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Research article

A detailed study on a solvable system related to the linear fractional difference equation

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Abstract: In this paper, we present a detailed study of the following system of difference equations

$$x_{n+1} = \frac{a}{1 + y_n x_{n-1}}, \ y_{n+1} = \frac{b}{1 + x_n y_{n-1}}, \ n \in \mathbb{N}_0,$$

where the parameters a, b, and the initial values $x_{-1}, \ x_0, \ y_{-1}, \ y_0$ are arbitrary real numbers such that x_n and y_n are defined. We mainly show by using a practical method that the general solution of the above system can be represented by characteristic zeros of the associated third-order linear equation. Also, we characterized the well-defined solutions of the system. Finally, we study long-term behavior of the well-defined solutions by using the obtained representation forms.

Keywords: behavior of solutions; characteristic equation; general solution; system of difference equations; periodic solution

1. Introduction and preliminaries

Nonlinear difference equations have long interested both mathematics and other sciences. Since these equations play a key role in many applications such as the natural model of a discrete process, they appear in many disciplines such as population biology, optics, economics, probability theory, genetics, psychology. See e.g., [1–6] and the references therein. For the last two decades, there has been interest in studying the global attractivity, the boundedness character and the periodic nature of nonlinear difference equations. For some recent results see, for example, [7–16]. However, for the last decade, some researchers have focused on the solvability of nonlinear difference equations and their systems. For some recent results see, for example, [17–33].

In this paper, we consider the following system of second-order nonlinear difference equations

$$x_{n+1} = \frac{a}{1 + y_n x_{n-1}}, \; y_{n+1} = \frac{b}{1 + x_n y_{n-1}}, \; n \in \mathbb{N}_0, \tag{1.1}$$

where the parameters a, b, and the initial values x_{-1}, x_0, y_{-1}, y_0 are arbitrary real numbers such that the solution $\{(x_n, y_n)\}_{n\geq -1}$ exists. System (1.1) can be obtained systematically as follows. First, we consider the following difference equation

$$x_{n+1} = \frac{a + bx_n}{c + dx_n}, \; ad \neq bc, \; d \neq 0, \; n \in \mathbb{N}_0, \tag{1.2}$$

where the parameters a, b, c, d, and the initial value x_0 are arbitrary such that x_n are defined. Equation (1.2) is called a first order linear fractional difference equation. Equation (1.2) is solvable by virtue of several changes of variables. The most common method is to transform Eq (1.2) into the second order linear equation by using the change of variables $c + dx_n = z_n/z_{n-1}$. For a detailed background on Eq (1.2), see e.g, [34, 35]. Also, for other equations related to Eq (1.2), see [36–39]. A different case occurs when we get $b = 0$. This case yields the following difference equation

$$x_{n+1} = \frac{a}{c + dx_n}, \; acd \neq 0, \; n \in \mathbb{N}_0. \tag{1.3}$$

Equation (1.3) can also be transformed into the second order linear equation by using the change of variables $x_n = z_{n-1}/z_n$ and so is solvable. Some generalizations of Eq (1.3) can inherit its solvability property. For example, the following difference equation

$$x_{n+1} = \frac{a}{c + dx_n x_{n-1}}, \; acd \neq 0, \; n \in \mathbb{N}_0, \tag{1.4}$$

where the parameters a, c, d, and the initial values x_{-1}, x_0 are arbitrary such that x_n are defined, is also solvable by using the change of variables $x_n = z_{n-1}/z_n$. Hence, the general solutions of (1.2)–(1.4) follow from the general solutions of the associated linear equations and the corresponding changes of variables. Note that both Eq (1.2) and Eq (1.3) can be reduced equations with one parameter or two parameters. If we choose $x_n = \frac{c}{d}u_n$, $\frac{ad}{c^2} = \alpha$, and $x_n = \sqrt{\frac{c}{d}}v_n$, $\frac{a}{c}\sqrt{\frac{d}{c}} = \beta$, then they are reduced equations with one parameter in u_n and v_n, respectively. Therefore, we can take $c = d = 1$ under favorable conditions.

Based on the above considerations, we investigate a two-dimensional generalization that maintains the solvability characteristic of Eq (1.4). So, we get a further generalization of (1.4), that is, the system given in (1.1). System (1.1) can also be transformed into a system of third-order linear equations by using the changes of variables $x_n = u_{n-1}/v_n$, $y_n = v_{n-1}/u_n$, and so can be solved. But, we will use a more practical method introduced firstly in [40] to solve the system.

We need to the following two results in the sequel of our study.

Lemma 1.1. *[41] Consider the cubic equation*

$$P(z) = z^3 - \alpha z^2 - \beta z - \gamma = 0. \tag{1.5}$$

Equation (1.5) has the discriminant

$$\Delta = -\alpha^2\beta^2 - 4\beta^3 + 4\alpha^3\gamma + 27\gamma^2 + 18\alpha\beta\gamma. \tag{1.6}$$

Then, the following statements are true:

(i) If $\Delta < 0$, then the polynomial P has three distinct real zeros ρ_1, ρ_2, ρ_3.

(ii) If $\Delta = 0$, then there are two subcases:

(a) if $\beta = \frac{-\alpha^2}{3}$ and $\gamma = \frac{\alpha^3}{27}$, then the polynomial P has the triple root $\rho = \frac{\alpha}{3}$,

(b) if $\beta \neq -\frac{\alpha^2}{3}$ or $\gamma \neq \frac{\alpha^3}{27}$, then the polynomial P has the double root r and the simple root ρ.

(iii) If $\Delta > 0$, then the polynomial P has one real root p and two complex roots $re^{\pm i\theta}$, $\theta \in (0, \pi)$.

Theorem 1.1 (Kronecker's theorem)**.** *[42] If θ is irrational, the set of points $u_n = n\theta - [n\theta]$ is dense in the interval* $(0, 1)$.

In the above theorem, n is an integer and $[n\theta]$ is greatest integer function of $n\theta$.

2. Main results

This section, which contains our main results, is examined in three subsections.

