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Research article

A spatial SIS model in heterogeneous environments with vary advective rate

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Abstract: We study a spatial susceptible-infected-susceptible(SIS) model in heterogeneous environments with vary advective rate. We establish the asymptotic stability of the unique disease-free equilibrium(DFE) when $\mathcal{R}_0 < 1$ and the existence of the endemic equilibrium when $\mathcal{R}_0 > 1$. Here $\mathcal{R}_0$ is the basic reproduction number. We also discuss the effect of diffusion on the stability of the DFE.

Keywords: spatial SIS model; vary advective rate; disease-free equilibrium; endemic equilibrium

1. Introduction

In this paper, we are concerned with the following susceptible-infected-susceptible(SIS) model

$$\begin{cases} S_t = (d_S S_x - a'(x)S)_x - \beta(x)\frac{SI}{S+I} + \gamma(x)I, & 0 < x < L,\ t > 0, \\ I_t = (d_I I_x - a'(x)I)_x + \beta(x)\frac{SI}{S+I} - \gamma(x)I, & 0 < x < L,\ t > 0, \\ d_S S_x - a'(x)S = d_I I_x - a'(x)I = 0, & x = 0, L,\ t > 0, \\ S(x,0) = S_0(x), \quad I(x,0) = I_0(x), & 0 < x < L. \end{cases} \tag{1.1}$$

Here $S(x,t)$ and $I(x,t)$ denote the density of susceptible and infected individuals in a given spatial interval $(0, L)$, d_S and d_I are positive constants which stand for the diffusion coefficients for the susceptible and infected populations, $a'(x)$ is a smooth nonnegative function which represents the advection speed rate, while $\beta(x)$ and $\gamma(x)$ represent the rates of disease transmission and recovery at location x, which are Hölder continuous functions on $(0, L)$. In addition, $S_0(x)$ and $I_0(x)$ are continuous and satisfy

$$(A1) \qquad S_0(x) \geq 0 \text{ and } I_0(x) \geq 0 \text{ for } x \in (0, L), \qquad \int_0^L I_0(x)dx > 0.$$

We would like to give the survey of some results on SIS model. In [1], Allen et al. investigated a discrete SIS model, in [2], they also proposed the SIS model with no advection in a given spatial region Ω, where they dealt with the existence, uniqueness and asymptotic behaviors of the endemic equilibrium as the diffusion rate of the susceptible individuals approaches to zero. Many authors also considered the SIS reaction–diffusion model, including the global stability of the endemic equilibrium, the effects of large and small diffusion rates of the susceptible and infected population on the persistence and extinction of the disease, discuss how the disease vanish or spreading in high-risk or low-risk domain, and so on. For the dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, we can see[3]. For A SIS reaction-diffusion-advection model in a low-risk and high-risk domain, we can see [4]. For Dynamics of an SIS reaction-diffusion epidemic model for disease transmission, we can see[5], For Concentration profile of endemic equilibrium of a reaction-diffusion-advection SIS epidemic model, we can see [6]. For the varying total population enhances disease persistence, we can see [7]; For the asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model, we can see [8]. For the global stability of the steady states of an SIS epidemic reaction-diffusion model, we can see [9]. For the asymptotic profile of the positive steady state for an SIS epidemic reaction- diffusion model: effects of epidemic risk and population movement, we can see [10]; For reaction-diffusion SIS epidemic model in a time-periodic environment, we can see [11]. For the global dynamics and traveling waves for a periodic and diffusive chemostat model with two nutrients and one microorganism, we can see [12]. For more information about dynamical systems in population biology, we also can refer to see [13] and the references therein. Recently, Cui and Lou studied (1.1) when $a'(x) \equiv q$ for $x \in [0, L]$ in [14], that is, it is a constant advection. Besides establishing the asymptotic stability of the unique disease-free equilibrium(DFE) when $\mathcal{R}_0 < 1$ and the existence of the endemic equilibrium when $\mathcal{R}_0 > 1$, they found that the DFE changes its stability at most once as d_I varies from zero to infinity, which is strong contrast with the case of no advection. Since (1.1) has vary advection, an natural and interesting question is whether we can establish the similar results on (1.1) to those in the case of no advection or not.

Since the functions $a'(x), \beta(x), \gamma(x), S_0(x)$ and $I_0(x)$ are continuous in $(0, L)$, by the standard theory for a system of semilinear parabolic equations, (1.1) is locally wellposedness in $(0, T_{\max})$. Noticing (A1), by the maximum principle, $S(x, t)$ and $I(x, t)$ are positive and bounded for $x \in [0, L]$ and $t \in (0, T_{\max})$. Hence, by the results in [15], $T_{\max} = \infty$ and (1.1) posses a unique classical solution $(S(x, t), I(x, t))$ for all time.

It is easy to verify that

$$\int_0^L [S(x,t) + I(x,t)]dx = \int_0^L [S(x,0) + I(x,0)]dx := N > 0, \quad t > 0. \tag{1.2}$$

Inspired by [2] and [14], we say that $(0, L)$ is a low-risk domain if $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$ and high-risk domain if $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$.

The corresponding equilibrium system of (1.1) is

$$\begin{cases} (d_S \tilde{S}_x - a'(x)\tilde{S})_x - \beta(x)\frac{\tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} + \gamma(x)\tilde{I} = 0, & 0 < x < L, \\ (d_I \tilde{I}_x - a'(x)\tilde{I})_x + \beta(x)\frac{\tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} - \gamma(x)\tilde{I} = 0, & 0 < x < L, \\ d_S \tilde{S}_x - a'(x)\tilde{S} = d_I \tilde{I}_x - a'(x)\tilde{I} = 0, & x = 0, L. \end{cases} \tag{1.3}$$

The half trivial solution $(\tilde{S}(x), 0)$ of (1.3) is called a disease-free equilibrium(DFE), while the solution $(\tilde{S}(x), \tilde{I}(x))$ of (1.3) is called endemic equilibrium(EE) if $\tilde{I}(x) > 0$ for some $x \in (0, L)$.

We also introduce the following basic reproduction number as those in literatures [2] and [14]. We also can refer to [16] and see the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, refer to [17] and see reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, see basic reproduction numbers for reaction-diffusion epidemic models [18].

$$\mathcal{R}_0 = \sup_{\varphi\in H^1((0,L)),\varphi\neq 0}\left\{\frac{\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\varphi^2 dx}{d_I\int_0^L e^{\frac{a(x)}{d_I}}\varphi_x^2 dx+\int_0^L \gamma(x)e^{\frac{a(x)}{d_I}}\varphi^2 dx}\right\}. \tag{1.4}$$

Our first result is concerned with the qualitative properties for $\mathcal{R}_0$.

Theorem 1.1. *Let $\hat{\mathcal{R}}_0$ be the basic reproduction number when $a(x)\equiv 0$ which was introduced in [2]. Then the following conclusions hold.*

(1) For any given $a'(x)>0$, $\mathcal{R}_0\to\frac{\beta(L)}{\gamma(L)}$ as $d_I\to 0$ and $\mathcal{R}_0\to\frac{\int_0^L\beta(x)dx}{\int_0^L\gamma(x)dx}$ as $d_I\to+\infty$;

(2) For any given $d_I>0$, $\mathcal{R}_0\to\hat{\mathcal{R}}_0$ as $\max_{x\in[0,L]}a'(x)\to 0$ and $\mathcal{R}_0\to\frac{\beta(L)}{\gamma(L)}$ as $\min_{x\in[0,L]}a'(x)\to+\infty$;

(3) If $\beta(x)>(<)\gamma(x)$ on $[0,L]$, then $\mathcal{R}_0>(<1)$ for any given $d_I>0$ and $a'(x)>0$.

Our second result deals with the stability of DFE, which will extend those of [2] and [14].

Theorem 1.2. *The DFE is unstable if $\mathcal{R}_0>1$ while it is globally asymptotically stable if $\mathcal{R}_0<1$.*

We will analyze (1.1) under the following assumptions on $\beta(x)$ and $\gamma(x)$:

(**C1**) $\beta(0)-\gamma(0)<0<\beta(L)-\gamma(L)$, i.e., $\beta(x)-\gamma(x)$ changes sign from negative to positive,

or

(**C2**) $\beta(0)-\gamma(0)>0>\beta(L)-\gamma(L)$, i.e., $\beta(x)-\gamma(x)$ changes sign from positive to negative.

In the point view of biological,

(**C1**) all lower-risk sites are located at the upstream and all high-risk sites are at the downstream,

or

(**C2**) all high-risk sites are distributed at the upstream and lower-risk sites are at the downstream.

To state other results, in convenience, let $q=\max_{x\in[0,L]}a'(x)$ and denote $a(x)=q\tilde{a}(x)$ sometimes in the sequels.

We can get further properties of $\mathcal{R}_0$ when $\int_0^L\beta(x)dx>\int_0^L\gamma(x)dx$.

Theorem 1.3. *Assume that $\int_0^L\beta(x)dx>\int_0^L\gamma(x)dx$. Denote $\mathcal{R}_0=\mathcal{R}_0(d_I,q)$.*

(i) If (C1) holds, then the DFE is unstable for any $q>\min_{x\in[0,L]}a'(x)>0$ and $d_I>0$;

(ii) If (C2) holds, then there exists a unique curve in d_I–q plane

$$\Gamma_1=\{(d_I,\rho_1(d_I)):\mathcal{R}_0(d_I,\rho_1(d_I))=1,\quad d_I\in(0,+\infty)\}$$

with the function $\rho_1=\rho_1(d_I):(0,+\infty)\to(0,+\infty)$ satisfying

$$\lim_{d_I\to 0+}\rho_1(d_I)=0,\qquad \lim_{d_I\to+\infty}\frac{\rho_1(d_I)}{d_I}=\theta_1,$$

and such that for every $d_I>0$, the DFE is unstable for $0<\min_{x\in[0,L]}a'(x)<q<\rho_1(d_I)$ and it is globally and asymptotically stable for $q>\min_{x\in[0,L]}a'(x)>\rho_1(d_I)$.

Here θ_1 is the unique positive solution of

$$\int_0^L[\beta(x)-\gamma(x)]e^{\theta_1\tilde{a}(x)}dx=0.$$

Similarly, we can get further properties of $\mathcal{R}_0$ when $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$.

Theorem 1.4. *Assume that* $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$. *Let* d_I^* *is the unique positive root of the equation* $\hat{\mathcal{R}}_0 = 1$, *where* $\hat{\mathcal{R}}_0$ *was introduced in [2].*

(1) If (C1) holds, then the DFE is unstable for any $q > \min_{x\in[0,L]} a'(x) > 0$ *and* $d_I \in (0, d_I^*]$, *while for* $d_I \in (d_I^*, +\infty)$ *there exists a unique curve in* d_I*–*q *plane*

$$\Gamma_2 = \{(d_I, \rho_2(d_I)) : \mathcal{R}_0(d_I, \rho_2(d_I)) = 1, \quad d_I \in (d_I^*, +\infty)\}$$

with the monotone function $\rho_2 = \rho_2(d_I) : (d_I^*, +\infty) \to (0, +\infty)$ *satisfying*

$$\lim_{d_I \to d_I^* +} \rho_2(d_I) = 0, \quad \lim_{d_I \to +\infty} \frac{\rho_2(d_I)}{d_I} = \theta_2,$$

and such that the DFE is unstable for $0 < \min_{x\in[0,L]} a'(x) < q < \rho_2(d_I)$ *and it is globally asymptotically stable for* $q > \min_{x\in[0,L]} a'(x) > \rho_2(d_I)$.

Here θ_2 *is the unique positive solution of*

$$\int_0^L [\beta(x) - \gamma(x)] e^{\theta_2 \tilde{a}(x)} dx = 0.$$

(2) If (C2) holds, then for $d_I \in (0, d_I^*)$, *there exists a unique curve in* $d_I - q$ *plane*

$$\Gamma_3 = \{(d_I, \rho_3(d_I)) : \mathcal{R}_0(d_I, \rho_3(d_I)) = 1, \quad d_I \in (0, d_I^*)\}$$

with the function $\rho_3 = \rho_3(d_I) : (0, d_I^*) \to (0, +\infty)$ *satisfying*

$$\lim_{d_I \to 0+} \rho_3(d_I) = 0, \quad \lim_{d_I \to d_I^* -} \rho_3(d_I) = 0,$$

and such that the DFE is unstable for $0 < \min_{x\in[0,L]} a'(x) < q < \rho_3(d_I)$ *and it is globally and asymptotically stable for* $q > \min_{x\in[0,L]} a'(x) > \rho_3(d_I)$, *while for* $d_I \in (d_I^*, +\infty)$, *the DFE is globally and asymptotically stable for any* $q > \min_{x\in[0,L]} a'(x) > 0$.

The following theorem deals with the existence of *EE*.

Theorem 1.5. *Assume that* $\beta(x) - \gamma(x)$ *changes sign once in* $(0, L)$. *If* $\mathcal{R}_0 > 1$, *then problem (1.3) possesses at least one EE.*

The last theorem will consider the results on (1.1) when $\beta(x) - \gamma(x)$ changes sign twice in $(0, L)$.

Theorem 1.6. *Assume that* $\beta(x) - \gamma(x)$ *changes sign twice in* $(0, L)$.

(1) If $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$ *and* $\beta(L) < \gamma(L)$, *then there exists some positive constant* Λ *which is independent of* d_I *and* q *such that for every* $d_I > \Lambda$, *we can find a positive constant* Q *which depends on* d_I *such that* $\mathcal{R}_0 > 1$ *when* $0 < \min_{x\in[0,L]} a'(x) < q < Q$ *and* $\mathcal{R}_0 < 1$ *when* $q > Q$.

(2) If $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$ *and* $\beta(L) > \gamma(L)$, *then there exists some positive constant* Λ *which is independent of* d_I *and* q *such that for every* $d_I > \Lambda$ *one of the following conclusions holds:*

(i) $\mathcal{R}_0 > 1$ *for any* $q > \min_{x\in[0,L]} a'(x) > 0$;

(ii) There exists a positive constant $\hat{Q}$ *which is independent of* d_I *and satisfies that* $\mathcal{R}_0 > 1$ *for* $q \neq \hat{Q}$ *and* $\mathcal{R}_0 = 1$ *when* $q = \hat{Q}$;

(iii) There exist two positive constants $Q_2 > Q_1$ *both depending on* d_I *such that* $\mathcal{R}_0 > 1$ *when* $q \in (0, Q_1) \cup (Q_2, +\infty)$ *while* $\mathcal{R}_0 < 1$ *when* $q \in (Q_1, Q_2)$.

(3) If $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$ *and* $\beta(L) > \gamma(L)$*, then there exists some positive constant* $\Lambda > d_I^*$ *which is independent of* d_I *and* q *such that for every* $d_I > \Lambda$*, we can find a positive constant* Q *which depends on* d_I *such that* $\mathcal{R}_0 < 1$ *when* $0 < \min_{x\in[0,L]} a'(x) < q < Q$ *and* $\mathcal{R}_0 > 1$ *when* $q > Q$*.*

(4) If $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$ *and* $\beta(L) < \gamma(L)$*, then there exists some positive constant* $\Lambda > d_I^*$ *which is independent of* d_I *and* q *such that for every* $d_I > \Lambda$ *one of the following conclusions holds:*

(iv) $\mathcal{R}_0 < 1$ *for any* $q > \min_{x\in[0,L]} a'(x) > 0$*;*

(v) There exists a positive constant $\hat{Q}$ *which is independent of* d_I *and satisfies that* $\mathcal{R}_0 < 1$ *for* $q \neq \hat{Q}$ *and* $\mathcal{R}_0 = 1$ *when* $q = \hat{Q}$*;*

(vi) There exist two positive constants $Q_2 > Q_1$ *both depending on* d_I *and satisfy that* $\mathcal{R}_0 < 1$ *when* $q \in (0, Q_1) \cup (Q_2, +\infty)$ *while* $\mathcal{R}_0 > 1$ *when* $q \in (Q_1, Q_2)$*.*

The rest of this paper is organized as follows. In Section 2, we give the proofs of Theorem 1.1 and Theorem 1.2. In Section 3, we will prove Theorem 1.3. In Section 4, we will prove Theorem 1.4. In Section 5, we will prove Theorem 1.5. In Section 6, we will prove Theorem 1.6.

