*Mathematical Biosciences
and Engineering*

Research article

Stability and optimal harvesting of a predator-prey system combining prey refuge with fuzzy biological parameters

Qinglong Wang, Shuqi Zhai, Qi Liu and Zhijun Liu[*]

School of Mathematics and Statistics, Hubei Minzu University, Enshi, Hubei 445000, China

* **Correspondence:** Email: zjliu@hbmzu.edu.cn; Tel: +86(0)7188437732.

Abstract: In this manuscript, a novel predator-prey system combining prey refuge with fuzzy parameters is formulated. Sufficient conditions for the existence and stability of biological equilibria are derived. The existence of bionomic equilibria is discussed under fuzzy biological parameters. The optimal harvesting policy, by Pontryagin's maximal principle, is also investigated under imprecise inflation and discount in fuzzy environment. Meticulous numerical simulations are performed to validate our theoretical analysis in detail.

Keywords: predator-prey system; prey refuge; fuzzy number; stability; bionomic equilibrium; fuzzy optimal harvesting

1. Introduction

Predator-prey system is one of the most important systems in ecology to study the relationship between prey and predator. For the sake of avoiding predation and increasing the survival rate, prey takes different strategies, for example, changing colour, releasing smell, injecting poison, seeking refuges and so on. Since Gause et al. [1] and Smith [2] introduced a quantity involving prey refuge, the study of prey refuge changing the dynamics of predator-prey system has been widely concerned by researchers, see [3–6] in details. Olivares and Jiliberto [7] put forward that the quantity on prey refuge can be expressed as a constant number or depending on the density of prey. Kar [8], Huang et al. [9] and Tripathi [10] separately studied the impact of refuge on dynamical behavior of predator-prey systems with Holling type II, Holling type III and Beddington-DeAngelis type function responses. Han et al. [11] studied the dynamical behavior of a spatiotemporal predator-prey model with refuge and diffusion including boundedness, permanence, coexisting equilibrium and Turing instability parameter space. Qi and Meng [12] investigated the threshold behavior of a stochastic predator-prey system with refuge and fear effect, and obtained the thresholds determining the extinction and persistence in the mean of prey and predator and the existence of a unique ergodic

stationary distribution.

Natural resources are the material basis for human survival and development, among which biological resources are the resources with the most unique development mechanism. They can be continuously renewed via their own reproduction, which is the material embodiment of earth's biological diversity. However, biological resources are also a kind of exhaustible natural resources, and the environmental deterioration or human unreasonable use or excessive development will not only make its quantity and quality decline, but may even lead to the extinction of species. With the demand for food and energy for the development of human beings being rapidly increasing, biological resources are being exploited more and more. Scientific management of the use of marketable biological resources such as fishery, forestry and wildlife is necessary to protect ecosystems. The exploitation of renewable resources used to base on maximum sustainable yield (MSY), but it is impractical to completely ignore the cost operation of resource exploitation. In the face of the insufficiency of MSY, people tried to replace MSY with the optimal sustainable yield (OSY). Lately the research on optimal management of renewable resources has aroused the interest of many scientists and researchers [13–17] and the references therein. Clark [18, 19] laid the foundation in the area of work. Kar and Chaudhuri [20] researched a harvesting model on two competing preys and a predator. He and Zhou [21] studied optimal harvesting for a nonlinear hierarchical age-structured population model.

In the study of biological mathematical models, many works focus on the deterministic model, which is mainly based on the law of large numbers and the assumption that the number of biological individuals is sufficiently large, the behaviors of the system will present a relatively stable statistical regularity, so as to be approximately regarded as a deterministic model. However, any species in nature will inevitably be affected by the complexity of the ecosystem itself and the limitations of human cognition, for example, the uncertainty caused by nature environment or human society such as forest fire, flood disaster, debris flow, volcanic eruption, earthquake, climate warming and so on, the uncertainty caused by simplified hypothesis, the uncertainty caused by replacing infinite samples with limited experimental samples in statistical analysis, the uncertainty caused by test results restricted by objective experimental conditions, even the uncertainty caused by some unknown factors. To tackle these uncertainty factors, many researchers used to adopt interval approach [22–26] and stochastic approach [27–31]. In the interval approach the uncertain parameters are characterized by interval-valued functions. In the stochastic approach the uncertain parameters are replaced as stochastic variables with known probability distribution functions. Nevertheless, the interval approach is too simple to do not achieve ideal effect on complex problems when characterizing imprecise parameters. Also a question arising for stochastic approach is that for imprecise parameters, appropriate probability distribution functions will make the model more complex. To overcome these difficulties, we apply fuzzy set theory to depict imprecise parameters. Fuzzy set theory was first come up with by professor Zadeh [32] and he also considered the appliance of fuzzy differential equations is an inherent way to model dynamic systems under possibilistic uncertainty [33]. And Chang and Zadeh [34] first initiated the concept of the fuzzy derivative. And then Kaleva [35] first put forward the concept of differential equations in a fuzzy environment. All derivatives are considered as Hukuhara derivative or generalized derivative in FDE (fuzzy differential equation). Bede et al. [36] studied first order linear fuzzy differential equations under generalized differentiability, and Khastan and Nieto [37] analysed a boundary value problem for second order fuzzy differential equations.

Stefanini and Bede [38] proposed generalization of the Hukuhara difference for compact convex sets, to introduce and study a generalization of the Hukuhara differentiability to the case of interval-valued functions. Puri and Ralescu [39] generalized the concept of Hukuhara differentiability (H-derivative) for set-valued mappings to fuzzy mappings classes. Kaleva [35] developed the theory for FDE by using the H-derivative. Of course, different ways to solve FDE have a huge number of works in this field. Bede and Gal [40] studied the solutions of fuzzy differential equations based on generalized differentiability. Hullermeier [41] proposed an approach to modelling and simulation of uncertain dynamical systems, and showed that all (reasonable) fuzzy functions can be approximated to any degree of accuracy in this way. Allahviranloo and Ahmadi [42] applied Laplace transform method to solve linear FDE. Also the existence and uniqueness of solution for fuzzy initial value problems are investigated by Villamizar-Roa et al. [43]. Different forms of FDEs are come up with in dynamics and good results have been achieved [23, 44–47]. Bassanezi et al. [48] is the cornerstone of studying the stability of dynamical system by employing fuzzy differential equations. Mizukoshi et al. [49] studied the stability of fuzzy dynamical systems considering initial conditions under fuzziness. Guo et al. [50] established fuzzy impulsive functional differential equations and applied them in logistic model and Gompertz model. The existence and uniqueness of solution for fuzzy functional differential equations were obtained by Lupulescu [51]. Recently, Sadhukhan et al. [52] considered optimal harvesting of a food chain model in fuzzy environment, it is worth mentioning that based on the fuzzy instantaneous annual rate of discount, the optimal harvesting policy is discussed. Pal and Mahapatra [23] drew interval number and fuzzy number into prey and predator interaction's model. The existence and stability of biological and bionomic equilibria are investigated under interval parameters, and the optimal harvesting policy considering instantaneous annual rate of discount in fuzzy environment is studied. Besides, Pal et al. [44] discussed the stability of a fuzzy predator-prey harvesting model with toxicity. And Pal et al. [53] also put forward a predator-prey model with fuzzy optimal harvesting in the environment of toxicity. The former is a major consideration on fuzzy number applied in the parameters of model, the latter mainly considers the inflation and discount rates as fuzzy number, and obtains the fuzzy optimal harvesting. However, for all we know, no one tried to consider both fuzzy biological parameters and fuzzy optimal harvesting in a model. This paper first establishes a predator-prey model with refuge, simultaneously considers both fuzzy biological parameters and fuzzy optimal harvesting into model. The dynamical behaviors covering the existence and stability of biological equilibria are performed. Moreover, there exist bionomic equilibria under fuzzy biological parameters. The optimal harvesting policy is obtained under imprecise inflation and discount in fuzzy environment.

The remaining part of this article is emerged as follows. Firstly, the concepts on fuzzy set and weighted sum method are put forward in Section 2. Section 3 presents the suitable model under fuzzy biological parameters in imprecise environment. The positivity and boundedness of system, as well as the existence and stability of equilibria are investigated in Sections 4 and 5, respectively. In Section 6, we discuss all kinds of bionomic equilibria. The fuzzy optimal harvesting is presented in Section 7. Some numerical examples verifying the theoretical results are displayed in Section 8. The last section gives a brief summary of the work of this paper.

2. Preliminaries

This section provides definitions, lemmas and methods that will be used in latter sections.

2.1. Concept of fuzzy set

Definition 2.1. *[32] Fuzzy set: A fuzzy set $\tilde{A}$ in a universe of discourse X is defined as the following set of pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. The mapping $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ is called the membership function of the fuzzy set $\tilde{A}$ and $\mu_{\tilde{A}}$ is called the membership value or degree of membership of $x \in X$ in the fuzzy set $\tilde{A}$.*

Definition 2.2. *[54] α-cut of fuzzy set: The α-cut of a fuzzy set $\tilde{A}$ is a crisp set and it is defined by $A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\}, \alpha \in (0, 1]$. For $\alpha = 0$ the support of $\tilde{A}$ is defined as $A_0 = Supp(\tilde{A}) = \overline{\{x \in R, \mu_{\tilde{A}}(x) > 0\}}$.*

Definition 2.3. *[32] Convex fuzzy set: A convex fuzzy set $\tilde{A}$ is a fuzzy set on a continuous universe such that for all α, A_α is a convex classical set.*

Definition 2.4. *[55] Fuzzy number: A fuzzy number is a convex fuzzy set with $X = R$.*

Definition 2.5. *[56] Triangular fuzzy number: A triangular fuzzy number (TFN) $\tilde{A} \equiv (a_1, a_2, a_3)$ is fuzzy set of the real line R characterized by the membership function $\mu_{\tilde{A}} : R \rightarrow [0, 1]$ as follows*

$$\mu_{\tilde{A}} = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } a_1 \leq x \leq a_2, \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{if } a_2 \leq x \leq a_3, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the α-cut of triangular fuzzy number is a bounded and closed internal $[A_L(\alpha), A_R(\alpha)]$, where $A_L(\alpha) = \inf\{x : \mu_{\tilde{A}}(x) \geq \alpha\} = a_1 + \alpha(a_2 - a_1)$ and $A_R(\alpha) = \sup\{x : \mu_{\tilde{A}}(x) \geq \alpha\} = a_3 - \alpha(a_3 - a_2)$.

The arithmetic operations on fuzzy numbers are provided by the following lemmas. Firstly, let us define E as the set of all upper semi-continuous normal convex fuzzy numbers with $\tilde{A} \in E$, then the α-level set A_α is a bounded closed interval which can be expressed as $A_\alpha = [A_L^\alpha, A_R^\alpha]$, where $A_L^\alpha = \inf\{x : \mu_{\tilde{A}}(x) \geq \alpha\}$ and $A_R^\alpha = \sup\{x : \mu_{\tilde{A}}(x) \geq \alpha\}$.

Lemma 2.1. *[57] If $\tilde{A}, \tilde{B} \in E$, then for $\alpha \in (0, 1]$,*
(1) $[\tilde{A} + \tilde{B}]_\alpha = [A_L^\alpha + B_L^\alpha, A_R^\alpha + B_R^\alpha]$,
(2) $[\tilde{A} \cdot \tilde{B}]_\alpha = [\min\{A_L^\alpha B_L^\alpha, A_L^\alpha B_R^\alpha, A_R^\alpha B_L^\alpha, A_R^\alpha B_R^\alpha\}, \max\{A_L^\alpha B_L^\alpha, A_L^\alpha B_R^\alpha, A_R^\alpha B_L^\alpha, A_R^\alpha B_R^\alpha\}]$,
(3) $[\tilde{A} - \tilde{B}]_\alpha = [A_L^\alpha - B_R^\alpha, A_R^\alpha - B_L^\alpha]$.

Lemma 2.2. *[58] $[A_L^\alpha, A_R^\alpha], 0 < \alpha \leq 1$ is a family of nonempty intervals. If*
(1) $[A_L^\alpha, A_R^\alpha] \supset [A_L^\beta, A_R^\beta]$ *for $0 < \alpha \leq \beta$ and*
(2) $[\lim_{k \to \infty} A_L^{\alpha_k}, \lim_{k \to \infty} A_R^{\alpha_k}] = [A_L^\alpha, A_R^\alpha]$,
whenever non-decreasing sequence α_k converges to $\alpha \in (0, 1]$; then, the family $[A_L^\alpha, A_R^\alpha], 0 < \alpha \leq 1$, represents the α-level sets of a fuzzy number $\tilde{A}$ in E. Conversely, if $[A_L^\alpha, A_R^\alpha], 0 < \alpha \leq 1$ are the α-level sets of a fuzzy number $\tilde{A} \in E$, then the conditions (1) and (2) hold true.

2.2. Weighted sum method

In weighted sum method [59], a simple utility function is in the form of $w_i g_i$ for ith objective, where w_i stands for the weight of ith objective. And the total utility function U is expressed as follows

$$U = \sum_{i}^{l} w_i g_i, \quad i = 1, 2, \cdots, l, \tag{2.1}$$

where $w_i > 0$ and $\sum_{i}^{l} w_i = 1$ are satisfied.