2.1. Representation forms of the general solution

In this subsection, by using an interesting and practical method, we solve system (1.1). If we take $a = 0$ in system (1.1), then we have $x_n = 0$ for every $n \geq 1$ and $y_n = b$ for every $n \geq 2$. If we take $b = 0$ in system (1.1), then we have $x_n = a$ for every $n \geq 2$ and $y_n = 0$ for every $n \geq 1$. So, to enable the use of the method, we suppose $ab \neq 0$ in the sequel of our study.

We start by writing system (1.1) in the following

$$\frac{1}{x_{2n+1}} = \frac{1}{a} + \frac{y_{2n}x_{2n-1}}{a}, \tag{2.1}$$

$$\frac{1}{x_{2n+2}} = \frac{1}{a} + \frac{y_{2n+1}x_{2n}}{a}, \tag{2.2}$$

$$\frac{1}{y_{2n+1}} = \frac{1}{b} + \frac{x_{2n}y_{2n-1}}{b}, \tag{2.3}$$

$$\frac{1}{y_{2n+2}} = \frac{1}{b} + \frac{x_{2n+1}y_{2n}}{b} \tag{2.4}$$

for every $n \geq 0$. By multiplying (2.1), (2.2), (2.3) and (2.4) by

$$\frac{1}{\prod_{k=0}^{n} x_{2k-1} \prod_{k=0}^{n} y_{2k}}, \tag{2.5}$$

$$\frac{1}{\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n+1} y_{2k-1}}, \tag{2.6}$$

$$\frac{1}{\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n} y_{2k-1}}, \tag{2.7}$$

$$\frac{1}{\prod_{k=0}^{n+1} x_{2k-1} \prod_{k=0}^{n} y_{2k}}, \tag{2.8}$$

we have the followings

$$\frac{1}{\prod_{k=0}^{n+1} x_{2k-1} \prod_{k=0}^{n} y_{2k}} = \frac{1}{a\prod_{k=0}^{n} x_{2k-1} \prod_{k=0}^{n} y_{2k}} + \frac{1}{a\prod_{k=0}^{n-1} x_{2k-1} \prod_{k=0}^{n-1} y_{2k}}, \tag{2.9}$$

$$\frac{1}{\prod_{k=0}^{n+1} x_{2k} \prod_{k=0}^{n+1} y_{2k-1}} = \frac{1}{a\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n+1} y_{2k-1}} + \frac{1}{a\prod_{k=0}^{n-1} x_{2k} \prod_{k=0}^{n} y_{2k-1}}, \tag{2.10}$$

$$\frac{1}{\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n+1} y_{2k-1}} = \frac{1}{b\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n} y_{2k-1}} + \frac{1}{b\prod_{k=0}^{n-1} x_{2k} \prod_{k=0}^{n-1} y_{2k-1}} \tag{2.11}$$

$$\frac{1}{\prod_{k=0}^{n+1} x_{2k-1} \prod_{k=0}^{n+1} y_{2k}} = \frac{1}{b\prod_{k=0}^{n+1} x_{2k-1} \prod_{k=0}^{n} y_{2k}} + \frac{1}{b\prod_{k=0}^{n} x_{2k-1} \prod_{k=0}^{n-1} y_{2k}} \tag{2.12}$$

for every $n \geq 0$, respectively. In fact, the equalities (2.9)–(2.12) constitute a linear system with respect to (2.5)–(2.8). Hence, we should solve (2.9)–(2.12). By using (2.9) in (2.12), (2.12) in (2.9) and similarly by using (2.10) in (2.11), (2.11) in (2.10), we have the following statements

$$\frac{1}{\prod_{k=0}^{n+1} x_{2k-1} \prod_{k=0}^{n+1} y_{2k}} = \frac{1}{ab\prod_{k=0}^{n} x_{2k-1} \prod_{k=0}^{n} y_{2k}} + \frac{2}{ab\prod_{k=0}^{n-1} x_{2k-1} \prod_{k=0}^{n-1} y_{2k}} + \frac{1}{ab\prod_{k=0}^{n-2} x_{2k-1} \prod_{k=0}^{n-2} y_{2k}}, \tag{2.13}$$

$$\frac{1}{\prod_{k=0}^{n+1} x_{2k-1} \prod_{k=0}^{n} y_{2k}} = \frac{1}{ab\prod_{k=0}^{n} x_{2k-1} \prod_{k=0}^{n-1} y_{2k}} + \frac{2}{ab\prod_{k=0}^{n-1} x_{2k-1} \prod_{k=0}^{n-2} y_{2k}} + \frac{1}{ab\prod_{k=0}^{n-2} x_{2k-1} \prod_{k=0}^{n-3} y_{2k}}, \tag{2.14}$$

$$\frac{1}{\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n+1} y_{2k-1}} = \frac{1}{ab\prod_{k=0}^{n-1} x_{2k} \prod_{k=0}^{n} y_{2k-1}} + \frac{2}{ab\prod_{k=0}^{n-2} x_{2k} \prod_{k=0}^{n-1} y_{2k-1}} + \frac{1}{ab\prod_{k=0}^{n-3} x_{2k} \prod_{k=0}^{n-2} y_{2k-1}}, \tag{2.15}$$

and

$$\frac{1}{\prod_{k=0}^{n+1} x_{2k} \prod_{k=0}^{n+1} y_{2k-1}} = \frac{1}{ab\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n} y_{2k-1}} + \frac{2}{ab\prod_{k=0}^{n-1} x_{2k} \prod_{k=0}^{n-1} y_{2k-1}} + \frac{1}{ab\prod_{k=0}^{n-2} x_{2k} \prod_{k=0}^{n-2} y_{2k-1}} \tag{2.16}$$

for every $n \geq 2$. Note that the equations in (2.13)–(2.16) are linear with respect to (2.5), (2.8), (2.6) and (2.7), respectively, and they can be represented by the following third-order difference equation

$$z_{n+1} - \frac{1}{ab} z_n - \frac{2}{ab} z_{n-1} - \frac{1}{ab} z_{n-2} = 0,\ n \geq 2, \tag{2.17}$$

whose characteristic equation is the following equation

$$P(\lambda) = \lambda^3 - \frac{1}{ab}\lambda^2 - \frac{2}{ab}\lambda - \frac{1}{ab} = 0. \tag{2.18}$$

By Lemma 1.1, we see that there are three cases to be considered.