2. Materials and method

2.1. The proofs of Theorem 1.1 and Theorem 1.2

In this section, we first give some qualitative properties of $\mathcal{R}_0$, then we deal with the stability of DFE, and we can finish the proofs of Theorem 1.1 and Theorem 1.2.

By the definition of $\mathcal{R}_0$, there exits some positive function $\Phi(x) \in C^2([0, L])$ such that

$$\begin{cases} -[d_I\Phi_x - a'(x)\Phi]_x + \gamma(x)\Phi = \frac{1}{\mathcal{R}_0}\beta(x)\Phi, \quad 0 < x < L, \\ d_I\Phi_x(0) - a'(0)\Phi(0) = 0, \quad d_I\Phi_x(L) - a'(L)\Phi(L) = 0. \end{cases} \tag{2.1}$$

Letting $\varphi(x) = e^{-\frac{a(x)}{d_I}}\Phi(x)$, we have

$$\begin{cases} -d_I\varphi_{xx} - a'(x)\varphi_x + \gamma(x)\varphi = \frac{1}{\mathcal{R}_0}\beta(x)\varphi, \quad 0 < x < L, \\ \varphi_x(0) = 0, \quad \varphi_x(L) = 0. \end{cases} \tag{2.2}$$

Linearizing (1.1) around $(\hat{S}, 0)$ and letting $\bar{\xi}(x,t) = S(x,t) - \hat{S}(x,t)$, $\bar{\eta}(x,t) = I(x,t)$, we have

$$\begin{cases} \bar{\xi}_t &= (d_S\bar{\xi}_x - a'(x)\bar{\xi})_x - [\beta(x) - \gamma(x)]\bar{\eta}, \quad 0 < x < L,\ t > 0, \\ \bar{\eta}_t &= (d_I\bar{\eta}_x - a'(x)\bar{\eta})_x + [\beta(x) - \gamma(x)]\bar{\eta}, \quad 0 < x < L,\ t > 0. \end{cases}$$

For the linear system, seeking for the solution which is separation of variables, i.e., $\bar{\xi}(x,t) = e^{-\lambda t}\xi(x)$ and $\bar{\eta}(x,t) = e^{-\lambda t}\eta(x)$, we have

$$\begin{cases} (d_S\xi_x - a'(x)\xi)_x - [\beta(x) - \gamma(x)]\eta + \lambda\xi = 0, \quad 0 < x < L, \\ (d_I\eta_x - a'(x)\eta)_x + [\beta(x) - \gamma(x)]\eta + \lambda\eta = 0, \quad 0 < x < L, \end{cases} \tag{2.3}$$

subject to boundary conditions

$$\begin{cases} d_S\xi_x(0) - a'(0)\xi(0) = 0, \quad d_S\xi_x(L) - a'(L)\xi(L) = 0, \\ d_S\eta_x(0) - a'(0)\eta(0) = 0, \quad d_S\eta_x(L) - a'(L)\eta(L) = 0. \end{cases} \tag{2.4}$$

By the conservation of total population, we need to impose that

$$\int_0^L [\xi(x) + \eta(x)]dx = 0. \tag{2.5}$$

Noticing that the second equation of (2.3) is independent of ξ, letting $\zeta(x) = e^{-\frac{a(x)}{d_I}}\eta(x)$, we only need to consider the following eigenvalue problem

$$\begin{cases} d_I\zeta_{xx} + a'(x)\zeta_x + [\beta(x) - \gamma(x)]\zeta(x) + \lambda\zeta(x) = 0, \quad 0 < x < L, \\ \zeta_x(0) = \zeta_x(L) = 0. \end{cases} \tag{2.6}$$

By the results of [19], all the eigenvalues are real, the smallest eigenvalue $\lambda_1(d_I, q)$ is simple, and its corresponding eigenfunction ϕ_1 can be chosen positive.

We will show a fact below.

Lemma 2.1.1. *For any d_I and $q > \min_{x\in[0,L]} a'(x) > 0$, $\lambda_1(d_I, q) < 0$ if $\mathcal{R}_0 > 1$, $\lambda_1(d_I, q) = 0$ if $\mathcal{R}_0 = 1$ and $\lambda_1(d_I, q) > 0$ if $\mathcal{R}_0 < 1$.*

Proof. Note that $(\lambda_1(d_I, q), \phi_1)$ satisfies

$$\begin{cases} -d_I(\phi_1)_{xx} - a'(x)(\phi_1)_x + [\gamma(x) - \beta(x)]\phi_1(x) = \lambda_1(d_I, q)\phi_1(x), \ 0 < x < L, \\ \\ (\phi_1)_x(0) = (\phi_1)_x(L) = 0. \end{cases} \tag{2.7}$$

Multiplying (2.1) by $e^{\frac{a(x)}{d_I}}\phi_1$ and (2.7) by $e^{\frac{a(x)}{d_I}}\Phi$, integrating by parts in $(0, L)$, and subtracting the resulting equations, we get

$$\int_0^L (\frac{1}{\mathcal{R}_0} - 1)\beta(x)\Phi(x)\phi_1(x)dx = \int_0^L \lambda_1(d_I, q)\Phi(x)\phi_1(x)dx.$$

Using the mean value theorem of integrating, we have

$$(\frac{1}{\mathcal{R}_0} - 1)\beta(x_1)\Phi(x_1)\phi_1(x_1) = \lambda_1(d_I, q)\Phi(x_2)\phi_1(x_2)$$

for some $0 \le x_1 \le L$ and $0 \le x_2 \le L$. Using $\beta(x_1)\Phi(x_1)\phi_1(x_1) > 0$ and $\Phi(x_2)\phi_1(x_2) > 0$, we know that

$$(\frac{1}{\mathcal{R}_0} - 1) \quad \text{has the same sign of} \quad \lambda_1(d_I, q),$$

which implies the conclusions are true. □

Lemma 2.1.2. *If $\frac{d_I}{q} \to 0$ and $\frac{d_I}{q^2} \to 0$, $\tilde{a}'(x) > \delta > 0$ for some constant δ, then $\mathcal{R}_0 \to \frac{\beta(L)}{\gamma(L)}$.*

Proof. Let $w(x) = e^{-\frac{q}{d_I}A\tilde{a}(x)}\Phi(x)$, where $\Phi(x)$ is the solution of (2.1), A is a constant which will be chosen later. It is easy to verify that w satisfies

$$\begin{cases} \quad [\frac{q^2A(A-1)}{d_I}(\tilde{a}'(x))^2 + q(A-1)\tilde{a}''(x) + \frac{1}{\mathcal{R}_0}\beta(x) - \gamma(x)]w \\ = -d_Iw_{xx} + (1-2A)a'(x)w_x, \qquad 0 < x < L, \ t > 0, \\ d_Iw_x(0) = a'(0)(1-A)w(0), \quad d_Iw_x(L) = a'(L)(1-A)w(L). \end{cases} \tag{2.8}$$

First we chose $A = 1 + \frac{C_1 d_I}{q^2}$, where C_1 is a positive constant to be chosen later. Then (2.8) becomes

$$\begin{cases} \quad [C_1(1 + \frac{C_1 d_I}{q^2})(\tilde{a}'(x))^2 + q(1 + \frac{C_1 d_I}{q^2})\tilde{a}''(x) + \frac{1}{\mathcal{R}_0}\beta(x) - \gamma(x)]w \\ = -d_I w_{xx} - (1 + \frac{2C_1 d_I}{q^2})a'(x)w_x, \qquad 0 < x < L,\ t > 0, \\ d_I w_x(0) = -\frac{C_1 d_I}{q}\tilde{a}'(0)w(0), \quad d_I w_x(L) = -\frac{C_1 d_I}{q}\tilde{a}'(L)w(L). \end{cases}$$

Assume that $w(x_*) = \min_{x\in[0,L]} w(x)$. We will show that $x_* = L$ below. $w_x(0) < 0$ implies that $x_* \neq 0$. If $x_* \in (0, L)$, then $w_{xx}(x_*) \geq 0$ and $w_x(x_*) = 0$, (2.9) means that

$$[C_1(1 + \frac{C_1 d_I}{q^2})(\tilde{a}'(x_*))^2 + q(1 + \frac{C_1 d_I}{q^2})\tilde{a}''(x_*) + \frac{1}{\mathcal{R}_0}\beta(x_*) - \gamma(x_*)] \leq 0$$

Taking $C_1 = Kq$ with K large enough, we can get a contradiction. Therefore, $x_* = L$ and $w(x) \geq w(L)$ for $x \in [0, L]$, which implies that

$$\frac{\Phi(x)}{\Phi(L)} \geq e^{-\frac{q}{d_I}(1+\frac{C_1 d_I}{q^2})[\tilde{a}(L)-\tilde{a}(x)]}. \tag{2.9}$$

Next, we chose $A = 1 - \frac{C_2 d_I}{q^2}$, where C_2 is a positive constant to be chosen later. Then (2.8) becomes

$$\begin{cases} \quad [C_2(1 - \frac{C_2 d_I}{q^2})(\tilde{a}'(x))^2 + q(1 - \frac{C_2 d_I}{q^2})\tilde{a}''(x) + \frac{1}{\mathcal{R}_0}\beta(x) - \gamma(x)]w \\ = -d_I w_{xx} - (1 - \frac{2C_2 d_I}{q^2})a'(x)w_x, \qquad 0 < x < L,\ t > 0, \\ d_I w_x(0) = \frac{C_2 d_I}{q}\tilde{a}'(0)w(0), \quad d_I w_x(L) = \frac{C_2 d_I}{q}\tilde{a}'(L)w(L). \end{cases}$$

Assume that $w(x^*) = \max_{x\in[0,L]} w(x)$. We will show that $x^* = L$ below. $w_x(0) > 0$ implies that $x^* \neq 0$. If $x^* \in (0, L)$, then $w_{xx}(x^*) \geq 0$ and $w_x(x^*) = 0$, (2.10) means that

$$[C_2(1 - \frac{C_2 d_I}{q^2})(\tilde{a}'(x^*))^2 + q(1 - \frac{C_2 d_I}{q^2})\tilde{a}''(x^*) + \frac{1}{\mathcal{R}_0}\beta(x^*) - \gamma(x^*)] \leq 0$$

Taking $C_2 = K'q$ with K' large enough, we can get a contradiction. Therefore, $x^* = L$ and $w(x) \leq w(L)$ for $x \in [0, L]$, which implies that

$$\frac{\Phi(x)}{\Phi(L)} \leq e^{-\frac{q}{d_I}(1-\frac{C_2 d_I}{q^2})[\tilde{a}(L)-\tilde{a}(x)]}. \tag{2.10}$$

Dividing (2.1) by $\Phi(L)$ and integrating the result in $(0, L)$, we have

$$\int_0^L \gamma(x)\frac{\Phi(x)}{\Phi(L)}dx = \frac{1}{\mathcal{R}_0}\int_0^L \beta(x)\frac{\Phi(x)}{\Phi(L)}dx. \tag{2.11}$$

Letting $y = \frac{q[\tilde{a}(L)-\tilde{a}(x)]}{d_I}$, i.e., $x = \tilde{a}^{-1}[\tilde{a}(L) - \frac{d_I y}{q}]$, we have

$$e^{-(1+\frac{C_1 d_I}{q})y} \leq \frac{\Phi(\tilde{a}^{-1}[\tilde{a}(L) - \frac{d_I y}{q}])}{\Phi(L)} \leq e^{-(1-\frac{C_2 d_I}{q})y} \tag{2.12}$$

and

$$\int_0^{\frac{q[\tilde{a}(L)-\tilde{a}(0)]}{d_I}} \gamma(\tilde{a}^{-1}[\tilde{a}(L) - \frac{d_I y}{q}])\frac{\Phi(\tilde{a}^{-1}[\tilde{a}(L) - \frac{d_I y}{q}])}{\tilde{a}'(\tilde{a}^{-1}[\tilde{a}(L) - \frac{d_I y}{q}])\Phi(L)}dy$$

$$= \frac{1}{\mathcal{R}_0}\int_0^{\frac{q[\tilde{a}(L)-\tilde{a}(0)]}{d_I}} \beta(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])\frac{\Phi(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])}{\tilde{a}'(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])\Phi(L)}dy. \tag{2.13}$$

Using (2.12), by Lebesgue dominant convergence theorem, then passing to the limit in (2.13), we get

$$\lim_{d_I/q\to 0, d_I/q^2\to 0}\mathcal{R}_0 = \lim_{d_I/q\to 0, d_I/q^2\to 0}\frac{\int_0^{\frac{q[\tilde{a}(L)-\tilde{a}(0)]}{d_I}} \beta(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])\frac{\Phi(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])}{\tilde{a}'(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])\Phi(L)}dy}{\int_0^{\frac{q[\tilde{a}(L)-\tilde{a}(0)]}{d_I}} \gamma(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])\frac{\Phi(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])}{\tilde{a}'(\tilde{a}^{-1}[\tilde{a}(L)-\frac{d_I y}{q}])\Phi(L)}dy}$$

$$= \frac{\int_0^\infty \frac{\beta(L)}{\tilde{a}'(L)}e^{-y}dy}{\int_0^\infty \frac{\gamma(L)}{\tilde{a}'(L)}e^{-y}dy} = \frac{\beta(L)}{\gamma(L)}. \tag{2.14}$$

□

We have the following corollary.

Corollary 2.1.1. *The following statements hold.*

(i) Given $d_I > 0$, $\mathcal{R}_0 \to \hat{\mathcal{R}}_0$ *as* $q \to 0$*;*

(ii) Given $d_I > 0$, $\mathcal{R}_0 \to \frac{\beta(L)}{\gamma(L)}$ *as* $q \to +\infty$*;*

(iii) Given $q > 0$, $\mathcal{R}_0 \to \frac{\beta(L)}{\gamma(L)}$ *as* $d_I \to 0$*;*

(iv) Given $q > 0$, $\mathcal{R}_0 \to \frac{\int_0^L \beta(x)dx}{\int_0^L \gamma(x)dx}$ *as* $d_I \to +\infty$.

Proof. (i) For any fixed $\varphi \in H^1((0,L))$, $\varphi \neq 0$, we have

$$\lim_{q\to 0}\frac{d_I\int_0^L e^{\frac{a(x)}{d_I}}\varphi_x^2dx + \int_0^L \gamma(x)e^{\frac{a(x)}{d_I}}\varphi^2dx}{\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\varphi^2dx} = \frac{d_I\int_0^L \varphi_x^2dx + \int_0^L\gamma(x)\varphi^2dx}{\int_0^L\beta(x)\varphi^2dx}.$$

Taking $\inf_{\varphi\in H^1((0,L)),\varphi\neq 0}$ both sides, we have $\frac{1}{\mathcal{R}_0} \to \frac{1}{\hat{\mathcal{R}}_0}$ as $q \to 0$.