3. Model foundation

Motivated by the predator-prey system (3) in Wang et al. [5] as follows

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{K}\right) - c(1 - m)xy - q_1 E_1 x, \\ \dfrac{dy}{dt} = -dy - sy^2 + e(1 - m)xy - q_2 E_2 y, \end{cases} \tag{3.1}$$

where r, K, c, m, d, s and e are positive constants. $x(t)$ and $y(t)$ depict the scale of prey and predator at t with initial densities $x_1(0) > 0$ and $x_2(0) > 0$. r is the growth rate of prey, K stands for the carrying capacity of prey, c is the predation rate for prey, m shows the proportion of quantity of prey using refuges, d expresses as the death rate of predator and s is the intraspecies competition rate of predator, c is the predation rate, e represents the product of predation rate and conversion rate, q_1, q_2 and E_1, E_2 respectively say the catchability coefficients and the harvesting efforts of prey and predator.

On account of the concept of fuzzy set, we regard the imprecise parameters $\tilde{r}, \tilde{c}, \tilde{d}, \tilde{s}$ and $\tilde{e}$ as triangular fuzzy numbers. Throughout Hukuhara's concept of derivative [39], we present fuzzy differential equations to describe predator-prey interaction with harvesting items as follows

$$\begin{cases} \dfrac{\widetilde{dx}}{dt} = \tilde{r}x\left(1 - \dfrac{x}{K}\right) - \tilde{c}(1 - m)xy - q_1 E_1 x, \\ \dfrac{dy}{dt} = -\tilde{d}y - \tilde{s}y^2 + \tilde{e}(1 - m)xy - q_2 E_2 y. \end{cases} \tag{3.2}$$

Here, prey increases in the absence of predator with fuzzy growth rate $\tilde{r}$ while predator decreases with fuzzy mortality $\tilde{d}$ for lack of prey. $\tilde{s}$ expresses as fuzzy decrease rate of predator attributed to intraspecific competition. The fuzzy quantity of the prey predated per one predator under unit time yields to $\tilde{c}(1 - m)x$, and $\tilde{e}$ represents the fuzzy growth rate of predator attacking on prey.

Applying α-level to cut triangular fuzzy numbers $\tilde{r}, \tilde{c}, \tilde{d}, \tilde{s}$ and $\tilde{e}$, system (3.2) can be expressed as follows

$$\begin{cases} \left(\dfrac{dx}{dt}\right)_L^\alpha = (r_L)^\alpha x - (r_R)^\alpha \dfrac{x^2}{K} - (c_R)^\alpha (1 - m)xy - q_1 E_1 x, \\ \left(\dfrac{dx}{dt}\right)_R^\alpha = (r_R)^\alpha x - (r_L)^\alpha \dfrac{x^2}{K} - (c_L)^\alpha (1 - m)xy - q_1 E_1 x, \\ \left(\dfrac{dy}{dt}\right)_L^\alpha = -(d_R)^\alpha y - (s_R)^\alpha y^2 + (e_L)^\alpha (1 - m)xy - q_2 E_2 y, \\ \left(\dfrac{dy}{dt}\right)_R^\alpha = -(d_L)^\alpha y - (s_L)^\alpha y^2 + (e_R)^\alpha (1 - m)xy - q_2 E_2 y. \end{cases} \tag{3.3}$$

By the method of weighted sum, the above system translates into

$$
\begin{cases}
\dfrac{dx}{dt} = w_1\left(\dfrac{dx}{dt}\right)_L^\alpha + w_2\left(\dfrac{dx}{dt}\right)_R^\alpha, \\
\dfrac{dy}{dt} = w_1\left(\dfrac{dy}{dt}\right)_L^\alpha + w_2\left(\dfrac{dy}{dt}\right)_R^\alpha,
\end{cases}
\tag{3.4}
$$

where w_1 and w_2 express as weight satisfying $w_1 + w_2 = 1$, and $w_1, w_2 \geq 0$. The simplified form of system (3.4) is given by

$$
\begin{cases}
\dfrac{dx}{dt} = A_1 x - A_2 \dfrac{x^2}{K} - A_3(1-m)xy - q_1 E_1 x, \\
\dfrac{dy}{dt} = -B_1 y - B_2 y^2 + B_3(1-m)xy - q_2 E_2 y,
\end{cases}
\tag{3.5}
$$

where

$$
\begin{aligned}
A_1 &= w_1(r_L)^\alpha + w_2(r_R)^\alpha, \quad A_2 = w_1(r_R)^\alpha + w_2(r_L)^\alpha, \quad A_3 = w_1(c_R)^\alpha + w_2(c_L)^\alpha, \\
B_1 &= w_1(d_R)^\alpha + w_2(d_L)^\alpha, \quad B_2 = w_1(s_R)^\alpha + w_2(s_L)^\alpha, \quad B_3 = w_1(e_L)^\alpha + w_2(e_R)^\alpha.
\end{aligned}
\tag{3.6}
$$

Of course if we consider imprecise biological parameters with interval parameters rather than fuzzy parameters, then system (3.5) can be translated into system (5) in Wang [5]. Moreover, if we neglect the intraspecific competition rates of prey and predator, as well as refuges, system (3.5) can be simplified to Eqs (7) and (8) in Pal et al. [22].

4. Positivity and boundedness of system

Throughout the following theorem, in this section, we make sure the positivity and boundedness of system (3.5).

Theorem 4.1. *Any solution of system* (3.5) *is positive and bounded satisfying initial conditions* $x(0) > 0$ *and* $y(0) > 0$ *for all* $t > 0$.

Proof. The right hand of system (3.5) satisfies completely continuous and locally Lipschitzian on C, which guarantees that there exists a unique solution $(x(t), y(t))$ of system (3.5) under initial conditions $x(0) > 0$ and $y(0) > 0$ on $[0, \xi)$, where $0 < \xi < +\infty$. It follows from system (3.5) with $x(0) > 0$ and $y(0) > 0$ that

$$
\begin{aligned}
x(t) &= x(0) \exp\left[\int_0^t \left(A_1 - \frac{A_2 x(s)}{K} - A_3(1-m)y(s) - q_1 E_1\right) ds\right] > 0, \\
y(t) &= y(0) \exp\left[\int_0^t \left(-B_1 - B_2 y(s) + B_3(1-m)x(s) - q_2 E_2\right) ds\right] > 0,
\end{aligned}
\tag{4.1}
$$

which ensures the solution of system (3.5) is positive.

On the other side, the first differential equation of system (3.5) provides

$$
\frac{dx}{dt} \leq A_1 x\left(1 - \frac{x}{\frac{A_1 K}{A_2}}\right),
\tag{4.2}
$$

which implies $\limsup_{t \to \infty} x(t) \leq \frac{A_1 K}{A_2} \equiv k_1$. From the second differential equation of system (3.5), one has

$$
\frac{dy}{dt} \leq k_1 B_3(1-m)y - B_2 y^2,
$$

which can be written as

$$\frac{dy}{dt} \leq k_1 B_3 (1-m) y \left(1 - \frac{y}{\frac{k_1 B_3 (1-m)}{B_2}} \right).$$

(4.3)

It is easy to see $\limsup_{t \to \infty} y(t) \leq \frac{k_1 B_3 (1-m)}{B_2}$. Therefore, the solution of system (3.5) is bounded.

5. Existence and stability of equilibria

We firstly put forward the existence of equilibria of system (3.5). Besides the local stability and global stability of the above equilibria are investigated by the method of Jacobian matrix and Lyapunov function, respectively.

By a simple calculation, the equilibria of system (3.5) are given by

(1) Trivial equilibrium: $S_0 = (0, 0)$.
(2) Axial equilibrium: $S_1 = (\frac{K(A_1 - q_1 E_1)}{A_2}, 0)$ exists if $A_1 > q_1 E_1$.
(3) Axial equilibrium: $S_2 = (0, -\frac{q_2 E_2 + B_1}{B_2})$ does not exist since $-\frac{q_2 E_2 + B_1}{B_2} < 0$.
(4) Interior equilibrium: $S^* = (x^*, y^*)$, where

$$
\begin{aligned}
x^* &= \frac{KA_1 B_2 + KA_3 B_1 (1-m) + KA_3 (1-m) q_2 E_2 - q_1 E_1 B_2 K}{A_2 B_2 + KA_3 B_3 (1-m)^2}, \\
y^* &= \frac{B_3 (1-m) x^* - (B_1 + q_2 E_2)}{B_2}
\end{aligned}
$$

(5.1)

exists if $B_3 (1-m) x^* > B_1 + q_2 E_2$ and $A_1 B_2 + A_3 (1-m)(B_1 + q_2 E_2) > B_2 q_1 E_1$ are satisfied.

5.1. Local stability

On account of the following theorem, the local stability of equilibria S_0, S_1, S^* for system (3.5) is discussed.

Theorem 5.1. *The following conclusions hold:*
(1) *Trivial equilibrium S_0 becomes stable node when $A_1 < q_1 E_1$ or saddle point when $A_1 > q_1 E_1$.*
(2) *Axial equilibrium S_1 becomes stable node when*

$$A_1 > q_1 E_1 \quad and \quad A_2(B_1 + q_2 E_2) > KB_3(1-m)(A_1 - q_1 E_1),$$

(5.2)

or saddle point when

$$A_1 > q_1 E_1 \quad and \quad A_2(B_1 + q_2 E_2) < KB_3(1-m)(A_1 - q_1 E_1).$$

(5.3)

(3) *Interior equilibrium S^* is stable node or focus when*

$$B_3(1-m)x^* > B_1 + q_2 E_2 \quad and \quad A_1 B_2 + A_3(1-m)(B_1 + q_2 E_2) > B_2 q_1 E_1.$$

(5.4)

Proof. The following Jacobian matrix of system (3.5) is expressed as

$$
M = \begin{pmatrix} A_1 - q_1 E_1 - 2A_2 \frac{x}{K} - A_3(1-m)y & -A_3(1-m)x \\ B_3(1-m)y & -(B_1 + q_2 E_2) - 2B_2 y + B_3(1-m)x \end{pmatrix}.
$$

(5.5)

(1) The Jacobian matrix M_0 at S_0 is represented as

$$M_0 = \begin{pmatrix} A_1 - q_1 E_1 & 0 \\ 0 & -B_1 - q_2 E_2 \end{pmatrix}.$$

According to the characteristic equation about the above matrix

$$(\lambda - A_1 + q_1 E_1)(\lambda + B_1 + q_2 E_2) = 0,$$

one has $\lambda_1 = A_1 - q_1 E_1$, $\lambda_2 = -B_1 - q_2 E_2$, hence A_0 is stable node when $A_1 < q_1 E_1$ or saddle point when $A_1 > q_1 E_1$.

(2) The Jacobian matrix M_1 at S_1 is depicted as

$$M_1 = \begin{pmatrix} -(A_1 - q_1 E_1) & -\frac{KA_3(1-m)(A_1 - q_1 E_1)}{A_2} \\ 0 & -(B_1 + q_2 E_2) + \frac{KB_3(1-m)(A_1 - q_1 E_1)}{A_2} \end{pmatrix}.$$

In accordance with the characteristic equation of the above matrix

$$(\lambda + A_1 - q_1 E_1)\left(\lambda + B_1 + q_2 E_2 - \frac{KB_3(1-m)(A_1 - q_1 E_1)}{A_2}\right) = 0,$$

we have $\lambda_1 = -(A_1 - q_1 E_1)$ and $\lambda_2 = -(B_1 + q_2 E_2) + \frac{KB_3(1-m)(A_1-q_1E_1)}{A_2}$. Obviously, S_1 exists if $A_1 > q_1 E_1$ holds, hence S_1 is a stable node when $A_1 > q_1 E_1$ and $A_2(B_1 + q_2 E_2) > KB_3(1-m)(A_1 - q_1 E_1)$ or saddle point when $A_1 > q_1 E_1$ and $A_2(B_1 + q_2 E_2) < KB_3(1 - m)(A_1 - q_1 E_1)$.

(3) The Jacobian matrix M^* at S^* is written as

$$M^* = \begin{pmatrix} -A_2 \frac{x^*}{K} & -A_3(1-m)x^* \\ B_3(1-m)y^* & -B_2 y^* \end{pmatrix}.$$

The characteristic equation is expressed as

$$\begin{vmatrix} \lambda + A_2 \frac{x^*}{K} & A_3(1-m)x^* \\ -B_3(1-m)y^* & \lambda + B_2 y^* \end{vmatrix} = \lambda^2 + \left(B_2 y^* + \frac{A_2 x^*}{K}\right)\lambda + A_3 B_3(1-m)^2 x^* y^* = 0. \tag{5.6}$$

It is easy to see the sum of the roots of Eq (5.6) is negative, i.e., $\lambda_1 + \lambda_2 = -\left(B_2 y^* + \frac{A_2 x^*}{K}\right) < 0$ and the product of the roots of Eq (5.6) is positive, i.e., $\lambda_1 \lambda_2 = A_3 B_3 (1 - m)^2 x^* y^* > 0$. Thus the roots of quadratic Eq (5.6) are negative real number or complex conjugates with negative real parts. Therefore, (x^*, y^*) is stable node or focus when the existence conditions $B_3(1-m)x^* > B_1 + q_2 E_2$ and $A_1 B_2 + A_3(1 - m)(B_1 + q_2 E_2) > B_2 q_1 E_1$ are satisfied.

