)\Delta C^I+\frac{\mu_2d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y\\
&=-\alpha\frac{(S-S_5)^2}{S}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1S_5V_5\left(4-\frac{S_5}{S}-\frac{SVL_5}{S_5V_5L}-\frac{LI_5}{L_5I}-\frac{IV_5}{I_5V}\right)\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2S_5I_5\left(3-\frac{S_5}{S}-\frac{SIL_5}{S_5I_5L}-\frac{LI_5}{L_5I}\right)+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1S_5V_5\\
&\times\left(3-\frac{S_5}{S}-\frac{SVI_5}{S_5V_5I}-\frac{IV_5}{I_5V}\right)+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2S_5I_5\left(2-\frac{S_5}{S}-\frac{S}{S_5}\right)\\
&+\varkappa_3S_5Y_5\left(3-\frac{S_5}{S}-\frac{SYE_5}{S_5Y_5E}-\frac{EY_5}{E_5Y}\right)+\frac{r}{\varphi}Y_5\left(2-\frac{YE_5}{Y_5E}-\frac{EY_5}{E_5Y}\right)\\
&+\frac{\mu_2(\psi+\omega)}{\varphi\psi}\left(Y_5-\frac{\pi_2}{\sigma_2}\right)C^Y+d_S\left(1-\frac{S_5}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_5}{L}\right)\Delta L\\
&+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_5}{I}\right)\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_5}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_5}{Y}\right)\Delta Y\\
&+\frac{d_V\varkappa_1S_5}{\varepsilon}\left(1-\frac{V_5}{V}\right)\Delta V+\frac{\mu_1d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_5^I}{C^I}\right)\Delta C^I+\frac{\mu_2d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y. \qquad (5.12)
\end{aligned}
$$

Then, Eq (5.12) will be reduced to the form

$$
\begin{aligned}
\frac{\partial\Phi_5}{\partial t}=&-\left[\alpha+\frac{\beta\varkappa_2I_5(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\frac{(S-S_5)^2}{S}-\frac{r}{\varphi}\frac{(YE_5-EY_5)^2}{EE_5Y}\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1S_5V_5\left(4-\frac{S_5}{S}-\frac{SVL_5}{S_5V_5L}-\frac{LI_5}{L_5I}-\frac{IV_5}{I_5V}\right)\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2S_5I_5\left(3-\frac{S_5}{S}-\frac{SIL_5}{S_5I_5L}-\frac{LI_5}{L_5I}\right)\\
&+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1S_5V_5\left(3-\frac{S_5}{S}-\frac{SVI_5}{S_5V_5I}-\frac{IV_5}{I_5V}\right)\\
&+\varkappa_3S_5Y_5\left(3-\frac{S_5}{S}-\frac{SYE_5}{S_5Y_5E}-\frac{EY_5}{E_5Y}\right)\\
&+\frac{\mu_2(\psi+\omega)\left[\pi_1\sigma_2(\varkappa_1b+\varkappa_2\varepsilon)+\pi_2\varkappa_3\varepsilon\sigma_1+\alpha\varepsilon\sigma_1\sigma_2\right]}{\varphi\psi\varkappa_3\varepsilon\sigma_1\sigma_2}(\Re_8-1)C^Y\\
&+d_S\left(1-\frac{S_5}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_5}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_5}{I}\right)\Delta I
\end{aligned}
$$

$$+\frac{d_E}{\varphi}\left(1-\frac{E_5}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_5}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_5}{\varepsilon}$$
$$\times\left(1-\frac{V_5}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_5^I}{C^I}\right)\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\Delta C^Y.$$

Calculating $\frac{d\hat{\Phi}_5}{dt}$ along the solution trajectories of system (2.4) and using equality (5.5) to get

$$\frac{d\hat{\Phi}_5}{dt}=-\left[\alpha+\frac{\beta\varkappa_2 I_5(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\int_\Gamma\frac{(S-S_5)^2}{S}\,\mathrm{d}x-\frac{r}{\varphi}\int_\Gamma\frac{(YE_5-EY_5)^2}{EE_5Y}\,\mathrm{d}x$$
$$+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_5V_5\int_\Gamma\left(4-\frac{S_5}{S}-\frac{SVL_5}{S_5V_5L}-\frac{LI_5}{L_5I}-\frac{IV_5}{I_5V}\right)\mathrm{d}x$$
$$+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_5I_5\int_\Gamma\left(3-\frac{S_5}{S}-\frac{SIL_5}{S_5I_5L}-\frac{LI_5}{L_5I}\right)\mathrm{d}x$$
$$+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_5V_5\int_\Gamma\left(3-\frac{S_5}{S}-\frac{SVI_5}{S_5V_5I}-\frac{IV_5}{I_5V}\right)\mathrm{d}x$$
$$+\varkappa_3 S_5Y_5\int_\Gamma\left(3-\frac{S_5}{S}-\frac{SYE_5}{S_5Y_5E}-\frac{EY_5}{E_5Y}\right)\mathrm{d}x$$
$$+\frac{\mu_2(\psi+\omega)\left[\pi_1\sigma_2(\varkappa_1 b+\varkappa_2\varepsilon)+\pi_2\varkappa_3\varepsilon\sigma_1+\alpha\varepsilon\sigma_1\sigma_2\right](\Re_8-1)}{\varphi\psi\varkappa_3\varepsilon\sigma_1\sigma_2}\int_\Gamma C^Y\,\mathrm{d}x$$
$$-d_SS_5\int_\Gamma\frac{\|\nabla S\|^2}{S^2}\,\mathrm{d}x-\frac{\lambda d_L L_5}{\beta\gamma+\lambda}\int_\Gamma\frac{\|\nabla L\|^2}{L^2}\,\mathrm{d}x-\frac{d_I I_5(\gamma+\lambda)}{\beta\gamma+\lambda}\int_\Gamma\frac{\|\nabla I\|^2}{I^2}\,\mathrm{d}x$$
$$-\frac{d_E E_5}{\varphi}\int_\Gamma\frac{\|\nabla E\|^2}{E^2}\,\mathrm{d}x-\frac{d_Y Y_5(\psi+\omega)}{\varphi\psi}\int_\Gamma\frac{\|\nabla Y\|^2}{Y^2}\,\mathrm{d}x$$
$$-\frac{d_V\varkappa_1 S_5V_5}{\varepsilon}\int_\Gamma\frac{\|\nabla V\|^2}{V^2}\,\mathrm{d}x-\frac{\mu_1 d_{C^I}C_5^I(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\int_\Gamma\frac{\|\nabla C^I\|^2}{(C^I)^2}\,\mathrm{d}x.$$

Hence, if $\Re_8\le 1$, then from inequalities (5.1)–(5.4) we obtain $\frac{d\hat{\Phi}_5}{dt}\le 0$ for all $S,L,I,E,Y,V,C^I,C^Y>0$. We have also $\frac{d\hat{\Phi}_5}{dt}=0$ at $(S,L,I,E,Y,V,C^I,C^Y)=(S_5,L_5,I_5,E_5,Y_5,V_5,C_5^I,0)$. The trajectories of system (2.4) converge to Υ_5' and hence, $\Upsilon_5'=\{Đ_5\}$. Applying Lyapunov-LaSalle asymptotic stability theorem we get $Đ_5$ is GAS.

Theorem 7. If $\Re_6>1$, $\Re_7\le 1$ and $\Re_2/\Re_1>1$, then $Đ_6$ is GAS.

Proof. Define $\Phi_6(x,t)$ as:

$$\Phi_6(x,t)=S_6F\left(\frac{S}{S_6}\right)+\frac{\lambda}{\beta\gamma+\lambda}L_6F\left(\frac{L}{L_6}\right)+\frac{\gamma+\lambda}{\beta\gamma+\lambda}I_6F\left(\frac{I}{I_6}\right)+\frac{1}{\varphi}E_6F\left(\frac{E}{E_6}\right)+\frac{\psi+\omega}{\varphi\psi}Y_6F\left(\frac{Y}{Y_6}\right)$$
$$+\frac{\varkappa_1S_6}{\varepsilon}V_6F\left(\frac{V}{V_6}\right)+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}C_6^YF\left(\frac{C^Y}{C_6^Y}\right).$$