2.1.1. The case $4/ab < -27$

In this case P has three real distinct zeros denoted by ρ_1, ρ_2, ρ_3, respectively. Hence, from (2.13) and (2.17), we can write

$$z_n = C_1\rho_1^n + C_2\rho_2^n + C_3\rho_3^n = \frac{1}{\prod_{k=0}^{n} x_{2k-1} \prod_{k=0}^{n} y_{2k}} \tag{2.19}$$

from which it follows that

$$\frac{1}{x_{2n-1}y_{2n}} = \frac{C_1\rho_1^n + C_2\rho_2^n + C_3\rho_3^n}{C_1\rho_1^{n-1} + C_2\rho_2^{n-1} + C_3\rho_3^{n-1}}, \tag{2.20}$$

where C_1, C_2, C_3 are arbitrary real constants given by

$$\begin{aligned}
C_1(x_{-1}, y_0) &= \frac{\rho_2\rho_3 x_1 y_2 x_3 y_4 - (\rho_2 + \rho_3) x_3 y_4 + 1}{(\rho_1 - \rho_2)(\rho_1 - \rho_3) x_{-1} y_0 x_1 y_2 x_3 y_4}, \\
C_2(x_{-1}, y_0) &= \frac{\rho_1\rho_3 x_1 y_2 x_3 y_4 - (\rho_1 + \rho_3) x_3 y_4 + 1}{(\rho_2 - \rho_1)(\rho_2 - \rho_3) x_{-1} y_0 x_1 y_2 x_3 y_4}, \\
C_3(x_{-1}, y_0) &= \frac{\rho_1\rho_2 x_1 y_2 x_3 y_4 - (\rho_1 + \rho_2) x_3 y_4 + 1}{(\rho_3 - \rho_1)(\rho_3 - \rho_2) x_{-1} y_0 x_1 y_2 x_3 y_4},
\end{aligned}$$

for every $n \geq 0$. By using (2.20) in the first equation of system (1.1), we have

$$x_{2n+1} = \frac{a\left(C_1\rho_1^n + C_2\rho_2^n + C_3\rho_3^n\right)}{C_1(\rho_1 + 1)\rho_1^{n-1} + C_2(\rho_2 + 1)\rho_2^{n-1} + C_3(\rho_3 + 1)\rho_3^{n-1}} \tag{2.21}$$

for every $n \geq -1$. On the other hand, the first equation of system (1.1) can be written as follows

$$y_{2n} = \frac{a - x_{2n+1}}{x_{2n+1}x_{2n-1}} \tag{2.22}$$

for every $n \geq 0$. By using (2.21) and its backward shifted one from n to $n-1$ in (2.22), we have

$$y_{2n} = \frac{C_1(\rho_1 + 1)\rho_1^{n-2} + C_2(\rho_2 + 1)\rho_2^{n-2} + C_3(\rho_3 + 1)\rho_3^{n-2}}{a\left(C_1\rho_1^n + C_2\rho_2^n + C_3\rho_3^n\right)} \tag{2.23}$$

for every $n \geq 0$. Now, we consider Eq (2.16), which is linear with respect to (2.7) and (2.17). Hence, we have

$$z_n = C_1'\rho_1^n + C_2'\rho_2^n + C_3'\rho_3^n = \frac{1}{\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n} y_{2k-1}} \tag{2.24}$$

from which it follows that

$$\frac{1}{x_{2n}y_{2n-1}} = \frac{C_1'\rho_1^n + C_2'\rho_2^n + C_3'\rho_3^n}{C_1'\rho_1^{n-1} + C_2'\rho_2^{n-1} + C_3'\rho_3^{n-1}}, \tag{2.25}$$

where C_1', C_2', C_3' are arbitrary real constants given by

$$C_1' = C_1(y_{-1}, x_0),\ C_2' = C_2(y_{-1}, x_0),\ C_3' = C_3(y_{-1}, x_0),$$

for every $n \geq 0$. By using (2.25) in the second equation of system (1.1), we have

$$y_{2n+1} = \frac{b\left(C_1'\rho_1^n + C_2'\rho_2^n + C_3'\rho_3^n\right)}{C_1'(\rho_1+1)\rho_1^{n-1} + C_2'(\rho_2+1)\rho_2^{n-1} + C_3'(\rho_3+1)\rho_3^{n-1}} \tag{2.26}$$

for every $n \geq -1$. On the other hand, the second equation of system (1.1) can be written as follows

$$x_{2n} = \frac{b - y_{2n+1}}{y_{2n+1}y_{2n-1}} \tag{2.27}$$

for every $n \geq 0$. By using (2.26) and its backward shifted one from n to $n-1$ in (2.27), we have

$$x_{2n} = \frac{C_1'(\rho_1+1)\rho_1^{n-2} + C_2'(\rho_2+1)\rho_2^{n-2} + C_3'(\rho_3+1)\rho_3^{n-2}}{b\left(C_1'\rho_1^n + C_2'\rho_2^n + C_3'\rho_3^n\right)} \tag{2.28}$$

for every $n \geq 0$. Consequently, in the case $\frac{4}{ab} < -27$, the representation forms of the general solution of system (1.1) are given by (2.21), (2.23), (2.26) and (2.28).