5.2. Global stability

We next discuss the global stability of interior equilibrium S^*.

Theorem 5.2. *If interior equilibrium S^* exists, then it is globally asymptotically stable.*

Proof. The Lyapunov function is established as follows

$$V(x, y) = x - x^* - x^* \ln\left(\frac{x}{x^*}\right) + l\left[y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right], \tag{5.7}$$

where $l > 0$ is a constant. Differentiating both sides of the above equation in regard to time t, one has

$$\begin{aligned}
\frac{dV}{dt} &= \frac{x - x^*}{x}\frac{dx}{dt} + l\frac{y - y^*}{y}\frac{dy}{dt} \\
&= (x - x^*)\left[A_1 - A_2\frac{x}{K} - A_3(1 - m)y - q_1 E_1\right] \\
&\quad + l(y - y^*)[-B_1 - B_2 y + B_3(1 - m)x - q_2 E_2] \\
&= -\frac{A_2}{K}(x - x^*)^2 - A_3(1 - m)(x - x^*)(y - y^*) - lB_2(y - y^*)^2 \\
&\quad + lB_3(1 - m)(x - x^*)(y - y^*).
\end{aligned}$$

Choose $l = \frac{A_3}{B_3}$, it yields to

$$\frac{dV}{dt} = -\frac{A_2}{K}(x - x^*)^2 - \frac{A_3 B_2}{B_3}(y - y^*)^2 < 0.$$

Since $\frac{dV}{dt}$ in some neighborhood of (x^*, y^*) is negative semidefinite, the equilibrium (x^*, y^*) is globally asymptotically stable.

6. Bionomic equilibria

As we have discussed in Section 5, the biological equilibrium yields to $\frac{dx}{dt} \equiv 0$ and $\frac{dy}{dt} \equiv 0$. And economic equilibrium is expressed by the total revenue equaling the total cost. Then bionomic equilibrium is a combination of biological equilibrium and economic equilibrium. In this section, we analyse the existence of all bionomic equilibria.

Firstly, c_1, c_2 denote unit capture cost for prey x and predator y, respectively. And p_1, p_2 show unit biomass price for prey x and predator y, respectively. Define the profit function as follows

$$N = (p_1 q_1 x - c_1)E_1 + (p_2 q_2 y - c_2)E_2 = N_1 + N_2, \tag{6.1}$$

where $N_1 = (p_1 q_1 x - c_1)E_1, N_2 = (p_2 q_2 y - c_2)E_2$. N_1 and N_2 separately represent the profit function of prey x and predator y. And the bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$ is governed by

$$\begin{cases}
A_1 x - A_2\dfrac{x^2}{K} - A_3(1 - m)xy - q_1 E_1 x = 0, \\
-B_1 y - B_2 y^2 + B_3(1 - m)xy - q_2 E_2 y = 0, \\
(p_1 q_1 x - c_1)E_1 + (p_2 q_2 y - c_2)E_2 = 0.
\end{cases} \tag{6.2}$$

The following we consider four cases to excavate the bionomic equilibria of system (3.5).
Case I Boundary bionomic equilibria with no catching species x: $E_{1\infty} = 0$. One has

$$\begin{cases}
p_2 q_2 y_\infty - c_2 = 0, \\
A_1 x_\infty - A_2\dfrac{x_\infty^2}{K} - A_3(1 - m)x_\infty y_\infty = 0, \\
-B_1 - B_2 y_\infty + B_3(1 - m)x_\infty - q_2 E_{2\infty} = 0.
\end{cases} \tag{6.3}$$

Clearly, $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty}) = (0, \frac{c_2}{p_2 q_2}, 0, -\frac{p_2 q_2 B_1 + B_2 c_2}{p_2 q_2^2})$ satisfies Eq (6.3), we have to omit the bionomic equilibrium owing to $E_{2\infty} < 0$.

Then solving the second equation for a nonzero x_∞ yields

$$x_\infty = \frac{A_1 K p_2 q_2 - A_3 (1 - m) c_2 K}{A_2 p_2 q_2}, \tag{6.4}$$

which exists if satisfying $A_1 p_2 q_2 > A_3 (1 - m) c_2$. Substituting Eq (6.4) into the third equation of Eq (6.3) gives

$$E_{2\infty} = \frac{A_1 B_3 K p_2 q_2 (1 - m) - A_3 B_3 c_2 K (1 - m)^2 - A_2 B_1 p_2 q_2 - A_2 B_2 c_2}{A_2 p_2 q_2^2}, \tag{6.5}$$

which is positive provided that

$$A_1 B_3 K p_2 q_2 (1 - m) > A_3 B_3 c_2 K (1 - m)^2 + A_2 B_1 p_2 q_2 + A_2 B_2 c_2 \tag{6.6}$$

holds. Therefore, there exists a boundary bionomic equilibrium $(x_\infty, y_\infty, 0, E_{2\infty})$.

Case II Boundary bionomic equilibria with no catching species y: $E_{2\infty} = 0$. It follows from

$$\begin{cases} p_1 q_1 x_\infty - c_1 = 0, \\ A_1 - A_2 \dfrac{x_\infty}{K} - A_3 (1 - m) y_\infty - q_1 E_{1\infty} = 0, \\ -B_1 y_\infty - B_2 y_\infty^2 + B_3 (1 - m) x_\infty y_\infty = 0, \end{cases} \tag{6.7}$$

that $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty}) = (\frac{c_1}{p_1 q_1}, 0, \frac{A_1 p_1 q_1 K - A_2 c_1}{p_1 q_1^2 K}, 0)$ and we get a nonegative semi-trivial bionomic equilibrium $(x_\infty, 0, E_{1\infty}, 0)$ with condition $A_1 p_1 q_1 K > A_2 c_1$.

Then solving the third equation for a nonzero y_∞ yields

$$y_\infty = \frac{B_3 c_1 (1 - m) - B_1 p_1 q_1}{B_2 p_1 q_1} > 0 \tag{6.8}$$

satisfying $B_3 c_1 (1 - m) > B_1 p_1 q_1$. Substituting Eq (6.8) into the second equation of Eq (6.7) gives

$$E_{1\infty} = \frac{K A_1 B_2 p_1 q_1 + K A_3 B_1 (1 - m) p_1 q_1 - A_2 B_2 c_1 - K A_3 B_3 c_1 (1 - m)^2}{K B_2 p_1 q_1^2}, \tag{6.9}$$

which is positive provided that

$$K A_1 B_2 p_1 q_1 + K A_3 B_1 (1 - m) p_1 q_1 > A_2 B_2 c_1 + K A_3 B_3 c_1 (1 - m)^2 \tag{6.10}$$

holds. Thus, there exists a boundary bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, 0)$.

Case III Boundary bionomic equilibria with no catching species x and y: $E_{1\infty} = 0$ and $E_{2\infty} = 0$.

In this case, the cost exceeds revenue for both species x and y, we have to stop harvesting both of them. The bionomic equilibrium is equal to the biological equilibrium with $E_{1\infty} = E_{2\infty} = 0$.

Case IV Positive bionomic equilibrium with effort fishing on both species x and y: $E_{1\infty} > 0$ and $E_{2\infty} > 0$.

Solving the third equation of Eq (6.2), we obtain $x_\infty = \frac{c_1}{p_1 q_1}, y_\infty = \frac{c_2}{p_2 q_2}$. Then, it follows from the second and third equations of Eq (6.2) that

$$
\begin{aligned}
E_{1\infty} &= \frac{KA_1 p_1 p_2 q_1 q_2 - A_2 c_1 p_2 q_2 - KA_3 c_2 (1-m) p_1 q_1}{K p_1 p_2 q_1^2 q_2}, \\
E_{2\infty} &= \frac{B_3 c_1 (1-m) p_2 q_2 - B_1 p_1 p_2 q_1 q_2 - B_2 c_2 p_1 q_1}{p_1 p_2 q_1 q_2^2},
\end{aligned}
\tag{6.11}
$$

which exist if satisfying the following conditions

$$
\begin{cases}
KA_1 p_1 p_2 q_1 q_2 > A_2 c_1 p_2 q_2 + KA_3 c_2 (1-m) p_1 q_1, \\
B_3 c_1 (1-m) p_2 q_2 > B_1 p_1 p_2 q_1 q_2 + B_2 c_2 p_1 q_1.
\end{cases}
\tag{6.12}
$$

Therefore, there exists the positive bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$.

Summarizing the above, we put forward the theorem as follows.

Theorem 6.1. *The following conclusions hold:*

(1) *The boundary bionomic equilibrium $(x_\infty, y_\infty, 0, E_{2\infty})$ exists if $A_1 p_2 q_2 > A_3 (1-m) c_2$ and Eq (6.6) are satisfied, where $y_\infty = \frac{c_2}{p_2 q_2}$ and x_∞ and $E_{2\infty}$ are defined in Eqs (6.4) and (6.5), respectively.*

(2) *The boundary bionomic equilibrium $(x_\infty, 0, E_{1\infty}, 0)$ exists if $A_1 p_1 q_1 K > A_2 c_1$ holds, where $x_\infty = \frac{c_1}{p_1 q_1}$ and $E_{1\infty} = \frac{A_1 p_1 q_1 K - A_2 c_1}{p_1 q_1^2 K}$.*

(3) *The boundary bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, 0)$ exists if $B_3 c_1 (1-m) > B_1 p_1 q_1$ and Eq (6.10) are satisfied, where $x_\infty = \frac{c_1}{p_1 q_1}$ and y_∞ and $E_{1\infty}$ are defined in Eqs (6.8) and (6.9), respectively.*

(4) *The boundary bionomic equilibrium $(x_\infty, y_\infty, 0, 0)$ is equal to the biological equilibrium with $E_{1\infty} = E_{2\infty} = 0$ described in Section 5.*

(5) *The positive bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$ exists if Eq (6.12) holds, where $x_\infty = \frac{c_1}{p_1 q_1}$, $y_\infty = \frac{c_2}{p_2 q_2}$ and $E_{1\infty}$ and $E_{2\infty}$ are defined in Eq (6.11).*

7. Fuzzy optimal harvesting

This section studies the optimal harvesting strategy with fuzzy net discount rate of inflation. Previously, denote $\tilde{k}$ and $\tilde{r}$ the inflation and discount rates and they are considered as fuzzy number essentially. The net discount rate of inflation $\tilde{\delta}$ is the difference of the above two fuzzy numbers, that is $\tilde{\delta} = \tilde{r} - \tilde{k}$, and it can be regarded as triangular fuzzy number, that is $\tilde{\delta} = (\delta_1, \delta_2, \delta_3)$. Our target is to maximize the value $\tilde{J}$ as follows

$$
\tilde{J}(E_1, E_2) = \int_0^\infty e^{-\tilde{\delta}t} [(p_1 q_1 x - c_1) E_1(t) + (p_2 q_2 y - c_2) E_2(t)] dt.
\tag{7.1}
$$

Using Pontryagin's maximal principle [60] to solve the above fuzzy optimization problem, the harvesting strategies not only guarantee profit maximization, but also maintain an optimal level for species. The control variable $E_i(t)$ subject to $0 \le E_i(t) \le E_i^{\max}, i = 1, 2$. According to Maity and Maiti [61], Sadhukhan et al. [52] and Pal and Mahapatra [23], using α' to cut triangular fuzzy number $\tilde{\delta} = (\delta_1, \delta_2, \delta_3)$, that is it regarded as an interval number $[\delta_L, \delta_R]$, the optimization problem Maximize

$\tilde{J}(E_1, E_2)$ is equal to Maximize $[J_L(E_1, E_2), J_R(E_1, E_2)]$, where

$$
\begin{aligned}
J_L(E_1, E_2) &= \int_0^\infty e^{-\delta_R t}[(p_1 q_1 x - c_1)E_1(t) + (p_2 q_2 y - c_2)E_2(t)]dt, \\
J_R(E_1, E_2) &= \int_0^\infty e^{-\delta_L t}[(p_1 q_1 x - c_1)E_1(t) + (p_2 q_2 y - c_2)E_2(t)]dt, \\
\delta_L &= \delta_1 + \alpha'(\delta_2 - \delta_1), \quad \delta_R = \delta_3 - \alpha'(\delta_3 - \delta_2).
\end{aligned}
\tag{7.2}
$$