Calculating $\frac{\partial\Phi_6}{\partial t}$ as:

$$\frac{\partial\Phi_6}{\partial t}=\left(1-\frac{S_6}{S}\right)\left[d_S\Delta S+\rho-\alpha S-\varkappa_1SV-\varkappa_2SI-\varkappa_3SY\right]$$

$$
\begin{aligned}
&+\frac{\lambda}{\beta\gamma+\lambda}\left(1-\frac{L_6}{L}\right)\left[d_L\Delta L+(1-\beta)\left(\varkappa_1 SV+\varkappa_2 SI\right)-(\lambda+\gamma)L\right]\\
&+\frac{\gamma+\lambda}{\beta\gamma+\lambda}\left(1-\frac{I_6}{I}\right)\left[d_I\Delta I+\beta\left(\varkappa_1 SV+\varkappa_2 SI\right)+\lambda L-aI-\mu_1 C^I I\right]\\
&+\frac{1}{\varphi}\left(1-\frac{E_6}{E}\right)\left[d_E\Delta E+\varphi\varkappa_3 SY+rY-(\psi+\omega)E\right]\\
&+\frac{\psi+\omega}{\varphi\psi}\left(1-\frac{Y_6}{Y}\right)\left[d_Y\Delta Y+\psi E-\delta Y-\mu_2 C^Y Y\right]\\
&+\frac{\varkappa_1 S_6}{\varepsilon}\left(1-\frac{V_6}{V}\right)\left[d_V\Delta V+bI-\varepsilon V\right]+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left[d_{C^I}\Delta C^I+\sigma_1 C^I I-\pi_1 C^I\right]\\
&+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_6^Y}{C^Y}\right)\left[d_{C^Y}\Delta C^Y+\sigma_2 C^Y Y-\pi_2 C^Y\right]\\
&=\left(1-\frac{S_6}{S}\right)(\rho-\alpha S)+\varkappa_2 S_6 I+\varkappa_3 S_6 Y-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\left(\varkappa_1 SV+\varkappa_2 SI\right)\frac{L_6}{L}+\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_6\\
&-\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\left(\varkappa_1 SV+\varkappa_2 SI\right)\frac{I_6}{I}-\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L\frac{I_6}{I}+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_6\\
&+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C^I I_6+\frac{r}{\varphi}Y-\varkappa_3 SY\frac{E_6}{E}-\frac{r}{\varphi}Y\frac{E_6}{E}+\frac{\psi+\omega}{\varphi}E_6-\frac{\delta(\psi+\omega)}{\varphi\psi}Y\\
&-\frac{\psi+\omega}{\varphi}E\frac{Y_6}{Y}+\frac{\delta(\psi+\omega)}{\varphi\psi}Y_6+\frac{\mu_2(\psi+\omega)}{\varphi\psi}C^Y Y_6+\varkappa_1 S_6\frac{bI}{\varepsilon}-\varkappa_1 S_6 V_6\frac{bI}{\varepsilon V}+\varkappa_1 S_6 V_6\\
&-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y-\frac{\mu_2(\psi+\omega)}{\varphi\psi}C_6^Y Y+\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C_6^Y\\
&+d_S\left(1-\frac{S_6}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_6}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_6}{I}\right)\Delta I\\
&+\frac{d_E}{\varphi}\left(1-\frac{E_6}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_6}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_6}{\varepsilon}\left(1-\frac{V_6}{V}\right)\Delta V\\
&+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_6^Y}{C^Y}\right)\Delta C^Y.
\end{aligned}
$$

Using the steady state conditions for Đ$_6$:

$$
\begin{aligned}
&\rho=\alpha S_6+\varkappa_1 S_6 V_6+\varkappa_2 S_6 I_6+\varkappa_3 S_6 Y_6,\\
&\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\left(\varkappa_1 S_6 V_6+\varkappa_2 S_6 I_6\right)=\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_6,\quad Y_6=\frac{\pi_2}{\sigma_2},\quad V_6=\frac{bI_6}{\varepsilon},\\
&\varkappa_1 S_6 V_6+\varkappa_2 S_6 I_6=\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_6,\\
&\varkappa_3 S_6 Y_6+\frac{r}{\varphi}Y_6=\frac{\psi+\omega}{\varphi}E_6=\frac{\delta(\psi+\omega)}{\varphi\psi}Y_6+\frac{\mu_2(\psi+\omega)}{\varphi\psi}C_6^Y Y_6.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\frac{\partial\Phi_6}{\partial t}&=\left(1-\frac{S_6}{S}\right)(\alpha S_6-\alpha S)+\left(\varkappa_1 S_6 V_6+\varkappa_2 S_6 I_6+\varkappa_3 S_6 Y_6\right)\left(1-\frac{S_6}{S}\right)\\
&-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_6 V_6\frac{SVL_6}{S_6 V_6 L}-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_6 I_6\frac{SIL_6}{S_6 I_6 L}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\varkappa_1 S_6 V_6+\varkappa_2 S_6 I_6\right)-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_6 V_6\frac{SVI_6}{S_6V_6I}-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2 S_6 I_6\frac{S}{S_6}\\
&-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\left(\varkappa_1 S_6 V_6+\varkappa_2 S_6 I_6\right)\frac{LI_6}{L_6I}+\varkappa_1 S_6 V_6+\varkappa_2 S_6 I_6-\varkappa_3 S_6 Y_6\frac{SYE_6}{S_6Y_6E}\\
&-\frac{r}{\varphi}Y_6\frac{YE_6}{Y_6E}+\varkappa_3 S_6 Y_6+\frac{r}{\varphi}Y_6-\varkappa_3 S_6 Y_6\frac{EY_6}{E_6Y}-\frac{r}{\varphi}Y_6\frac{EY_6}{E_6Y}+\varkappa_3 S_6 Y_6\\
&+\frac{r}{\varphi}Y_6-\varkappa_1 S_6 V_6\frac{IV_6}{I_6V}+\varkappa_1 S_6 V_6+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}\left(I_6-\frac{\pi_1}{\sigma_1}\right)C^I\\
&+d_S\left(1-\frac{S_6}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_6}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_6}{I}\right)\Delta I\\
&+\frac{d_E}{\varphi}\left(1-\frac{E_6}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_6}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_6}{\varepsilon}\left(1-\frac{V_6}{V}\right)\Delta V\\
&+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_6^Y}{C^Y}\right)\Delta C^Y\\
&=-\alpha\frac{(S-S_6)^2}{S}+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_6 V_6\left(4-\frac{S_6}{S}-\frac{SVL_6}{S_6V_6L}-\frac{LI_6}{L_6I}-\frac{IV_6}{I_6V}\right)\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_6 I_6\left(3-\frac{S_6}{S}-\frac{SIL_6}{S_6I_6L}-\frac{LI_6}{L_6I}\right)+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_6 V_6\\
&\times\left(3-\frac{S_6}{S}-\frac{SVI_6}{S_6V_6I}-\frac{IV_6}{I_6V}\right)+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2 S_6 I_6\left(2-\frac{S_6}{S}-\frac{S}{S_6}\right)\\
&+\varkappa_3 S_6 Y_6\left(3-\frac{S_6}{S}-\frac{SYE_6}{S_6Y_6E}-\frac{EY_6}{E_6Y}\right)+\frac{r}{\varphi}Y_6\left(2-\frac{YE_6}{Y_6E}-\frac{EY_6}{E_6Y}\right)\\
&+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}\left(I_6-\frac{\pi_1}{\sigma_1}\right)C^I+d_S\left(1-\frac{S_6}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_6}{L}\right)\Delta L\\
&+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_6}{I}\right)\Delta I+\frac{d_E}{\varphi}\left(1-\frac{E_6}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_6}{Y}\right)\Delta Y\\
&+\frac{d_V\varkappa_1 S_6}{\varepsilon}\left(1-\frac{V_6}{V}\right)\Delta V+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_6^Y}{C^Y}\right)\Delta C^Y. \qquad (5.13)
\end{aligned}
$$

Then, Eq (5.13) will be reduced to the form

$$
\begin{aligned}
\frac{\partial\Phi_6}{\partial t}&=-\left[\alpha+\frac{\beta\varkappa_2 I_6(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\frac{(S-S_6)^2}{S}-\frac{r}{\varphi}\frac{(YE_6-EY_6)^2}{EE_6Y}\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_6 V_6\left(4-\frac{S_6}{S}-\frac{SVL_6}{S_6V_6L}-\frac{LI_6}{L_6I}-\frac{IV_6}{I_6V}\right)\\
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_6 I_6\left(3-\frac{S_6}{S}-\frac{SIL_6}{S_6I_6L}-\frac{LI_6}{L_6I}\right)\\
&+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_6 V_6\left(3-\frac{S_6}{S}-\frac{SVI_6}{S_6V_6I}-\frac{IV_6}{I_6V}\right)\\
&+\varkappa_3 S_6 Y_6\left(3-\frac{S_6}{S}-\frac{SYE_6}{S_6Y_6E}-\frac{EY_6}{E_6Y}\right)
\end{aligned}
$$

$$+\frac{\mu_1(\gamma+\lambda)[\pi_1\sigma_2(\varkappa_1 b+\varkappa_2\varepsilon)+\pi_2\varkappa_3\varepsilon\sigma_1+\alpha\varepsilon\sigma_1\sigma_2]}{\sigma_1\sigma_2(\beta\gamma+\lambda)(\varkappa_1 b+\varkappa_2\varepsilon)}(\Re_7-1)C^I$$
$$+d_S\left(1-\frac{S_6}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_6}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_6}{I}\right)\Delta I$$
$$+\frac{d_E}{\varphi}\left(1-\frac{E_6}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_6}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_6}{\varepsilon}\left(1-\frac{V_6}{V}\right)\Delta V$$
$$+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_6^Y}{C^Y}\right)\Delta C^Y.$$