(ii) and (iii) are the direct conclusions of Lemma 2.2.

(iv) By the definition of $\frac{1}{\mathcal{R}_0}$, for $\varphi \equiv 1$, we have

$$\frac{1}{\mathcal{R}_0} \leq \frac{\int_0^L \gamma(x)e^{\frac{a(x)}{d_I}}dx}{\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}dx} \leq \frac{\max_{x\in[0,L]}\gamma(x)}{\min_{x\in[0,L]}\beta(x)},$$

which implies that $\frac{1}{\mathcal{R}_0}$ is uniformly bounded for $d_I > 0$, passing to a subsequence if necessary, it has a finite limit $\frac{1}{\bar{\mathcal{R}}_0}$ as $d_I \to \infty$.

On the other hand, by the standard elliptic regularity and the Sobolev embedding theorem, Φ is uniformly bounded for all $d_I \geq 1$. Dividing both sides of (2.1) by d_I and letting $d_I \to +\infty$, we have $\Phi_{xx} \to 0$ for $x \in (0,L)$ and $\Phi_x(0) \to 0$, $\Phi_x(L) \to 0$. Consequently, there exists a positive constant $\bar{\Phi}$ such that $\Phi(x) \to \bar{\Phi}$ as $d_I \to +\infty$. Integrating (2.1) by parts over $(0,L)$, we can get

$$\frac{q}{d_I}\int_0^L e^{-\frac{a(x)}{d_I}}[d_I\Phi_x - a'(x)\Phi(x)]dx + \int_0^L e^{-\frac{a(x)}{d_I}}\gamma(x)\Phi(x)dx$$

$$= \frac{1}{\mathcal{R}_0}\int_0^L e^{-\frac{a(x)}{d_I}}\beta(x)\Phi(x)dx.$$

Letting $d_I \to +\infty$, we obtain $\bar{\mathcal{R}}_0 = \frac{\int_0^L \beta(x)dx}{\int_0^L \gamma(x)dx}$. □

Lemma 2.1.3. *The following statements hold.*
(i) If $\beta(x) > \gamma(x)$ on $[0, L]$, then $\mathcal{R}_0 > 1$ for any $d_I > 0$ and $q > \min_{x\in[0,L]} a'(x) > 0$;
(i) If $\beta(x) < \gamma(x)$ on $[0, L]$, then $\mathcal{R}_0 < 1$ for any $d_I > 0$ and $q > \min_{x\in[0,L]} a'(x) > 0$.

Proof. (i) If $\beta(x) > \gamma(x)$ on $[0, L]$, by the definition of $\frac{1}{\mathcal{R}_0}$, for $\varphi \equiv 1$, we have

$$\frac{1}{\mathcal{R}_0} \le \frac{\int_0^L \gamma(x)e^{\frac{a(x)}{d_I}}dx}{\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}dx} < 1,$$

i.e., $\mathcal{R}_0 > 1$.

(ii) Subtracting both sides of (2.2) by $\beta(x)\varphi$, multiplying by $e^{\frac{a(x)}{d_I}}\varphi$, we have

$$-d_I\varphi_{xx}e^{\frac{a(x)}{d_I}}\varphi - a'(x)\varphi_x e^{\frac{a(x)}{d_I}}\varphi + [\gamma(x) - \beta(x)]e^{\frac{a(x)}{d_I}}\varphi^2 = (\frac{1}{\mathcal{R}_0} - 1)\beta(x)e^{\frac{a(x)}{d_I}}\varphi^2.$$

Integrating it by parts over $(0, L)$, using $\varphi_x(0) = \varphi_x(L) = 0$, we obtain

$$d_I\int_0^L e^{\frac{a(x)}{d_I}}(\varphi_x)^2dx + \int_0^L [\gamma(x) - \beta(x)]e^{\frac{a(x)}{d_I}}\varphi^2dx = (\frac{1}{\mathcal{R}_0} - 1)\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\varphi^2dx.$$

Since $\beta(x) < \gamma(x)$ on $[0, L]$, the left side of the above equality is positive, and

$$(\frac{1}{\mathcal{R}_0} - 1)\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\varphi^2dx > 0,$$

which implies that $\mathcal{R}_0 < 1$. □

Proof. Theorem 1.1 is the direct results of Lemma 2.1.2, Corollary 2.1.1 and Lemma 2.1.3. □

Next we will consider the stability of DFE.

Lemma 2.1.4. *The DFE is stable if $\mathcal{R}_0 < 1$, while it is unstable if $\mathcal{R}_0 > 1$.*

Proof. 1. Assume contradictorily the DFE is unstable if $\mathcal{R}_0 < 1$. Then we can find (λ, ξ, η) which is a solution of (2.3)–(2.4) subject to (2.5), with at least one of ξ and η is not identical zero, and $\Re(\lambda) \le 0$. Suppose that $\eta \equiv 0$, then $\xi \not\equiv 0$ on $[0, L]$. Using (2.3)–(2.4), we have

$$\begin{cases} -(d_S\xi_x - a'(x)\xi)_x = \lambda\xi, & 0 < x < L, \\ d_S\xi_x(0) - a'(0)\xi(0) = 0, & d_S\xi_x(L) - a'(L)\xi(L) = 0. \end{cases} \tag{2.15}$$

It is easy to see that λ is real and nonnegative, and therefore $\lambda = 0$. We find that $\xi = \xi_0 e^{\frac{q}{d_I}\tilde{a}(x)}$, where ξ_0 is some constant to be determined later. By (1.2), we impose that $\int_0^L[\xi(x) + \eta(x)]dx = 0$, $\xi_0 = 0$, i.e., $\xi \equiv 0$ on $[0, L]$. This is a contradiction. Then we conclude that $\eta \equiv 0$ on $[0, L]$. From (2.6), λ must be real and $\lambda \le 0$. Since $\lambda_1(d_I, q)$ is the principal eigenvalue, then $\lambda_1(d_I, q) \le \lambda \le 0$. Lemma 2.1 implies that $\mathcal{R}_0 \ge 1$, which is a contradiction. Then we conclude that if (λ, ξ, η) is a solution of (2.3)–(2.4), with at least one of ξ and η not identical zero on $[0, L]$, then $\Re(\lambda) > 0$. This proves the linear stability of the DFE.

2. Suppose that $\mathcal{R}_0 > 1$. Since $(\lambda_1(d_I, q), \phi_1)$ is the principal eigen-pair of (2.6), $(\lambda_1(d_I, q), e^{\frac{a(x)}{d_I}}\phi_1)$ satisfies

$$\begin{cases} [d_I(\phi_1)_x - a'(x)\phi_1]_x + [\beta(x) - \gamma(x)]\phi_1 + \lambda_1(d_I, q)\phi_1 = 0, \quad 0 < x < L, \\ d_I(\phi_1)_x - a'(x)\phi_1 = 0, \quad x = 0,\ L. \end{cases}$$

By the result of Lemma 2.1.1, $\lambda_1(d_I, q) < 0$. On the other hand,

$$\begin{cases} (d_S\xi_x - a'(x)\xi)_x + \lambda\xi = [\beta(x) - \gamma(x)]e^{\frac{a(x)}{d_I}}\phi_1, \quad 0 < x < L, \\ d_S\xi_x(0) - a'(0)\xi(0) = 0, \quad d_S\xi_x(L) - a'(L)\xi(L) = 0. \end{cases} \tag{2.16}$$

There exists a unique solution ξ_1 of (2.16). And (2.5) becomes

$$\int_0^L [\xi_1(x) + e^{\frac{a(x)}{d_I}}\phi_1(x)]dx = 0,$$

which implies that (2.3)–(2.4) has a solution $(\lambda_1(d_I, q), \xi_1, e^{\frac{a(x)}{d_I}}\phi_1(x))$ satisfying $\lambda_1(d_I, q) < 0$ and $e^{\frac{a(x)}{d_I}}\phi_1(x) > 0$ in $(0, L)$. Therefore, the DFE is linearly unstable.

□

Lemma 2.1.5. *If $\mathcal{R}_0 < 1$, then $(S, I) \to (\hat{S}, 0)$ in $C([0, L])$ as $t \to +\infty$.*

Proof. If $\mathcal{R}_0 < 1$, letting $u(x, t) = Me^{-\lambda_1(d_I,q)t}e^{\frac{a(x)}{d_I}}\phi_1(x)$, then we have

$$\begin{cases} u_t = [d_Iu_x - a'(x)u]_x + [\beta(x) - \gamma(x)]u, \quad 0 < x < L, \quad t > 0, \\ d_Iu_x(0, t) - a'(0)u(0, t) = 0, \quad d_Iu_x(L, t) - a'(L)u(L, t) = 0,\ t > 0. \end{cases}$$

Here $(\lambda_1(d_I, q), \phi_1)$ is the principal eigen-pair, $\lambda_1(d_I, q) > 0$ and $\phi_1(x) > 0$ on $[0, L]$. M is large enough such that $I(x, 0) \le u(x, 0)$ for every $x \in (0, L)$. Noticing that

$$\begin{cases} I_t = [d_II_x - a'(x)I]_x + [\beta(x) - \gamma(x)]I, \quad 0 < x < L, \quad t > 0, \\ d_Iu_x(0, t) - a'(0)u(0, t) = 0, \quad d_Iu_x(L, t) - a'(L)u(L, t) = 0,\ t > 0. \end{cases}$$

By the comparison principle, we have $I(x, t) \le u(x, t)$ for every $x \in (0, L)$ and $t \ge 0$. Obviously, $u(x, t) \to 0$ for every $x \in (0, L)$ as $t \to \infty$, which implies that $I(x, t) \to 0$ for every $x \in (0, L)$ as $t \to \infty$.

Now we will show that $S \to \hat{S}$ as $t \to +\infty$. Since

$$S_t = (d_SS_x - a'(x)S)_x - \beta(x)\frac{SI}{S + I} + \gamma(x)I,\ 0 < x < L,\ t > 0,$$

we have

$$|S_t - (d_SS_x - a'(x)S)_x| \le (\|\beta\|_\infty + \|\gamma\|_\infty)I \le Ce^{-\lambda_1(d_I,q)t},$$

for $0 < x < L$, $t > 0$. Noticing that

$$\lim_{t\to+\infty} e^{-\lambda_1(d_I,q)t} \to 0$$

as $t \to +\infty$, we know that there exists a positive function $\tilde{S}(x)$ such that

$$\lim_{t\to+\infty} S(x,t) = \tilde{S}(x), \quad \int_0^L \tilde{S}(x)dx = N.$$

Therefore, $\lim_{t\to+\infty} S(x,t) = \tilde{S}(x) = \hat{S}(x)$.

□

Proof. Theorem 1.2 is the direct results of Lemma 2.1.4 and Lemma 2.1.5.

□

2.2. Further properties of $\mathcal{R}_0$: $\beta(x) - \gamma(x)$ changing sign once

In this section, we will study further properties of $\mathcal{R}_0$ in the case of $\beta(x) - \gamma(x)$ changing sign once.

Lemma 2.2.1. *Assume that ϕ_1 is a positive eigenfunction corresponding to $\mathcal{R}_0 = 1$, $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$. If assumption (C1)(or (C2)) holds, then $(\phi_1)_x > 0$(or $(\phi_1)_x < 0$) in $(0, L)$.*

Proof. If $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and assumption (C1) holds, then there exists some $x_0 \in (0, L)$ such that $\beta(x) - \gamma(x) < 0$ in $(0, x_0)$, $\beta(x_0) = \gamma(x_0)$ and $\beta(x) - \gamma(x) > 0$ in (x_0, L).

By the definition of ϕ_1, we have

$$\begin{cases} -d_I(\phi_1)_{xx} - a'(x)(\phi_1)_x = [\beta(x) - \gamma(x)]\phi_1, \quad 0 < x < L, \\ (\phi_1)_x(0) = (\phi_1)_x(L) = 0. \end{cases} \tag{2.17}$$

Multiplying (2.17) by $e^{\frac{a(x)}{d_I}}$, we obtain

$$-d_I(e^{\frac{a(x)}{d_I}}(\phi_1)_x)_x = [\beta(x) - \gamma(x)]e^{\frac{a(x)}{d_I}}\phi_1.$$

Under the assumptions on $\beta(x)$ and $\gamma(x)$, we can obtain $(e^{\frac{a(x)}{d_I}}(\phi_1)_x)_x > 0$ in $(0, x_0)$, $(e^{\frac{a(x)}{d_I}}(\phi_1)_x)_x = 0$ at x_0 and $(e^{\frac{a(x)}{d_I}}(\phi_1)_x)_x < 0$ in (x_0, L). That is, $e^{\frac{a(x)}{d_I}}(\phi_1)_x$ is strictly increasing in $(0, x_0)$ and strictly decreasing in (x_0, L). Noticing that $(\phi_1)_x(0) = (\phi_1)_x(L) = 0$, we can get $e^{\frac{a(x)}{d_I}}(\phi_1)_x > 0$ in $(0, L)$. So $(\phi_1)_x > 0$ in $(0, L)$.

Similarly, if $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and assumption (C2) holds, $(\phi_1)_x < 0$ in $(0, L)$. We omit the details here. □

Now we prove two general lemmas below.

For any continuous function $m(x)$ on $[0, L]$, define

$$F(\eta) = \int_0^L \tilde{a}'(x)e^{\eta\tilde{a}(x)}m(x)dx, \quad 0 \le \eta < \infty.$$

Lemma 2.2.2. *Assume that $m(x) \in C^1([0, L])$ and $m(L) > 0$(or $m(L) < 0$). Then there exists some positive constant M such that $F(\eta) > 0$(or $F(\eta) < 0$) for any $\eta > M$.*

Proof. Since $m'(x)$ and $\tilde{a}'(x)$ is uniformly bounded independent of η, we have

$$\begin{aligned}\lim_{\eta\to+\infty}\eta e^{-\eta\tilde{a}(L)}F(\eta) &= \lim_{\eta\to+\infty}\int_0^L \eta\tilde{a}'(x)e^{-\eta[\tilde{a}(x)-\tilde{a}(L)]}m(x)dx\\ &= m(L) - \lim_{\eta\to+\infty}\left(m(0)e^{\eta[\tilde{a}(0)-\tilde{a}(L)]} + \int_0^L m'(x)e^{\eta[\tilde{a}(x)-\tilde{a}(L)]}dx\right)\\ &= m(L) - \lim_{\eta\to+\infty}\left(m(0)e^{\eta[\tilde{a}(0)-\tilde{a}(L)]} + \int_0^L m'(x)e^{\tilde{a}'(\xi)[x-L]}dx\right)\\ &= m(L) > 0(< 0).\end{aligned}$$

Therefore, there exists some positive constant M such that $F(\eta) > 0(< 0)$ for $\eta > M$. □

Lemma 2.2.3. *Assume that $m(x)$ changes sign once in $(0, L)$. Then*
(i) If $m(L) > 0$ and $\int_0^L \tilde{a}'(x)m(x)dx > 0$, then $F(\eta) > 0$ for any $\eta > 0$;
(ii) If $m(L) < 0$ and $\int_0^L \tilde{a}'(x)m(x)dx < 0$, then $F(\eta) < 0$ for any $\eta > 0$;
(iii) If $m(L) > 0$ and $\int_0^L \tilde{a}'(x)m(x)dx < 0$, then there exists a unique $\eta_1 \in (0, +\infty)$ such that $F(\eta_1) = 0$ and $F'(\eta_1) > 0$;
(iv) If $m(L) < 0$ and $\int_0^L \tilde{a}'(x)m(x)dx > 0$, then there exists a unique $\eta_1 \in (0, +\infty)$ such that $F(\eta_1) = 0$ and $F'(\eta_1) < 0$.

