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Theory article

Persistence and extinction of a modified Leslie-Gower Holling-type II predator-prey stochastic model in polluted environments with impulsive toxicant input

Yongxin Gao and Shuyuan Yao*

College of Science, Civil Aviation University of China, 2898 Jinbei Road, Tianjin 300300, China

* **Correspondence:** Email: yshuyuan0222@163.com.

Abstract: In this paper, a modified Leslie-Gower Holling-type II two-predator one-prey stochastic model in polluted environments with impulsive toxicant input is proposed where we use an Ornstein-Uhlenbeck process to improve the stochasticity of the environment. The sharp sufficient conditions for persistence in the mean and extinction are established. The results reveal that the persistence and extinction of the species have close relationships with the toxicant and environmental stochasticity. In addition, the theoretical results are verified by numerical simulation.

Keywords: stochastic models; Itô′s formula; persistence in the mean; extinction; intensity of white noise; toxicant effect

1. Introduction

The relationship between predator and prey has long been one of the important topics concerned by scholars. In recent years, several scholars have proposed more realistic models which should take the functional response into account [1, 2]. As a result, Aziz-Alaoui and Okiye [3] proposed the famous predator-prey model with modified Leslie-Gower and Holling-type II schemes, which is described as follows:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(r_1 - ax(t) - \frac{cy(t)}{h+x(t)}), \\ \frac{dy(t)}{dt} = y(t)(r_2 - \frac{fy(t)}{h+x(t)}), \end{cases}$$

where $x(t)$ and $y(t)$ stand for the sizes of the prey population and the predator population respectively; a represents the intraspecific competition strength; c means the per capita reduction rate of the prey due to the capture of the predator; h stands for the safeguard of the environment; f has the like signification of c. A growing number of scholars based on the above model have studied the possibility of a reciprocal relationship between the decline of predator populations and the per capita availability of prey (see e.g.

[4–9]).

In recent years, the world economy has grown rapidly with the development of industry and agriculture. At the same time, environmental pollution is becoming more and more serious, and even poses a threat to the survival of biological populations and human beings. For example, serious soil erosion and exhaust emissions from cars on the road are destroying the biological population structure. In order to better control and understand the effects of toxic substances on species, we must assess the population survival risk of exposure to toxic substances.

Hallam and his colleagues [10–12] have opened the door to the study of environmental toxins by publishing three papers in a row that suggest the effects of toxins on deterministic models of ecosystems. From then on, many deterministic population models with toxic effects have been proposed and studied (see e.g. [13–18]). Particularly, consider that toxins are often released into the environment in pulses of regularity. For example, pesticides and heavy metals [19, 20]. Therefore, based on the study of deterministic population models in polluted environments with impulsive toxin inputs, several authors explored the effects of toxins on population (see e.g. [21–24]).

In particular, suppose that the living organisms absorb environmental toxicant into their bodies. $C_{10}(t)$, $C_{20}(t)$ and $C_e(t)$ denote the concentration of the toxicant in the organism of the prey species, the predator species and the environment at time t, respectively. Suppose that the growth rate, r_i, is an affine function of C_{i0}:

$$r_i \to r_{i0} - r_{i1}C_{i0}(t), \quad i = 1, 2.$$

Therefore, the following model of predator and prey with modified Leslie-Gower and Holling-type II schemes in the presence of toxins is proposed.

$$\begin{cases} dx(t) = x(t)[r_{10} - r_{11}C_{10}(t) - ax(t) - \dfrac{cy(t)}{h + x(t)}]dt, \\ dy(t) = y(t)[r_{20} - r_{21}C_{20}(t) - \dfrac{fy(t)}{h + x(t)}]dt. \end{cases} \tag{1.1}$$

In fact, the rate of species growth is often disturbed by random perturbations [25]. In general, random perturbations in the environment can be represented by white noise [26, 27]. Therefore, we consider the perturbations of white noise to the population growth rate with $r_{i0} \to r_{i0} + \sigma_i\dot{B}_i(t)$, we obtain the following stochastic model:

$$\begin{cases} dx(t) = x(t)[r_{10} - r_{11}C_{10}(t) - ax(t) - \dfrac{cy(t)}{h + x(t)}]dt + \sigma_1 x(t)dB_1(t), \\ dy(t) = y(t)[r_{20} - r_{21}C_{20}(t) - \dfrac{fy(t)}{h + x(t)}]dt + \sigma_2 y(t)dB_2(t), \end{cases} \tag{1.2}$$

where σ_i^2, $i = 1, 2$ is the intensity of white noise; $B_1(t)$ and $B_2(t)$ are mutually independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in R_+}$. Liu et al. [28] probed into several dynamical characteristics of model (1.2) and offered extinct and persistent conditions for the model (1.2).