We calculate $\frac{d\hat{\Phi}_6}{dt}$ along the solution trajectories of system (2.4) and then we use equality (5.5) to obtain

$$\frac{d\hat{\Phi}_6}{dt}=-\left[\alpha+\frac{\beta\varkappa_2 I_6(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\int_\Gamma\frac{(S-S_6)^2}{S}\,\mathrm{d}x-\frac{r}{\varphi}\int_\Gamma\frac{(YE_6-EY_6)^2}{EE_6Y}\,\mathrm{d}x$$
$$+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_6V_6\int_\Gamma\left(4-\frac{S_6}{S}-\frac{SVL_6}{S_6V_6L}-\frac{LI_6}{L_6I}-\frac{IV_6}{I_6V}\right)\mathrm{d}x$$
$$+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_6I_6\int_\Gamma\left(3-\frac{S_6}{S}-\frac{SIL_6}{S_6I_6L}-\frac{LI_6}{L_6I}\right)\mathrm{d}x$$
$$+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_6V_6\int_\Gamma\left(3-\frac{S_6}{S}-\frac{SVI_6}{S_6V_6I}-\frac{IV_6}{I_6V}\right)\mathrm{d}x$$
$$+\varkappa_3 S_6Y_6\int_\Gamma\left(3-\frac{S_6}{S}-\frac{SYE_6}{S_6Y_6E}-\frac{EY_6}{E_6Y}\right)\mathrm{d}x$$
$$+\frac{\mu_1(\gamma+\lambda)[\pi_1\sigma_2(\varkappa_1 b+\varkappa_2\varepsilon)+\pi_2\varkappa_3\varepsilon\sigma_1+\alpha\varepsilon\sigma_1\sigma_2](\Re_7-1)}{\sigma_1\sigma_2(\beta\gamma+\lambda)(\varkappa_1 b+\varkappa_2\varepsilon)}\int_\Gamma C^I\,\mathrm{d}x$$
$$-d_S S_6\int_\Gamma\frac{\|\nabla S\|^2}{S^2}\,\mathrm{d}x-\frac{\lambda d_L L_6}{\beta\gamma+\lambda}\int_\Gamma\frac{\|\nabla L\|^2}{L^2}\,\mathrm{d}x-\frac{d_I I_6(\gamma+\lambda)}{\beta\gamma+\lambda}\int_\Gamma\frac{\|\nabla I\|^2}{I^2}\,\mathrm{d}x$$
$$-\frac{d_E E_6}{\varphi}\int_\Gamma\frac{\|\nabla E\|^2}{E^2}\,\mathrm{d}x-\frac{d_Y Y_6(\psi+\omega)}{\varphi\psi}\int_\Gamma\frac{\|\nabla Y\|^2}{Y^2}\,\mathrm{d}x$$
$$-\frac{d_V\varkappa_1 S_6V_6}{\varepsilon}\int_\Gamma\frac{\|\nabla V\|^2}{V^2}\,\mathrm{d}x-\frac{\mu_2 d_{C^Y}C_6^Y(\psi+\omega)}{\varphi\psi\sigma_2}\int_\Gamma\frac{\|\nabla C^Y\|^2}{(C^Y)^2}\,\mathrm{d}x.$$

Hence, if $\Re_7\le 1$, then using inequalities (5.1)–(5.4) we get $\frac{d\hat{\Phi}_6}{dt}\le 0$ for all $S,L,I,E,Y,V,C^I,C^Y>0$, where $\frac{d\hat{\Phi}_6}{dt}=0$ when $(S,L,I,E,Y,V,C^Y,C^I)=(S_6,L_6,I_6,E_6,Y_6,V_6,C_6^Y,0)$. The solutions of model (2.4) tend to Υ_6' and hence $\Upsilon_6'=\{Đ_6\}$. Applying Lyapunov-LaSalle asymptotic stability theorem we get $Đ_6$ is GAS.

Theorem 8. If $\Re_7>1$ and $\Re_8>1$, then $Đ_7$ is GAS.

Proof. Define $\Phi_7(x,t)$ as:

$$\Phi_7(x,t)=S_7F\left(\frac{S}{S_7}\right)+\frac{\lambda}{\beta\gamma+\lambda}L_7F\left(\frac{L}{L_7}\right)+\frac{\gamma+\lambda}{\beta\gamma+\lambda}I_7F\left(\frac{I}{I_7}\right)+\frac{1}{\varphi}E_7F\left(\frac{E}{E_7}\right)+\frac{\psi+\omega}{\varphi\psi}Y_7F\left(\frac{Y}{Y_7}\right)$$
$$+\frac{\varkappa_1 S_7}{\varepsilon}V_7F\left(\frac{V}{V_7}\right)+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C_7^IF\left(\frac{C^I}{C_7^I}\right)+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}C_7^YF\left(\frac{C^Y}{C_7^Y}\right).$$

Calculating $\frac{\partial\Phi_7}{\partial t}$ as:

$$\begin{aligned}
\frac{\partial\Phi_7}{\partial t} &= \left(1-\frac{S_7}{S}\right)[d_S\Delta S+\rho-\alpha S-\varkappa_1 SV-\varkappa_2 SI-\varkappa_3 SY]\\
&+\frac{\lambda}{\beta\gamma+\lambda}\left(1-\frac{L_7}{L}\right)[d_L\Delta L+(1-\beta)(\varkappa_1 SV+\varkappa_2 SI)-(\lambda+\gamma)L]\\
&+\frac{\gamma+\lambda}{\beta\gamma+\lambda}\left(1-\frac{I_7}{I}\right)\left[d_I\Delta I+\beta(\varkappa_1 SV+\varkappa_2 SI)+\lambda L-aI-\mu_1 C^I I\right]\\
&+\frac{1}{\varphi}\left(1-\frac{E_7}{E}\right)[d_E\Delta E+\varphi\varkappa_3 SY+rY-(\psi+\omega)E]\\
&+\frac{\psi+\omega}{\varphi\psi}\left(1-\frac{Y_7}{Y}\right)\left[d_Y\Delta Y+\psi E-\delta Y-\mu_2 C^Y Y\right]\\
&+\frac{\varkappa_1 S_7}{\varepsilon}\left(1-\frac{V_7}{V}\right)[d_V\Delta V+bI-\varepsilon V]\\
&+\frac{\mu_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C^I_7}{C^I}\right)\left[d_{C^I}\Delta C^I+\sigma_1 C^I I-\pi_1 C^I\right]\\
&+\frac{\mu_2(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C^Y_7}{C^Y}\right)\left[d_{C^Y}\Delta C^Y+\sigma_2 C^Y Y-\pi_2 C^Y\right]\\
&=\left(1-\frac{S_7}{S}\right)(\rho-\alpha S)+\varkappa_2 S_7 I+\varkappa_3 S_7 Y-\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 SV+\varkappa_2 SI)\frac{L_7}{L}\\
&+\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_7-\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I-\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}(\varkappa_1 SV+\varkappa_2 SI)\frac{I_7}{I}-\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L\frac{I_7}{I}\\
&+\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_7+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C^I I_7+\frac{r}{\varphi}Y-\varkappa_3 SY\frac{E_7}{E}-\frac{r}{\varphi}Y\frac{E_7}{E}+\frac{\psi+\omega}{\varphi}E_7\\
&-\frac{\delta(\psi+\omega)}{\varphi\psi}Y-\frac{\psi+\omega}{\varphi}E\frac{Y_7}{Y}+\frac{\delta(\psi+\omega)}{\varphi\psi}Y_7+\frac{\mu_2(\psi+\omega)}{\varphi\psi}C^Y Y_7+\varkappa_1 S_7\frac{bI}{\varepsilon}\\
&-\varkappa_1 S_7 V_7\frac{bI}{\varepsilon V}+\varkappa_1 S_7 V_7-\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I-\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C^I_7 I\\
&+\frac{\mu_1\pi_1(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}C^I_7-\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y-\frac{\mu_2(\psi+\omega)}{\varphi\psi}C^Y_7 Y+\frac{\mu_2\pi_2(\psi+\omega)}{\varphi\psi\sigma_2}C^Y_7\\
&+d_S\left(1-\frac{S_7}{S}\right)\Delta S+\frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_7}{L}\right)\Delta L+\frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_7}{I}\right)\Delta I\\
&+\frac{d_E}{\varphi}\left(1-\frac{E_7}{E}\right)\Delta E+\frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_7}{Y}\right)\Delta Y+\frac{d_V\varkappa_1 S_7}{\varepsilon}\left(1-\frac{V_7}{V}\right)\Delta V\\
&+\frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C^I_7}{C^I}\right)\Delta C^I+\frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C^Y_7}{C^Y}\right)\Delta C^Y.
\end{aligned}$$