2.1.2. Case $4/ab = -27$

In this case $P(\lambda)$ has the simple root ρ and the double root r. Moreover, since $ab = -\frac{4}{27}$, we have $\rho = -3/4$ and $r = -3$. Hence, from (2.13) and (2.17), we have

$$z_n = C_1\rho^n + r^n(C_2 + C_3 n) = \frac{1}{\prod_{k=0}^{n} x_{2k-1} \prod_{k=0}^{n} y_{2k}} \tag{2.29}$$

from which it follows that

$$\frac{1}{x_{2n-1}y_{2n}} = \frac{C_1\rho^n + r^n(C_2 + C_3 n)}{C_1\rho^{n-1} + r^{n-1}(C_2 + C_3(n-1))}, \tag{2.30}$$

where C_1, C_2, C_3 are arbitrary real constants given by

$$\begin{aligned}
C_1(x_{-1}, y_0) &= \frac{r^2 x_1y_2x_3y_4 - 2rx_3y_4 + 1}{(\rho - r)^2 x_{-1}y_0x_1y_2x_3y_4}, \\
C_2(x_{-1}, y_0) &= \frac{\rho(\rho - 2r)x_1y_2x_3y_4 + 2rx_3y_4 - 1}{(\rho - r)^2 x_{-1}y_0x_1y_2x_3y_4}, \\
C_3(x_{-1}, y_0) &= \frac{\rho r x_1y_2x_3y_4 - (\rho + r)x_3y_4 + 1}{(r - \rho)rx_{-1}y_0x_1y_2x_3y_4},
\end{aligned}$$

for every $n \geq 0$. By substituting (2.30) in the first equation of system (1.1), we have

$$x_{2n+1} = \frac{a\left(C_1\rho^n + r^n(C_2 + C_3 n)\right)}{C_1(\rho+1)\rho^{n-1} + r^{n-1}\left(C_2(r+1) + C_3(nr + n - 1)\right)} \tag{2.31}$$

for every $n \geq -1$. On the other hand, by using (2.31) and its backward shifted one from n to $n-1$ in (2.22), we have

$$y_{2n} = \frac{C_1(\rho+1)\rho^{n-2} + r^{n-2}(C_2(r+1) + C_3((n-1)r + n - 2))}{a(C_1\rho^n + r^n(C_2 + C_3 n))} \tag{2.32}$$

for every $n \geq 0$. Now, by considering Eqs (2.16) and (2.17). we have

$$z_n = C_1'\rho^n + r^n(C_2' + C_3'n) = \frac{1}{\prod_{k=0}^{n} x_{2k} \prod_{k=0}^{n} y_{2k-1}} \tag{2.33}$$

from which it follows that

$$\frac{1}{x_{2n}y_{2n-1}} = \frac{C_1'\rho^n + r^n\left(C_2' + C_3'n\right)}{C_1'\rho^{n-1} + r^{n-1}\left(C_2' + C_3'(n-1)\right)}, \tag{2.34}$$

where C_1', C_2', C_3' are arbitrary real constants given by

$$C_1' = C_1(y_{-1}, x_0),\ C_2' = C_2(y_{-1}, x_0),\ C_3' = C_3(y_{-1}, x_0),$$

for every $n \geq 0$. By substituting (2.34) in the second equation of system (1.1), we have

$$y_{2n+1} = \frac{b\left(C_1'\rho^n + r^n\left(C_2' + C_3'n\right)\right)}{C_1'(\rho+1)\rho^{n-1} + r^{n-1}\left(C_2'(r+1) + C_3'(nr + n - 1)\right)} \tag{2.35}$$

for every $n \geq -1$. On the other hand, by using (2.35) and its backward shifted one from n to $n-1$ in (2.27), it follows that

$$x_{2n} = \frac{C_1'(\rho+1)\rho^{n-2} + r^{n-2}\left(C_2'(r+1) + C_3'((n-1)r + n - 2)\right)}{b\left(C_1'\rho^n + r^n\left(C_2' + C_3'n\right)\right)} \tag{2.36}$$

for every $n \geq 0$. Consequently, in the case $\frac{4}{ab} = -27$, the representation forms of the general solution of system (1.1) are given by (2.31), (2.32), (2.35) and (2.36).

2.1.3. Case $4/ab > -27$

In this case $P(\lambda)$ has one real root and two complex roots denoted by ρ and $re^{\pm i\theta}$, $\theta \in (0, \pi)$, respectively. Hence, from (2.13) and (2.17), we have

$$z_n = C_1\rho^n + r^n(C_2\cos n\theta + C_3\sin n\theta) = \frac{1}{\prod_{k=0}^{n} x_{2k-1} \prod_{k=0}^{n} y_{2k}} \tag{2.37}$$

from which it follows that

$$\frac{1}{x_{2n-1}y_{2n}} = \frac{C_1\rho^n + r^n(C_2\cos n\theta + C_3\sin n\theta)}{C_1\rho^{n-1} + r^{n-1}(C_2\cos(n-1)\theta + C_3\sin(n-1)\theta)}, \tag{2.38}$$

where C_1, C_2, C_3 are arbitrary real constants given by

$$\begin{aligned} C_1(x_{-1},y_0) &= \frac{r^2x_1y_2x_3y_4 - 2r\cos\theta x_3y_4 + 1}{(\rho^2-2\rho r\cos\theta+r^2)\,x_{-1}y_0x_1y_2x_3y_4}, \\ C_2(x_{-1},y_0) &= \frac{\rho(\rho-2r\cos\theta)\,x_1y_2x_3y_4 + 2r\cos\theta x_3y_4 - 1}{(\rho^2-2\rho r\cos\theta+r^2)\,x_{-1}y_0x_1y_2x_3y_4}, \\ C_3(x_{-1},y_0) &= \frac{\rho r(r\cos 2\theta-\rho\cos\theta)\,x_1y_2x_3y_4 + \left(\rho^2-r^2\cos 2\theta\right)x_3y_4 + r\cos\theta-\rho}{r\sin\theta\,(\rho^2-2\rho r\cos\theta+r^2)\,x_{-1}y_0x_1y_2x_3y_4}, \end{aligned}$$

for every $n \geq 0$. By using (2.38) in the first equation of system (1.1), we have

$$x_{2n+1} = \frac{a\,(C_1\rho^n + r^n(C_2\cos n\theta + C_3\sin n\theta))}{C_1(\rho+1)\rho^{n-1} + r^{n-1}(C_4\cos n\theta + C_5\sin n\theta)}, \tag{2.39}$$