Model (1.2) assumes that the growth rate in the random environments is linear with respect to the Gaussian white noise

$$\hat{r}_{i0}(t) = r_{i0} + \sigma_i \frac{dB_i(t)}{dt}, \quad i = 1, 2.$$

Integrating on the interval $[0, T]$ results in

$$\bar{r}_{i0} = \frac{1}{T}\int_0^T \hat{r}_{i0}(t)dt \to r_{i0} + \sigma_i \frac{B_i(T)}{T} \sim N(r_{i0}, \sigma_i^2/T).$$

Hence, the variance of the average per capita growth rate $\bar{r}_{i0}$ over an interval of length T tends to ∞ as $T \to 0$. This is insufficient to describe the actual situation. Several authors [29,30] have claimed that using the mean-reverting Ornstein-Uhlenbeck process is a more appropriate approach to incorporate the environment perturbations. On account of this method [31], one has

$$d\hat{r}_{i0}(t) = \alpha_i(r_{i0} - \hat{r}_{i0}(t))dt + \xi_i dB_i(t), \quad i = 1, 2,$$

i.e.

$$\begin{aligned} \hat{r}_{i0}(t) &= r_{i0} + (\tilde{r}_{i0} - r_{i0})e^{-\alpha_i t} + \xi_i \int_0^t e^{-\alpha_i(t-s)}dB_i(s) \\ &= r_{i0} + (\tilde{r}_{i0} - r_{i0})e^{-\alpha_i t} + \sigma_i(t)\frac{dB_i(t)}{dt}, \quad i = 1, 2, \end{aligned}$$

where $\tilde{r}_{i0} = \hat{r}_{i0}(0)$, $\sigma_i(t) = \frac{\xi_i}{\sqrt{2\alpha_i}}\sqrt{1 - e^{-2\alpha_i t}}$, $\alpha_i > 0$ represents the speed of reversion, ξ_i^2 is the intensity of stochastic perturbations. Based on the ideas above, a three-species predator-prey model can be expressed as follows:

$$\begin{cases} dy_1(t) = y_1(t)[r_{10} + (\tilde{r}_{10} - r_{10})e^{-\alpha_1 t} - r_{11}C_{10}(t) - ay_1(t) - \frac{c_2 y_2(t)}{h_2 + y_1(t)} - \frac{c_3 y_3(t)}{h_3 + y_1(t)}]dt \\ \qquad + \sigma_1(t)y_1(t)dB_1(t), \\ dy_2(t) = y_2(t)[r_{20} + (\tilde{r}_{20} - r_{20})e^{-\alpha_2 t} - r_{21}C_{20}(t) - \frac{f_2 y_2(t)}{h_2 + y_1(t)}]dt \\ \qquad + \sigma_2(t)y_2(t)dB_2(t), \\ dy_3(t) = y_3(t)[r_{30} + (\tilde{r}_{30} - r_{30})e^{-\alpha_3 t} - r_{31}C_{30}(t) - \frac{f_3 y_3(t)}{h_3 + y_1(t)}]dt \\ \qquad + \sigma_3(t)y_3(t)dB_3(t), \end{cases} \tag{1.3}$$

where $y_1(t)$ is the population size of the prey at time t, $y_i(t), i = 2, 3$ is the population size of the predator at time t.

Now let us introduce the model of the concentration of toxicant. Suppose that $C_{i0}(t)$ satisfies the following model:

$$\frac{dC_{i0(t)}}{dt} = k_i C_e(t) - l_i C_{i0}(t),$$

where k_i stands for the uptake rate of toxicant from the environment; l_i denotes the loss rate of the toxicant from the species. $C_e(t)$ denotes the concentrations of the toxicant in the environment at time t and satisfies the following model:

$$\begin{cases} \frac{dC_e(t)}{dt} = -hC_e(t), \quad t \neq n\gamma, \quad n \in Z^+, \\ \triangle C_e(t) = q, \quad t = n\gamma, \quad n \in Z^+, \end{cases}$$

where $\triangle\zeta(t) = \zeta(t^+) - \zeta(t)$, h is the loss rate of toxicant from environment, q is the toxicant input amount at every time, γ stands for the period of the impulsive input of toxicant.