Again, by the method of weighted sum, it yields to

$$
Max\tilde{J} = Max[J_L, J_R] = Max(w_1' J_L + w_2' J_R),
\tag{7.3}
$$

where $w_1', w_2' \geq 0$ are two weights such that $w_1' + w_2' = 1$. The Hamiltonian can be written as

$$
\begin{aligned}
H = {}& (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})[(p_1 q_1 x - c_1)E_1 + (p_2 q_2 y - c_2)E_2] \\
& + \lambda_1\Big[A_1 x - A_2 \frac{x^2}{K} - A_3(1 - m)xy - q_1 E_1 x\Big] \\
& + \lambda_2[-B_1 y - B_2 y^2 + B_3(1 - m)xy - q_2 E_2 y],
\end{aligned}
\tag{7.4}
$$

where $\lambda_i = \lambda_i(t)(i = 1, 2)$ are the adjoint variables. Applying Pontryagin's maximum principle [60], we get the adjoint equations

$$
\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y}.
\tag{7.5}
$$

It follows from Eqs (7.4) and (7.5) that

$$
\begin{aligned}
\frac{d\lambda_1}{dt} = {}& -\Big\{p_1 q_1(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})E_1 + \lambda_2 B_3(1 - m)y \\
& + \lambda_1\Big[A_1 - \frac{2A_2}{K}x - A_3(1 - m)y - q_1 E_1\Big]\Big\}, \\
\frac{d\lambda_2}{dt} = {}& -\Big\{p_2 q_2(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})E_2 - \lambda_1 A_3(1 - m)x \\
& + \lambda_2[-B_1 - 2B_2 y + B_3(1 - m)x - q_2 E_2]\Big\}.
\end{aligned}
\tag{7.6}
$$

Substituting the interior equilibrium into Eq (7.6) yields

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \frac{\lambda_1 A_2}{K}x - \lambda_2 B_3(1 - m)y - p_1 q_1(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})E_1, \\
\frac{d\lambda_2}{dt} &= \lambda_1 A_3(1 - m)x + \lambda_2 B_2 y - p_2 q_2(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})E_2.
\end{aligned}
\tag{7.7}
$$

Now, we neglect λ_1 in Eq (7.7) to get a reduced differential equation for λ_2

$$
a_0 \frac{d^2\lambda_2}{dt^2} + a_1 \frac{d\lambda_2}{dt} + a_2\lambda_2 = M_{2L} e^{-\delta_R t} + M_{2R} e^{-\delta_L t},
\tag{7.8}
$$

where

$$
\begin{aligned}
a_0 = {}& 1, \quad a_1 = -\Big(\frac{A_2}{K}x + B_2 y\Big), \quad a_2 = \Big[A_3 B_3(1 - m)^2 + \frac{A_2 B_2}{K}\Big]xy, \\
M_{2L} = {}& -A_3(1 - m)p_1 q_1 x w_1' E_1 + \frac{A_2}{K}p_2 q_2 x w_1' E_2 + p_2 q_2 w_1' \delta_R E_2, \\
M_{2R} = {}& -A_3(1 - m)p_1 q_1 x w_2' E_1 + \frac{A_2}{K}p_2 q_2 x w_2' E_2 + p_2 q_2 w_2' \delta_L E_2.
\end{aligned}
$$

The solution of Eq (7.8) as follows

$$\lambda_2 = C_1 e^{m_1 t} + C_2 e^{m_2 t} + \frac{M_{2L}}{N_L} e^{-\delta_R t} + \frac{M_{2R}}{N_R} e^{-\delta_L t}, \tag{7.9}$$

where $C_i (i = 1, 2)$ are arbitrary constants and $m_i (i = 1, 2)$ are the roots of the auxiliary equations $a_0 m^2 + a_1 m + a_2 = 0$ and

$$N_L = a_0 \delta_R^2 - a_1 \delta_R + a_2 \neq 0, \quad N_R = a_0 \delta_L^2 - a_1 \delta_L + a_2 \neq 0.$$

From Eq (7.9), λ_2 is bounded if and only if $m_i < 0 (i = 1, 2)$ or $C_i = 0$. Since it is difficult to check whether $m_i < 0$, we consider $C_i = 0$. Eq (7.9) is simplified as

$$\lambda_2 = \frac{M_{2L}}{N_L} e^{-\delta_R t} + \frac{M_{2R}}{N_R} e^{-\delta_L t}. \tag{7.10}$$

Similarly, we get

$$\lambda_1 = \frac{M_{1L}}{N_L} e^{-\delta_R t} + \frac{M_{1R}}{N_R} e^{-\delta_L t}, \tag{7.11}$$

where

$$\begin{aligned}
M_{1L} &= B_3 (1 - m) y p_2 q_2 w_1' E_2 + p_1 q_1 w_1' \delta_R E_1 + B_2 y p_1 q_1 w_1' E_1, \\
M_{1R} &= B_3 (1 - m) y p_2 q_2 w_2' E_2 + p_1 q_1 w_2' \delta_L E_1 + B_2 y p_1 q_1 w_2' E_1.
\end{aligned} \tag{7.12}$$

It is easy to see the shadow prices $\lambda_i e^{\delta_L t} (i = 1, 2)$ of the two species satisfy the transversality condition at ∞ [18], i.e., they are bounded as $t \to \infty$. The Hamiltonian in Eq (7.4) must be maximized for $E_i \in [0, E_i^{\max}]$. Suppose that the optimal equilibrium does not appear at either $E_i = 0$ or $E_i = E_i^{\max}$, we therefore consider the singular controls

$$\begin{aligned}
\frac{\partial H}{\partial E_1} &= (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})(p_1 q_1 x - c_1) - \lambda_1 q_1 x = 0, \\
\frac{\partial H}{\partial E_2} &= (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})(p_2 q_2 y - c_2) - \lambda_2 q_2 y = 0,
\end{aligned} \tag{7.13}$$

that is,

$$\begin{aligned}
(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})(p_1 q_1 x - c_1) &= \lambda_1 q_1 x, \\
(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})(p_2 q_2 y - c_2) &= \lambda_2 q_2 y.
\end{aligned} \tag{7.14}$$

We substitute λ_1 and λ_2 in Eq (7.11) and Eq (7.10) into Eq (7.14) and obtain

$$\begin{aligned}
P_L e^{-\delta_R t} + P_R e^{-\delta_L t} &= (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t}) c_1, \\
Q_L e^{-\delta_R t} + Q_R e^{-\delta_L t} &= (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t}) c_2,
\end{aligned} \tag{7.15}$$

where

$$\begin{aligned}
P_L &= \left(p_1 w_1' - \frac{M_{1L}}{N_L} \right) q_1 x, \quad P_R = \left(p_1 w_2' - \frac{M_{1R}}{N_R} \right) q_1 x, \\
Q_L &= \left(p_2 w_1' - \frac{M_{2L}}{N_L} \right) q_2 y, \quad Q_R = \left(p_2 w_2' - \frac{M_{2R}}{N_R} \right) q_2 y.
\end{aligned} \tag{7.16}$$

Equation (7.15) is equivalent to the following equations

$$\begin{aligned}
(P_L - c_1 w_1') e^{-(\delta_R - \delta_L) t} &= -(P_R - c_1 w_2'), \\
(Q_L - c_2 w_1') e^{-(\delta_R - \delta_L) t} &= -(Q_R - c_2 w_2').
\end{aligned} \tag{7.17}$$

To eliminate $e^{-(\delta_R - \delta_L)t}$ in Eq (7.17), we consider the following two cases:

Case I If $P_L \neq c_1 w_1'$ and $Q_L \neq c_2 w_1'$, then Eq (7.17) is equal to

$$(P_L - c_1 w_1')(Q_R - c_2 w_2') = (Q_L - c_2 w_1')(P_R - c_1 w_2'). \tag{7.18}$$

Case II If $P_L = c_1 w_1'$ or $Q_L = c_2 w_1'$, we have

$$P_L - c_1 w_1' = P_R - c_1 w_2' = 0, \quad \text{or} \quad Q_L - c_2 w_1' = Q_R - c_2 w_2' = 0. \tag{7.18$'$}$$

Motivated by the method in [25], firstly transforming Eq (7.14) into the following form

$$\lambda_1 = (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})\Big(p_1 - \frac{c_1}{q_1 x}\Big),$$
$$\lambda_2 = (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})\Big(p_2 - \frac{c_2}{q_2 y}\Big). \tag{7.19}$$

Differentiating $\lambda_i (i = 1, 2)$ in Eq (7.19) with respect to t, it derives that

$$\frac{d\lambda_1}{dt} = (-\delta_R w_1' e^{-\delta_R t} - \delta_L w_2' e^{-\delta_L t})\Big(p_1 - \frac{c_1}{q_1 x}\Big)$$
$$+ \frac{c_1}{q_1 x}(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})\Big[A_1 - \frac{A_2}{K}x - A_3(1 - m)y - q_1 E_1\Big],$$
$$\frac{d\lambda_2}{dt} = (-\delta_R w_1' e^{-\delta_R t} - \delta_L w_2' e^{-\delta_L t})\Big(p_2 - \frac{c_2}{q_2 y}\Big)$$
$$+ \frac{c_2}{q_2 y}(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})\Big[-B_1 - B_2 y + B_3(1 - m)x - q_2 E_2\Big]. \tag{7.20}$$

Substituting $\lambda_i (i = 1, 2)$ in Eq (7.19) into Eq (7.6) yields

$$\frac{d\lambda_1}{dt} = -p_1 q_1 E_1 (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t}) - B_3(1 - m)y\Big(p_2 - \frac{c_2}{q_2 y}\Big)(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})$$
$$- \Big(p_1 - \frac{c_1}{q_1 x}\Big)\Big[A_1 - \frac{2A_2}{K}x - A_3(1 - m)y - q_1 E_1\Big](w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t}),$$
$$\frac{d\lambda_2}{dt} = -p_2 q_2 E_2 (w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t}) + A_3(1 - m)x\Big(p_1 - \frac{c_1}{q_1 x}\Big)(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})$$
$$- \Big(p_2 - \frac{c_2}{q_2 y}\Big)\Big[-B_1 - 2B_2 y + B_3(1 - m)x - q_2 E_2\Big](w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t}). \tag{7.21}$$

Combined Eqs (7.20) and (7.21) with E_1 and E_2 omitted, one has

$$\Big(p_1 - \frac{c_1}{q_1 x}\Big)(\delta_R w_1' e^{-\delta_R t} + \delta_L w_2' e^{-\delta_L t}) = B_3(1 - m)y\Big(p_2 - \frac{c_2}{q_2 y}\Big)(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})$$
$$+ p_1\Big[A_1 - \frac{2A_2}{K}x - A_3(1 - m)y\Big](w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})$$
$$+ \frac{c_1 A_2}{q_1 K}(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t}),$$
$$\Big(p_2 - \frac{c_2}{q_2 y}\Big)(\delta_R w_1' e^{-\delta_R t} + \delta_L w_2' e^{-\delta_L t}) = -A_3(1 - m)x\Big(p_1 - \frac{c_1}{q_1 x}\Big)(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})$$
$$+ p_2\Big[-B_1 - 2B_2 y + B_3(1 - m)x\Big](w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t})$$
$$+ \frac{c_2 B_2}{q_2}(w_1' e^{-\delta_R t} + w_2' e^{-\delta_L t}). \tag{7.22}$$

Consider $N_1 = (p_1 q_1 x - c_1)E_1 > 0$ and $N_2 = (p_2 q_2 y - c_2)E_2 > 0$, we divide both sides of the first equation by that of the second equation of Eq (7.22)

$$\frac{p_1 - \frac{c_1}{q_1 x}}{p_2 - \frac{c_2}{q_2 y}} = \frac{B_3(1 - m)y\left(p_2 - \frac{c_2}{q_2 y}\right) + p_1\left[A_1 - \frac{2A_2}{K}x - A_3(1 - m)y\right] + \frac{c_1 A_2}{q_1 K}}{-A_3(1 - m)x\left(p_1 - \frac{c_1}{q_1 x}\right) + p_2\left[-B_1 - 2B_2 y + B_3(1 - m)x\right] + \frac{c_2 B_2}{q_2}}. \tag{7.23}$$

Besides, the values of E_1 and E_2 at the interior equilibrium turn to

$$E_1 = \frac{A_1}{q_1} - \frac{A_2}{q_1 K}x - \frac{A_3(1 - m)}{q_1}y,$$
$$E_2 = -\frac{B_1}{q_2} - \frac{B_2}{q_2}y + \frac{B_3(1 - m)}{q_2}x. \tag{7.24}$$

Solving Eqs (7.18) (or (7.18)'), (7.23) and (7.24) to get the optimal equilibrium solution $x = x_{\tilde{\delta}}, y = y_{\tilde{\delta}}$ and the optimal harvesting efforts $E_1 = E_{1\tilde{\delta}}, E_2 = E_{2\tilde{\delta}}$.

8. Numerical simulations

In this section, some hypothetical data are taken to illustrate analytical results of the previous sections by mainly using the software MATLAB. Due to the features characterized by simulations should be analysed from qualitative perspective rather than quantitative one, then the parameters in system are not in view of real observed values. We consider three numerical examples to account for the stability of equilibria, the existence of bionomic equilibria as well as fuzzy optimal harvesting.

Example 1 Consider the parameter values as follows: $\tilde{r} = (2.00, 2.20, 2.40)$, $\tilde{c} = (1.20, 1.35, 1.50)$, $\tilde{d} = (0.30, 0.40, 0.50)$, $\tilde{s} = (0.06, 0.07, 0.08)$, $\tilde{e} = (0.60, 0.70, 0.80)$, $K = 5$ and $m = 0.1$.