where $C_4 = C_2(r+\cos\theta) - C_3\sin\theta$, $C_5 = C_3(r+\cos\theta) + C_2\sin\theta$, for every $n \geq -1$. On the other hand, by using (2.39) and its backward shifted one from n to $n-1$ in (2.22), we have

$$y_{2n} = \frac{C_1(\rho+1)\rho^{n-2} + r^{n-2}(C_4\cos(n-1)\theta + C_5\sin(n-1)\theta)}{a\,(C_1\rho^n + r^n(C_2\cos n\theta + C_3\sin n\theta))} \tag{2.40}$$

for every $n \geq 0$. Now, by considering Eqs (2.16) and (2.17), we have

$$z_n = C_1'\rho^n + r^n(C_2'\cos n\theta + C_3'\sin n\theta) = \frac{1}{\prod_{k=0}^{n} x_{2k}\prod_{k=0}^{n} y_{2k-1}} \tag{2.41}$$

from which it follows that

$$\frac{1}{x_{2n}y_{2n-1}} = \frac{C_1'\rho^n + r^n\left(C_2'\cos n\theta + C_3'\sin n\theta\right)}{C_1'\rho^{n-1} + r^{n-1}\left(C_2'\cos(n-1)\theta + C_3'\sin(n-1)\theta\right)}, \tag{2.42}$$

where C_1', C_2', C_3' are arbitrary real constants given by

$$C_1' = C_1(y_{-1},x_0),\; C_2' = C_2(y_{-1},x_0),\; C_3' = C_3(y_{-1},x_0),$$

for every $n \geq 0$. By using (2.38) in the second equation of system (1.1), we have

$$y_{2n+1} = \frac{b\left(C_1'\rho^n + r^n\left(C_2'\cos n\theta + C_3'\sin n\theta\right)\right)}{C_1'(\rho+1)\rho^{n-1} + r^{n-1}\left(C_4'\cos n\theta + C_5'\sin n\theta\right)} \tag{2.43}$$

where $C_4' = C_2'(r+\cos\theta) - C_3'\sin\theta$, $C_5' = C_3'(r+\cos\theta) + C_2'\sin\theta$, for every $n \geq -1$. On the other hand, by using (2.43) and its backward shifted one from n to $n-1$ in (2.27), we have

$$x_{2n} = \frac{C_1'(\rho+1)\rho^{n-2} + r^{n-2}\left(C_4'\cos(n-1)\theta + C_5'\sin(n-1)\theta\right)}{b\left(C_1'\rho^n + r^n\left(C_2'\cos n\theta + C_3'\sin n\theta\right)\right)} \tag{2.44}$$

for every $n \geq 0$. Consequently, in the case $\frac{4}{ab} > -27$, the representation forms of the general solution of system (1.1) are given by (2.39), (2.40), (2.43) and (2.44).

2.2. Forbidden set and well-defined solutions

The representation forms given in the previous subsection are valid where the denominators are not zero. That is, we can obtain the set of initial values that make the solutions of the system undefined from the forms by equating their denominators to zero. This operation enables us to obtain a set of initial values that produce the well-defined solutions of system (1.1). In the following we give a theorem that helps us characterize such solutions.

Theorem 2.1. *Consider system (1.1). Then, the following statements are true:*

(a) If $\frac{4}{ab} < -27$*, then the forbidden set of system (1.1) is given by*

$$F = \{(x_{-1}, x_0, y_{-1}, y_0) : \alpha'_n = 0 \text{ or } \beta_n = 0 \text{ or } \alpha_n = 0 \text{ or } \beta'_n = 0\},$$

where

$$\begin{aligned}
\alpha'_n &= C'_1\rho_1^n + C'_2\rho_2^n + C'_3\rho_3^n,\ n \geq 0,\\
\beta_n &= C_1(\rho_1+1)\rho_1^{n-1} + C_2(\rho_2+1)\rho_2^{n-1} + C_3(\rho_3+1)\rho_3^{n-1},\ n \geq -1,\\
\alpha_n &= C_1\rho_1^n + C_2\rho_2^n + C_3\rho_3^n,\ n \geq 0,\\
\beta'_n &= C'_1(\rho_1+1)\rho_1^{n-1} + C'_2(\rho_2+1)\rho_2^{n-1} + C'_3(\rho_3+1)\rho_3^{n-1},\ n \geq -1.
\end{aligned}$$

(b) If $\frac{4}{ab} = -27$*, then the forbidden set of system (1.1) is given by*

$$F = \{(x_{-1}, x_0, y_{-1}, y_0) : \alpha'_n = 0 \text{ or } \beta_n = 0 \text{ or } \alpha_n = 0 \text{ or } \beta'_n = 0\},$$

where

$$\begin{aligned}
\alpha'_n &= C'_1\rho^n + r^n(C'_2 + C'_3 n),\ n \geq 0,\\
\beta_n &= C_1(\rho+1)\rho^{n-1} + r^{n-1}(C_2(r+1) + C_3(nr+n-1)),\ n \geq -1,\\
\alpha_n &= C_1\rho^n + r^n(C_2 + C_3 n),\ n \geq 0,\\
\beta'_n &= C'_1(\rho+1)\rho^{n-1} + r^{n-1}(C'_2(r+1) + C'_3(nr+n-1)),\ n \geq -1,
\end{aligned}$$

and $\rho = -3/4$, $r = -3$.

(c) If $\frac{4}{ab} > -27$*, then the forbidden set of system (1.1) is given by*

$$F = \{(x_{-1}, x_0, y_{-1}, y_0) : \alpha'_n = 0 \text{ or } \beta_n = 0 \text{ or } \alpha_n = 0 \text{ or } \beta'_n = 0\},$$

where

$$\begin{aligned}
\alpha'_n &= C'_1\rho^n + r^n(C'_2\cos n\theta + C'_3\sin n\theta),\ n \geq 0\\
\beta_n &= C_1(\rho+1)\rho^{n-1} + r^{n-1}(C_4\cos n\theta + C_5\sin n\theta),\ n \geq -1\\
\alpha_n &= C_1\rho^n + r^n(C_2\cos n\theta + C_3\sin n\theta),\ n \geq 0\\
\beta'_n &= C'_1(\rho+1)\rho^{n-1} + r^{n-1}(C'_4\cos n\theta + C'_5\sin n\theta),\ n \geq -1.
\end{aligned}$$

Proof. The proof is simple and follows by equalizing denominators of the representation forms obtained in the previous section to zero. □

By considering this theorem, we say that a well-defined solution of system (1.1) is a solution $\{(x_n, y_n)\}_{n\geq -1}$ obtained using the initial values such that $(x_{-1}, x_0, y_{-1}, y_0) \in \mathbb{R}^4 \backslash F$.