Therefore, we have the following one-prey two-predator system in polluted environments with pulse toxicant input:

$$\begin{cases} dy_1(t) &= y_1(t)[r_{10} + (\tilde{r}_{10} - r_{10})e^{-\alpha_1 t} - r_{11}C_{10}(t) - ay_1(t) - \frac{c_2 y_2(t)}{h_2 + y_1(t)} - \frac{c_3 y_3(t)}{h_3 + y_1(t)}]dt \\ & \quad +\sigma_1(t)y_1(t)dB_1(t), \\ dy_2(t) &= y_2[r_{20} + (\tilde{r}_{20} - r_{20})e^{-\alpha_2 t} - r_{21}C_{20}(t) - \frac{f_2 y_2(t)}{h_2 + y_1(t)}]dt \\ & \quad +\sigma_2(t)y_2(t)dB_2(t), \\ dy_3(t) &= y_3[r_{30} + (\tilde{r}_{30} - r_{30})e^{-\alpha_3 t} - r_{31}C_{30}(t) - \frac{f_3 y_3(t)}{h_3 + y_1(t)}]dt \\ & \quad +\sigma_3(t)y_3(t)dB_3(t), \\ \frac{dC_{i0(t)}}{dt} &= k_i C_e(t) - l_i C_{i0}(t), i = 1, 2, 3, \\ \frac{dC_e(t)}{dt} &= -hC_e(t), \quad t \neq n\gamma, \quad n \in Z^+, \\ \triangle C_e(t) &= q, \quad t = n\gamma, \quad n \in Z^+. \end{cases} \tag{1.4}$$

Remark 1 In model (1.4), the parameters $r_{i0}, \tilde{r}_{i0}, k_i, l_i (i = 1, 2, 3)$, $f_i, h_i, c_i (i = 2, 3)$ and a satisfy $0 < r_{i0}, \tilde{r}_{i0}, k_i, l_i, f_i, h_i, c_i, a \leq 1$. The parameter σ_i^2 is the intensity of the white noise on the growth rate of species i, thus $\sigma_i^2 > 0, i = 1, 2, 3$. The parameter $\gamma, q > 0$. r_{i1} represents the decreasing rate of the intrinsic growth rate associated with the uptake of the toxicant. Thus $r_{i1} > 0$. In addition, C_{i0} and C_e stand for the concentrations of the toxicant, therefore $0 \leq C_{i0}(t) \leq 1$ and $0 \leq C_e(t) \leq 1$ for $t \geq 0$. As a result, the following conditions need to be met: $k_i \leq l_i$, $q \leq 1 - e^{-h\gamma}$, $i = 1, 2, 3$.

To the best of our knowledge, Liu et al. [28] only studied the persistence and extinction of two species system in polluted environment, little research has been done on the dynamics of corresponding three species system. Therefore, we deeply analyze the properties of model (1.4) in the stochastic environment improved by the Ornstein-Uhlenbeck process.

The arrangement of this paper is as follows. In section 2, the persistence and extinction threshold for each species are proposed. In section 3, we carry out some numerical simulations to verify the theoretical results. Finally, we give some conclusions in section 4.

2. Main results

For the sake of convenience and simplicity, we define the following notations:

$$R_+^3 = \{z \in R^3 | z_i > 0, i = 1, 2, 3\}, \quad b_i(t) = r_{i0} - \frac{\xi_i^2}{4\alpha_i} + \frac{\xi_i^2}{4\alpha_i} e^{-2\alpha_i t} - r_{i1}C_{i0}(t),$$

$$K_i = \frac{qk_i}{hl_i}, \qquad \overline{b}_i(t) = \lim_{t \to +\infty} t^{-1} \int_0^t b_i(s)ds = r_{i0} - \frac{\xi_i^2}{4\alpha_i} - \frac{r_{i1}K_i}{\gamma}, i = 1, 2, 3.$$