Proof. We only prove part (i) and part (iii). The proofs of part (ii) and part (iv) are similar.

(i) If $m(L) > 0$ and $m(x)$ changes sign once in $(0, L)$, then there exists $x_1 \in (0, L)$ such that $m(x) < 0$ for $x \in (0, x_1)$ and $m(x) > 0$ for $x \in (x_1, L)$. Since $\tilde{a}(x)$ is increasing, we have $m(x)[\tilde{a}(x) - \tilde{a}(x_1)] > 0$ for $x \in (0, L)$ and $x \neq x_1$. And

$$\begin{aligned}[e^{-\tilde{a}(x_1)\eta}F(\eta)]' &= e^{-\tilde{a}(x_1)\eta}[F'(\eta) - \tilde{a}(x_1)F(\eta)]\\ &= e^{-\tilde{a}(x_1)\eta}\int_0^L [\tilde{a}(x) - \tilde{a}(x_1)]m(x)\tilde{a}'(x)e^{\eta\tilde{a}(x)}dx > 0,\end{aligned} \tag{2.18}$$

which implies that $e^{-\tilde{a}(x_1)\eta}F(\eta)$ is strictly increasing in $\eta \in (0, \infty)$, $e^{-\tilde{a}(x_1)\eta}F(\eta) > F(0) = \int_0^L \tilde{a}'(x)m(x)dx > 0$. Consequently, $F(\eta) > 0$ for any $\eta > 0$. Here the prime notation denotes differentiation by η. Part (i) is proved.

(iii) $\int_0^L \tilde{a}'(x)m(x)dx < 0$ means that $F(0) < 0$, while, by the result of Lemma 2.2.2, $m(L) > 0$ means that $F(\eta) > 0$ for $\eta > M$ with M large enough. By continuity, there at least exists a positive root for $F(\eta) = 0$. But $e^{-\tilde{a}(x_1)\eta}F(\eta)$ is increasing in $\eta \in (0, \infty)$, so $F(\eta) = 0$ only has a unique positive root η_1. By (2.18), we have $F'(\eta_1) > a(x_1)F(\eta_1) = 0$. Part (iii) is proved.

□

2.3. The stability of DFE

In this section, we consider the stability of DFE. First we have

Lemma 2.3.1. *Assume that $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$.*
(i) If $\beta(x)$ and $\gamma(x)$ satisfy (C1), then $\mathcal{R}_0 > 1$ for $d_I > 0$ and $q > \min_{x\in[0,L]} a'(x) > 0$;
(ii) If $\beta(x)$ and $\gamma(x)$ satisfy (C2), then for every $d_I > 0$, there exists a unique $\bar{q} = \bar{q}(d_I)$ such that $\mathcal{R}_0 > 1$ for $0 < \min_{x\in[0,L]} a'(x) < q < \bar{q}$, $\mathcal{R}_0 = 1$ for $q = \bar{q}$ and $\mathcal{R}_0 < 1$ for $q > \bar{q}$.

Proof. (i) Subtracting both sides of (2.2) by $\beta(x)\varphi$, multiplying by $\frac{e^{\frac{a(x)}{d_I}}}{\varphi}$, we have

$$[-d_I\varphi_{xx} - a'(x)\varphi_x]\frac{e^{\frac{a(x)}{d_I}}}{\varphi} + [\gamma(x) - \beta(x)]e^{\frac{a(x)}{d_I}} = (\frac{1}{\mathcal{R}_0} - 1)\beta(x)e^{\frac{a(x)}{d_I}}.$$

Integrating it by parts over $(0, L)$, using $\varphi_x(0) = \varphi_x(L) = 0$, we obtain

$$d_I \int_0^L \frac{e^{\frac{a(x)}{d_I}}(\varphi_x)^2}{\varphi^2}dx + \int_0^L [\beta(x) - \gamma(x)]e^{\frac{a(x)}{d_I}}dx = (1 - \frac{1}{\mathcal{R}_0})\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}dx.$$

Using Lemma 2.2.3(i) with $m(x) = \frac{[\beta(x)-\gamma(x)]}{\tilde{a}'(x)}$, $\int_0^L [\beta(x) - \gamma(x)]e^{\frac{a(x)}{d_I}}dx > 0$, and

$$(1 - \frac{1}{\mathcal{R}_0})\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\varphi^2 dx > 0,$$

which implies that $\mathcal{R}_0 > 1$.

(ii) Differentiating both sides of (2.2) with respect to q, denoting the differentiation with respect to q by the dot notation, we obtain

$$\begin{cases} -d_I\dot{\varphi}_{xx} - \tilde{a}'(x)\varphi_x - \tilde{a}'(x)\dot{\varphi}_x + \gamma(x)\dot{\varphi} = -\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}\beta(x)\varphi + \frac{1}{\mathcal{R}_0}\beta(x)\dot{\varphi}, & 0 < x < L, \\ \dot{\varphi}_x(0) = \dot{\varphi}_x(L) = 0. \end{cases} \tag{2.19}$$

Multiplying (2.19) by $e^{\frac{a(x)}{d_I}}\varphi$ and integrating the resulting equation in $(0, L)$, we have

$$\begin{aligned} &d_I \int_0^L e^{\frac{a(x)}{d_I}}\dot{\varphi}_x\varphi_x dx - \int_0^L e^{\frac{a(x)}{d_I}}\varphi_x\varphi\tilde{a}'(x)dx + \int_0^L \gamma(x)e^{\frac{a(x)}{d_I}}\dot{\varphi}\varphi dx \\ &= -\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\varphi^2 dx + \frac{1}{\mathcal{R}_0}\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\dot{\varphi}\varphi dx. \end{aligned} \tag{2.20}$$

Multiplying (2.2) by $e^{\frac{a(x)}{d_I}}\dot{\varphi}$ and integrating the resulting equation in $(0, L)$, we get

$$d_I \int_0^L e^{\frac{a(x)}{d_I}}\dot{\varphi}_x\varphi_x dx + \int_0^L \gamma(x)e^{\frac{a(x)}{d_I}}\dot{\varphi}\varphi dx = \frac{1}{\mathcal{R}_0}\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\dot{\varphi}\varphi dx. \tag{2.21}$$

Subtracting (2.20) and (2.21), we obtain

$$\frac{\partial\mathcal{R}_0}{\partial q} = \frac{\mathcal{R}_0^2\int_0^L e^{\frac{a(x)}{d_I}}\varphi_x\varphi\tilde{a}'(x)dx}{\int_0^L \beta(x)e^{\frac{a(x)}{d_I}}\varphi^2 dx}. \tag{2.22}$$

By the result of Corollary 2.1.1, we know that

$$\lim_{q\to\infty}\mathcal{R}_0 = \frac{\beta(L)}{\gamma(L)} < 1.$$

Meanwhile, we have

$$\lim_{q\to 0}\mathcal{R}_0 = \hat{\mathcal{R}}_0 > 1$$

for any d_I. Then there must exist at least some $\bar{q}$ such that $\mathcal{R}_0(\bar{q}) = 1$. By Lemma 2.1.1, for any $\bar{q} > 0$ satisfying $\mathcal{R}_0(\bar{q}) = 1$, $(\phi_1)_x < 0$ in $(0, L)$. Recalling (2.22), we have

$$\frac{\partial \mathcal{R}_0}{\partial \bar{q}} = \frac{\int_0^L e^{\frac{\bar{q}}{d_I}\tilde{a}(x)}(\phi_1)_x \phi_1 dx}{\int_0^L \beta(x) e^{\frac{\bar{q}}{d_I}\tilde{a}(x)}(\phi_1)^2 dx} < 0,$$

which implies that $\bar{q}$ is the unique point satisfying $\mathcal{R}_0(\bar{q}) = 1$.

□

The following lemma will tell us that there exists a function $q = \rho_1(d_I)$ such that $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$ and give the asymptotic profile of $\rho_1(d_I)$ if $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$.

Lemma 2.3.2. *Assume that* $\beta(x) - \gamma(x)$ *changes sign once in* $(0, L)$, $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$, *and* θ_1 *is the unique solution of*

$$\int_0^L [\beta(x) - \gamma(x)] e^{\theta_1 \tilde{a}(x)} dx = 0.$$

Suppose that $\beta(x)$ *and* $\gamma(x)$ *satisfy (C2). Then there exists a function* $\rho_1 : (0, \infty) \to (0, \infty)$ *such that* $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$. *And* ρ_1 *satisfies*

$$\lim_{d_I \to 0} \rho_1(d_I) = 0, \quad \lim_{d_I \to \infty} \frac{\rho_1(d_I)}{d_I} = \theta_1.$$

Proof. 1. Let's first consider the limit of $\frac{\rho_1(d_I)}{d_I}$ as $d_I \to \infty$. Assume that $\frac{\rho_1(d_I)}{d_I} \to \infty$ as $d_I \to \infty$. Under the assumption (C2), by Lemma 2.1.4, we have

$$\lim_{\rho_1(d_I) \to \infty, \frac{\rho_1(d_I)}{d_I} \to \infty} \mathcal{R}_0(d_I, \rho_1(d_I)) = \frac{\beta(L)}{\gamma(L)} < 1,$$

which is a contradiction to $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$.

Next, we will prove that $\frac{\rho_1(d_I)}{d_I} \to \theta_1$ as $d_I \to \infty$. Here θ_1 is the unique positive root of $\int_0^L [\beta(x) - \gamma(x)] e^{\theta_1 \tilde{a}(x)} dx = 0$. By the discussions above, we know that $\frac{\rho_1(d_I)}{d_I}$ is bounded for large d_I. Passing to a subsequence if necessary, we suppose that $\frac{\rho_1(d_I)}{d_I} \to \theta_*$ for some nonnegative number θ_* as $d_I \to \infty$. Let $\tilde{\varphi}$ be the unique normalized eigenfunction of the eigenvalue $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$. Then

$$\begin{cases} -d_I(e^{\frac{\rho_1(d_I)}{d_I}\tilde{a}(x)}\tilde{\varphi}_x)_x + [\gamma(x) - \beta(x)] e^{\frac{\rho_1(d_I)}{d_I}\tilde{a}(x)}\tilde{\varphi} = 0, & 0 < x < L, \\ \tilde{\varphi}_x(0) = \tilde{\varphi}_x(L) = 0. \end{cases} \tag{2.23}$$

Integrating (2.23) in $(0, L)$, we get

$$\int_0^L [\beta(x) - \gamma(x)] e^{\frac{\rho_1(d_I)}{d_I}\tilde{a}(x)} \tilde{\varphi} dx = 0. \tag{2.24}$$

Recalling that, up to a subsequence if necessary, $\tilde{\varphi} \to 1$ in $C([0, 1])$ as $d_I \to \infty$. Letting $d_I \to \infty$ in (2.24), we have

$$\int_0^L [\beta(x) - \gamma(x)] e^{\theta_* \tilde{a}(x)} dx = 0.$$

By Lemma 2.2.3 with $m(x) = \frac{[\beta(x)-\gamma(x)]}{\tilde{a}'(x)}$, $F(\eta)$ has a unique positive root, i.e., $\theta_* = \theta_1$.

2. Contradictorily, assume that $q = \rho_1(d_I) \to q^* > 0$ or $q = \rho_1(d_I) \to \infty$ as $d_I \to 0$. By Lemma 2.1.4, we know that

$$\lim_{\rho_1(d_I)\to q^*, \frac{\rho_1(d_I)}{d_I}\to\infty} \mathcal{R}_0(d_I, \rho_1(d_I)) = \frac{\beta(L)}{\gamma(L)} < 1$$

or

$$\lim_{\rho_1(d_I)\to \infty, \frac{\rho_1(d_I)}{d_I}\to\infty} \mathcal{R}_0(d_I, \rho_1(d_I)) = \frac{\beta(L)}{\gamma(L)} < 1,$$

which is a contradiction to $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$. Therefore, we have $\lim_{d_I\to 0} \rho_1(d_I) = 0$.

□

To study the properties of $\mathcal{R}_0$ when $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$, we need the following results which were stated in [2]:

Proposition 2.3.1. *Assume that $\beta(x) - \gamma(x)$ changes sign in $(0, L)$.*

(i) $\hat{\mathcal{R}}_0$ is a monotone decreasing function of d_I with $\hat{\mathcal{R}}_0 \to \max\{\beta(x)/\gamma(x) : x \in [0, L]\}$ as $d_I \to 0$ and $\hat{\mathcal{R}}_0 \to \int_0^L \beta(x)dx / \int_0^L \gamma(x)dx$ as $d_I \to +\infty$;

(ii) $\hat{\mathcal{R}}_0 > 1$ for all $d_I > 0$ if $\int_0^L \beta(x)dx \geq \int_0^L \gamma(x)dx$;

(iii) There exists a threshold value $d_I^* \in (0, +\infty)$ such that $\hat{\mathcal{R}}_0 > 1$ for $d_I < d_I^*$ and $\hat{\mathcal{R}}_0 < 1$ for $d_I > d_I^*$ if $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$.

Lemma 2.3.3. *Assume that $\beta(x)-\gamma(x)$ changes sign once in $(0, L)$ and $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$. Then there exists some constant $d_I^* > 0$ such that d_I^* is the unique positive root of the equation $\hat{\mathcal{R}}_0(d_I) = 1$ and the following statements hold.*

1. If $\beta(x)$ and $\gamma(x)$ satisfy (C1), then

(i) for $d_I \in (0, d_I^*]$, $\mathcal{R}_0 > 1$ for any $q > \min_{x\in[0,L]} a'(x) > 0$;

(ii) for $d_I \in (d_I^*, \infty)$, there exists a unique $\bar{q} = \bar{q}(d_I)$ such that $\mathcal{R}_0 < 1$ for any $0 < \min_{x\in[0,L]} a'(x) < q < \bar{q}$ and $\mathcal{R}_0 > 1$ for any $q > \bar{q}$.

2. If $\beta(x)$ and $\gamma(x)$ satisfy (C2), then

(iii) for $d_I \in (0, d_I^*]$, there exists a unique $\bar{q} = \bar{q}(d_I)$ such that $\mathcal{R}_0 > 1$ for any $0 < \min_{x\in[0,L]} a'(x) < q < \bar{q}$ and $\mathcal{R}_0 < 1$ for any $q > \bar{q}$;

(iv) for $d_I \in (d_I^*, \infty)$, $\mathcal{R}_0 < 1$ for any $q > \min_{x\in[0,L]} a'(x) > 0$.

Proof. (i) Noticing that $\beta(x)$ and $\gamma(x)$ satisfy (C1), similar to the proof of (ii) in Lemma 2.1.4, we can prove that there exists a unique $\bar{q} > 0$ satisfying $\mathcal{R}_0(\bar{q}) = 1$ and $\mathcal{R}_0'(\bar{q}) > 0$. Hence, the conclusion is true for $d_I \in (d_I^*, +\infty)$.

For $d_I \in (0, d_I^*]$, by the results of Proposition 2.3.1, we have $\lim_{q\to 0} \mathcal{R}_0 = \hat{\mathcal{R}}_0 \geq 1$. By the results of Corollary 2.1.1, $\lim_{q\to+\infty} \mathcal{R}_0 = \beta(L)/\gamma(L) > 1$ under the condition (C1). Hence $\mathcal{R}_0 > 1$ for any $q > 0$.

(ii) The proof of Lemma 2.3.3 under the condition (C2) is similar to that of Lemma 2.1.4, we omit the details here.