Using the steady state conditions for Đ_7:

$$\rho=\alpha S_7+\varkappa_1 S_7 V_7+\varkappa_2 S_7 I_7+\varkappa_3 S_7 Y_7,\quad \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 S_7 V_7+\varkappa_2 S_7 I_7)=\frac{\lambda(\gamma+\lambda)}{\beta\gamma+\lambda}L_7,$$
$$\varkappa_1 S_7 V_7+\varkappa_2 S_7 I_7=\frac{a(\gamma+\lambda)}{\beta\gamma+\lambda}I_7+\frac{\mu_1(\gamma+\lambda)}{\beta\gamma+\lambda}C^I_7 I_7,\quad I_7=\frac{\pi_1}{\sigma_1},\quad Y_7=\frac{\pi_2}{\sigma_2},\quad V_7=\frac{bI_7}{\varepsilon}$$

$$\varkappa_3 S_7 Y_7 + \frac{r}{\varphi} Y_7 = \frac{\psi+\omega}{\varphi} E_7 = \frac{\delta(\psi+\omega)}{\varphi\psi} Y_7 + \frac{\mu_2(\psi+\omega)}{\varphi\psi} C_7^Y Y_7.$$

We obtain

$$\begin{aligned}
\frac{\partial \Phi_7}{\partial t} &= \left(1-\frac{S_7}{S}\right)(\alpha S_7 - \alpha S) + (\varkappa_1 S_7 V_7 + \varkappa_2 S_7 I_7 + \varkappa_3 S_7 Y_7)\left(1-\frac{S_7}{S}\right) \\
&- \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_7 V_7 \frac{SVL_7}{S_7 V_7 L} - \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_7 I_7 \frac{SIL_7}{S_7 I_7 L} + \frac{\lambda(1-\beta)}{\beta\gamma+\lambda} \\
&\times(\varkappa_1 S_7 V_7 + \varkappa_2 S_7 I_7) - \frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_7 V_7 \frac{SVI_7}{S_7 V_7 I} - \frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_2 S_7 I_7 \frac{S}{S_7} \\
&- \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}(\varkappa_1 S_7 V_7 + \varkappa_2 S_7 I_7)\frac{LI_7}{L_7 I} + \varkappa_1 S_7 V_7 + \varkappa_2 S_7 I_7 \\
&- \varkappa_3 S_7 Y_7 \frac{SYE_7}{S_7 Y_7 E} - \frac{r}{\varphi} Y_7 \frac{YE_7}{Y_7 E} + \varkappa_3 S_7 Y_7 + \frac{r}{\varphi} Y_7 - \varkappa_3 S_7 Y_7 \frac{EY_7}{E_7 Y} \\
&- \frac{r}{\varphi} Y_7 \frac{EY_7}{E_7 Y} + \varkappa_3 S_7 Y_7 + \frac{r}{\varphi} Y_7 - \varkappa_1 S_7 V_7 \frac{IV_7}{I_7 V} + \varkappa_1 S_7 V_7 \\
&+ d_S\left(1-\frac{S_7}{S}\right)\Delta S + \frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_7}{L}\right)\Delta L + \frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_7}{I}\right)\Delta I \\
&+ \frac{d_E}{\varphi}\left(1-\frac{E_7}{E}\right)\Delta E + \frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_7}{Y}\right)\Delta Y + \frac{d_V \varkappa_1 S_7}{\varepsilon}\left(1-\frac{V_7}{V}\right)\Delta V \\
&+ \frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_7^I}{C^I}\right)\Delta C^I + \frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_7^Y}{C^Y}\right)\Delta C^Y \\
&= -\left[\alpha + \frac{\beta\varkappa_2 I_7(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\frac{(S-S_7)^2}{S} - \frac{r}{\varphi}\frac{(YE_7 - EY_7)^2}{EE_7 Y} \\
&+ \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_7 V_7\left(4 - \frac{S_7}{S} - \frac{SVL_7}{S_7 V_7 L} - \frac{LI_7}{L_7 I} - \frac{IV_7}{I_7 V}\right) \\
&+ \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_7 I_7\left(3 - \frac{S_7}{S} - \frac{SIL_7}{S_7 I_7 L} - \frac{LI_7}{L_7 I}\right) + \frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_7 V_7 \\
&\times\left(3 - \frac{S_7}{S} - \frac{SVI_7}{S_7 V_7 I} - \frac{IV_7}{I_7 V}\right) + \varkappa_3 S_7 Y_7\left(3 - \frac{S_7}{S} - \frac{SYE_7}{S_7 Y_7 E} - \frac{EY_7}{E_7 Y}\right) \\
&+ d_S\left(1-\frac{S_7}{S}\right)\Delta S + \frac{\lambda d_L}{\beta\gamma+\lambda}\left(1-\frac{L_7}{L}\right)\Delta L + \frac{d_I(\gamma+\lambda)}{\beta\gamma+\lambda}\left(1-\frac{I_7}{I}\right)\Delta I \\
&+ \frac{d_E}{\varphi}\left(1-\frac{E_7}{E}\right)\Delta E + \frac{d_Y(\psi+\omega)}{\varphi\psi}\left(1-\frac{Y_7}{Y}\right)\Delta Y + \frac{d_V \varkappa_1 S_7}{\varepsilon}\left(1-\frac{V_7}{V}\right)\Delta V \\
&+ \frac{\mu_1 d_{C^I}(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\left(1-\frac{C_7^I}{C^I}\right)\Delta C^I + \frac{\mu_2 d_{C^Y}(\psi+\omega)}{\varphi\psi\sigma_2}\left(1-\frac{C_7^Y}{C^Y}\right)\Delta C^Y.
\end{aligned}$$

Calculating $\frac{d\hat{\Phi}_7}{dt}$ and using equality (5.5) we obtain

$$\begin{aligned}
\frac{d\hat{\Phi}_7}{dt} &= -\left[\alpha + \frac{\beta\varkappa_2 I_7(\gamma+\lambda)}{\beta\gamma+\lambda}\right]\int_\Gamma \frac{(S-S_7)^2}{S}\,\mathrm{d}x - \frac{r}{\varphi}\int_\Gamma \frac{(YE_7 - EY_7)^2}{EE_7 Y}\,\mathrm{d}x \\
&+ \frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_1 S_7 V_7 \int_\Gamma \left(4 - \frac{S_7}{S} - \frac{SVL_7}{S_7 V_7 L} - \frac{LI_7}{L_7 I} - \frac{IV_7}{I_7 V}\right)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda(1-\beta)}{\beta\gamma+\lambda}\varkappa_2 S_7 I_7\int_\Gamma\left(3-\frac{S_7}{S}-\frac{SIL_7}{S_7I_7L}-\frac{LI_7}{L_7I}\right)\mathrm{d}x+\frac{\beta(\gamma+\lambda)}{\beta\gamma+\lambda}\varkappa_1 S_7V_7\\
&\times\int_\Gamma\left(3-\frac{S_7}{S}-\frac{SVI_7}{S_7V_7I}-\frac{IV_7}{I_7V}\right)\mathrm{d}x+\varkappa_3 S_7Y_7\int_\Gamma\left(3-\frac{S_7}{S}-\frac{SYE_7}{S_7Y_7E}-\frac{EY_7}{E_7Y}\right)\mathrm{d}x\\
&-d_S S_7\int_\Gamma\frac{\|\nabla S\|^2}{S^2}\mathrm{d}x-\frac{\lambda d_L L_7}{\beta\gamma+\lambda}\int_\Gamma\frac{\|\nabla L\|^2}{L^2}\mathrm{d}x-\frac{d_I I_7(\gamma+\lambda)}{\beta\gamma+\lambda}\int_\Gamma\frac{\|\nabla I\|^2}{I^2}\mathrm{d}x\\
&-\frac{d_E E_7}{\varphi}\int_\Gamma\frac{\|\nabla E\|^2}{E^2}\mathrm{d}x-\frac{d_Y Y_7(\psi+\omega)}{\varphi\psi}\int_\Gamma\frac{\|\nabla Y\|^2}{Y^2}\mathrm{d}x-\frac{d_V\varkappa_1 S_7V_7}{\varepsilon}\int_\Gamma\frac{\|\nabla V\|^2}{V^2}\mathrm{d}x\\
&-\frac{\mu_1 d_{C^I}C_7^I(\gamma+\lambda)}{\sigma_1(\beta\gamma+\lambda)}\int_\Gamma\frac{\|\nabla C^I\|^2}{(C^I)^2}\mathrm{d}x-\frac{\mu_2 d_{C^Y}C_7^Y(\psi+\omega)}{\varphi\psi\sigma_2}\int_\Gamma\frac{\|\nabla C^Y\|^2}{(C^Y)^2}\mathrm{d}x.
\end{aligned}
$$

Inequalities (5.1)–(5.4) imply that $\frac{d\hat{\Phi}_7}{dt}\le 0$ for all $S, L, I, E, Y, V, C^I, C^Y > 0$. Moreover, $\frac{d\hat{\Phi}_7}{dt} = 0$ when $(S, L, I, E, Y, V, C^I, C^Y) = (S_7, L_7, I_7, E_7, Y_7, V_7, C_7^I, C_7^Y)$. The solutions of model (2.4) converge to $\Upsilon_7' = \{Đ_7\}$. Applying Lyapunov-LaSalle asymptotic stability theorem we get $Đ_7$ is GAS.