Lemma 1 *Consider the following subsystem of model* (1.4) *[32]:*

$$\begin{cases} \frac{dC_{10}(t)}{dt} = k_1 C_e(t) - l_1 C_{10}(t), \\ \frac{dC_{20}(t)}{dt} = k_2 C_e(t) - l_2 C_{20}(t), \\ \frac{dC_{30}(t)}{dt} = k_3 C_e(t) - l_3 C_{30}(t), \\ \frac{dC_e(t)}{dt} = -hC_e(t), t \neq n\gamma, n \in Z^+, \\ \Delta C_{i0}(t) = 0, \Delta C_e(t) = q, t = n\gamma, n \in Z^+, \\ 0 \leq C_{i0}(0) \leq 1, 0 \leq C_e(0) \leq 1, i = 1, 2, 3. \end{cases}$$

The model has a unique positive γ-periodic solution $(\widetilde{C}_{10}(t), \widetilde{C}_{20}(t), \widetilde{C}_{30}(t), \widetilde{C}_e(t))^T$ *which satisfies*

$$\lim_{t\to+\infty} t^{-1}\int_0^t \widetilde{C}_{i0}(s)ds = \frac{k_i q}{hl_i\gamma} = \frac{K_i}{\gamma}, \quad i = 1, 2, 3.$$

Lemma 2 *For arbitrary* $(y_1(0), y_2(0), y_3(0)) \in R^3_+$, *model* (1.3) *possesses a unique solution* $(y_1(t), y_2(t), y_3(t)) \in R^3_+$ *for all* $t \geq 0$ *a.s.(almost surely).*

Proof. Pay attention to the following system:

$$\begin{cases} du(t) &= [b_1(t) + (\tilde{r}_{10} - r_{10})e^{-\alpha_1 t} - ae^{u(t)} - \frac{c_2 e^{v(t)}}{h_2+e^{u(t)}} - \frac{c_3 e^{w(t)}}{h_3+e^{u(t)}}]dt \\ & \quad +\sigma_1(t)dB_1(t), \\ dv(t) &= [b_2(t) + (\tilde{r}_{20} - r_{20})e^{-\alpha_2 t} - \frac{f_2 e^{v(t)}}{h_2+e^{u(t)}}]dt + \sigma_2(t)dB_2(t), \\ dw(t) &= [b_3(t) + (\tilde{r}_{30} - r_{30})e^{-\alpha_3 t} - \frac{f_3 e^{w(t)}}{h_3+e^{u(t)}}]dt + \sigma_3(t)dB_3(t), \end{cases} \tag{2.1}$$

and $u(0) = lny_1(0)$, $v(0) = lny_2(0)$, $w(0) = lny_3(0)$. Due to the fact that the coefficients of model (2.1) satisfy the local Lipschitz condition, model (2.1) possesses a unique local solution $(u(t), v(t), w(t))^T$ on $[0, \tau_*)$, where τ_* means the explosion time. From the Itô's formula, we derive that $(y_1(t), y_2(t), y_3(t)) = (e^{u(t)}, e^{v(t)}, e^{w(t)})$ is the unique local positive solution of model (1.3).

Now let us verify that $\tau_* = +\infty$. Consider the following systems:

$$d\phi(t) = \phi(t)[r_{10} + (\tilde{r}_{10} - r_{10})e^{-\alpha_1 t} - r_{11}C_{10}(t) - a\phi(t)]dt + \sigma_1\phi(t)dB_1(t), \quad \phi(0) = y_1(0); \tag{2.2}$$

$$dn(t) = n(t)[r_{20} + (\tilde{r}_{20} - r_{20})e^{-\alpha_2 t} - r_{21}C_{20}(t) - \frac{f_2}{h_2}n(t)]dt + \sigma_2 n(t)dB_2(t), \quad n(0) = y_2(0); \tag{2.3}$$

$$dN(t) = N(t)[r_{20} + (\tilde{r}_{20} - r_{20})e^{-\alpha_2 t} - r_{21}C_{20}(t) - \frac{f_2}{h_2 + \phi(t)}N(t)]dt + \sigma_2 N(t)dB_2(t), N(0) = y_2(0); \tag{2.4}$$