2.3. Long-term behavior of the solutions

In this subsection we study the long-term behavior of the solutions of system (1.1) by using the representation forms obtained in the first subsection. We analyze the solutions in the following cases of the parameter ab:

i) Case $\frac{4}{ab} < -27$: in this case we have $ab \in \left(-\frac{4}{27}, 0\right)$.

ii) Case $\frac{4}{ab} = -27$: in this case we have $ab = \frac{-4}{27}$.

iii) Case $\frac{4}{ab} > -27$: in this case we have $ab \in \left(-\infty, -\frac{4}{27}\right) \cup (0, +\infty)$.

2.3.1. Case $4/ab < -27$

This case yields the following result.

Theorem 2.2. *Let $\{(x_n, y_n)\}_{n\geq -1}$ be a well-defined solution of system (1.1). Suppose that $\frac{4}{ab} < -27$. Then, the following statements are true:*

(a) If $C_i \neq 0$ for $i \in \{1,2,3\}$ and $|\rho| = \max\{|\rho_1|, |\rho_2|, |\rho_3|\}$, then $x_{2n+1} \to \frac{a\rho}{\rho+1}$ and $y_{2n} \to \frac{b\rho}{\rho+1}$ as $n \to \infty$.

(b) If $C_i' \neq 0$ for $i \in \{1,2,3\}$ and $|\rho| = \max\{|\rho_1|, |\rho_2|, |\rho_3|\}$, then $x_{2n} \to \frac{a\rho}{\rho+1}$ and $y_{2n+1} \to \frac{b\rho}{\rho+1}$ as $n \to \infty$.

(c) If $C_i = 0$ and $C_jC_k \neq 0$ for $i, j, k \in \{1,2,3\}$ with $i \neq j \neq k$ and $|\rho| = \max\left\{\left|\rho_j\right|, |\rho_k|\right\}$, then $x_{2n+1} \to \frac{a\rho}{\rho+1}$ and $y_{2n} \to \frac{b\rho}{\rho+1}$ as $n \to \infty$.

(d) If $C_i' = 0$ and $C_j'C_k' \neq 0$ for $i, j, k \in \{1,2,3\}$ with $i \neq j \neq k$ and $|\rho| = \max\left\{\left|\rho_j\right|, |\rho_k|\right\}$, then $x_{2n} \to \frac{a\rho}{\rho+1}$ and $y_{2n+1} \to \frac{b\rho}{\rho+1}$ as $n \to \infty$.

Proof. (a)–(b) Let us assume without losing generality that $|\rho_1| = \max\{|\rho_1|, |\rho_2|, |\rho_3|\}$. Then, we have the following limits

$$\begin{aligned}\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{\rho_1^{n-2}}{\rho_1^n} \frac{C_1'(\rho_1+1) + C_2'(\rho_2+1)\left(\frac{\rho_2}{\rho_1}\right)^{n-2} + C_3'(\rho_3+1)\left(\frac{\rho_3}{\rho_1}\right)^{n-2}}{b\left(C_1' + C_2'\left(\frac{\rho_2}{\rho_1}\right)^n + C_3'\left(\frac{\rho_3}{\rho_1}\right)^n\right)} \\ &= \frac{\rho_1+1}{b\rho_1^2}.\end{aligned}$$

Since ρ_1 is a zero of the polynomial P, we have the relation

$$\frac{\rho_1+1}{b\rho_1^2} = \frac{a\rho_1}{\rho_1+1}$$

from (2.18). Hence, we have

$$\lim_{n\to\infty} x_{2n} = \frac{a\rho_1}{\rho_1+1}$$

and

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \frac{\rho_1^n}{\rho_1^{n-1}} \frac{a\left(C_1 + C_2\left(\frac{\rho_2}{\rho_1}\right)^n + C_3\left(\frac{\rho_3}{\rho_1}\right)^n\right)}{C_1(\rho_1+1) + C_2(\rho_2+1)\left(\frac{\rho_2}{\rho_1}\right)^{n-1} + C_3(\rho_3+1)\left(\frac{\rho_3}{\rho_1}\right)^{n-1}}$$

$$= \frac{a\rho_1}{\rho_1+1},$$

$$\begin{aligned}\lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty}\frac{\rho_1^{n-2}}{\rho_1^n}\frac{C_1(\rho_1+1)+C_2(\rho_2+1)\left(\frac{\rho_2}{\rho_1}\right)^{n-2}+C_3(\rho_3+1)\left(\frac{\rho_3}{\rho_1}\right)^{n-2}}{a\left(C_1+C_2\left(\frac{\rho_2}{\rho_1}\right)^n+C_3\left(\frac{\rho_3}{\rho_1}\right)^n\right)}\\ &= \frac{\rho_1+1}{a\rho_1^2}\\ &= \frac{b\rho_1}{\rho_1+1},\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} y_{2n+1} &= \lim_{n\to\infty}\frac{\rho_1^{n}}{\rho_1^{n-1}}\frac{b\left(C_1'+C_2'\left(\frac{\rho_2}{\rho_1}\right)^n+C_3'\left(\frac{\rho_3}{\rho_1}\right)^n\right)}{C_1'(\rho_1+1)+C_2'(\rho_2+1)\left(\frac{\rho_2}{\rho_1}\right)^{n-1}+C_3'(\rho_3+1)\left(\frac{\rho_3}{\rho_1}\right)^{n-1}}\\ &= \frac{b\rho_1}{\rho_1+1}.\end{aligned}$$