We discuss equilibria S_0, S_1 and S^* of the predator-prey model under different values of α, w_1 and w_2, which are showed from Table 1 to Table 3.

Table 1. The trivial equilibrium S_0 for $q_1 = 0.2, q_2 = 0.2, E_1 = 15$ and $E_2 = 10$.

| w_1	w_2	S_0 at $\alpha = 0$	S_0 at $\alpha = 0.3$	S_0 at $\alpha = 0.6$	S_0 at $\alpha = 0.9$
0	1	(0,0)	(0,0)	(0,0)	(0,0)
0.2	0.8	(0,0)	(0,0)	(0,0)	(0,0)
0.4	0.6	(0,0)	(0,0)	(0,0)	(0,0)
0.6	0.4	(0,0)	(0,0)	(0,0)	(0,0)
0.8	0.2	(0,0)	(0,0)	(0,0)	(0,0)
1	0	(0,0)	(0,0)	(0,0)	(0,0)

Tables 1–3 demonstrate the trivial equilibrium is always fixed at $(0,0)$ with different combinations of w_1, w_2 and α, the values of prey x and predator y stay at zero; for the axial equilibrium, the value of prey x decreases under the same α with increasing w_1, predator y is invariant in zero; for the interior equilibrium, the value of prey x increases under the same α with increasing w_1, and predator y decreases under the same α with increasing w_1.

Table 2. The axial equilibrium S_1 for $q_1 = 0.2, q_2 = 0.2, E_1 = 5$ and $E_2 = 15$.

w_1	w_2	S_1 at $\alpha = 0$	S_1 at $\alpha = 0.3$	S_1 at $\alpha = 0.6$	S_1 at $\alpha = 0.9$
0	1	(3.5000,0)	(3.2524,0)	(3.0189,0)	(2.7982,0)
0.2	0.8	(3.1731,0)	(3.0340,0)	(2.8996,0)	(2.7697,0)
0.4	0.6	(2.8704,0)	(2.8269,0)	(2.7839,0)	(2.7413,0)
0.6	0.4	(2.5893,0)	(2.6302,0)	(2.6715,0)	(2.7132,0)
0.8	0.2	(2.3276,0)	(2.4431,0)	(2.5623,0)	(2.6854,0)
1	0	(2.0833,0)	(2.2650,0)	(2.4561,0)	(2.6577,0)

Table 3. The interior equilibrium S^* for $q_1 = 0.2, q_2 = 0.2, E_1 = 5$ and $E_2 = 2$.

| w_1	w_2	S^* at $\alpha = 0$	S^* at $\alpha = 0.3$	S^* at $\alpha = 0.6$	S^* at $\alpha = 0.9$
0	1	(1.0479,0.9082)	(1.1245,0.7824)	(1.2067,0.6618)	(1.2952,0.5454)
0.2	0.8	(1.1513,0.7417)	(1.2011,0.6697)	(1.2531,0.5992)	(1.3075,0.5301)
0.4	0.6	(1.2650,0.5838)	(1.2830,0.5607)	(1.3013,0.5378)	(1.3199,0.5150)
0.6	0.4	(1.3907,0.4323)	(1.3710,0.4547)	(1.3516,0.4772)	(1.3325,0.4998)
0.8	0.2	(1.5307,0.2853)	(1.4658,0.3510)	(1.4041,0.4175)	(1.3452,0.4847)
1	0	(1.6875,0.1407)	(1.5682,0.2490)	(1.4588,0.3584)	(1.3580,0.4697)
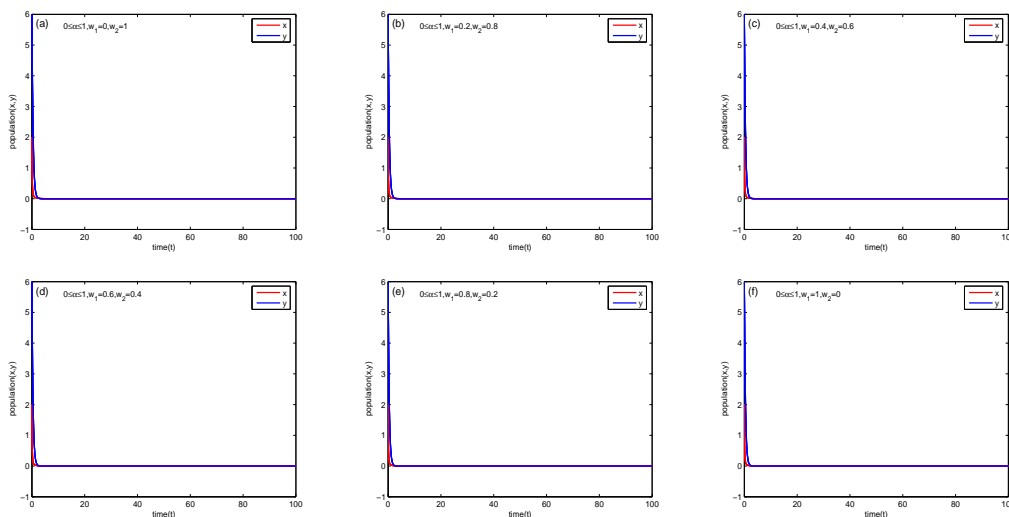
Figure 1. (a)–(f) Time series diagram of prey x and predator y with initial values $(2, 6)$ for $q_1 = 0.2, q_2 = 0.2, E_1 = 15, E_2 = 10$ and different w_1, w_2 and α, respectively, $t \in [0, 100]$.
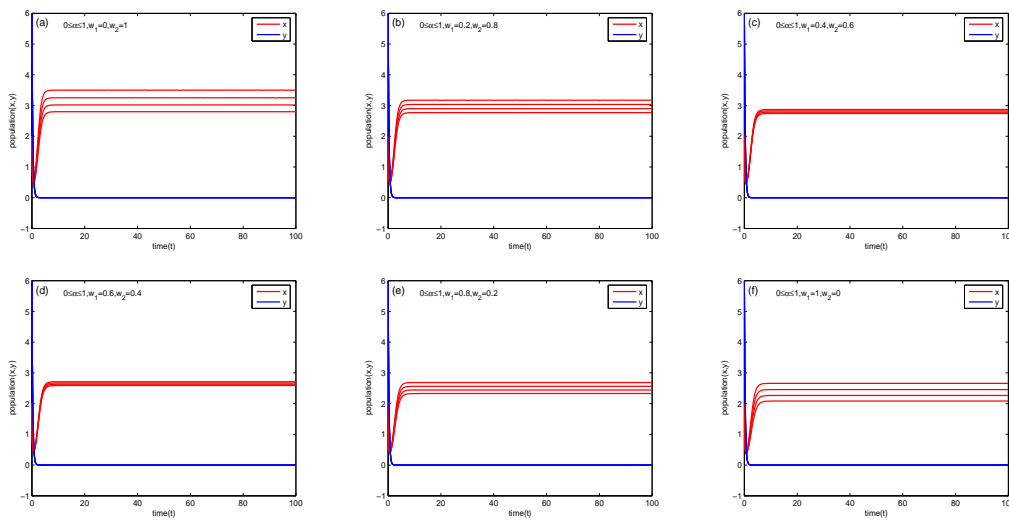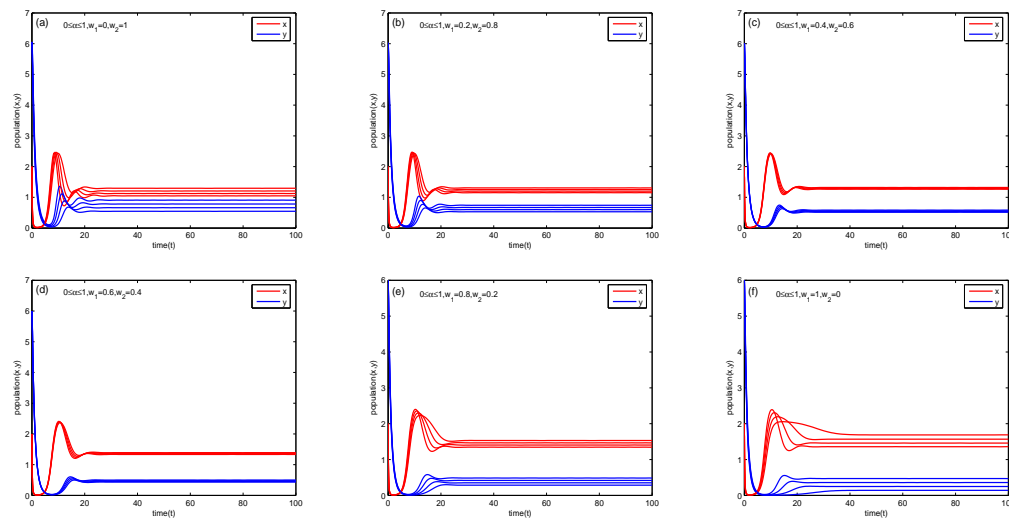
Figure 2. (a)–(f) Time series diagram of prey x and predator y with initial values $(2, 6)$ for $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 15$ and different w_1, w_2 and α, respectively, $t \in [0, 100]$.

Figure 3. (a)–(f) Time series diagram of prey x and predator y with initial values $(2, 6)$ for $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 2$ and different w_1, w_2 and α, respectively, $t \in [0, 100]$.

Figures 1–3 are time series of prey x and predator y with initial values $(2, 6)$ for different w_1, w_2 and α, which show the same results as what Tables 1–3 reflect. The initial fluctuate for both species gradually saturate to a steady state level over time.

The phase portrait of prey x and predator y corresponding to stabilities of different equilibria for all kinds of combinations of w_1, w_2 and α are shown in Figures 4–6, respectively. Meanwhile, the equilibria are stable under different initial conditions.
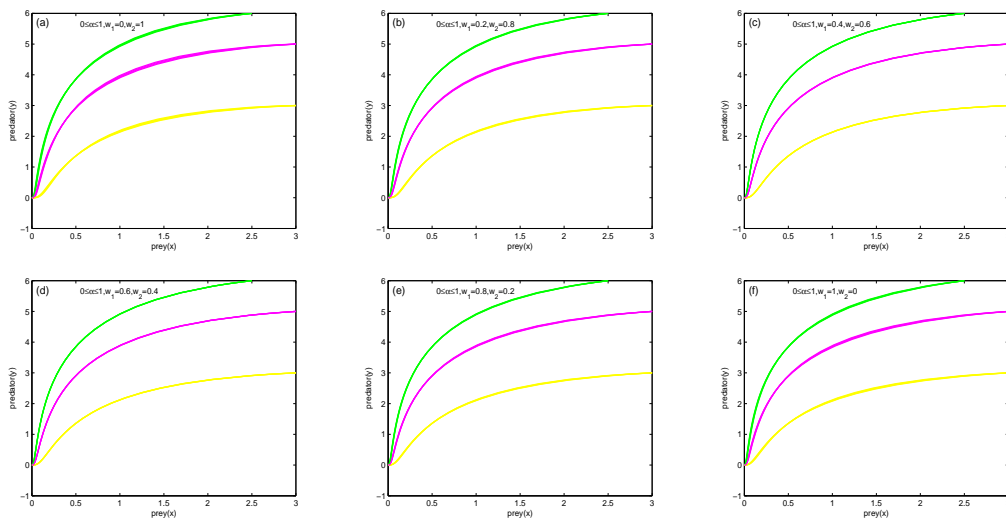
Figure 4. (a)–(f) Phase portrait of prey x and predator y with initial values $(2.5, 6), (3, 3)$ and $(3, 5)$ for $q_1 = 0.2, q_2 = 0.2, E_1 = 15, E_2 = 10$ and different w_1, w_2 and α, respectively, $t \in [0, 100]$.