$$dm(t) = m(t)[r_{30} + (\tilde{r}_{30} - r_{30})e^{-\alpha_3 t} - r_{31}C_{30}(t) - \frac{f_3}{h_3}m(t)]dt + \sigma_3 m(t)dB_3(t), \quad m(0) = y_3(0); \tag{2.5}$$

$$dM(t) = M(t)[r_{30} + (\tilde{r}_{30} - r_{30})e^{-\alpha_3 t} - r_{31}C_{30}(t) - \frac{f_3}{h_3 + \phi(t)}M(t)]dt + \sigma_3 M(t)dB_3(t), M(0) = y_3(0). \tag{2.6}$$

On the basis of the comparison theorem for stochastic differential equations [33], we get for $t \in [0, \tau_*)$,

$$y_1(t) \leq \phi(t), \quad n(t) \leq y_2(t) \leq N(t), \quad m(t) \leq y_3(t) \leq M(t), \quad a.s. \tag{2.7}$$

According to Theorem 2.2 in Jiang and Shi [34], we get

$$\phi(t) = \frac{e^{\int_0^t b_1(s)ds - \frac{\tilde{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 t}-1) + \int_0^t \sigma_1(s)dB_1(s)}}{y_1^{-1}(0) + a\int_0^t e^{\int_0^s b_1(\tau)d\tau - \frac{\tilde{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 s}-1) + \int_0^s \sigma_1(\tau)dB_1(\tau)}ds}, \tag{2.8}$$

$$n(t) = \frac{e^{\int_0^t b_2(s)ds - \frac{\tilde{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 t}-1) + \int_0^t \sigma_2(s)dB_2(s)}}{y_2^{-1}(0) + \frac{f_2}{h_2}\int_0^t e^{\int_0^s b_2(\tau)d\tau - \frac{\tilde{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 s}-1) + \int_0^s \sigma_2(\tau)dB_2(\tau)}ds}, \tag{2.9}$$

$$N(t)=\frac{e^{\int_0^t b_2(s)ds-\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 t}-1)+\int_0^t \sigma_2(s)dB_2(s)}}{y_2^{-1}(0)+\int_0^t \frac{f_2}{h_2+\phi(s)}e^{\int_0^s b_2(\tau)d\tau-\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 s}-1)+\int_0^s \sigma_2(\tau)dB_2(\tau)}ds}, \tag{2.10}$$

$$m(t)=\frac{e^{\int_0^t b_3(s)ds-\frac{\bar{r}_{30}-r_{30}}{\alpha_3}(e^{-\alpha_3 t}-1)+\int_0^t \sigma_3(s)dB_3(s)}}{y_3^{-1}(0)+\frac{f_3}{h_3}\int_0^t e^{\int_0^s b_3(\tau)d\tau-\frac{\bar{r}_{30}-r_{30}}{\alpha_3}(e^{-\alpha_3 s}-1)+\int_0^s \sigma_3(\tau)dB_3(\tau)}ds}, \tag{2.11}$$

$$M(t)=\frac{e^{\int_0^t b_3(s)ds-\frac{\bar{r}_{30}-r_{30}}{\alpha_3}(e^{-\alpha_3 t}-1)+\int_0^t \sigma_3(s)dB_3(s)}}{y_3^{-1}(0)+\int_0^t \frac{f_3}{h_3+\phi(s)}e^{\int_0^s b_3(\tau)d\tau-\frac{\bar{r}_{30}-r_{30}}{\alpha_3}(e^{-\alpha_3 s}-1)+\int_0^s \sigma_3(\tau)dB_3(\tau)}ds}. \tag{2.12}$$

Due to the fact that $\phi(t), n(t), N(t), m(t)$ and $M(t)$ are global, we can know that $\tau_* = +\infty$.