The proofs of other cases are similar and so they will be omitted. □

Remark 2.1. *Note that the cases* $|C_1|+|C_2|+|C_3|=|C_i|$ *and* $\left|C_1'\right|+\left|C_2'\right|+\left|C_3'\right|=\left|C_j\right|$, $i,j\in\{1,2,3\}$ *are impossible. Because, for example, if* $C_1=C_2=0$*, then we need to the common solution of the system*

$$\rho_2\rho_3x_1y_2x_3y_4-(\rho_2+\rho_3)x_3y_4+1=0,\ \rho_1\rho_3x_1y_2x_3y_4-(\rho_1+\rho_3)x_3y_4+1=0.$$

This case requires that $\rho_1=\rho_2$ *which is a contradiction.*

Corollary 2.1. *Suppose that* $\frac{4}{ab}<-27$*. Then, every well-defined solution of system (1.1) has a finite limit point.*

2.3.2. Case $4/ab=-27$

This case yields the following result.

Theorem 2.3. *Let* $\{(x_n,y_n)\}_{n\geq -1}$ *be a well-defined solution of system (1.1). Suppose that* $\frac{4}{ab}=-27$*. Then, the following statements are true:*

(a) If $|C_2|+|C_3|\neq 0$*, then* $x_{2n+1}\to\frac{3a}{2}$ *and* $y_{2n}\to\frac{3b}{2}$ *as* $n\to\infty$*.*
(b) If $\left|C_2'\right|+\left|C_3'\right|\neq 0$*, then* $x_{2n}\to\frac{3a}{2}$ *and* $y_{2n+1}\to\frac{3b}{2}$ *as* $n\to\infty$*.*
(c) If $C_2=C_3=0$ *and* $C_1\neq 0$*, then* $x_{2n+1}\to -3a$ *and* $y_{2n}\to -3b$ *as* $n\to\infty$*.*
(d) If $C_2'=C_3'=0$ *and* $C_1'\neq 0$*, then* $x_{2n}\to -3a$*, and* $y_{2n+1}\to -3b$ *as* $n\to\infty$*.*

Proof. (a)–(b) Since $\rho=-3/4$ and $r=-3$, we have $|\rho|<|r|$. So, from (2.31), (2.32), (2.35) and (2.36), we have the following limits

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty}\frac{r^{n-2}}{br^n}\frac{C_1'(\rho+1)\left(\frac{\rho}{r}\right)^{n-2}+C_2'(r+1)+C_3'((n-1)r+n-2)}{C_1'\left(\frac{\rho}{r}\right)^n+\left(C_2'+C_3'n\right)}$$

$$
\begin{aligned}
&= \frac{r+1}{br^2}\\
&= \frac{-2}{9b}\\
&= \frac{3a}{2},
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty} \frac{ar^n}{r^{n-1}} \frac{C_1\left(\frac{\rho}{r}\right)^n + C_2 + C_3 n}{C_1(\rho+1)\left(\frac{\rho}{r}\right)^{n-1} + C_2(r+1) + C_3(nr+n-1)}\\
&= \frac{ar}{r+1}\\
&= \frac{3a}{2},
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} \frac{r^{n-2}}{ar^n} \frac{C_1(\rho+1)\left(\frac{\rho}{r}\right)^{n-2} + C_2(r+1) + C_3((n-1)r+n-2)}{C_1\left(\frac{\rho}{r}\right)^n + C_2 + C_3 n}\\
&= \frac{r+1}{ar^2}\\
&= \frac{-2}{9a}\\
&= \frac{3b}{2},
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} y_{2n+1} &= \lim_{n\to\infty} \frac{br^n}{r^{n-1}} \frac{C_1'\left(\frac{\rho}{r}\right)^n + C_2' + C_3' n}{C_1'(\rho+1)\left(\frac{\rho}{r}\right)^{n-1} + C_2'(r+1) + C_3'(nr+n-1)}\\
&= \frac{br}{r+1}\\
&= \frac{3b}{2}.
\end{aligned}
$$

The proofs of (c) and (d) are clear from the forms in (2.31), (2.32), (2.35) and (2.36). □

Corollary 2.2. *Suppose that* $\frac{4}{ab} = -27$*. Then, every well-defined solution of system (1.1) has a finite limit point.*

2.3.3. Case $4/ab > -27$

For this case we first prove the following lemma.

Lemma 2.1. *Suppose that* $\frac{4}{ab} > -27$ *and the zeros of the polynomial* $P(\lambda)$ *are* ρ *and* $re^{\pm i\theta}$*,* $r > 0$*,* $\theta \in (0, \pi)$*. Then, the following statements are true:*

(a) If $ab \in (0, +\infty)$*, then* $r < \rho$

(b) If $ab \in \left(-\infty, -\frac{4}{27}\right)$*, then* $r > |\rho|$

Proof. Let $\frac{4}{ab} > -27$. Then, since ρ and $re^{\pm i\theta}$ are the zeros of the polynomial $P(\lambda)$, the relations

$$\rho r^2 = \frac{1}{ab} \text{ and } \rho^3 - \frac{1}{ab}(\rho+1)^2 = 0$$

are satisfied. We conclude from these relations that $ab\rho > 0$. This implies that if $ab < 0$, then $\rho < 0$ and if $ab > 0$, then $\rho > 0$. Also, from (2.18), we have

$$\rho^2 - \frac{1}{ab\rho}\rho^2 - \frac{2}{ab\rho}\rho - \frac{1}{ab\rho} = \rho^2 - r^2\rho^2 - 2r^2\rho - r^2 = 0,$$

which implies

$$r = \left|\frac{\rho}{\rho+1}\right|. \tag{2.45}$$

We must consider the following two cases:

(a) If $ab \in (0, +\infty)$, then $\rho > 0$ and so, from (2.45), we have

$$r = \frac{\rho}{\rho+1} < \rho.$$

(b) If $ab \in \left(-\infty, -\frac{4}{27}\right)$, then we see from (2.45) that $\rho < -1$. So, from (2.45), we have

$$r = \left|\frac{\rho}{\rho+1}\right| > |\rho|.$$

So, the proof is completed. □

Remark 2.2. *Note that the equality $r = |\rho|$ is impossible. Because, in this case we have $\frac{1}{ab} = \rho^3$ which yields $ab = -\frac{1}{8} \notin \left(-\infty, -\frac{4}{27}\right)$.*

Theorem 2.4. *Let $\{(x_n, y_n)\}_{n\geq -1}$ be a well-defined solution of system (1.1). Suppose that $\frac{4}{ab} > -27$. Then, the following statements are true:*

(a) If $ab \in (0, +\infty)$ and $C_1 \neq 0$, then $x_{2n+1} \to \frac{a\rho}{\rho+1}$ and $y_{2n} \to \frac{b\rho}{\rho+1}$ as $n \to \infty$.