In Table 1, we summarize the global stability results given in Theorems 1–8.

Table 1. Global stability conditions of the steady states of model (2.4).

Steady state	Global stability conditions
$Đ_0$	$\Re_1 \le 1$ and $\Re_2 \le 1$
$Đ_1$	$\Re_1 > 1$, $\Re_2/\Re_1 \le 1$ and $\Re_3 \le 1$
$Đ_2$	$\Re_2 > 1$, $\Re_1/\Re_2 \le 1$ and $\Re_4 \le 1$
$Đ_3$	$\Re_3 > 1$ and $\Re_5 \le 1$
$Đ_4$	$\Re_4 > 1$ and $\Re_6 \le 1$
$Đ_5$	$\Re_5 > 1$, $\Re_8 \le 1$ and $\Re_1/\Re_2 > 1$
$Đ_6$	$\Re_6 > 1$, $\Re_7 \le 1$ and $\Re_2/\Re_1 > 1$
$Đ_7$	$\Re_7 > 1$ and $\Re_8 > 1$

6. Numerical simulations

In this section, we numerically show the global stability of steady states using the values of the parameters given in Table 2. Moreover, we present comparison between single and dual infections. We choose the spatial domain as $\Gamma = [0, 2]$ with a step size 0.02. The step size for time is given by 0.1. Further, we choose the following initial conditions for system (2.1):

$$
\begin{aligned}
&S(x,0)=500\left[1+0.2\cos^2(\pi x)\right],\quad L(x,0)=1.5\left[1+0.5\cos^2(\pi x)\right],\\
&I(x,0)=1.5\left[1+0.5\cos^2(\pi x)\right],\quad E(x,0)=30\left[1+0.5\cos^2(\pi x)\right],\\
&Y(x,0)=0.3\left[1+0.5\cos^2(\pi x)\right],\quad V(x,0)=5\left[1+0.5\cos^2(\pi x)\right],\\
&C^I(x,0)=1+0.5\cos^2\left(\frac{3\pi}{2}x\right),\quad C^Y(x,0)=3\left[1+0.5\cos^2\left(\frac{3\pi}{2}x\right)\right],\quad x\in[0,2]. \qquad (6.1)
\end{aligned}
$$

In addition, we consider the homogeneous Neumann boundary conditions:

$$\frac{\partial S}{\partial \vec{V}} = \frac{\partial L}{\partial \vec{V}} = \frac{\partial I}{\partial \vec{V}} = \frac{\partial E}{\partial \vec{V}} = \frac{\partial Y}{\partial \vec{V}} = \frac{\partial V}{\partial \vec{V}} = \frac{\partial C^I}{\partial \vec{V}} = \frac{\partial C^Y}{\partial \vec{V}} = 0, \quad t > 0, \quad x = 0, 2. \tag{6.2}$$

Table 2. The values of the model's parameters.

Parameter	Value	Source	Parameter	Value	Source	Parameter	Value	Source
ρ	10	[44,65]	π_1	0.1	[67]	ψ	0.003	[44]
α	0.01	[6,44,66]	π_2	0.1	Assumed	d_S	0.1	[71]
$\varkappa_1$	Varied		μ_1	0.2	[68]	d_L	0.1	Assumed
$\varkappa_2$	Varied		μ_2	0.2	[46]	d_I	0.01	Assumed
$\varkappa_3$	Varied		ε	2	[68]	d_E	0.01	Assumed
a	0.5	[4]	β	0.7	[69]	d_Y	0.2	Assumed
φ	0.2	[35]	γ	0.02	Assumed	d_V	0.01	[72]
κ	0.9	[35]	σ_1	Varied		d_{C^I}	0.2	Assumed
r^*	0.008	Assumed	σ_2	Varied		d_{C^Y}	0.2	Assumed
δ^*	0.2	[46]	λ	0.2	[70]			
b	5	Assumed	ω	0.01	[44]			

6.1. Stability of the steady states

In this subsection, we select different values of $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, σ_1, and σ_2 under the above initial and boundary conditions which leads to the following strategies:

Strategy 1 (Stability of $Đ_0$): $\varkappa_1 = \varkappa_2 = 0.0001$, $\varkappa_3 = 0.001$, and $\sigma_1 = \sigma_2 = 0.2$. For this set of parameters, we have $\Re_1 = 0.68 < 1$ and $\Re_2 = 0.23 < 1$. Figure 1 shows that the solution of system (2.1) converges the steady state $Đ_0 = (1000, 0, 0, 0, 0, 0, 0, 0)$. This shows that $Đ_0$ is GAS according to Theorem 1. In this case both HTLV-I and HIV will be cleared.

Strategy *2* (Stability of $Đ_1$)*:* $\varkappa_1 = 0.0005$, $\varkappa_2 = 0.0003$, $\varkappa_3 = 0.0005$, $\sigma_1 = 0.003$, and $\sigma_2 = 0.2$. With such choice we get $\Re_2 = 0.12 < 1 < 3.02 = \Re_1$, $\Re_3 = 0.49 < 1$ and hence $\Re_2/\Re_1 = 0.04 < 1$. Theorem 2 implies that $Đ_1 = (331.63, 9.11, 13, 0, 0, 32.51, 0, 0)$ is GAS. This will lead to the situation of persistent HIV single infection but with an ineffective CTL immune response.

Strategy *3* (Stability of $Đ_2$)*:* $\varkappa_1 = 0.0001$, $\varkappa_2 = 0.0002$, $\varkappa_3 = 0.01$, $\sigma_1 = 0.001$, and $\sigma_2 = 0.05$. Then, we calculate $\Re_1 = 0.88 < 1 < 2.33 = \Re_2$, $\Re_4 = 0.78 < 1$ and then $\Re_1/\Re_2 = 0.38 < 1$. The numerical results show that $Đ_2 = (428, 0, 0, 88.74, 1.34, 0, 0, 0)$ exists and is GAS according to Theorem 3. It means that, a persistent HTLV single infection with an ineffective CTL immune response will be reached.

Strategy *4* (Stability of $Đ_3$)*:* $\varkappa_1 = 0.001$, $\varkappa_2 = 0.0001$, $\varkappa_3 = 0.005$, and $\sigma_1 = \sigma_2 = 0.01$. Then, we calculate $\Re_3 = 1.41 > 1$ and $\Re_5 = 0.32 < 1$. The numerical results show that $Đ_3 = (277.78, 9.85, 10, 0, 0, 25, 1.01, 0)$ is GAS based on Theorem 4. Hence, a persistent HIV single infection with effective HIV-specific CTL immune response is attained.

Strategy *5* (Stability of $Đ_4$)*:* $\varkappa_1 = 0.0007$, $\varkappa_2 = 0.0001$, $\varkappa_3 = 0.1$, $\sigma_1 = 0.05$, and $\sigma_2 = 0.3$. Then, we calculate $\Re_4 = 5.38 > 1$ and $\Re_6 = 0.83 < 1$. According to these data $Đ_4$ exists with

$Ð_4 = (230.77, 0, 0, 118.53, 0.33, 0, 0, 4.34)$ and it is GAS based on Theorem 5. In this case, a persistent HTLV single infection with effective HTLV-specific CTL immunity is reached.

Strategy *6* (Stability of $Ð_5$)*:* $\varkappa_1 = 0.001$, $\varkappa_2 = 0.0001$, $\varkappa_3 = 0.01$, $\sigma_1 = 0.05$, and $\sigma_2 = 0.08$. Then, we calculate $\Re_5 = 1.53 > 1$, $\Re_8 = 0.84 < 1$ and $\Re_1/\Re_2 = 2.17 > 1$. The numerical results show that $Ð_5 = (428, 3.03, 2, 54.21, 0.82, 5, 2.91, 0)$ exists and it is GAS and this supports Theorem 6. This case leads to a persistent dual infection with HTLV and HIV where the HIV-specific CTL immunity is effective and the HTLV-specific CTL immunity is ineffective.

Strategy 7 (Stability of $Ð_6$): $\varkappa_1 = 0.0006$, $\varkappa_2 = 0.0001$, $\varkappa_3 = 0.04$, $\sigma_1 = 0.01$, and $\sigma_2 = 0.5$. We compute $\Re_6 = 1.73 > 1$, $\Re_7 = 0.92 < 1$ and $\Re_2/\Re_1 = 2.99 > 1$. The numerical outcomes show that $Ð_6 = (321.26, 5.75, 8.2, 39.65, 0.2, 20.51, 0, 1.98)$ is GAS which support Theorem 7. This situation leads to a persistent dual infection with HTLV and HIV where the HTLV-specific CTL immunity is effective and the HIV-specific CTL immunity is not working.