Lemma 3 *Let $X(t) \in C(\Omega \times [0,+\infty), R_+)$, where $C(\Omega \times [0,+\infty), R_+)$ denotes the family of all positive-valued functions defined on $\Omega \times [0,+\infty)$. [35]*
(i) If there exist three positive constants t_0, β and β_0 such that for all $t \geq t_0$,

$$\ln X(t) \leq \beta t - \beta_0 \int_0^t X(s)ds + F(t),$$

where $F(t)/t \to 0$ as $t \to +\infty$, then
$\limsup_{t\to+\infty} t^{-1}\int_0^t X(s)ds \leq \beta/\beta_0$*, a.s.*
(ii) If there exist three positive constants t_0,β and β_0 such that for all $t \geq t_0$,

$$\ln X(t) \geq \beta t - \beta_0 \int_0^t X(S)ds + F(t),$$

where $F(t)/t \to 0$ as $t \to +\infty$, then
$\liminf_{t\to+\infty} t^{-1}\int_0^t X(s)ds \geq \beta/\beta_0$ *, a.s.*

Lemma 4 *Suppose that $\bar{b}_1 > 0$, then*
(i) If $\bar{b}_2 > 0$, then $\lim_{t\to+\infty}\frac{\ln y_2(t)}{t} = 0$, a.s.
(ii) If $\bar{b}_3 > 0$, then $\lim_{t\to+\infty}\frac{\ln y_3(t)}{t} = 0$, a.s.

Proof (i). Choose sufficiently large T which fulfils that, for $t \geq T$,

$$(\bar{b}_i-\varepsilon)t \leq \int_0^t b_i(s)ds \leq (\bar{b}_i+\varepsilon)t, \quad e^{(\bar{b}_i-\varepsilon)t} \geq 2e^{(\bar{b}_i-\varepsilon)T}.$$

For $t \geq T$, one can deduce from (2.8) that

$$\begin{aligned}
\phi(t) &= \frac{e^{\int_0^t b_1(s)ds-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)dB_1(s)}}{y_1^{-1}(0)+a\int_0^t e^{\int_0^s b_1(\tau)d\tau-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 s}-1)+\int_0^s \sigma_1(\tau)dB_1(\tau)}ds} \\
&\leq \frac{e^{\int_0^t b_1(s)ds-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)dB_1(s)}}{a\int_0^t e^{\int_0^s b_1(\tau)d\tau-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 s}-1)+\int_0^s \sigma_1(\tau)dB_1(\tau)}ds} \\
&\leq \frac{e^{(\bar{b}_1+\varepsilon)t-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)dB_1(s)}}{ae^{min_{0\leq v\leq t}\{\int_0^v \sigma_1(\tau)dB_1(\tau)-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 v}-1)\}}\int_T^t e^{(\bar{b}_1-\varepsilon)s}ds}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\bar{b}_1-\varepsilon)e^{(\bar{b}_1+\varepsilon)t-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)dB_1(s)}}{a(e^{(\bar{b}_1-\varepsilon)t}-e^{(\bar{b}_1-\varepsilon)T})e^{min_{0\le v\le t}\{\int_0^v \sigma_1(\tau)dB_1(\tau)-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 v}-1)\}}}\\
&\le \frac{2(\bar{b}_1-\varepsilon)e^{(\bar{b}_1+\varepsilon)t-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 t}-1)+\int_0^t \sigma_1(s)dB_1(s)}}{ae^{(\bar{b}_1-\varepsilon)t}e^{min_{0\le v\le t}\{\int_0^v \sigma_1(\tau)dB_1(\tau)-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 v}-1)\}}}\\
&= \frac{2(\bar{b}_1-\varepsilon)}{a}e^{2\varepsilon t}L_1(t),
\end{aligned}
$$

where

$$
L_1(t)=\frac{e^{\int_0^t \sigma_1(s)dB_1(s)-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 t}-1)}}{e^{min_{0\le v\le t}\{\int_0^v \sigma_1(\tau)dB_1(\tau)-\frac{\bar{r}_{10}-r_{10}}{\alpha_1}(e^{-\alpha_1 v}-1)\}}}.
$$