Figure 5. (a)–(f) Phase portrait of prey x and predator y with initial values $(2.5, 6), (3, 3)$ and $(3, 5)$ for $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 15$ and different w_1, w_2 and α, respectively, $t \in [0, 100]$.
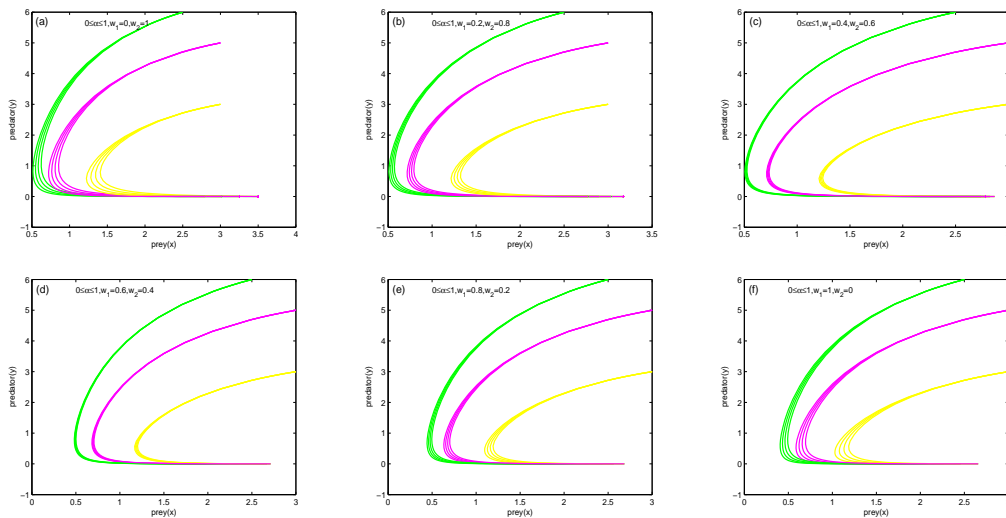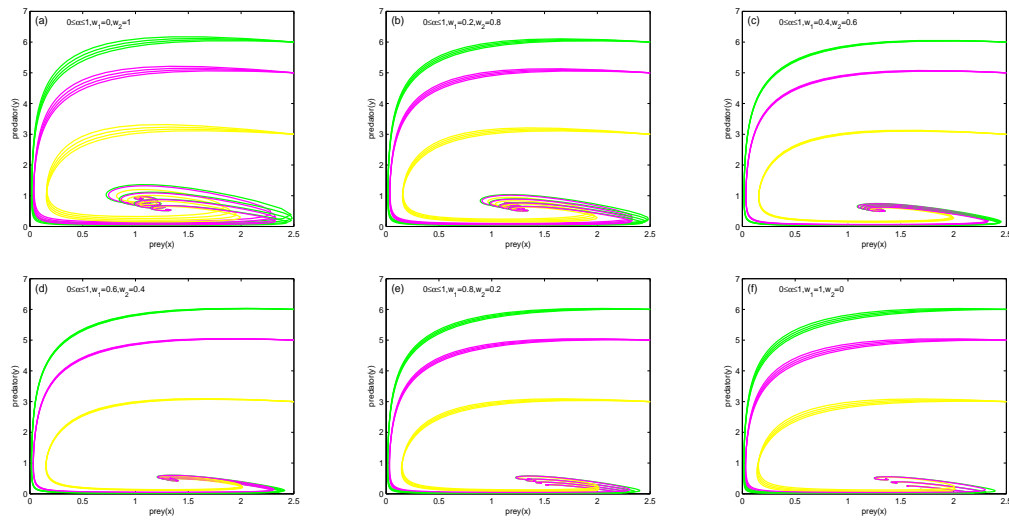
Figure 6. (a)–(f) Phase portrait of prey x and predator y with initial values $(2.5, 6), (2.5, 3)$ and $(2.5, 5)$ for $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 2$ and different w_1, w_2 and α, respectively, $t \in [0, 100]$.
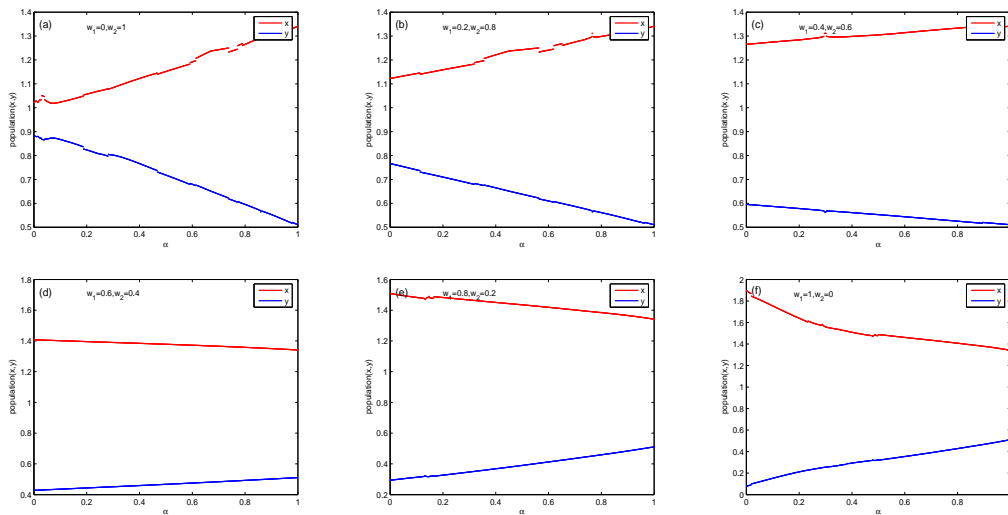

Figure 7. (a)–(f) Dynamical behavior of prey x and predator y with respect to α for initial condition $(x(0), y(0)) = (2.5, 6)$, and $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 2$.
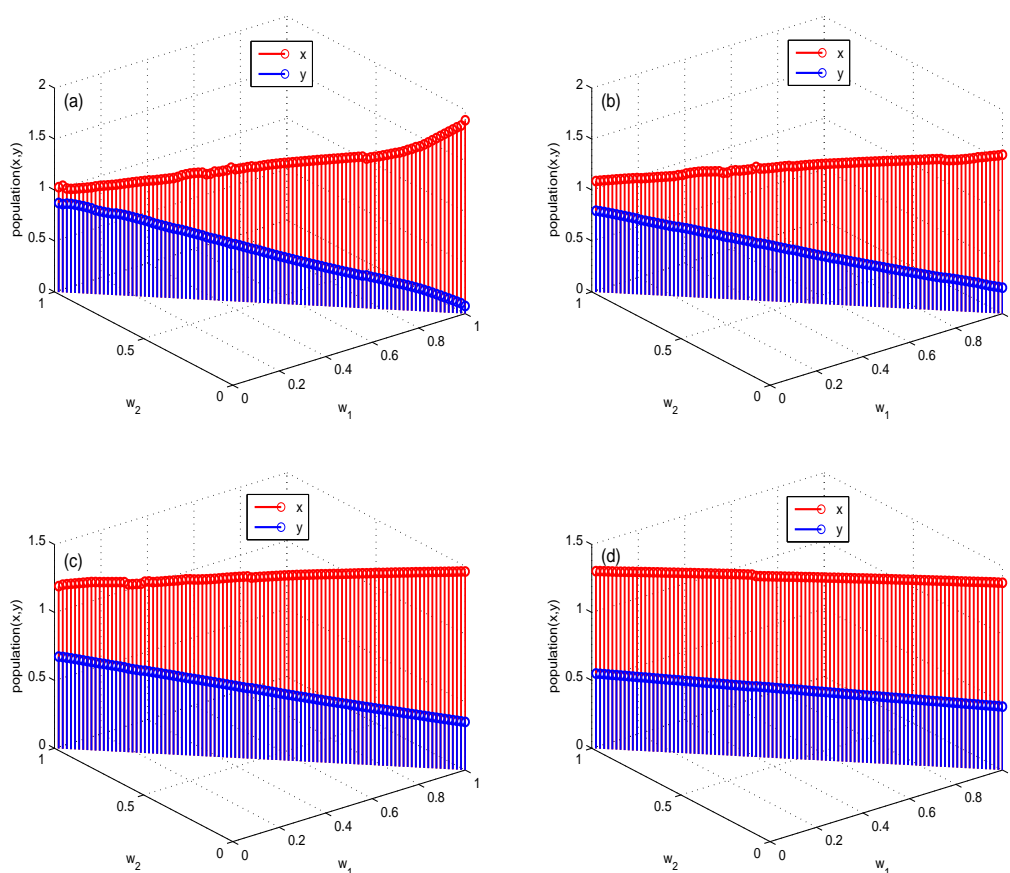
Figure 8. (a)–(d) Dynamical behavior of prey x and predator y with respect to w_1 and w_2 under initial condition $(x(0), y(0)) = (2.5, 6)$ for $\alpha = 0, 0.3, 0.6$ and $\alpha = 0.9$ respectively, and $q_1 = 0.2$, $q_2 = 0.2$, $E_1 = 5$, $E_2 = 2$.

Dynamical behavior of prey x and predator y under interior steady state level with respect to α for initial condition $(x(0), y(0)) = (2.5, 6)$, and $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 2$ are presented in Figure 7.

The figures in Figure 7 reflect the interior equilibrium changes for different α which is in accordance with the results in Table 3. As α increases, prey x increases and predator y decreases for $w_1 = 0, w_2 = 1$ and $w_1 = 0.2, w_2 = 0.8$ and $w_1 = 0.4, w_2 = 0.6$. However, in the cases $w_1 = 0.6, w_2 = 0.4$ and $w_1 = 0.8, w_2 = 0.2$ and $w_1 = 1, w_2 = 0$, as α increases, prey x decreases and predator y increases. Therefore, the parameter α plays an major part in the stability of interior equilibrium.

In the mean time, the dynamical behavior of prey x and predator y about w_1 and w_2 under initial condition $(x(0), y(0)) = (2.5, 6)$ for $\alpha = 0, 0.3, 0.6$ and $\alpha = 0.9$, and $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 2$ are presented in Figure 8. These figures depict the same results as that in Table 3, i.e., for the interior equilibrium, the value of prey x increases under fixed α with increasing w_1, and predator y decreases under fixed α with increasing w_1.

Example 2 Consider the parameter values as follows: $\tilde{r} = (1.50, 1.55, 1.60)$, $\tilde{c} = (0.250, 0.275, 0.300)$, $\tilde{d} = (0.450, 0.475, 0.500)$, $\tilde{s} = (0.150, 0.175, 0.200)$, $\tilde{e} = (1.300, 1.325, 1.350)$, $K = 10, m = 0.1, q_1 = 0.92, q_2 = 0.95, p_1 = 20, p_2 = 25, c_1 = 30$ and $c_2 = 15$.

Table 4. Positive bionomic equilibrium for different w_1, w_2 and α.

w_1	w_2	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
		(x_∞, y_∞)	$(E_{1\infty}, E_{2\infty})$	(x_∞, y_∞)	$(E_{1\infty}, E_{2\infty})$	(x_∞, y_∞)	$(E_{1\infty}, E_{2\infty})$	(x_∞, y_∞)	$(E_{1\infty}, E_{2\infty})$
0	1	(1.6304,0.6316)	(1.3188,1.5118)	(1.6304,0.6316)	(1.2952,1.4874)	(1.6304,0.6316)	(1.2716,1.4629)	(1.6304,0.6316)	(1.2480,1.4384)
0.2	0.8	(1.6304,0.6316)	(1.2874,1.4792)	(1.6304,0.6316)	(1.2732,1.4645)	(1.6304,0.6316)	(1.2591,1.4499)	(1.6304,0.6316)	(1.2449,1.4352)
0.4	0.6	(1.6304,0.6316)	(1.2559,1.4466)	(1.6304,0.6316)	(1.2512,1.4417)	(1.6304,0.6316)	(1.2465,1.4368)	(1.6304,0.6316)	(1.2418,1.4319)
0.6	0.4	(1.6304,0.6316)	(1.2245,1.4140)	(1.6304,0.6316)	(1.2292,1.4189)	(1.6304,0.6316)	(1.2339,1.4238)	(1.6304,0.6316)	(1.2386,1.4287)
0.8	0.2	(1.6304,0.6316)	(1.1930,1.3814)	(1.6304,0.6316)	(1.2071,1.3960)	(1.6304,0.6316)	(1.2213,1.4107)	(1.6304,0.6316)	(1.2355,1.4254)
1	0	(1.6304,0.6316)	(1.1615,1.3487)	(1.6304,0.6316)	(1.1851,1.3732)	(1.6304,0.6316)	(1.2087,1.3977)	(1.6304,0.6316)	(1.2323,1.4221)

The bionomic equilibria for different combinations of w_1, w_2 and α are emerged in Table 4. It is observed that x_∞ and y_∞ are invariable with respect to w_1, w_2 and α, and $E_{1\infty}$ and $E_{2\infty}$ are decreasing with fixed α and increasing w_1. As well as, together with the value of α increasing, $E_{1\infty}$ and $E_{2\infty}$ decreases for $w_1 = 0, w_2 = 1$ and $w_1 = 0.2, w_2 = 0.8$ and $w_1 = 0.4, w_2 = 0.6$. Nevertheless, in the cases $w_1 = 0.6, w_2 = 0.4$ and $w_1 = 0.8, w_2 = 0.2$ and $w_1 = 1, w_2 = 0$, as the value of α increases, $E_{1\infty}$ and $E_{2\infty}$ are increasing.

Example 3 Consider the parameter values as follows: $\tilde{r} = (2.80, 2.83, 2.85), \tilde{c} = (2.45, 2.55, 2.65), \tilde{d} = (0.012, 0.013, 0.015), \tilde{s} = (0.008, 0.009, 0.010), \tilde{e} = (0.20, 0.21, 0.22), K = 10, m = 0.1, q_1 = 0.45, q_2 = 0.55, p_1 = 30, p_2 = 25, c_1 = 25, c_2 = 12$ and $\tilde{\delta} = (0.8, 0.9, 1.0)$.