Strategy *8* (Stability of $Ð_7$)*:* $\varkappa_1 = 0.0006$, $\varkappa_2 = 0.0002$, $\varkappa_3 = 0.04$, $\sigma_1 = 0.05$, and $\sigma_2 = 0.5$. These data give $\Re_7 = 1.55 > 1$ and $\Re_8 = 4.35 > 1$. Figure 2 illustrates that $Ð_7 = (467.29, 2.17, 2, 57.62, 0.2, 5, 1.36, 3.33)$ is GAS which confirms Theorem 8. In this case, a persistent dual infection with HTLV and HIV is reached where both the immune response is well working.

6.2. Comparison study

In this subsection, we compare between single and dual infections dynamics

Influence of HTLV infection on the dynamics of HIV single infection

To study the effect of HTLV infection on the dynamics of HIV single infection, we make a comparison between model (2.1) and the following HIV single infection model:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_S \Delta S(x,t) + \rho - \alpha S(x,t) - \varkappa_1 S(x,t)V(x,t) - \varkappa_2 S(x,t)I(x,t), \\ \frac{\partial L(x,t)}{\partial t} = d_L \Delta L(x,t) + (1-\beta) S(x,t) [\varkappa_1 V(x,t) + \varkappa_2 I(x,t)] - (\lambda + \gamma) L(x,t), \\ \frac{\partial I(x,t)}{\partial t} = d_I \Delta I(x,t) + \beta S(x,t) [\varkappa_1 V(x,t) + \varkappa_2 I(x,t)] + \lambda L(x,t) - aI(x,t) - \mu_1 C^I(x,t)I(x,t), \\ \frac{\partial V(x,t)}{\partial t} = d_V \Delta V(x,t) + bI(x,t) - \varepsilon V(x,t), \\ \frac{\partial C^I(x,t)}{\partial t} = d_{C^I} \Delta C^I(x,t) + \sigma_1 C^I(x,t)I(x,t) - \pi_1 C^I(x,t). \end{cases} \tag{6.3}$$

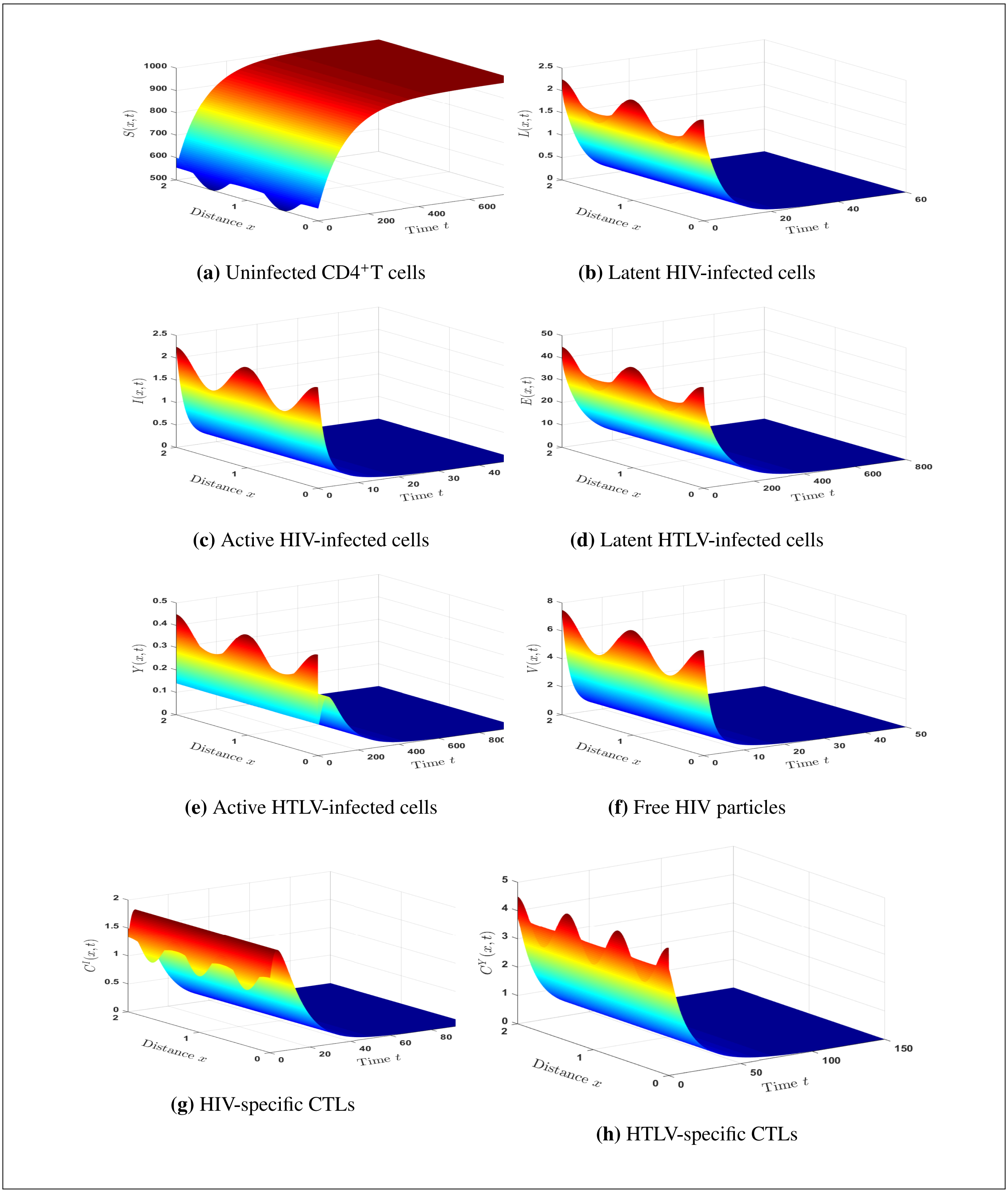
We fix the parameters $\varkappa_1 = 0.0006$, $\varkappa_2 = 0.0001$, $\sigma_1 = 0.05$, and $\sigma_2 = 0.5$ and consider initial conditions (6.1) and boundary conditions (6.2). We choose $\varkappa_3 = 0.04$ (HTLV/HIV dual infection). Figure 3 shows that when an individual who has only HIV infection is dually infected with HTLV then the numbers of uninfected (and latent) $CD4^+T$ cells and HIV-specific CTLs are declined. In contrast, the numbers of free HIV particles in both HIV single infection and HTLV/HIV dual infection limits to a same value. In fact, this observation is consistent with the recent study [73], where it has found that there is no worthy differences in the concentration of HIV particles in comparison between HIV single infected and HTLV/HIV dual infected patients.

Influence of HIV infection on the dynamics of HTLV single infection

To see the effect of HIV infection on the dynamics of HTLV single infection, we perform a comparison between model (2.1) and the following HTLV single infection model:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = d_S \Delta S(x,t) + \rho - \alpha S(x,t) - \varkappa_3 S(x,t) Y(x,t), \\ \frac{\partial E(x,t)}{\partial t} = d_E \Delta E(x,t) + \varphi \varkappa_3 S(x,t) Y(x,t) + \kappa r^* Y(x,t) - (\psi + \omega) E(x,t), \\ \frac{\partial Y(x,t)}{\partial t} = d_Y \Delta Y(x,t) + \psi E(x,t) + (1-\kappa) r^* Y(x,t) - \delta^* Y(x,t) - \mu_2 C^Y(x,t) Y(x,t), \\ \frac{\partial C^Y(x,t)}{\partial t} = d_{C^Y} \Delta C^Y(x,t) + \sigma_2 C^Y(x,t) Y(x,t) - \pi_2 C^Y(x,t). \end{cases} \tag{6.4}$$

We fix parameters $\varkappa_3 = 0.01$; $\sigma_1 = 0.05$, and $\sigma_2 = 0.5$ and consider initial conditions (6.1) and boundary conditions (6.2). We choose $\varkappa_1 = 0.001$ and $\varkappa_2 = 0.0002$ (HTLV/HIV dual infection). Figure 4 displays the solutions of two systems (2.1) and (6.4). We observe that the concentrations of uninfected $CD4^+T$ cells, latent HTLV-infected cells and HTLV-specific CTLs are smaller in case of dual infection than that of HTLV single infection. In contrast, the concentration of active HTLV-infected cells reaches the same value in both HTLV single and HTLV/HIV dual infections.