Note that $L_1(t)\ge 1$, consequently,

$$
\begin{aligned}
&\int_T^t \frac{f_2}{h_2+\phi(s)}e^{\int_0^s b_2(\tau)d\tau-\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 s}-1)+\int_0^s \sigma_2(\tau)dB_2(\tau)}ds\\
\ge\ & \int_T^t \frac{f_2e^{(\bar{b}_2-\varepsilon)s-\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 s}-1)+\int_0^s \sigma_2(\tau)dB_2(\tau)}}{h_2+\frac{2(\bar{b}_1-\varepsilon)}{a}e^{2\varepsilon s}L_1(s)}ds\\
\ge\ & \int_T^t \frac{f_2e^{(\bar{b}_2-\varepsilon)s-\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 s}-1)+\int_0^s \sigma_2(\tau)dB_2(\tau)}}{(h_2+\frac{2(\bar{b}_1-\varepsilon)}{a})e^{2\varepsilon s}L_1(s)}ds\\
=\ & \frac{af_2}{ah_2+2(\bar{b}_1-\varepsilon)}\int_T^t e^{(\bar{b}_2-3\varepsilon)s-\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 s}-1)+\int_0^s \sigma_2(\tau)dB_2(\tau)}L_1^{-1}(s)ds\\
\ge\ & \frac{af_2}{ah_2+2(\bar{b}_1-\varepsilon)}\frac{1}{\bar{b}_2-3\varepsilon}(e^{(\bar{b}_2-3\varepsilon)t}-e^{(\bar{b}_2-3\varepsilon)T})\min_{0\le v\le t}\{L_2(v)\}\\
=\ & L_3(t)(e^{(\bar{b}_2-3\varepsilon)t}-e^{(\bar{b}_2-3\varepsilon)T}),
\end{aligned}
$$

where

$$
\begin{aligned}
L_2(t) &= L_1^{-1}(t)e^{\int_0^t \sigma_2(\tau)dB_2(\tau)-\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2 t}-1)},\\
L_3(t) &= \frac{af_2}{ah_2+2(\bar{b}_1-\varepsilon)}\frac{1}{\bar{b}_2-3\varepsilon}\min_{0\le v\le t}\{L_2(v)\}.
\end{aligned}
$$

Then (2.10) implies that

$$
\begin{aligned}
\frac{1}{N(t)} &\ge e^{-\int_T^t b_2(s)ds+\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2(t-T)}-1)-\int_T^t \sigma_2(s)dB_2(s)}\\
&\quad \times L_3(t)(e^{(\bar{b}_2-3\varepsilon)t}-e^{(\bar{b}_2-3\varepsilon)T})\\
&\ge e^{-\int_T^t b_2(s)ds+\frac{\bar{r}_{20}-r_{20}}{\alpha_2}(e^{-\alpha_2(t-T)}-1)-\int_T^t \sigma_2(s)dB_2(s)}\times\frac{1}{2}L_3(t)e^{(\bar{b}_2-3\varepsilon)t}
\end{aligned}
$$

$$\geq \quad L_4(t) \times e^{-4\varepsilon t},$$

where

$$L_4(t) = \frac{1}{2} L_3(t) e^{\int_0^T b_2(s)ds + \frac{\tilde{r}_{20} - r_{20}}{\alpha_2}(e^{-\alpha_2(t-T)}-1) - \int_T^t \sigma_2(s)dB_2(s)}.$$

Therefore,

$$t^{-1} \ln N(t) < -t^{-1} \ln L_4(t) + 4\varepsilon. \tag{2.13}$$

Note that $\lim_{t\to+\infty} t^{-1} \int_0^t \sigma_i(s)dB_i(s) = 0 (i = 1, 2, 3)$, we then deduce that if $\bar{b}_2 > 0$, $\lim_{t\to+\infty} t^{-1} \ln L_4(t) = 0$, a.s.

This together with (2.13), indicates

$$\limsup_{t\to+\infty} t^{-1} \ln y_2(t) \leq \limsup_{t\to+\infty} t^{-1} \ln N(t) \leq 0, a.s.$$

Using Itô′s formula to (2.3) deduces

$$d \ln n(t) = (b_2(t) + (\tilde{r}_{20} - r_{20})e^{-\alpha_2 t} - \frac{f_2}{h_2} n(t))dt + \sigma_2(t)dB_2(t).$$

In other words,

$$\begin{aligned} t^{-1} \ln n(t) = t^{-1} \ln y_2(0) + t^{-1} \int_0^t b_2(s)ds - \frac{\tilde{r}_{20} - r_{20}}{\alpha_2 t}(e^{-\alpha_2 t} - 1) \\ - \frac{f_2}{h_2} t^{-1} \int_0^t n(s)ds + t^{-1} \int_0^t \sigma_2(s)dB_2(s). \end{aligned} \tag{2.14}$$