Table 5. Optimal equilibrium and optimal harvesting effort for $w_1 = 0, w_2 = 1, \alpha = 0$.

w_1'	w_2'	$\alpha' = 0$		$\alpha' = 0.3$		$\alpha' = 0.6$		$\alpha' = 0.9$	
		$(x_{\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$
0.2	0.8	(1.8964,1.0333)	(0.0902,0.6461)	(1.8966,1.0332)	(0.0906,0.6460)	(1.8966,1.0331)	(0.0908,0.6460)	(1.8967,1.0331)	(0.0909,0.6460)
0.4	0.6	(1.8964,1.0333)	(0.0900,0.6463)	(1.8965,1.0332)	(0.0904,0.6461)	(1.8966,1.0331)	(0.0907,0.6460)	(1.8967,1.0331)	(0.0909,0.6460)
0.5	0.5	(1.8964,1.0333)	(0.0900,0.6463)	(1.8965,1.0332)	(0.0904,0.6461)	(1.8967,1.0331)	(0.0908,0.6460)	(1.8967,1.0331)	(0.0908,0.6460)
0.6	0.4	(1.8964,1.0333)	(0.0900,0.6463)	(1.8965,1.0332)	(0.0904,0.6461)	(1.8966,1.0331)	(0.0907,0.6460)	(1.8967,1.0331)	(0.0908,0.6460)
0.8	0.2	(1.8964,1.0333)	(0.0902,0.6461)	(1.8966,1.0332)	(0.0906,0.6460)	(1.8966,1.0331)	(0.0908,0.6460)	(1.8967,1.0331)	(0.0909,0.6460)

Table 6. Optimal equilibrium and optimal harvesting effort for $w_1 = 0, w_2 = 1, \alpha = 0.5$.

w_1'	w_2'	$\alpha' = 0$		$\alpha' = 0.3$		$\alpha' = 0.6$		$\alpha' = 0.9$	
		$(x_{\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$	$(x_{\tilde{\delta}}, y_{\tilde{\delta}})$	$(E_{1\tilde{\delta}}, E_{2\tilde{\delta}})$
0.2	0.8	(1.9230,0.9964)	(0.1262,0.6384)	(1.9232,0.9963)	(0.1264,0.6385)	(1.9234,0.9963)	(0.1265,0.6386)	(1.9235,0.9963)	(0.1266,0.6386)
0.4	0.6	(1.9230,0.9964)	(0.1262,0.6384)	(1.9232,0.9963)	(0.1264,0.6385)	(1.9234,0.9963)	(0.1265,0.6386)	(1.9235,0.9963)	(0.1266,0.6386)
0.5	0.5	(1.9230,0.9964)	(0.1262,0.6384)	(1.9232,0.9963)	(0.1264,0.6385)	(1.9234,0.9963)	(0.1265,0.6386)	(1.9235,0.9963)	(0.1266,0.6386)
0.6	0.4	(1.9230,0.9964)	(0.1262,0.6384)	(1.9232,0.9963)	(0.1264,0.6385)	(1.9234,0.9963)	(0.1265,0.6386)	(1.9235,0.9963)	(0.1266,0.6386)
0.8	0.2	(1.9230,0.9964)	(0.1262,0.6384)	(1.9232,0.9963)	(0.1264,0.6385)	(1.9234,0.9963)	(0.1265,0.6386)	(1.9235,0.9963)	(0.1266,0.6386)

The optimal equilibrium and harvesting effort for different combinations w_1, w_2, α and w_1', w_2', α' are appeared in Tables 5–9, respectively. For different combinations w_1, w_2, α and w_1', w_2', α', we see the existence of optimal equilibrium.

In Table 5, when $\alpha' = 0$ it is observed that prey $x_{\tilde{\delta}}$ and predator $y_{\tilde{\delta}}$ are invariant, optimal harvesting effort of the prey $E_{1\tilde{\delta}}$ decreases and then increases, but the optimal harvesting effort of the predator $E_{2\tilde{\delta}}$ increases and then decreases with increasing w_1'; when $\alpha' = 0.3$, $x_{\tilde{\delta}}$ decreases and then increases, $y_{\tilde{\delta}}$ is invariant, $E_{1\tilde{\delta}}$ decreases and then increases, and $E_{2\tilde{\delta}}$ increases and then decreases with increasing

Table 7. Optimal equilibrium and optimal harvesting effort for $w_1 = 0, w_2 = 1, \alpha = 0.9$.

w_1'	w_2'	$\alpha' = 0$		$\alpha' = 0.3$		$\alpha' = 0.6$		$\alpha' = 0.9$	
		$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$
0.2	0.8	(1.9361,0.9744)	(0.1272,0.6293)	(1.9366,0.9744)	(0.1270,0.6294)	(1.9369,0.9744)	(0.1268,0.6295)	(1.9370,0.9744)	(0.1267,0.6296)
0.4	0.6	(1.9361,0.9744)	(0.1272,0.6293)	(1.9366,0.9744)	(0.1270,0.6294)	(1.9369,0.9744)	(0.1268,0.6295)	(1.9370,0.9744)	(0.1267,0.6296)
0.5	0.5	(1.9361,0.9744)	(0.1272,0.6293)	(1.9366,0.9744)	(0.1270,0.6294)	(1.9369,0.9744)	(0.1268,0.6295)	(1.9370,0.9744)	(0.1267,0.6296)
0.6	0.4	(1.9361,0.9744)	(0.1272,0.6293)	(1.9366,0.9744)	(0.1270,0.6294)	(1.9369,0.9744)	(0.1268,0.6295)	(1.9370,0.9744)	(0.1267,0.6296)
0.8	0.2	(1.9361,0.9744)	(0.1272,0.6293)	(1.9366,0.9744)	(0.1270,0.6294)	(1.9369,0.9744)	(0.1268,0.6295)	(1.9370,0.9744)	(0.1267,0.6296)

Table 8. Optimal equilibrium and optimal harvesting effort for $w_1 = 0.5, w_2 = 0.5, \alpha = 0.9$.

w_1'	w_2'	$\alpha' = 0$		$\alpha' = 0.3$		$\alpha' = 0.6$		$\alpha' = 0.9$	
		$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$
0.2	0.8	(1.9406,0.9694)	(0.1236,0.6273)	(1.9417,0.9695)	(0.1224,0.6277)	(1.9429,0.9696)	(0.1212,0.6280)	(1.9440,0.9697)	(0.1198,0.6284)
0.4	0.6	(1.9406,0.9694)	(0.1236,0.6273)	(1.9417,0.9695)	(0.1224,0.6277)	(1.9429,0.9696)	(0.1212,0.6280)	(1.9440,0.9697)	(0.1198,0.6284)
0.5	0.5	(1.9406,0.9694)	(0.1236,0.6273)	(1.9417,0.9695)	(0.1224, 0.6277)	(1.9429,0.9696)	(0.1212,0.6280)	(1.9440,0.9697)	(0.1198,0.6284)
0.6	0.4	(1.9406,0.9694)	(0.1236,0.6273)	(1.9417,0.9695)	(0.1224,0.6277)	(1.9429,0.9696)	(0.1212,0.6280)	(1.9440,0.9697)	(0.1198,0.6284)
0.8	0.2	(1.9406,0.9694)	(0.1236,0.6273)	(1.9417,0.9695)	(0.1224,0.6277)	(1.9429,0.9696)	(0.1212,0.6280)	(1.9440,0.9697)	(0.1198,0.6284)

w_1'; when $\alpha' = 0.6$, $x_{\bar{\delta}}$ increases and then decreases, $y_{\bar{\delta}}$ is invariant, $E_{1\bar{\delta}}$ fluctuates between 0.0907 and 0.0908, and $E_{2\bar{\delta}}$ is invariant with increasing w_1'; when $\alpha' = 0.9$, $x_{\bar{\delta}}$ and $y_{\bar{\delta}}$ are invariant, $E_{1\bar{\delta}}$ decreases and then increases, and $E_{2\bar{\delta}}$ is invariant with increasing w_1'. Besides, when fixed w_1' and w_2', $x_{\bar{\delta}}$ increases, $y_{\bar{\delta}}$ decreases, $E_{1\bar{\delta}}$ increases, and $E_{2\bar{\delta}}$ decreases with the increase of α'.

In Table 6, when fixed α', $x_{\bar{\delta}}$, $y_{\bar{\delta}}$, $E_{1\bar{\delta}}$ and $E_{2\bar{\delta}}$ are invariant with the increase of w_1'; when fixed w_1' and w_2', $x_{\bar{\delta}}$ increases, $y_{\bar{\delta}}$ decreases, $E_{1\bar{\delta}}$ increases, and $E_{2\bar{\delta}}$ increases with the increase of α'.

In Table 7, when fixed α', $x_{\bar{\delta}}$, $y_{\bar{\delta}}$, $E_{1\bar{\delta}}$ and $E_{2\bar{\delta}}$ are invariant with the increase of w_1'; when fixed w_1' and w_2', $x_{\bar{\delta}}$ increases, $y_{\bar{\delta}}$ is invariant, $E_{1\bar{\delta}}$ decreases, and $E_{2\bar{\delta}}$ increases with the increase of α'.

In Table 8, when fixed α', $x_{\bar{\delta}}$, $y_{\bar{\delta}}$, $E_{1\bar{\delta}}$ and $E_{2\bar{\delta}}$ are invariant with the increase of w_1'; when fixed w_1' and w_2', $x_{\bar{\delta}}$ increases, $y_{\bar{\delta}}$ increases, $E_{1\bar{\delta}}$ decreases, and $E_{2\bar{\delta}}$ increases with the increase of α'.

In Table 9, when $\alpha' = 0$, $x_{\bar{\delta}}$ increases and then decreases and $y_{\bar{\delta}}$ is invariant, $E_{1\bar{\delta}}$ increases and then decreases, but $E_{2\bar{\delta}}$ decreases and then increases with increasing w_1'; when $\alpha' = 0.3$ or $\alpha' = 0.6$, we can see that $x_{\bar{\delta}}$ increases and then decreases, $y_{\bar{\delta}}$, $E_{1\bar{\delta}}$ and $E_{2\bar{\delta}}$ are invariant with increasing w_1'; when $\alpha' = 0.9$, $x_{\bar{\delta}}$, $y_{\bar{\delta}}$ and $E_{1\bar{\delta}}$ are invariant, $E_{2\bar{\delta}}$ decreases and then increases with increasing w_1'. When fixed w_1' and w_2', $x_{\bar{\delta}}$ decreases, $y_{\bar{\delta}}$ is invariant, $E_{1\bar{\delta}}$ is almost invariant, and $E_{2\bar{\delta}}$ increases with the increase of α'.

From Tables 5–7, it is easy to see that when fixed w_1, w_2 and w_1', w_2', α', the prey increases, the predator decreases, the optimal harvesting effort of the prey increases and the optimal harvesting effort of the predator decreases with increasing α.

Table 9. Optimal equilibrium and optimal harvesting effort for $w_1 = 1, w_2 = 0, \alpha = 0.9$.

w_1'	w_2'	$\alpha' = 0$		$\alpha' = 0.3$		$\alpha' = 0.6$		$\alpha' = 0.9$	
		$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$	$(x_{\bar{\delta}}, y_{\bar{\delta}})$	$(E_{1\bar{\delta}}, E_{2\bar{\delta}})$
0.2	0.8	(1.9435,0.9648)	(0.1196,0.6246)	(1.9434,0.9648)	(0.1197,0.6246)	(1.9433,0.9648)	(0.1197,0.6246)	(1.9433,0.9648)	(0.1197,0.6247)
0.4	0.6	(1.9436,0.9648)	(0.1197,0.6245)	(1.9434,0.9648)	(0.1197,0.6246)	(1.9434,0.9648)	(0.1197,0.6246)	(1.9433,0.9648)	(0.1197,0.6246)
0.5	0.5	(1.9436,0.9648)	(0.1197,0.6245)	(1.9435,0.9648)	(0.1197,0.6246)	(1.9434,0.9648)	(0.1197,0.6246)	(1.9433,0.9648)	(0.1197,0.6246)
0.6	0.4	(1.9436,0.9648)	(0.1197,0.6245)	(1.9434,0.9648)	(0.1197,0.6246)	(1.9434,0.9648)	(0.1197,0.6246)	(1.9433,0.9648)	(0.1197,0.6247)
0.8	0.2	(1.9435,0.9648)	(0.1196,0.6246)	(1.9434,0.9648)	(0.1197,0.6246)	(1.9433,0.9648)	(0.1197,0.6246)	(1.9433,0.9648)	(0.1197,0.6247)

Based on Tables 7–9, we observed that when fixed α and w_1', w_2', α', the prey increases, the predator decreases, the optimal harvesting effort of the prey decreases and the optimal harvesting effort of the predator decreases with increasing w_1.

9. Conclusions

In the field of biomathematics most harvesting models are on account of the assumption with precise parameters, but any species in nature will inevitably be affected by the complexity of the ecosystem itself and the limitations of people's cognition leading to the uncertainty of the biological parameters. These uncertainties can be roughly classified into three categories: fuzziness, randomness, and unascertainty. In view of this point, we have modified a fuzzy parameters predator-prey model with refuge. Here, the fuzzy parameters are introduced into biotic potential of the prey (r), predator mortality (d), intraspecific competition rate of the predator (s), predation rate (c) and the product of predation rate and conversion rate (e).