(a) Uninfected CD4$^+$T cells

(b) Latent HIV-infected cells

(c) Active HIV-infected cells

(d) Latent HTLV-infected cells

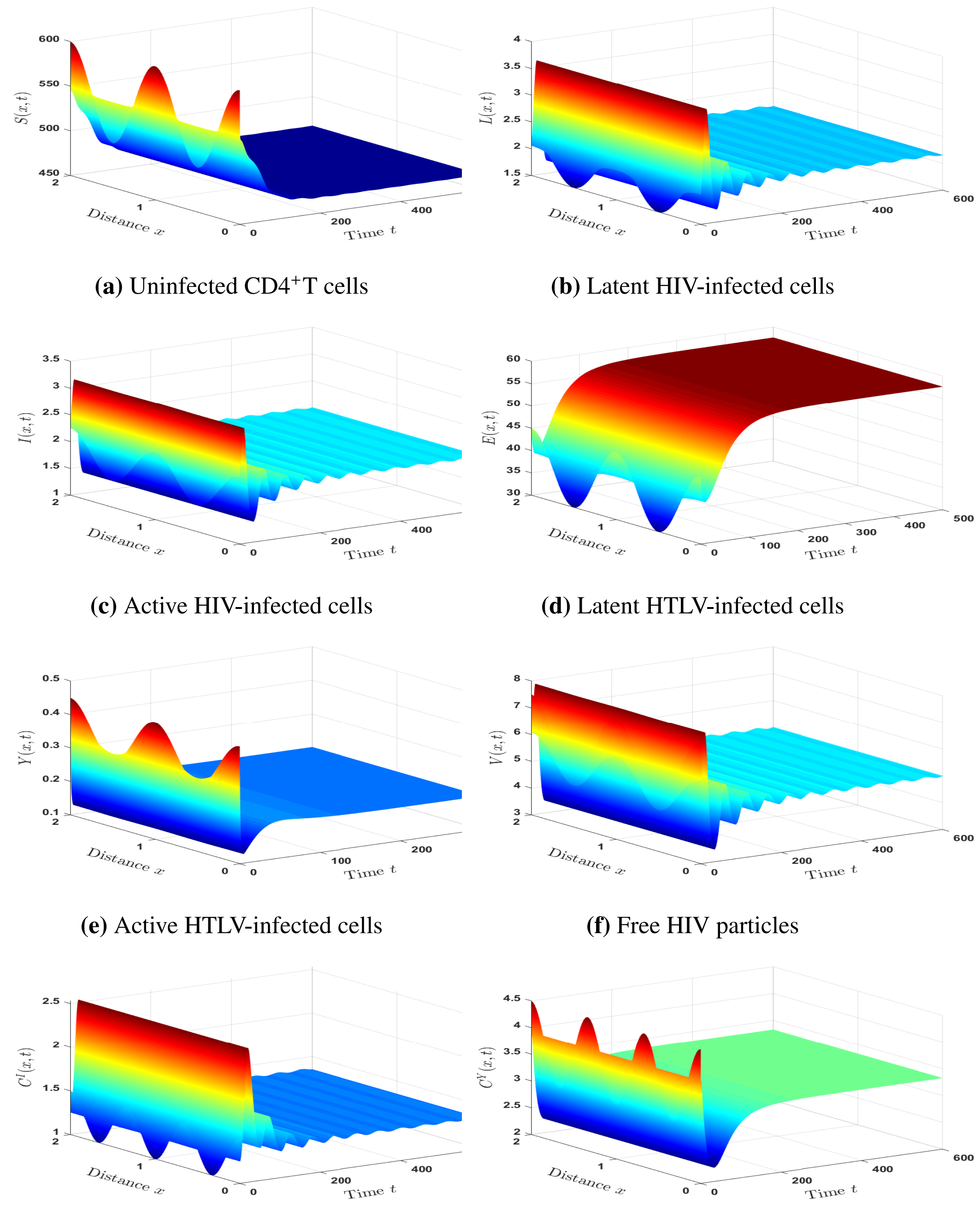
(e) Active HTLV-infected cells

(f) Free HIV particles

(g) HIV-specific CTLs

(h) HTLV-specific CTLs

Figure 1. Taking Strategy 1 ($\Re_1 \leq 1$ and $\Re_2 \leq 1$), the steady state $Đ_0 = (1000, 0, 0, 0, 0, 0, 0, 0)$ is asymptotically stable.



(a) Uninfected CD4^{+}T cells

(b) Latent HIV-infected cells

(c) Active HIV-infected cells

(d) Latent HTLV-infected cells

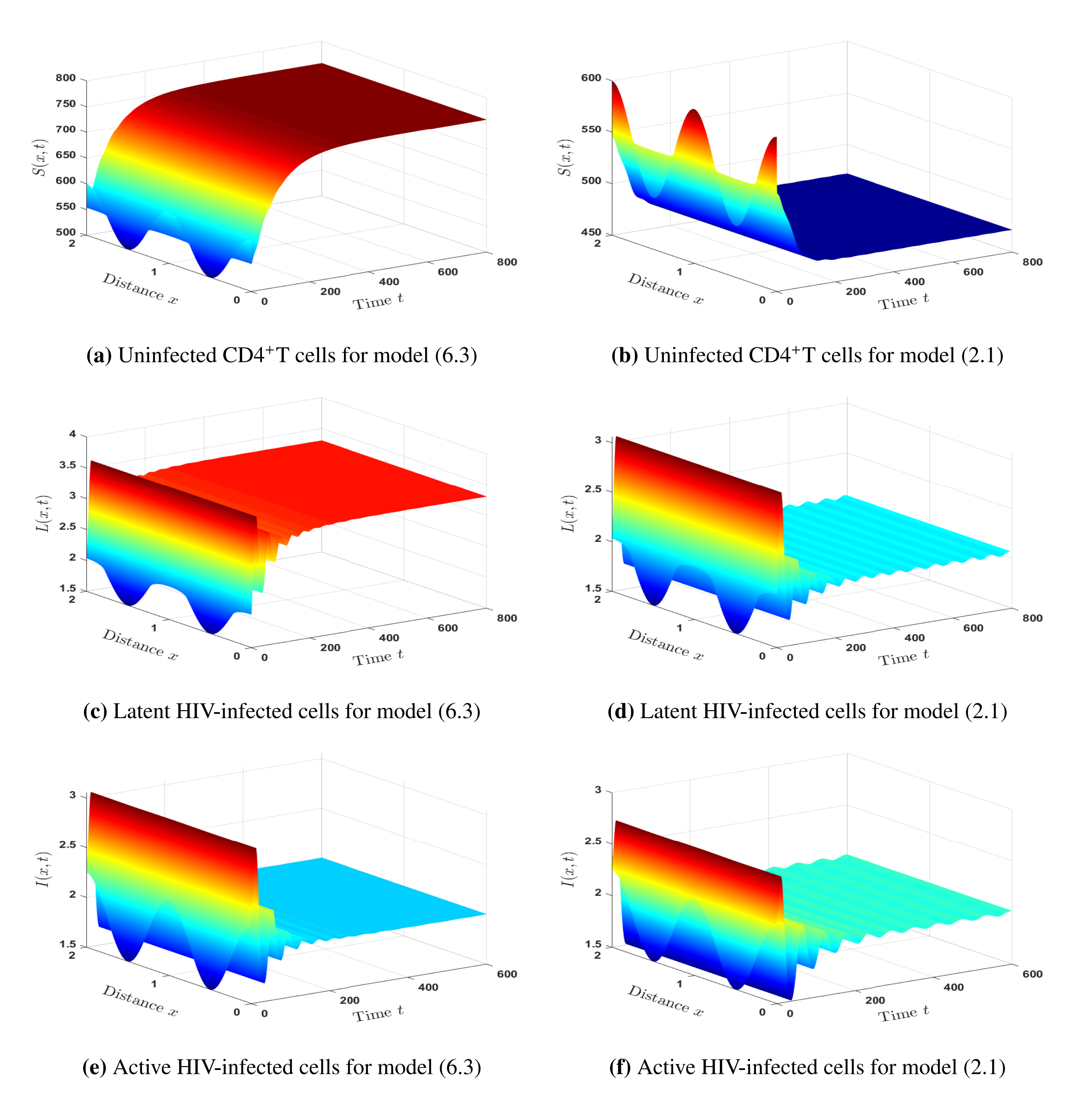
(e) Active HTLV-infected cells

(f) Free HIV particles

(g) HIV-specific CTLs

(h) HTLV-specific CTLs

Figure 2. Taking Strategy 8 ($\Re_7 > 1$ and $\Re_8 > 1$), the steady state $Đ_7 = (467.29, 2.17, 2, 57.62, 0.2, 5, 1.36, 3.33)$ is asymptotically stable.