□

Lemma 2.3.4. *Assume that $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$. Then there exists a constant $d_I^* > 0$ such that d_I^* is the unique positive root of the equation $\hat{\mathcal{R}}_0(d_I) = 1$ and the following statements hold.*

1. If $\beta(x)$ and $\gamma(x)$ satisfy (C1), then there exists a function $\rho_2 : (d_I^*, \infty) \to (0, \infty)$ such that ρ_2 is a monotone increasing function of d_I and $\mathcal{R}_0(d_I, \rho_2(d_I)) = 1$. Let θ_2 be the unique solution of

$$\int_0^L [\beta(x) - \gamma(x)] e^{\theta_2 \tilde{a}(x)} dx = 0.$$

Then

$$\lim_{d_I \to d_I^* +} \rho_2(d_I) = 0, \quad \lim_{d_I \to \infty} \frac{\rho_2(d_I)}{d_I} = \theta_2.$$

2. If $\beta(x)$ and $\gamma(x)$ satisfy (C2), then there exists a function $\rho_3 : (0, d_I^*) \to (0, \infty)$ such that $\mathcal{R}_0(d_I, \rho_3(d_I)) = 1$ and

$$\lim_{d_I \to 0+} \rho_3(d_I) = 0, \quad \lim_{d_I \to d_I^* -} \frac{\rho_3(d_I)}{d_I} = 0.$$

Proof. 1. If we can prove that $\rho_2'(d_I) > 0$ for $d_I \in (d_I^*, \infty)$, then $\rho_2(d_I)$ is a monotone increasing function of d_I. Here the prime notation denotes differentiation by d_I. Since $\mathcal{R}_0(d_I, \rho_2(d_I)) = 1$, we can get

$$\frac{\partial \mathcal{R}_0}{\partial q} \rho_2'(d_I) + \frac{\partial \mathcal{R}_0}{\partial d_I} = 0. \tag{2.25}$$

By Lemma 2.3.1, $\frac{\partial \mathcal{R}_0}{\partial q} > 0$ for $\mathcal{R}_0(d_I, \rho_2(d_I)) = 1$. So we need to prove that $\frac{\partial \mathcal{R}_0}{\partial d_I} < 0$.

Differentiating both sides of (2.2) with respect to d_I, denoting the differentiation with respect to d_I by the dot notation, we obtain

$$\begin{cases} -\varphi_{xx} - d_I \dot{\varphi}_{xx} - a'(x) \dot{\varphi}_x + \gamma(x) \dot{\varphi} = -\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2} \beta(x) \varphi + \frac{1}{\mathcal{R}_0} \beta(x) \dot{\varphi}, & 0 < x < L, \\ \dot{\varphi}_x(0) = \dot{\varphi}_x(L) = 0. \end{cases} \tag{2.26}$$

Multiplying (2.26) by $e^{\frac{a(x)}{d_I}} \varphi$ and integrating the resulting equation in $(0, L)$, we obtain

$$\begin{aligned} &- \int_0^L e^{\frac{a(x)}{d_I}} \varphi_{xx} \varphi dx + d_I \int_0^L e^{\frac{a(x)}{d_I}} \dot{\varphi}_x \varphi_x dx + \int_0^L \gamma(x) e^{\frac{a(x)}{d_I}} \dot{\varphi} \varphi dx \\ &= -\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2} \int_0^L \beta(x) e^{\frac{a(x)}{d_I}} \varphi^2 dx + \frac{1}{\mathcal{R}_0} \int_0^L \beta(x) e^{\frac{a(x)}{d_I}} \dot{\varphi} \varphi dx. \end{aligned} \tag{2.27}$$

Multiplying (2.2) by $e^{\frac{a(x)}{d_I}} \dot{\varphi}$ and integrating the resulting equation in $(0, L)$, we get

$$d_I \int_0^L e^{\frac{a(x)}{d_I}} \dot{\varphi}_x \varphi_x dx + \int_0^L \gamma(x) e^{\frac{a(x)}{d_I}} \dot{\varphi} \varphi dx = \frac{1}{\mathcal{R}_0} \int_0^L \beta(x) e^{\frac{a(x)}{d_I}} \dot{\varphi} \varphi dx. \tag{2.28}$$

Subtracting (2.27) and (2.28), we have

$$\frac{\partial \mathcal{R}_0}{\partial d_I} = \frac{\mathcal{R}_0^2 \int_0^L e^{\frac{a(x)}{d_I}} \varphi_{xx} \varphi dx}{\int_0^L \beta(x) e^{\frac{a(x)}{d_I}} \varphi^2 dx} = -\frac{\mathcal{R}_0^2 \int_0^L e^{\frac{a(x)}{d_I}} (\varphi_x)^2 dx}{\int_0^L \beta(x) e^{\frac{a(x)}{d_I}} \varphi^2 dx} - \frac{\mathcal{R}_0^2 \int_0^L e^{\frac{a(x)}{d_I}} \varphi_x \varphi a'(x) dx}{d_I \int_0^L \beta(x) e^{\frac{a(x)}{d_I}} \varphi^2 dx}. \tag{2.29}$$

By Lemma 2.2.1, for any d_I satisfying $\mathcal{R}_0(d_I, q) = 1$, $(\phi_1)_x > 0$, we can get

$$\frac{\partial \mathcal{R}_0}{\partial d_I} = -\frac{\mathcal{R}_0^2 \int_0^L e^{\frac{a(x)}{d_I}}[(\phi_1)_x]^2 dx}{\int_0^L \beta(x) e^{\frac{a(x)}{d_I}} \phi_1^2 dx} - \frac{\mathcal{R}_0^2 \int_0^L e^{\frac{a(x)}{d_I}}(\phi_1)_x \phi_1 a'(x) dx}{d_I \int_0^L \beta(x) e^{\frac{a(x)}{d_I}} \phi_1^2 dx} < 0. \tag{2.30}$$

(2.25) and (2.30) imply that $\rho_2'(d_I) > 0$ for $d_I \in (d_I^*, \infty)$.

The proof of $\lim_{d_I \to \infty} \frac{\rho_2(d_I)}{d_I} = \theta_2$ (θ_2 is the unique solution of $\int_0^L [\beta(x) - \gamma(x)] e^{\theta_2 a(x)} dx = 0$) is similar to the proof of Lemma 2.3.2, we omit the details here.

Now we will prove that $\lim_{d_I \to d_I^*+} \rho_2(d_I) = 0$. Assume that there exists q^* such that $q = \rho_2(d_I) \to q^*$ as $d_I \to d_I^*+$. Then there exists a positive function $\phi^*(x) \in C^2([0, L])$ such that

$$\begin{cases} -d_I^* \phi_{xx}^* - q^* \tilde{a}'(x) \phi_x^* + \gamma(x) \phi^* = \beta(x) \phi^*, & 0 < x < L, \\ \phi_x^*(0) = \phi_x^*(L) = 0. \end{cases} \tag{2.31}$$

Noticing that d_I^* is the unique positive root of $\hat{\mathcal{R}}_0 = 1$ and the definition of $\hat{\mathcal{R}}_0$ implies $q = 0$, there exists a positive function $\hat{\phi}(x) \in C^2([0, L])$ such that

$$\begin{cases} -d_I^* \hat{\phi}_{xx} + \gamma(x) \hat{\phi} = \beta(x) \hat{\phi}, & 0 < x < L, \\ \hat{\phi}_x(0) = \hat{\phi}_x(L) = 0. \end{cases} \tag{2.32}$$

Multiplying (2.31) by $\hat{\phi}$, (2.32) by ϕ^*, subtracting the two resulting equations, then integrating by parts over $(0, L)$, we get

$$q^* \int_0^L \tilde{a}'(x) \phi_x^* \hat{\phi} dx = 0.$$

Since ϕ_x^* is positive(by Lemma 2.2.1), we have $q^* = 0$. Therefore, $\lim_{d_I \to d_I^*+} \rho_2(d_I) = 0$.

2. Using the arguments above, similar to the proof of Lemma 2.3.2, we can obtain the conclusions.

□

2.4. The endemic equilibrium

In this section, we will show that: If the disease-free equilibrium is unstable, then we can use the bifurcation analysis and degree theory to study the existence of endemic equilibrium.

Letting $\tilde{S} = e^{\frac{a(x)}{d_S}} \bar{S}$, $\tilde{I} = e^{\frac{a(x)}{d_I}} \bar{I}$, we have

$$\begin{cases} d_S \bar{S}_{xx} + a'(x) \bar{S}_x - \beta(x) \frac{e^{\frac{a(x)}{d_I}} \bar{S} \bar{I}}{e^{\frac{a(x)}{d_S}} \bar{S} + e^{\frac{a(x)}{d_I}} \bar{I}} + \gamma(x) e^{(\frac{1}{d_I} - \frac{1}{d_S}) a(x)} \bar{I} = 0, & 0 < x < L, \\ d_I \bar{I}_{xx} + a'(x) \bar{I}_x + \beta(x) \frac{e^{\frac{a(x)}{d_S}} \bar{S} \bar{I}}{e^{\frac{a(x)}{d_S}} \bar{S} + e^{\frac{a(x)}{d_I}} \bar{I}} - \gamma(x) \bar{I} = 0, & 0 < x < L, \\ \bar{S}_x(0) = \bar{S}_x(L) = 0, \quad \bar{I}_x(0) = \bar{I}_x(L) = 0, \\ \int_0^L [e^{\frac{a(x)}{d_S}} \bar{S} + e^{\frac{a(x)}{d_I}} \bar{I}] dx = N. \end{cases} \tag{2.33}$$

Since the structure of the solution set of (2.33) is the same as that of (1.3), we study (2.33) instead of (1.3). Denote the unique disease-free equilibrium of (2.33) by $(\hat{\bar{S}}, 0) = (\frac{N}{\int_0^L e^{\frac{a(x)}{d_S}}}, 0)$. We will consider a branch of positive solutions of (2.33) bifurcating from the branch of semi-trivial solutions given by

$$\Gamma_S := \{(q, (\hat{\bar{S}}, 0)) : 0 < \min_{x \in [0, L]} a'(x) < q < \infty\}$$

through using the local and global bifurcation theorems. For fixed $d_S,\ d_I > 0$, we take q as the bifurcation parameter. Let

$$X = \{u \in W^{2,p}((0,L)) : u_x(0) = u_x(L) = 0\}, \quad Y = L^p((0,L))$$

for $p > 1$ and the set of positive solution of (2.33) to be

$$O = \{(q,(S,I)) \in \mathbb{R}^+ \times X \times X : q > \min_{x\in[0,L]} a'(x) > 0, S > 0, I > 0, (q,(S,I)) \text{ satisfies (2.33)}\}.$$

Lemma 2.4.1 *Assume that $d_S,\ d_I > 0$ and $\beta(x) - \gamma(x)$ changes sign once in $(0,L)$. Then*

1. $q_* > 0$ is a bifurcation point for the positive solutions of (2.33) from the semi-trivial branch Γ_S if and only if q_* satisfies $R_0(d_I, q_*) = 1$. That is,

(I) If $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$, then such q_* exists uniquely for any $d_I > 0$ if and only if assumption (C2) holds;

(II)If $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$, let d_I^* be the unique positive root of $\hat{R}_0 = 1$, then such q_* exists uniquely for any $d_I > 0$ if and only if either $\beta(x)$ and $\gamma(x)$ satisfy condition (C1) and $d > d_I^*$ or they satisfy condition (C2) and $0 < d < d_I^*$.

2. There exits some $\delta > 0$ such that all positive solutions of (2.33) near $(q_*, (\hat{\hat{S}}, 0))) \in \mathbb{R}\times X\times X$ can be parameterized as

$$\Gamma = \{(q(\tau), (\hat{\hat{S}} + \bar{S}_1(\tau), \bar{I}_1(\tau))) : \tau \in [0,\delta)\}, \tag{2.34}$$

where $(q(\tau), (\hat{\hat{S}} + \bar{S}_1(\tau), I_1(\tau)))$ is a smooth curve with respect to τ and satisfies $q(0) = q_*$, $\hat{S}_1(0) = I_1(0) = 0$.

3. There exists a connected component Σ of $\bar{O}$ satisfying $\Gamma \subseteq \Sigma$, and Σ possesses some properties as follows.

Case (I) Assume that $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$ and (C2) holds. Then there exists some endemic equilibrium $(\hat{S}_*, \hat{I}_*)$ of (2.33) when $q = 0$ such that for Σ, the projection of Σ to the q-axis satisfies $Proj_q\Sigma = [0, q_*]$ and the connected component Σ connects to $(0, (\hat{S}_*, \hat{I}_*))$.

Case (II) Assume that $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$. Then

(i) If (C1) holds and $d_I > d_I^*$, then (2.33) has no positive solution for $0 < \min_{x\in[0,L]} a'(x) < q < q_*$ and for Σ, the projection of Σ to the q-axis satisfies $Proj_q\Sigma = [q_*, \infty)$.

(ii) If (C2) holds and $0 < d_I < d_I^*$, then there exists some endemic equilibrium $(\hat{S}_*, \hat{I}_*)$ of (2.33) when $q = 0$ such that for Σ, the projection of Σ to the q-axis satisfies $Proj_q\Sigma = [0, q_*]$ and the connected component Σ connects to $(0, (\hat{S}_*, \hat{I}_*))$.

Proof. 1. Let $F : \mathbb{R}^+ \times X \times X \to Y \times Y \times \mathbb{R}$ be the mapping as follows.

$$F(q,(\bar{S},\bar{I})) = \begin{pmatrix} d_S\bar{S}_{xx} + a'(x)\bar{S}_x - \beta(x)\dfrac{e^{\frac{a(x)}{d_I}}\bar{S}\bar{I}}{e^{\frac{a(x)}{d_S}}\bar{S} + e^{\frac{a(x)}{d_I}}\bar{I}} + \gamma(x)e^{(\frac{1}{d_I}-\frac{1}{d_S})a(x)}\bar{I} \\ d_I\bar{I}_{xx} + a'(x)\bar{I}_x + \beta(x)\dfrac{e^{\frac{a(x)}{d_S}}\bar{S}\bar{I}}{e^{\frac{a(x)}{d_S}}\bar{S} + e^{\frac{a(x)}{d_I}}\bar{I}} - \gamma(x)\bar{I} \\ \int_0^L [e^{\frac{a(x)}{d_S}}\bar{S} + e^{\frac{a(x)}{d_I}}\bar{I}]dx - N \end{pmatrix}.$$

It is to verify that the pair $(\bar{S},\bar{I})$ is a solution of (2.33) if only if $F(q,(\bar{S},\bar{I}))=0$. Obviously, $F(q,(\hat{\hat{S}},0))=0$ for any $q>\min_{x\in[0,L]}a'(x)>0$. The Fréchet derivatives of F at $(\hat{\hat{S}},0)$ are given by

$$D_{(\bar{S},\bar{I})}F(q,(\hat{\hat{S}},0))\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}=\begin{pmatrix} d_S\Phi_{xx}+\tilde{a}'(x)\Phi_x+[\gamma(x)-\beta(x)]e^{(\frac{1}{d_I}-\frac{1}{d_S})a(x)}\Psi\\ d_I\Psi_{xx}+\tilde{a}'(x)\Psi_x+[\beta(x)-\gamma(x)]\Psi\\ \int_0^L[e^{\frac{a(x)}{d_S}}\Phi+e^{\frac{a(x)}{d_I}}\Psi]dx\end{pmatrix},$$

$$D_{q,(\bar{S},\bar{I})}F(q,(\hat{\hat{S}},0))\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}=\begin{pmatrix} \tilde{a}'(x)\Phi_x+(\frac{a(x)}{d_I}-\frac{a(x)}{d_S})[\gamma(x)-\beta(x)]e^{(\frac{1}{d_I}-\frac{1}{d_S})a(x)}\Psi\\ \tilde{a}'(x)\Psi_x\\ \int_0^L[\frac{a(x)}{d_S}e^{\frac{a(x)}{d_S}}\Phi+\frac{a(x)}{d_I}e^{\frac{a(x)}{d_I}}\Psi]dx\end{pmatrix},$$

$$D_{(\bar{S},\bar{I}),(\bar{S},\bar{I})}F(q,(\hat{\hat{S}},0))\begin{bmatrix}\Phi\\ \Psi\end{bmatrix}^2=\begin{pmatrix} \frac{2}{\hat{\hat{S}}}\beta(x)e^{2(\frac{q}{d_I}-\frac{q}{d_S})\tilde{a}(x)}\Psi^2\\ -\frac{2}{\hat{\hat{S}}}\beta(x)e^{(\frac{1}{d_I}-\frac{1}{d_S})a(x)}\Psi^2\\ 0\end{pmatrix}.$$