We have discussed the positivity and boundedness of system (Theorem 4.1) and investigated the existence and stability of equilibria (Theorems 5.1 and 5.2). The tables and graphs for different equilibria are showed from Table 1 to Table 3 and from Figure 1 to Figure 3, respectively. Figures 4–6 reflect phase portrait of prey and predator about different w_1, w_2 and α. Dynamical behaviors of prey and predator about α or w_1 and w_2, separately, are presented in Figures 7 and 8. We have also analysed the existence of bionomic equilibria (Theorem 6.1) and the mathematical results for positive bionomic equilibrium are displayed in Table 4.

The highlight of this article is to solve the optimal control problem with fuzzy inflation and discount, that is maximize the fuzzy objective functional $\tilde{J}$. The optimal steady state solution is displayed in Tables 5–9. We obtain optimal equilibrium and optimal harvesting effort as respect to w_1', w_2' and α' under five cases including $w_1 = 0$, $w_2 = 1$, $\alpha = 0$ and $w_1 = 0$, $w_2 = 1$, $\alpha = 0.5$ and $w_1 = 0$, $w_2 = 1$, $\alpha = 0.9$ and $w_1 = 0.5$, $w_2 = 0.5$, $\alpha = 0.9$ and $w_1 = 1$, $w_2 = 0$, $\alpha = 0.9$.

In this paper, we consider the fuzzification of biological parameters in model. However, the fuzzification of unit capture cost or unit biomass price for both species seems to be also feasible. Future work will focus on the imprecision of economic parameters to enhance our model. Finally, this paper uses α-cut of triangular fuzzy number to describe imprecise parameters in model, however other common forms of fuzzy number such as normal fuzzy number and trapezoidal fuzzy number are important in fuzzy set theory, and then we are going to apply other forms of fuzzy number depicting imprecise parameters in model.

Acknowledgements

Conflict of interest

The authors have no conflict of interest.

References

1. G. F. Gause, N. P. Smaragdova, A. A. Witt, Further studies of interaction between predators and prey, *J. Anim. Ecol.*, **5** (1936), 1–18.

2. J. M. Smith, *Models in Ecology*, Cambridge University Press, Cambridge, 1974.

3. C. Bianca, Modeling complex systems with particles refuge by thermostatted kinetic theory methods, *Abstr. Appl. Anal.*, **2013** (2013), 717–718.

4. C. Bianca, C. Dogba, L. Guerrini, A thermostatted kinetic framework with particle refuge for the modeling of tumors hiding, *Appl. Math. Inf. Sci.*, **8** (2014), 469–473.

5. Q. L. Wang, Z. J. Liu, X. A. Zhang, R. A. Cheke, Incorporating prey refuge into a predator-prey system with imprecise parameter estimates, *Comp. Appl. Math.*, **36** (2017), 1067–1084.

6. H. L. Li, L. Zhang, C. Hu, Y. L. Jiang, Z. D. Teng, Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge, *J. Appl. Math. Comput.*, **54** (2017), 435–449.

7. E. G. Olivares, R. R. Jiliberto, Dynamic consequences of prey refuges in a simple model system: more prey, fewer predators and enhanced stability, *Ecol. Model.*, **166** (2003), 135–146.

8. T. K. Kar, Stability analysis of a prey-predator model incorporating a prey refuge, *Commun. Nonlinear Sci. Numer. Simul.*, **10** (2005), 681–691.

9. Y. J. Huang, F. D. Chen, L. Zhong, Stability analysis of a prey-predator model with Holling type III response function incorporating a prey refuge, *Appl. Math. Comput.*, **182** (2006), 672–683.

10. J. P. Tripathi, S. Abbas, M. Thakur, Dynamical analysis of a prey-predator model with Beddington-DeAngelis type function response incorporating a prey refuge, *Nonlinear Dyn.*, **80** (2015), 177–196.

11. R. J. Han, L. N. Guin, B. X. Dai, Consequences of refuge and diffusion in a spatiotemporal predator-prey model, *Nonlinear Anal. Real World Appl.*, **60** (2021), 103311.

12. H. K. Qi, X. Z. Meng, Threshold behavior of a stochastic predator-prey system with prey refuge and fear effect, *Appl. Math. Lett.*, **113** (2021), 106846.

13. Y. Zhang, J. Zhang, Optimal harvesting for a stochastic competition system with stage structure and distributed delay, *Electron. J. Qual. Theory Differ. Equations*, **2021** (2021), 1–22.

14. G. S. Mahapatra, P. Santra, Prey-predator model for optimal harvesting with functional response incorporating prey refuge, *Int. J. Biomath.*, **9** (2016), 1650014.

15. W. X. Li, K. Wang, Optimal harvesting policy for general stochastic logistic population model, *J. Math. Anal. Appl.*, **368** (2010), 420–428.

16. W. X. Li, K. Wang, H. Su, Optimal harvesting policy for stochastic logistic population model, *Appl. Math. Comput.*, **218** (2011), 157–162.

17. A. R. Palma, E. G. Olivares, Optimal harvesting in a predator-prey model with Allee effect and Sigmoid functional response, *Appl. Math. Model.*, **5** (2012), 1864–1874.

18. C. W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, Wiley, New York, 1976.

19. C. W. Clark, *Bioeconomic Modelling and Fisheries Management*, John Wiley and Sons, New York, 1985.

20. T. K. Kar, K. S. Chaudhuri, Harvesting in a two-prey one predator fishery: a bioeconomic model, *ANZIAM J.*, **45** (2004), 443–456.

21. Z. R. He, N. Zhou, Optimal harvesting for a nonlinear hierarchical age-structured population model, *(Chin.) J. Syst. Sci. Math. Sci.*, **40** (2020), 2248–2263.

22. D. Pal, G. S. Mahaptra, G. P. Samanta, Optimal harvesting of prey-predator system with interval biological parameters: A bioeconomic model, *Math. Biosci.*, **241** (2013), 181–187.

23. D. Pal, G. S. Mahapatra, A bioeconomic modeling of two-prey and one-predator fishery model with optimal harvesting policy through hybridization approach, *Appl. Math. Comput.*, **242** (2014), 748–763.

24. D. Pal, G. S. Mahapatra, Dynamic behavior of a predator-prey system of combined harvesting with interval-valued rate parameters, *Nonlinear Dynam.*, **83** (2016), 2113–2123.

25. Q. Z. Xiao, B. X. Dai, L. Wang, Analysis of a competition fishery model with interval-valued parameters: extinction, coexistence, bionomic equilibria and optimal harvesting policy, *Nonlinear Dynam.*, **80** (2015), 1631–1642.

26. S. Y. Chen, Z. J. Liu, L. W. Wang, J. Hu, Stability of a delayed competitive model with saturation effect and interval biological parameters, *J. Appl. Math. Comput.*, **64** (2020), 1–15.

27. K. Qi, Z. J. Liu, L. W. Wang, Q. L. Wang, Survival and stationary distribution of a stochastic facultative mutualism model with distributed delays and strong kernels, *Math. Biosci. Eng.*, **18** (2021), 3160–3179.

28. W. X. Ning, Z. J. Liu, L. W. Wang, R. H. Tan, A stochastic mutualism model with saturation effect and impulsive toxicant input in a polluted environment, *J. Appl. Math. Comput.*, **65** (2021), 177–197.

29. M. Liu, K. Wang, Persistence and extinction of a stochastic single-specie model under regime switching in a polluted environment, *J. Theoret. Biol.*, **264** (2010), 934–944.

30. M. Liu, K. Wang, Dynamics of a Leslie-Gower Holling-type II predator-prey system with Lévy jumps, *Nonlinear Anal.*, **85** (2013), 204–213.

31. M. Liu, K. Wang, Population dynamical behavior of Lotka-Volterra cooperative systems with random perturbations, *Discrete Contin. Dyn. Syst.*, **33** (2013), 2495–2522.

32. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353.

33. L. A. Zadeh, Toward a generalized theory of uncertainty (GTU)-An outline, *Inf. Sci.*, **172** (2005), 1–40.

34. S. L. Chang, L. A. Zadeh, On fuzzy mapping and control, *IEEE Trans. Syst. Man Cybern.*, **2** (1972), 30–34.

35. O. Kaleva, Fuzzy differential equations, *Fuzzy Sets Syst.*, **24** (1987), 301–317.

36. B. Bede, I. J. Rudas, A. L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, *Inf. Sci.*, **177** (2007), 1648–1662.

37. A. Khastan, J. J. Nieto, A boundary value problem for second order fuzzy differential equations, *Nonlinear Anal.*, **72** (2010), 3583–3593.

38. L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, *Nonlinear Anal.*, **71** (2009), 1311–1328.

39. M. Puri, D. Ralescu, Differentials of fuzzy functions, *J. Math. Anal. Appl.*, **91** (1983), 552–558.

40. B. Bede, S.G. Gal, Solutions of fuzzy differential equations based on generalized differentiability, *Commun. Math. Anal.*, **9** (2010), 22–41.

41. E. Hullermeier, An approach to modelling and simulation of uncertain dynamical systems, *Int. J. Uncertain. Fuzziness Knowl. Based Syst.*, **5** (1997), 117–137.

42. T. Allahviranloo, M. B. Ahmadi, Fuzzy Laplace transforms, *Soft Comput.*, **14** (2010), 235–243.

43. E. J. Villamizar-Roa, V. Angulo-Castillo, Y. Chalco-Cano, Existence of solutions to fuzzy differential equations with generalized Hukuhara derivative via contractive-like mapping principles, *Fuzzy Sets Syst.*, **265** (2015), 24–38.

44. D. Pal, G. S. Mahapatra, G. P. Samanta, Stability and bionomic analysis of fuzzy prey-predator harvesting model in presence of toxicity: a dynamic approach, *Bull. Math. Biol.*, **78** (2016), 1493–1519.

45. X. W. Yu, S. L. Yuan, T. H. Zhang, About the optimal harvesting of a fuzzy predator-prey system: a bioeconomic model incorporating prey refuge and predator mutual interference, *Comput. Math. Appl.*, **94** (2018), 2143–2160.

46. X. Y. Meng, Y. Q. Wu, Dynamical analysis of a fuzzy phytoplankton-zooplankton model with refuge, fishery protection and harvesting, *J. Appl. Math. Comput.*, **63** (2020), 361–389.

47. D. Pal, G. S. Mahapatra, G. P. Samanta, A study of bifurcation of prey-predator model with time delay and harvesting using fuzzy parameters, *J. Biol. Syst.*, **26** (2018), 339–372.

48. R. C. Bassanezi, L. C. Barros, A. Tonelli, Attractors and asymptotic stability for fuzzy dynamical systems, *Fuzzy Sets Syst.*, **113** (2000), 473–483.

49. M. T. Mizukoshi, L. C. Barros, R. C. Bassanezi, Stability of fuzzy dynamic systems, *Int. J. Uncertain, Fuzziness Knowl. -Based Syst.*, **17** (2009), 69–83.

50. M. S. Guo, X. P. Xue, R. L. Li, Impulsive functional differential inclusions and fuzzy population models, *Fuzzy Sets Syst.*, **138** (2003), 601–615.

51. V. Lupulescu, On a class of fuzzy functional differential equations, *Fuzzy Sets Syst.*, **160** (2009), 1547–1562.

52. D. Sadhukhan, L. N. Sahoo, B. Mondal, M. Maiti, Food chain model with optimal harvesting in fuzzy environment, *J. Appl. Math. Comput.*, **34** (2010), 1–18.

53. D. Pal, G. S. Mahapatra, S. K. Mahato, G. P. Samanta, A mathematical model of a prey-predator type fishery in the presence of toxicity with fuzzy optimal harvesting, *J. Appl. Math. Inf.*, **38** (2020), 13–36.

54. D. Pal, G. S. Mahapatra, G. P. Samanta, Stability and bionomic analysis of fuzzy parameter based prey-predator harvesting model using UFM, *Nonlinear Dyn.*, **79** (2015), 1939–1955.

55. J. Dijkman, H. Haeringen, S. DeLange, Fuzzy numbers, *J. Math. Anal. Appl.*, **92** (1983), 301–341.

56. R. Jafari, W. Yu, Uncertainty nonlinear systems modeling with fuzzy equations, in *2015 IEEE 16th International Conference on Information Reuse and Integration*, San Francisco, (2015), 182–188.

57. M. Mizumoto, K. Tanaka, The four operations of arithmetic on fuzzy numbers, *Syst. Comput. Controls*, **7** (1976), 73–81.

58. D. Ralescu, A survey of the representation of fuzzy concepts and its applications, in *Advances in Fuzzy Set Theory and Applications, North-Holland, Amsterdam-New York*, (1979), 77–91.

59. K. M. Miettinen, *Non-Linear Multi-Objective*, Optimization, Kluwer's International Series, 1999.

60. L. S. Pontryagin, V. G. Boltyonsku, R. V. Gamkrelidre, E. F. Mishchenko, *Math. Theory Optim. Processes*, Wiley, New York, 1962.

61. K. Maity, M. Maiti, A numerical approach to a multi-objective optimal inventory control problem for deteriorating multi-items under fuzzy inflation and discounting, *Comput. Math. Appl.*, **55** (2008), 1794–1807.