(a) Uninfected CD4$^+$T cells for model (6.3)

(b) Uninfected CD4$^+$T cells for model (2.1)

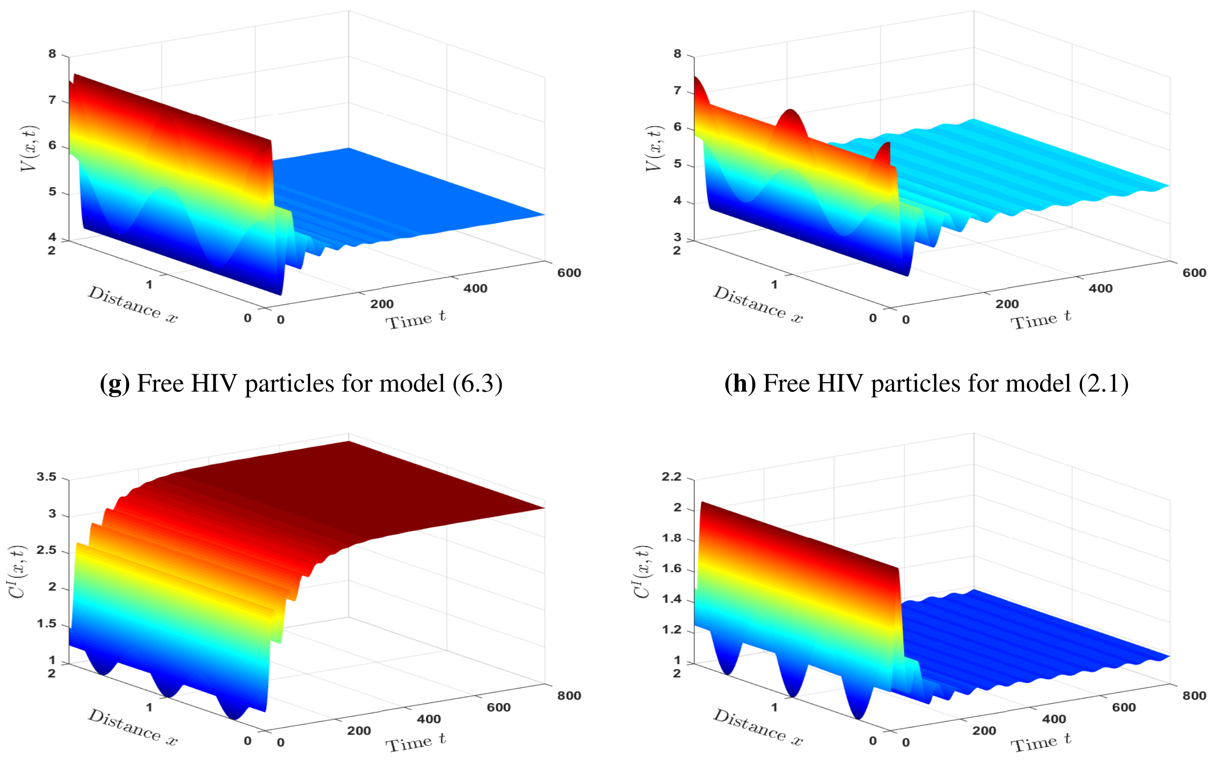
(c) Latent HIV-infected cells for model (6.3)

(d) Latent HIV-infected cells for model (2.1)

(e) Active HIV-infected cells for model (6.3)

(f) Active HIV-infected cells for model (2.1)

Figure 3. Comparison between the dynamics of HIV single infection and HTLV/HIV dual infection.



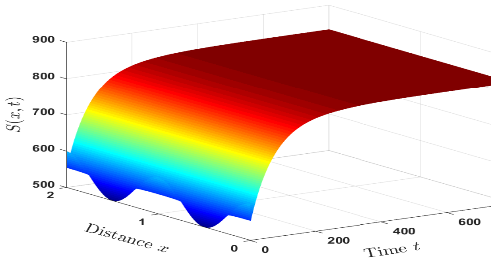
(g) Free HIV particles for model (6.3)

(h) Free HIV particles for model (2.1)

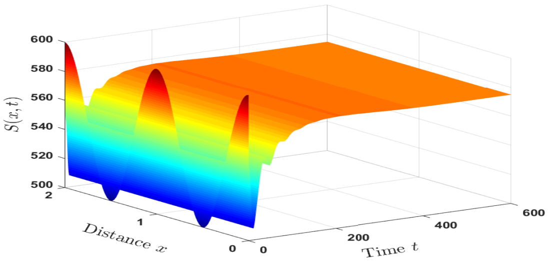
(i) HIV-specific CTLs for model (6.3)

(j) HIV-specific CTLs for model (2.1)

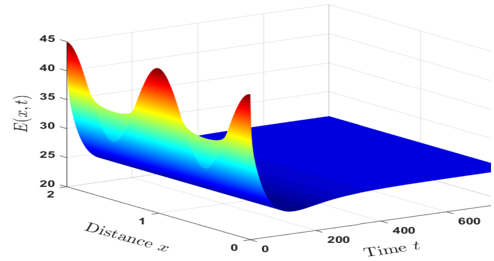
Figure 3. Comparison between the dynamics of HIV single infection and HTLV/HIV dual infection. (cont.)



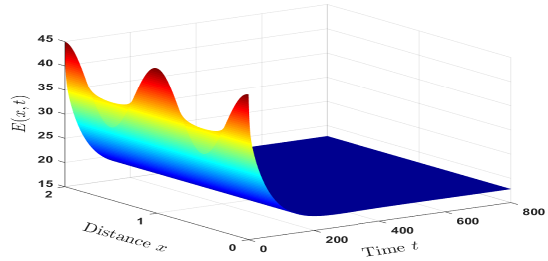
(a) Uninfected CD4$^+$T cells for model (6.4)



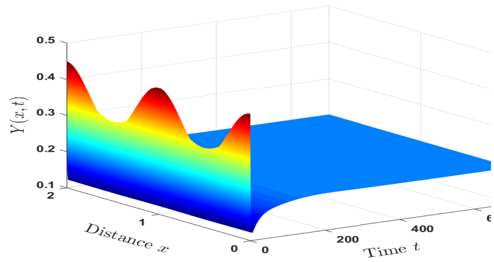
(b) Uninfected CD4$^+$T cells for model (2.1)



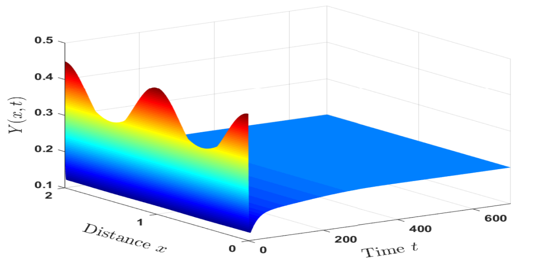
(c) Latent HTLV-infected cells for model (6.4)



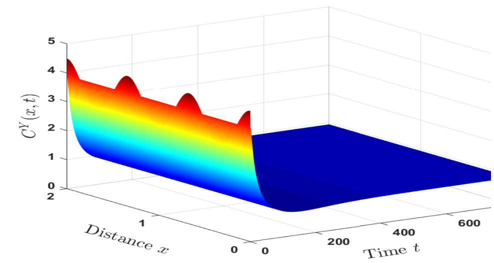
(d) Latent HTLV-infected cells for model (2.1)



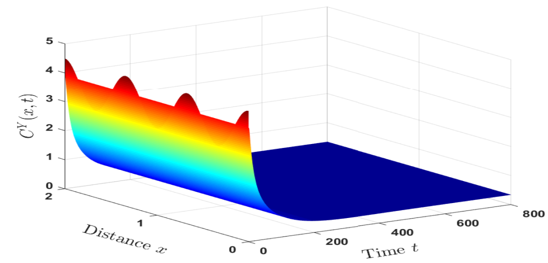
(e) Active HTLV-infected cells for model (6.4)



(f) Active HTLV-infected cells for model (2.1)



(g) HTLV-specific CTLs for model (6.4)



(h) HTLV-specific CTLs for model (2.1)

Figure 4. Comparison between the dynamics of HTLV single infection and HTLV/HIV dual infection.

7. Conclusions

This work proposes and investigates a within host HTLV/HIV dual infection model taking into account the mobility of viruses and cells. The model was given by 8-dimentional nonlinear PDEs which describe the evolution of eight compartments with respect to position and time; uninfected $CD4^{+}T$ cells, latent HIV-infected cells, active HIV-infected cells, latent HTLV-infected cells, active HTLV-infected cells, free HIV particles, HIV-specific CTLs, and HTLV-specific CTLs. We considered two ways of HIV transmission, free-to-cell and infected-to-cell. We also included two directions of HTLV transmission, horizontal via infected-to-cell contact, and vertical transmission through mitosis of active HTLV-infected cells. We first showed the existence of global solutions and the boundedness of the model's solutions. We showed that the model has eight steady states and their existence and stability are governed by eight threshold parameters. The global asymptotic stability of all steady states was investigated by formulating suitable Lyapunov functions and utilizing Lyapunov-LaSalle asymptotic stability theorem. We conducted some numerical simulations to clearify the theoretical results. We made a comparison between the dynamical behavior of dual HTLV/HIV infection and single HTLV (or HIV) infection. We found that HTLV/HIV dual infected patients have less uninfected $CD4^{+}T$ cells counts in comparison with HTLV or HIV single infected patients.

Conflict of interest

The authors declare that they have no conflict interests.

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