If (Φ_1,Ψ_1) is a nontrivial solution of the following problem

$$\begin{cases} d_S\Phi_{xx}+\tilde{a}'(x)\Phi_x+[\gamma(x)-\beta(x)]e^{(\frac{1}{d_I}-\frac{1}{d_S})a(x)}\Psi=0, & 0<x<L,\\ d_I\Psi_{xx}+\tilde{a}'(x)\Psi_x+[\beta(x)-\gamma(x)]\Psi=0, & 0<x<L,\\ \Phi_x(0)=\Phi_x(L)=\Psi_x(0)=\Psi_x(L)=0,\\ \int_0^L[e^{\frac{a(x)}{d_S}}\Phi+e^{\frac{a(x)}{d_I}}\Psi]dx=0,\end{cases}\tag{2.35}$$

then $(q_*,(\hat{\hat{S}},0)))$ is degenerate solution of (2.33). The second equation of (2.33) has a positive solution Ψ_1 only if $q=q_*$ satisfies $\mathcal{R}_0(d_I,q_*)=1$. And Φ_1 satisfies

$$\begin{cases} d_S(\Phi_1)_{xx}+\tilde{a}'(x)(\Phi_1)_x+[\gamma(x)-\beta(x)]e^{(\frac{1}{d_I}-\frac{1}{d_S})a(x)}\Psi_1=0, & 0<x<L,\\ (\Phi_1)_x(0)=(\Phi_1)_x(L)=0,\\ \int_0^L[e^{\frac{a(x)}{d_S}}\Phi_1+e^{\frac{a(x)}{d_I}}\Psi_1]dx=0,\end{cases}\tag{2.36}$$

Obviously, Φ_1 is uniquely determined by Ψ_1 in (2.36). Therefore, $q=q_*$ is the only possible bifurcation point along Γ_S where positive solutions of (2.33) bifurcates and such q_* exists if and only if $\mathcal{R}_0=1$. We can obtain the necessary and sufficient conditions for the occurrence of bifurcation by Lemma 2.3.1 and Lemma 2.3.3.

2. At $(q,(\bar{S},\bar{I}))=(q_*,(\hat{\hat{S}},0))$, the kernel

$$\mathrm{Ker}(D_{(\bar{S},\bar{I})}F(q_*,(\hat{\hat{S}},0)))=\mathrm{span}\{(\Phi_1,\Psi_1)\},$$

where (Φ_1,Ψ_1) is the solution of (2.35) with $q=q_*$. Up to a multiple of constant, (Φ_1,Ψ_1) is unique. And the range of $D_{(\bar{S},\bar{I})}F(q_*,(\hat{\hat{S}},0))$ is given by

$$\mathrm{Range}(D_{(\bar{S},\bar{I})}F(q_*,(\hat{\hat{S}},0)))=\{(f,g,k)\in Y\times Y\times\mathbb{R}^N:\int_0^L g\Psi_1e^{\frac{a(x)}{d_I}}dx=0\},$$

and it is co-dimension one. By the result of Lemma 2.1.1, $(\Psi_1)_x$ keeps one sign in $(0,L)$ and $\int_0^L(\Psi_1)_x\Psi_1e^{\frac{a(x)}{d_I}}dx\neq0$, which implies that

$$D_{q,(\bar{S},\bar{I})}F(q_*,(\hat{\hat{S}},0))[(\Phi_1,\Psi_1)]\notin\mathrm{Range}(D_{q,(\bar{S},\bar{I})}F(q_*,(\hat{\hat{S}},0))).$$

Therefore, using the local bifurcation theorem in [20] to $F(q,(\bar{S},\bar{I}))$ at $(q_*,(\hat{\bar{S}},0))$, we know that the set of positive solutions of (2.33) is a smooth curve

$$\Gamma = \{(q(\tau),(\hat{\bar{S}}+\bar{S}_1(\tau),\bar{I}_1(\tau))) : \tau \in [0,\delta)\}$$

satisfying $q(0) = q_*$, $\bar{S}_1(\tau) = \tau\hat{\bar{S}} + o(|\tau|)$ and $I_1(\tau) = o(|\tau|)$. Similar to the procedure in [21] and [22], (also see [23]), we can compute

$$q' = -\frac{< l, D_{(\bar{S},\bar{I}),(\bar{S},\bar{I})}F(q_*,(\hat{\bar{S}},0))[\Phi_1,\Psi_1]^2 >}{2 < l, D_{q,(\bar{S},\bar{I})}F(q_*,(\hat{\bar{S}},0))[(\Phi_1,\Psi_1)]} = \frac{\int_0^L \beta(x)e^{(\frac{1}{d_I}-\frac{1}{d_S})a(x)}\phi_1^3 dx}{\hat{\bar{S}}\int_0^L e^{\frac{a(x)}{d_I}}\phi_1(\phi_1)_x dx}.$$

Here l is the linear functional on $Y\times Y\times\mathbb{R}$ defined by $< l,[f,g,k] >= \int_0^L g\Psi_1 e^{\frac{a(x)}{d_I}}dx$.

3. By the global bifurcation theorem in [23] and [24], we can get the existence of the connected component Σ. Moreover, Σ is either unbounded, or connects to another $(q,(\hat{\bar{S}},0))$, or Σ connects to another point on the boundary of O.

Case (I) Assume that $\int_0^L \beta(x)dx > \int_0^L \gamma(x)dx$ and (C2) holds. By Lemma 2.2.1 and the proof of part 2, we see that there exits a unique q_* such that the local bifurcation occurs at $(q_*,(\hat{\bar{S}},0))$ and $q'(0) < 0$, which means that the bifurcation direction is subcritical. Therefore, there exists some small $\delta > 0$ such that (2.33) has a positive solution if $q_*-\delta < q < q_*$. By Lemma 2.1.4, $\mathcal{R}_0 > 1$ if $q_*-\delta < q < q_*$ for $\delta > 0$ small enough. By Lemma 2.1.5, (2.33) has no positive solution if $\mathcal{R}_0 < 1$, which implies that (2.33) has no positive solution if $q > q_*$. Consequently, the projection of Σ to the q-axis $Proj_q\Sigma \subset [0,q_*]$. And Σ must be bounded in $\bar{O}$ because the positive solutions are uniformly bounded in L^∞ for $0 \le q \le q_*$. So the third option must happen here. Hence Σ must connect to $(0,(\bar{S}_*,\bar{I}_*))$, so $0 \in Proj_q\Sigma$. Here $(\bar{S}_*,\bar{I}_*)$ is the unique endemic equilibrium of (2.33) when $q = 0$.

Case (II) Assume that $\int_0^L \beta(x)dx < \int_0^L \gamma(x)dx$.

(i) If (C1) holds and $d_I > d_I^*$, by Lemma 2.2.1 and the bifurcation analysis above, there exists unique bifurcation point q_* satisfying $q'(0) > 0$, which means the bifurcation direction is supercritical. Then there exists some small $\delta > 0$ such that (2.33) has a positive solution if $q_* < q < q_* + \delta$. By Lemma 2.3.3, $\mathcal{R}_0 > 1$ if $q_* < q < q_* + \delta$ for some $\delta > 0$ small enough. By Lemma 2.1.5, (2.33) has no positive solution if $\mathcal{R}_0 < 1$, which implies that (2.33) has no positive solution if $0 < q < q_*$. So the first option must happen here. If there exists some finite $q^* > q_*$ such that $Proj_q\Sigma = [q_*,q^*)$, then it contradicts to the fact that all positive solutions are uniformly bounded in L^∞ for $q = q^*$. Consequently, the projection of Σ to the q-axis $Proj_q\Sigma = [q_*,\infty)$.

(ii) If (C2) holds and $0 < d_I < d_I^*$, the proof is similar to that of Case (I), we omit the details here.

□

We will give the Leray-Schauder degree argument.

Lemma 2.4.2. *For any $\epsilon > 0$, there exist two constants $\underline{C}$ and $\bar{C}$ which depend on d_I, ϵ, $\|\beta\|_\infty$, $\|\gamma\|_\infty$ and N such that if $\mathcal{R}_0 \neq 1$, then for any positive solution of (2.33),*

$$\underline{C} \le \bar{S}(x), \bar{I}(x) \le \bar{C} \quad \text{for any } x \in [0,L] \tag{2.37}$$

for any $\epsilon \le d_S \le \frac{1}{\epsilon}$ and $0 \le q \le \frac{1}{\epsilon}$.

Proof. $\int_0^L [e^{\frac{a(x)}{d_S}}\bar{S} + e^{\frac{a(x)}{d_I}}\bar{I}]dx = N$ means that $\bar{S}(x)$ and $\bar{I}(x)$ are bounded in L^1 space. Using the standard theory of elliptic equation, it is easy to see that $\bar{S}$ and $\bar{I}$ have the upper bound $\bar{C}$ depending on d_I, ϵ, $\|\beta\|_\infty$, $\|\gamma\|_\infty$ and N.

Therefore, we just need to prove that $\bar{S}$ and $\bar{I}$ have lower bounds.

Suppose contradictorily that there exist a sequence of $\{(d_{S,i}, q_i)\}_{i=1}^\infty$ satisfies $\epsilon \le d_{S,i} \le \frac{1}{\epsilon}$ and $0 \le q_i \le \frac{1}{\epsilon}$ and $\mathcal{R}_0 \ne 1$, and $\{(\bar{S}_i(x), \bar{I}_i(x))\}_{i=1}^\infty$ are the corresponding positive solutions of (2.33) satisfying

$$\max_{x\in[0,L]} I_i(x) \to 0, \quad \text{as } i \to \infty,$$

and $(\bar{S}_i(x), \bar{I}_i(x))$ satisfies

$$\begin{cases} d_{S,i}(\bar{S}_i)_{xx} + q_i\tilde{a}'(x)(\bar{S}_i)_x - \beta(x)\dfrac{e^{\frac{q_i}{d_I}\tilde{a}(x)}\bar{S}_i\bar{I}_i}{e^{\frac{q_i}{d_{S,i}}\tilde{a}(x)}\bar{S}_i + e^{\frac{q_i}{d_I}\tilde{a}(x)}\bar{I}_i} + \gamma(x)e^{(\frac{q_i}{d_I}-\frac{q_i}{d_{S,i}})\tilde{a}(x)}\bar{I}_i = 0, & 0 < x < L, \\ d_I(\bar{I}_i)_{xx} + q_i\tilde{a}'(x)(\bar{I}_i)_x + \beta(x)\dfrac{e^{\frac{q_i}{d_{S,i}}\tilde{a}(x)}\bar{S}_i\bar{I}_i}{e^{\frac{q_i}{d_{S,i}}\tilde{a}(x)}\bar{S}_i + e^{\frac{q_i}{d_I}\tilde{a}(x)}\bar{I}_i} - \gamma(x)\bar{I}_i = 0, & 0 < x < L, \\ (\bar{S}_i)_x(0) = (\bar{S}_i)_x(L) = 0, \quad (\bar{I}_i)_x(0) = (\bar{I}_i)_x(L) = 0, \\ \int_0^L [e^{\frac{q_i}{d_{S,i}}\tilde{a}(x)}\bar{S}_i + e^{\frac{q_i}{d_I}\tilde{a}(x)}\bar{I}_i]dx = N. \end{cases} \tag{2.38}$$

Up to a subsequence, we assume that $d_{S,i} \to d_S > 0$ and $q_i \to q \ge 0$. Note that $\|\bar{I}_i\|_\infty$ are uniformly bounded. Letting $\bar{\bar{I}}_i = \frac{\bar{I}_i}{\|\bar{I}_i\|_\infty}$, we have

$$\begin{cases} d_I(\bar{\bar{I}}_i)_{xx} + q_i\tilde{a}'(x)(\bar{\bar{I}}_i)_x + \beta(x)\bar{\bar{I}}_i\dfrac{e^{\frac{q_i}{d_{S,i}}\tilde{a}(x)}\bar{S}_i}{e^{\frac{q_i}{d_{S,i}}\tilde{a}(x)}\bar{S}_i + e^{\frac{q_i}{d_I}\tilde{a}(x)}\bar{I}_i} - \gamma(x)\bar{\bar{I}}_i = 0, & 0 < x < L, \\ (\bar{\bar{I}}_i)_x(0) = (\bar{\bar{I}}_i)_x(L) = 0. \end{cases}$$

By standard regularity and Sobolev embedding theorem in [25], up to a subsequence, $\bar{I}_i \to 0$ in $C^1([0,L])$ and there exists $I^* > 0$ such that $\bar{\bar{I}}_i \to I^*$ in $C^1([0,L])$ and $\|I^*\|_\infty = 1$. Since $\bar{I}_i \to 0$ in $C^1([0,L])$ and $\int_0^L [e^{\frac{q_i}{d_{S,i}}\tilde{a}(x)}\bar{S}_i + e^{\frac{q_i}{d_I}\tilde{a}(x)}\bar{I}_i]dx = N$ implies that $\bar{S}_i$ is bounded in $L^1([0,L])$, using the equation of $\bar{S}_i$, we get $\bar{S}_i \to \hat{\bar{S}} > 0$ in $C^1([0,L])$. Letting $i \to \infty$ in the equation of $\bar{I}_i$, we have

$$\begin{cases} d_I I^*_{xx} + a'(x)I^*_x + [\beta(x) - \gamma(x)]I^* = 0, & 0 < x < L, \\ I^*_x(0) = I^*_x(L) = 0. \end{cases} \tag{2.39}$$

Since $I^* > 0$, (2.39) means that 0 is the principle eigenvalue, which is a contradiction of the assumption of $\mathcal{R}_0 \ne 1$ for any $d_I > 0$ and $0 \le q \le \frac{1}{\epsilon}$. Therefore, there must exist some positive constant $\underline{C}$ such that $\max_{x\in[0,L]} I(x) \ge \underline{C}$. Similar to the argument in [26], by Harnack inequality, we have

$$\max_{x\in[0,L]} \bar{I}(x) \le C^* \min_{x\in[0,L]} \bar{I}(x)$$

for some constant C^* depending on d_I, ϵ, $\|\beta\|_\infty$, $\|\gamma\|_\infty$ and N, which implies that $\bar{I}(x)$ has uniformly positive lower bound.