(b) If $ab \in (0, +\infty)$ and $C_1' \neq 0$, then $x_{2n} \to \frac{a\rho}{\rho+1}$ and $y_{2n+1} \to \frac{b\rho}{\rho+1}$ as $n \to \infty$.

(c) If $ab \in \left(-\infty, -\frac{4}{27}\right)$ and $|C_2| + |C_3| \neq 0$, then both x_{2n+1} and y_{2n} are periodic or converge to a periodic solution or dense in $\mathbb{R}$.

(d) If $ab \in \left(-\infty, -\frac{4}{27}\right)$ and $\left|C_2'\right| + \left|C_3'\right| \neq 0$, then both x_{2n} and y_{2n+1} are periodic or converge to a periodic solution or dense in $\mathbb{R}$.

Proof. (a)–(b) The proof follows from the formulas given in (2.39), (2.40), (2.43), (2.44) and Lemma 2.1 by taking the limit.

(c)–(d) Since the proof is similar for the forms given in (2.39), (2.40), (2.43), (2.44), we only prove for (2.44). Suppose that $C_1' = 0$. Then, from (2.44), we have

$$x_{2n} = \frac{1}{br^2}\frac{C_4'\cos(n-1)\theta + C_5'\sin(n-1)\theta}{C_2'\cos n\theta + C_3'\sin n\theta}, \tag{2.46}$$

where $C_4' = C_2'(r+\cos\theta) - C_3'\sin\theta$, $C_5' = C_3'(r+\cos\theta) + C_2'\sin\theta$. Also, we see that the form given in (2.46) can be written as

$$x_{2n} = \frac{1}{br^2}\left(r\cos\theta + \cos 2\theta + (r\sin\theta + \sin 2\theta)\tan(n\theta - \gamma)\right),$$

where γ is an arbitrary constant, which corresponds to the arbitrary constants C_2', C_3', and satisfies the equality

$$\frac{\cos\gamma}{C_2'} = \frac{\sin\gamma}{C_3'},\ -\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}.$$

Now, we consider the following two cases:

(i) If $\theta = \frac{p}{q}\pi$ such that p, q are co-prime integers, we have

$$\begin{aligned} x_{2n} &= \frac{1}{br^2}\left(r\cos\theta + \cos 2\theta + (r\sin\theta + \sin 2\theta)\tan(q\theta - \gamma)\right) \\ &= \frac{1}{br^2}\left(r\cos\theta + \cos 2\theta + (r\sin\theta + \sin 2\theta)\tan(p\pi - \gamma)\right) \\ &= \frac{1}{br^2}\left(r\cos\theta + \cos 2\theta + (r\sin\theta + \sin 2\theta)\tan(-\gamma)\right) \end{aligned}$$

which implies $x_{2n+q} = x_{2n}$. Suppose that $C_1' \neq 0$. Then, since the inequality $r > |\rho|$ holds, from (2.44), we have

$$x_{2n} = \frac{1}{br^2}\frac{C_1'(\rho+1)\left(\frac{\rho}{r}\right)^{n-2} + C_4'\cos(n-1)\theta + C_5'\sin(n-1)\theta}{C_1'\left(\frac{\rho}{r}\right)^{n} + C_2'\cos n\theta + C_3'\sin n\theta},$$

which leads to

$$x_{2n} \to \frac{1}{br^2}\frac{C_4'\cos(n-1)\theta + C_5'\sin(n-1)\theta}{C_2'\cos n\theta + C_3'\sin n\theta}$$

for large enough values of n. Hence, the sequence $(x_{2n})_{n\ge 0}$ converges to a periodic solution obtained in the case $C_1' = 0$.

(ii) If $\theta = t\pi$ such that t is irrational, then we have by virtue of the Kronecker's Theorem that the set $\{(nt - [nt])\pi : n \in \mathbb{N}_0 \text{ and } t \text{ is irrational}\}$ is dense in the interval $(0, \pi)$. Hence, we have

$$\tan(n\theta - \gamma) = \tan(nt\pi - \gamma) = \tan(nt\pi - [nt]\pi - \gamma)$$

which implies the sequence $(x_{2n})_{n\ge 0}$ is dense in $(-\infty, +\infty)$. □

3. Conclusions

In this paper we conducted a detailed analysis on all solutions of system (1.1). To do this analysis, we obtained the representation forms of general solution of the system by using a practical method. By using these forms, we characterized the well-defined solutions of system (1.1). Finally, we studied the long-term behavior of the well-defined solutions. We can summarize our results as follows:

Consider system (1.1). Then,

(a) If $ab \in \left(-\frac{4}{27}, 0\right)$, then every well-defined solution of system (1.1) has a finite limit point.

(b) If $ab \in (0, +\infty)$ and $C_1 C_1' \neq 0$, then every well-defined solution of system (1.1) has a finite limit point.

(c) If $ab \in \left(-\infty, -\frac{4}{27}\right)$ and $(|C_2| + |C_3|)\left(\left|C_2'\right| + \left|C_3'\right|\right) \neq 0$, then every well-defined solution of system (1.1) is periodic or converges to a periodic solution or dense in $\mathbb{R}^2$.

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Conflict of interest

The authors declare that there are no conflict of interest associated with this publication.

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