Now we prove that $S(x)$ has a uniform positive lower bound. Let $S(x_0) = \min_{x\in[0,L]} S(x)$. Using the minimum principle in [27], we have

$$\beta(x_0)\frac{e^{\frac{q}{d_I}\tilde{a}(x_0)}\bar{S}(x_0)}{e^{\frac{q}{d_S}\tilde{a}(x_0)}\bar{S}(x_0) + e^{\frac{q}{d_I}\tilde{a}(x_0)}\bar{I}(x_0)} - \gamma(x_0)e^{(\frac{q}{d_I}-\frac{q}{d_S})\tilde{a}(x_0)} \ge 0.$$

Consequently,

$$\beta(x_0)\frac{\bar S(x_0)}{\bar I(x_0)} \geq \beta(x_0)\frac{e^{\frac{q}{d_I}\tilde a(x_0)}\bar S(x_0)}{e^{\frac{q}{d_S}\tilde a(x_0)}\bar S(x_0)+e^{\frac{q}{d_I}\tilde a(x_0)}\bar I(x_0)} \geq \gamma(x_0)e^{(\frac{q}{d_I}-\frac{q}{d_S})\tilde a(x_0)}$$

and

$$\bar S(x_0) \geq \frac{\gamma(x_0)e^{(\frac{q}{d_I}-\frac{q}{d_S})\tilde a(x_0)}\bar I(x_0)}{\beta(x_0)}\bar I(x_0) \geq C\min_{x\in[0,L]}\bar I(x),$$

which completes the proof. □

Lemma 2.4.3. *Assume that $\beta(x)-\gamma(x)$ changes sign once in $(0,L)$ and one of the following conditions holds:*

(i) $d_I>0$, $q>\min_{x\in[0,L]}a'(x)>0$, $\int_0^L\beta(x)dx>\int_0^L\gamma(x)dx$ and (C2) holds;

(ii) $0<d_I<d_I^*$, $q>\min_{x\in[0,L]}a'(x)>0$, $\int_0^L\beta(x)dx<\int_0^L\gamma(x)dx$ and (C1) holds.

Then (2.33) has at least an endemic equilibrium.

Proof. Note that we can extend the ranges of f and g properly for any nonnegative pair $(f,g)\in C([0,L])\times C([0,L])$ such that the function $\frac{fg}{e^{\frac{\tau a(x)}{d_S}}f+e^{\frac{\tau a(x)}{d_I}}g}$ is Lipschitz continuous for $f,g\in\mathbb{R}$ and $\tau\in[0,1]$. Therefore we define the following compact operator family from $C([0,L])\times C([0,L])$ to $C([0,L])\times C([0,L])$:

$$\begin{cases} \quad(\tau d_S+(1-\tau)d_I)u_{xx}+\tau a'(x)u_x+\gamma(x)e^{(\frac{\tau}{d_I}-\frac{\tau}{d_S})a(x)}v \\ =\beta(x)\frac{fg\,e^{\frac{\tau a(x)}{d_I}}}{fe^{\frac{\tau a(x)}{d_S}}+ge^{\frac{\tau a(x)}{d_I}}}, & 0<x<L,\\ d_Iv_{xx}+\tau a'(x)v_x-\gamma(x)v=-\beta(x)\frac{fge^{\frac{\tau a(x)}{d_S}}}{fe^{\frac{\tau a(x)}{d_S}}+ge^{\frac{\tau a(x)}{d_I}}}, & 0<x<L,\\ u_x(0)=u_x(L)=0,\quad v_x(0)=v_x(L)=0,\\ \int_0^L[e^{\frac{\tau a(x)}{\tau d_S+(1-\tau)d_I}}u+e^{\frac{\tau a(x)}{d_I}}v]dx=N. \end{cases} \tag{2.40}$$

Since the operator $d_I\frac{d^2}{dx^2}+\tau a'(x)\frac{d}{dx}-\gamma(x)$ is invertible, then for any $\tau\in[0,1]$ and $(f,g)\in C([0,L])\times C([0,L])$, by the second equation of (2.40), v is uniquely determined. Substituting this v into the first and last equations of (2.40), u is also uniquely determined. Therefore, we can define $\mathcal{G}_\tau(f,g):=(u,v)$.

Under conditions (i) and (ii), $\mathcal{R}_{0,\tau}>1$ for any $\tau\in[0,1]$. Here

$$\mathcal{R}_{0,\tau}=\sup_{\varphi\in H^1((0,L)),\varphi\neq0}\left\{\frac{\int_0^L\beta(x)e^{\frac{\tau a(x)}{d_I}}\varphi^2dx}{d_I\int_0^L\beta(x)e^{\frac{\tau a(x)}{d_I}}\varphi_x^2dx+\int_0^L\gamma(x)e^{\frac{\tau a(x)}{d_I}}\varphi^2dx}\right\}.$$

By the result of Lemma 2.4.2, for any $\tau\in[0,1]$, there exist two positive constant $\bar C$ and $\underline{C}$ depending on d_S, d_I, q, $\|\beta\|_\infty$, $\|\gamma\|_\infty$ and N such that $\underline{C}\leq u,v\leq\bar C$ for any solution of (2.40).

Let

$$D=\{(u,v)\in C([0,L])\times C([0,L]):\frac{\underline{C}}{2}\leq u,v\leq 2\bar C\}.$$

Then $(\bar S,\bar I)\neq\mathcal{G}(\tau,(\bar S,\bar I))$ for any $\tau\in[0,1]$ and $(\bar S,\bar I)\in\partial D$, which implies that Leray-Schauder degree $deg(\mathbf{I}-\mathcal{G}(\tau,(\cdot,\cdot)),D,0)$ is well defined, and it is independent of τ. Here $\mathbf{I}$ is the identity map.

Moreover, $(\bar{S}, \bar{I})$ is a solution of (2.33) if and only if $(\bar{S}, \bar{I})$ satisfies $(\bar{S}, \bar{I}) = \mathcal{G}(1, (\bar{S}, \bar{I}))$. If $(\bar{S}, \bar{I}) \in D$ and $(\mathbf{I} - \mathcal{G}(0, (\cdot, \cdot)))(\bar{S}, \bar{I}) = 0$, then $(\bar{S}, \bar{I})$ is a positive solution of

$$\begin{cases} d_I \bar{S}_{xx} - \beta(x)\frac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} + \gamma(x)\bar{I} = 0, & 0 < x < L, \\ d_I \bar{I}_{xx} + \beta(x)\frac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} - \gamma(x)\bar{I} = 0, & 0 < x < L, \\ \bar{S}_x(0) = \bar{S}_x(L) = 0, \quad \bar{I}_x(0) = \bar{I}_x(L) = 0, \\ \int_0^L [\bar{S} + \bar{I}]dx = N. \end{cases} \tag{2.41}$$

By the result of [2], (2.41) has a unique positive solution (S_*, I_*) satisfying $S_* + I_* = \frac{N}{L}$ if the basic reproduction number $\hat{\mathcal{R}}_0 > 1$. Linearizing (2.41) around (S_*, I_*), we get

$$\begin{cases} -d_I \Phi_{xx} + \beta(x)\frac{I_*^2}{(S_*+I_*)^2}\Phi + \beta(x)\frac{S_*^2}{(S_*+I_*)^2}\Psi - \gamma(x)\Psi = \mu\Phi, & 0 < x < L, \\ -d_I \Psi_{xx} - \beta(x)\frac{S_*^2}{(S_*+I_*)^2}\Psi + \gamma(x)\Psi - \beta(x)\frac{I_*^2}{(S_*+I_*)^2}\Phi = \mu\Psi, & 0 < x < L, \\ \Phi_x(0) = \Phi_x(L) = 0, \quad \Psi_x(0) = \Psi_x(L) = 0, \\ \int_0^L [\Phi + \Psi]dx = N. \end{cases} \tag{2.42}$$

Adding the first two equations of (2.42) and using the boundary condition $\Phi_x = \Psi_x = 0$, $x = 0, L$, we get

$$\begin{aligned} &- d_I(\Phi_{xx} + \Psi_{xx}) = \mu(\Phi + \Psi), \quad x \in (0, L), \\ &(\Phi + \Psi)_x = 0, \quad x = 0, L. \end{aligned}$$

Solving it, we have $\Phi = -\Psi$. Substituting this relation into the first equation of (2.42), we obtain

$$-d_I \Phi_{xx} + \left(\frac{2L\beta(x)}{N} I_* + \gamma(x) - \beta(x)\right)\Phi = \mu\Phi.$$

Since I_* is a positive solution of (2.40), we know that $-d_I \frac{d^2}{dx^2} + \frac{2L}{N}\beta(x)I_* + \gamma(x) - \beta(x)$ is a positive operator, so $\mu > 0$. Hence the unique positive solution (S_*, I_*) is linearly stable. Using Leray-Schauder degree index (see Theorem 1.2.8.1 in [28]), we obtain

$$deg(\mathbf{I} - \mathcal{G}(0, (\cdot, \cdot)), D, 0) = 1.$$

Consequently, using the homotopy invariance of Leray-Schauder degree, we have

$$deg(\mathbf{I} - \mathcal{G}(1, (\cdot, \cdot)), D, 0) = deg(\mathbf{I} - \mathcal{G}(0, (\cdot, \cdot)), D, 0) = 1$$

for $(d_I, q) \in \Omega_{hh}^U \cup \Omega_{lh}^{U_1}$. By the properties of degree, $\mathcal{G}(1, (\cdot, \cdot)$ has a fixed point in D if $(d_I, q) \in \Omega_{hh}^U \cup \Omega_{lh}^{U_1}$, which implies that (2.33) has at least one positive solution.

□

2.5. Properties of $\mathcal{R}_0$ when $\beta(x) - \gamma(x)$ changes sign twice

In this section, we consider the properties of $\mathcal{R}_0$ when $\beta(x) - \gamma(x)$ changes sign twice. We also need the results on the positive roots of $F(\eta)$ which is defined as

$$F(\eta) = \int_0^L \tilde{a}'(x)m(x)e^{\eta\tilde{a}(x)}dx, \quad 0 \le \eta < \infty,$$

for any given continuous function $m(x)$ on $[0, L]$.

Lemma 2.5.1. *Assume that there exists* $0 < x_1 < x_2 < L$ *such that* $m(x_1) = m(x_2) = 0$*, i.e.,* $m(x)$ *change sign twice for* $x \in [0, L]$*. Then*

(i) If $m(L) < 0$ *and* $\int_0^L \tilde{a}'(x)m(x)dx > 0$*, then* $F(\eta)$ *has a unique positive root* η_1 *for* $\eta \in (0, +\infty)$ *satisfying* $F'(\eta_1) < 0$*;*

(ii) If $m(L) > 0$ *and* $\int_0^L \tilde{a}'(x)m(x)dx < 0$*, then* $F(\eta)$ *has a unique positive root* η_1 *for* $\eta \in (0, +\infty)$ *satisfying* $F'(\eta_1) > 0$*;*

(iii) If $m(L) > 0$ *and* $\int_0^L \tilde{a}'(x)m(x)dx > 0$*, then* $F(\eta)$ *has at most two positive roots for* $\eta \in (0, +\infty)$*;*

(iv) If $m(L) < 0$ *and* $\int_0^L \tilde{a}'(x)m(x)dx < 0$*, then* $F(\eta)$ *has at most two positive roots for* $\eta \in (0, +\infty)$*.*

Proof. We only prove part (i) and part (iii). The proofs of part (ii) and part (iv) are similar.

(i). Let $G_1(\eta) := e^{-\tilde{a}(x_2)\eta}[\tilde{a}(x_1)F(\eta) - F'(\eta)]$ and the prime notation denote differentiation with respect to η. Since $m(L) < 0$ and $m(x)$ changes sign twice, it is easy to see that $m(x) < 0$ for $x \in (0, x_1) \cup (x_2, L)$ and $m(x) > 0$ for $x \in (x_1, x_2)$. Note that $\tilde{a}(x)$ is increasing. We know that

$$m(x)[\tilde{a}(x) - \tilde{a}(x_1)][\tilde{a}(x) - \tilde{a}(x_2)] < 0$$

for $x \in (0, L)$ and $x \neq x_i (i = 1, 2)$. As a result, for any $\eta > 0$, we have

$$\begin{aligned} G_1'(\eta) &= -e^{-\tilde{a}(x_2)\eta}\left(F''(\eta) - [\tilde{a}(x_1) + \tilde{a}(x_2)]F'(\eta) + \tilde{a}(x_1)\tilde{a}(x_2)F(\eta)\right) \\ &= -\int_0^L e^{\eta[\tilde{a}(x) - \tilde{a}(x_2)]}\tilde{a}'(x)m(x)[\tilde{a}(x) - \tilde{a}(x_1)][\tilde{a}(x) - \tilde{a}(x_2)]dx > 0, \end{aligned}$$

which implies that $G_1'(\eta)$ is a strictly increasing function for $\eta \in (0, \infty)$. By Lemma 2.2.2 and $m(L) < 0$, $F(\eta) < 0$ for $\eta > M$ if M is large enough. But $F(0) = \int_0^L \tilde{a}'(x)m(x)dx > 0$, so there exits at least a positive root of $F(\eta)$. Let η_1 be the smallest positive one, then $F'(\eta_1) \leq 0$.

If $F'(\eta_1) = 0$, since

$$\begin{aligned} &F''(\eta) - [\tilde{a}(x_1) + \tilde{a}(x_2)]F'(\eta) + \tilde{a}(x_1)\tilde{a}(x_2)F(\eta) \\ = &\int_0^L e^{\eta[\tilde{a}(x) - \tilde{a}(x_2)]}\tilde{a}'(x)m(x)[\tilde{a}(x) - \tilde{a}(x_1)][\tilde{a}(x) - \tilde{a}(x_2)]dx < 0, \end{aligned}$$

then

$$F''(\eta_1) - [\tilde{a}(x_1) + \tilde{a}(x_2)]F'(\eta_1) + \tilde{a}(x_1)\tilde{a}(x_2)F(\eta_1) = F''(\eta_1) < 0.$$

That is, η_1 is a strict local maximum value point of $F(\eta)$, which is a contradiction. So $F'(\eta_1) < 0$. Now we will prove that η_1 is the unique positive root of $F(\eta)$. Assume contradictorily that $\eta_2 > \eta_1$ is the first number such that $F(\eta_2) = 0$. Since $F(\eta_1) = 0$ and $F'(\eta_1) < 0$, then $F(\eta) < 0$ in (η_1, η_2), which implies that $F'(\eta_2) \geq 0$. By the definition of $G_1(\eta)$, and noticing that $F(\eta_1) = F(\eta_2) = 0$, we have $G_1(\eta_1) = -\tilde{a}(x_1)e^{\tilde{a}(x_2)\eta_1}F'(\eta_1) > 0$ and $G_1(\eta_2) = -\tilde{a}(x_1)e^{\tilde{a}(x_2)\eta_2}F'(\eta_2) \leq 0$, which is a contradiction to the fact that $G_1(\eta)$ is strictly increasing.

(iii) By Lemma 2.2.2 and $m(L) > 0$, we see that $F(\eta) > 0$ for $\eta > M$ if M is large enough. Then either $F(\eta) > 0$ for any $\eta > 0$ or $F(\eta)$ has positive roots in $(0, \infty)$. Let $G_2(\eta) = e^{-\tilde{a}(x_2)\eta}[F'(\eta) - \tilde{a}(x_1)F(\eta)]$ and η_1 be the first positive root of $F(\eta) = 0$. Similar to the proof of part (i), it is easy to prove that $G_2(\eta)$ is strictly monotone increasing in $(0, +\infty)$ and $F'(\eta_1) \leq 0$. We discuss in two cases.

Case 1: $F'(\eta_1) = 0$. We will show that η_1 is the unique positive root of $F(\eta)$. Since

$$\begin{aligned} & F''(\eta) - [\tilde{a}(x_1) + \tilde{a}(x_2)]F'(\eta) + \tilde{a}(x_1)\tilde{a}(x_2)F(\eta) \\ =& \int_0^L e^{\eta[\tilde{a}(x)-\tilde{a}(x_2)]}\tilde{a}'(x)m(x)[\tilde{a}(x) - \tilde{a}(x_1)][\tilde{a}(x) - \tilde{a}(x_2)]dx > 0 \end{aligned}$$

then $F''(\eta_1) - [\tilde{a}(x_1) + \tilde{a}(x_2)]F'(\eta_1) + \tilde{a}(x_1)\tilde{a}(x_2)F(\eta_1) = F''(\eta_1) > 0$. That is, $F(\eta)$ attains a strict local minimum at η_1. Now we will prove that η_1 is the unique positive root of $F(\eta)$. Assume contradictorily that $\eta_2 > \eta_1$ is the first number such that $F(\eta_2) = 0$. Since η_1 is a strict local minimum value point, we have $F(\eta) > 0$ in (η_1, η_2), which implies that $F'(\eta_2) \leq 0$. By the definition of $G_2(\eta)$, and noticing that $F(\eta_1) = F(\eta_2) = 0$, we have $G_2(\eta_1) = 0$ and $G_2(\eta_2) = e^{a(x_2)\eta_2}F'(\eta_2) \leq 0$, which is a contradiction to the fact that $G_2(\eta)$ is strictly increasing. So $F(\eta)$ only has a unique positive root η_1 in this case.

Case 2. $F'(\eta_1) < 0$. Since $F(\eta_1) = 0$, so $F(\eta) < 0$ if $\eta > \eta_1$ and η close to η_1 enough. By Lemma 3.2 and $m(L) > 0$, $F(\eta) > 0$ for $\eta > M$ if M is large enough. Therefore, there exists at least a root of $F(\eta) = 0$ in (η_1, ∞). Assume that η_2 is the first root of $F(\eta) = 0$ in (η_1, ∞). Then $F(\eta) < 0$ in (η_1, η_2) and $F'(\eta_2) \geq 0$. If $F'(\eta_2) = 0$, then

$$\begin{aligned} F''(\eta_2) &= F''(\eta_2) - [\tilde{a}(x_1) + \tilde{a}(x_2)]F'(\eta_2) + \tilde{a}(x_1)\tilde{a}(x_2)F(\eta_2) \\ &= \int_0^L e^{\eta_2[\tilde{a}(x)-\tilde{a}(x_2)]}\tilde{a}'(x)m(x)[\tilde{a}(x) - \tilde{a}(x_1)][\tilde{a}(x) - \tilde{a}(x_2)]dx > 0. \end{aligned}$$

And $F(\eta)$ attains a strict local minimum at η_2, which is a contradiction. Hence $F'(\eta_2) > 0$.

We need to show that there is no positive root of $F(\eta)$ =) for $\eta > \eta_2$. Assume contradictorily that there exists $\eta_3 > \eta_2$ such that $F(\eta_3) = 0$ and $F(\eta) > 0$ in (η_2, η_3). Then $F'(\eta_3) < 0$. And $G_2(\eta_2) = e^{\tilde{a}(x_2)\eta_2}F'(\eta_2) > 0$ and $G_2(\eta_3) = e^{a(x_2)\eta_3}F'(\eta_3) < 0$, which contradicts the fact that $G_2(\eta)$ is strictly increasing. Therefore we have proved that there exists a unique $\eta_2 > \eta_1$ such that $F(\eta_2) = 0$ and $F'(\eta_2) > 0$.

□

Now we give the proof of Theorem 1.6 below.

Proof. We only prove part(i) and (iii). The proofs of (ii) and (iv) are similar.

Part (i): Similar to the proofs of Lemma 2.3.2 and 2.3.3, it is easy to prove that there exists some positive constant Λ which is independent of d_I and q and for each $d_I > \Lambda$, there exists some $\tilde{q} = \tilde{q}(d_I)$ which satisfies $\mathcal{R}_0(d_I, \tilde{q}) = 1$ and $\frac{\tilde{q}}{d_I} \to \eta_0$ as $d_I \to \infty$. Here η_0 is the unique positive root of $F(\eta) = 0$.

Next, we will prove that

$$\frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) < 0$$

for any $\tilde{q}$ satisfying $\mathcal{R}_0(d_I, \tilde{q}) = 1$ if d_I is large enough.

Let $\tilde{\varphi}$ be the unique normalized eigenfunction of the eigenvalue $\mathcal{R}_0(d_I, \tilde{q}) = 1$, i.e., $\max_{[0,L]} \tilde{\varphi} = 1$ and

$$\begin{cases} -d_I(e^{\frac{\tilde{q}}{d_I}\tilde{a}(x)}\tilde{\varphi}_x)_x + [\gamma(x) - \beta(x)]e^{\frac{\tilde{q}}{d_I}\tilde{a}(x)}\tilde{\varphi} = 0, \quad 0 < x < L, \\ \tilde{\varphi}_x(0) = \tilde{\varphi}_x(L) = 0. \end{cases} \tag{2.43}$$

By (2.22), we have

$$\frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) = \frac{\mathcal{R}_0^2 \int_0^L e^{\frac{\tilde{q}}{d_I}\tilde{a}(x)} \tilde{\varphi}_x \tilde{\varphi} \tilde{a}'(x) dx}{\int_0^L \beta(x) e^{\frac{\tilde{q}}{d_I}\tilde{a}(x)} \tilde{\varphi}^2 dx}. \tag{2.44}$$

Multiplying (2.43) by $\int_0^x \tilde{\varphi}(s)ds$ and integrating it over $(0, L)$, we get

$$d_I \int_0^L e^{\frac{\tilde{q}}{d_I}\tilde{a}(x)} \tilde{\varphi}_x \tilde{\varphi} \tilde{a}'(x) dx + \int_0^L [\gamma(x) - \beta(x)] e^{\frac{\tilde{q}}{d_I}\tilde{a}(x)} \tilde{\varphi} (\int_0^x \tilde{\varphi}(s)ds) dx = 0.$$

Substitute it into (2.44), we obtain

$$d_I \frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) = \frac{\int_0^L [\beta(x) - \gamma(x)] e^{\frac{\tilde{q}}{d_I}\tilde{a}(x)} \tilde{\varphi} (\int_0^x \tilde{\varphi}(s)ds) dx}{\int_0^L \beta(x) e^{\frac{\tilde{q}}{d_I}\tilde{a}(x)} \tilde{\varphi}^2 dx}.$$

As $d_I \to \infty$, $\frac{\tilde{q}}{d_I} \to \eta_0$ and $\tilde{\varphi} \to 1$, we have

$$\lim_{d_I \to \infty} d_I \frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) = \frac{\int_0^L x[\beta(x) - \gamma(x)] e^{\eta_0 \tilde{a}(x)} dx}{\int_0^L \beta(x) e^{\eta_0 \tilde{a}(x)} dx}.$$

By Lemma 2.5.1(i),

$$\int_0^L x[\beta(x) - \gamma(x)] e^{\eta_0 \tilde{a}(x)} dx = F'(\eta_0) < 0.$$

Hence, there exists some constant $Q > 0$(dependent on d_I) such that $\mathcal{R}_0 > 1$ for $0 < q < Q$ and $\mathcal{R}_0 < 1$ for $q > Q$.

Part (iii). According to the results of Lemma 2.5.1(iii), we divide into three cases to prove it.

Case 1. $F(\eta) > 0$ for any $\eta > 0$. It is easy to show that there exists some positive constant Λ independent of d_I and q such that $\mathcal{R}_0 > 1$ for every $d_I > \Lambda$ and any $q > 0$.

Case 2. $F(\eta)$ has a unique positive root η_1 for $\eta \in (0, +\infty)$ and $F'(\eta_1) = 0$. Similar to the proof of part (i), we can prove that there exists some positive constant Λ independent of d_I and q such that for every $d_I > \Lambda$, there exists some $\tilde{q} = \tilde{q}(d_I)$ such that $\mathcal{R}_0(d_I, \tilde{q}) = 1$ and $\frac{\tilde{q}}{d_I} \to \eta_0$ as $d_I \to \infty$, where η_0 is the unique positive root of $F(\eta) = 0$. Moreover, $\frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) = 0$. Therefore there exists some positive constant Λ which is independent of d_I and q such that for every $d_I > \Lambda$, there exists a constant $Q > 0$ dependent on d_I satisfying $\mathcal{R}_0 = 1$ for $q = Q$ and $\mathcal{R}_0 > 1$ for $q \in (0, Q) \cup (Q, \infty)$.

Case 3. $F(\eta)$ has two positive roots η_1 and $\eta_2 (\eta_1 < \eta_2)$ for $\eta \in (0, +\infty)$ and $F'(\eta_1) < 0$, $F'(\eta_2) > 0$. Similar to the discussion of part (i), for each $d_I > 0$, there exist $\tilde{q}_1 = \tilde{q}_1(d_I)$ and $\tilde{q}_2 = \tilde{q}_2(d_I)$ such that $\mathcal{R}_0(d_I, \tilde{q}_i) = 1 (i = 1, 2)$ and $\frac{\tilde{q}_1}{d_I} \to \eta_1$, $\frac{\tilde{q}_2}{d_I} \to \eta_2$ as $d_I \to \infty$. And

$$\frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}_1) < 0, \quad \frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}_2) > 0.$$

Consequently, there exist two constants $Q_2 > Q_1 > 0$ which depend on d_I and satisfy that $\mathcal{R}_0 > 1$ for $q \in (0, Q_1) \cup (Q_2, \infty)$, $\mathcal{R}_0 < 1$ for $q \in (Q_1, Q_2)$.

□

3. Results

In this section, we will summarize the main results of this paper.

Theorem 1.1 gives some properties for the basic reproduction number $\mathcal{R}_0$ and Theorem 1.2 says that $\mathcal{R}_0 = 1$ is the watershed for judging whether the DFE is stable or not. Theorem 1.3 and Theorem 1.4 deal with the stable and unstable regions of the DFE. Theorem 1.5 establishes the existence of EE. Theorem 1.6 considers the results on (1.1) when $\beta(x) - \gamma(x)$ changes sign twice in $(0, L)$.

4. Discussion

We only establish the results on (1.1) under the assumption of $a'(x) > 0$ in this paper. However, it is much more difficult to obtain the results on (1.1) if there exists some $x_0 \in (0, L)$ satisfying $a'(x_0) = 0$.

5. Conclusion

Biologically, the influence of advection is from the upstream to the downstream, small diffusion or large advection tends to force the individuals to concentrate at the downstream end. Therefore, the disease persists for arbitrary advection rate if the habitat is a high-risk domain and the downstream end is a high-risk site. While the advection transports the individuals to a favorable location and thus it can help eliminate the disease if the downstream end is a low-risk site. In conclusion, when advection is strong or the diffusion is small, the disease will be eliminated if the downstream end is a low–risk site, while the disease will persist if the downstream end is a high–risk site.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. L. J. S. Allen, B. M. Bolker, Y. Lou, A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic patch model, *SIAM J. Appl. Math.*, **67** (2007), 1283–1309.
2. L. J. S. Allen, B. M. Bolker, Y. Lou, A. L. Nevai, Asymptotic profiles of the steady states for an SIS epidemic reaction–diffusion model, *Discrete Contin. Dyn. Syst.*, **21** (2008), 1–20.
3. R. H. Cui, K. Y. Lam, Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, *J. Differ. Equations*, **263** (2017), 2343–2373.
4. J. Ge, K. I. Kim, Z. G. Lin, H. P. Zhu, A SIS reaction–diffusion–advection model in a low–risk and high–risk domain, *J. Differ. Equations*, **259** (2015), 5486–5509.

5. W. Z. Huang, M. A. Han, K. Y. Liu, Dynamics of an SIS reaction–diffusion epidemic model for disease transmission, *Math. Biosci. Eng.*, **7** (2010), 51–66.

6. K. Kuto, H. Matsuzawa, R. Peng, Concentration profile of endemic equilibrium of a reaction–diffusion–advection SIS epidemic model, *Calc. Var. Partial Differential Equations*, **56** (2017), Paper No. 112, 28 pp.

7. H. C. Li, R. Peng, F. B. Wang, Varying total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model, *J. Differ. Equations*, **262** (2017), 885–913.

8. R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction–diffusion model. I, *J. Differ. Equations*, **247** (2009), 1096–1119.

9. R. Peng, S. Q. Liu, Global stability of the steady states of an SIS epidemic reaction–diffusion model, *Nonlinear Anal.*, **71** (2009), 239–247.

10. R. Peng, F. Q. Yi, Asymptotic profile of the positive steady state for an SIS epidemic reaction–diffusion model: effects of epidemic risk and population movement, *Phys. D*, **259** (2013), 8–25.

11. R. Peng, X. Q. Zhao, A reaction–diffusion SIS epidemic model in a time–periodic environment, *Nonlinearity*, **25** (2012), 1451–1471.

12. W. Wang, W. Ma, Z. Feng, Global dynamics and traveling waves for a periodic and diffusive chemostat model with two nutrients and one microorganism, *Nonlinearity*, **33** (2020), 4338–4380.

13. X. Q. Zhao, *Dynamical systems in population biology* 2^{nd} edition. CMS Books in Mathematics/Ouvrages de Mathmatiques de la SMC. Springer, Cham, 2017. xv+413 pp. ISBN: 978-3-319-56432-6; 978-3-319-56433-3.

14. R. H. Cui, Y. Lou, A spatial SIS model in advective heterogeneous environments, *J. Differ. Equations*, **261** (2016), 3305–3343.

15. D. Henry, Geometric Theory of Semilinear Parabolic Equations, *Lecture Notes in Mathematics*, vol.840, Springer-Verlag, Berlin-New York, 1981.

16. O. Diekmann, J. A. P. Heesterbeek, J. A. J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, *J. Math. Biol.*, **28** (1990), 365–382.

17. P. van den Driessche, J. Watmough, Reproduction numbers and sub–threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.*, **180** (2002), 29–48.

18. W. D. Wang, X. Q. Zhao, Basic reproduction numbers for reaction–diffusion epidemic models, *SIAM J. Appl. Dyn. Syst.*, **11** (2012), 1652–1673.

19. H. L. Smith, *Monotone Dynamical Systems, Mathematical Surveys and Monographs*, vol.41, American Mathemat-ical Society, Providence, RI, 1995.

20. M. G. Crandall, P. H. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.*, **8** (1971), 321–340.

21. Y. H. Du, J. P. Shi, Allee effect and bistability in a spatially heterogeneous predator-prey model, *Trans. Amer. Math. Soc.*, **359** (2007), 4557–4593.

22. J. P. Shi, Persistence and bifurcation of degenerate solutions, *J. Funct. Anal.*, **169** (1999), 494–531.

23. J. P. Shi, X. F. Wang, On global bifurcation for quasilinear elliptic systems on bounded domains, *J. Differ. Equations*, **246** (2009), 2788–2812.

24. P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.*, **7** (1971), 487–513.

25. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2^{nd} edition, Springer-Verlag, Berlin, 1983.

26. C. S. Lin, W. M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, *J. Differ. Equations*, **72** (1988), 1–27.

27. Y. Lou, W. M. Ni, Diffusion, self–diffusion and cross–diffusion, *J. Differ. Equations*, **131** (1996), 79–131.

28. L. Nirenberg, *Topics in Nonlinear Functional Analysis*, New York University, Courant Institute of Mathematical Sciences/American Mathematical Society, New York/Providence, RI, 2001.