Clearly, for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that , for $t > T$,

$$\bar{b}_2 - 2\varepsilon \leq t^{-1}[\ln y_2(0) - \frac{\tilde{r}_{20} - r_{20}}{\alpha_2}(e^{-\alpha_2 t} - 1) + \int_0^t b_2(s)ds] \leq \bar{b}_2 + 2\varepsilon.$$

Then,we derive from (2.14) that,for $t \geq T$,

$$t^{-1} \ln n(t) \leq \bar{b}_2 + 2\varepsilon - \frac{f_2}{h_2} t^{-1} \int_0^t n(s)ds + t^{-1} \int_0^t \sigma_2(s)dB_2(s), \tag{2.15}$$

$$t^{-1} \ln n(t) \geq \bar{b}_2 - 2\varepsilon - \frac{f_2}{h_2} t^{-1} \int_0^t n(s)ds + t^{-1} \int_0^t \sigma_2(s)dB_2(s). \tag{2.16}$$

Choose ε be sufficiently small such that $0 \leq 2\varepsilon \leq \bar{b}_2$. Applying (i) and (ii) in Lemma 3 yields that

$$\frac{h_2(\bar{b}_2 - 2\varepsilon)}{f_2} \quad \leq \quad \liminf_{t\to+\infty} t^{-1} \int_0^t n(s)ds \leq \limsup_{t\to+\infty} t^{-1} \int_0^t n(s)ds$$

$$\leq \frac{h_2(\bar{b}_2 + 2\varepsilon)}{f_2}, \qquad a.s.$$

An application of the arbitrariness of ε ,we can obtain that

$$\lim_{t\to+\infty} t^{-1}\int_0^t n(s)ds = \frac{h_2\bar{b}_2}{f_2}, \quad a.s. \tag{2.17}$$

Which implies that $\lim_{t\to+\infty} t^{-1}\ln n(t) = 0$ a.s. Thanks to (2.7), one can derive that

$$\liminf_{t\to+\infty} t^{-1}\ln y_2(t) \geq \lim_{t\to+\infty} t^{-1}\ln n(t) = 0, \quad a.s. \tag{2.18}$$

The proof of (i) is completed.

The proof of (ii) is similar to that of (i), so it is omitted here.

Theorem 1 *For model* (1.3)*, the following conclusions hold:*

(I) If $\bar{b}_1 < 0, \bar{b}_2 < 0$ *and* $\bar{b}_3 < 0$*, then* y_1, y_2 *and* y_3 *become extinct, i.e.* $\lim_{t\to+\infty} y_i(t) = 0$*, a.s.,* $i = 1, 2, 3$.

(II) If $\bar{b}_1 < 0, \bar{b}_2 > 0$ *and* $\bar{b}_3 < 0$*, then* y_1 *and* y_3 *become extinct and* y_2 *is persistent in the mean , i.e.* $\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = h_2\bar{b}_2/f_2$*, a.s.*

(III) If $\bar{b}_1 < 0, \bar{b}_2 < 0$ *and* $\bar{b}_3 > 0$*, then* y_1 *and* y_2 *become extinct and* y_3 *is persistent in the mean , i.e.* $\lim_{t\to+\infty} t^{-1}\int_0^t y_3(s)ds = h_3\bar{b}_3/f_3$*, a.s.*

(IV) If $\bar{b}_1 > 0, \bar{b}_2 < 0$ *and* $\bar{b}_3 < 0$*, then* y_2 *and* y_3 *become extinct and* y_1 *is persistent in the mean, i.e.* $\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \bar{b}_1/a$*, a.s.*

(V) If $\bar{b}_1 < 0, \bar{b}_2 > 0$ *and* $\bar{b}_3 > 0$*, then* y_1 *becomes extinct and* y_2 *and* y_3 *are persistent in the mean, i.e.* $\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = h_2\bar{b}_2/f_2, \lim_{t\to+\infty} t^{-1}\int_0^t y_3(s)ds = h_3\bar{b}_3/f_3$*, a.s.*

(VI) If $\bar{b}_1 > 0, \bar{b}_2 > 0$ *and* $\bar{b}_3 < 0$*, then* y_3 *becomes extinct, and*

(i) If $\frac{\bar{b}_1}{c_2} < \frac{\bar{b}_2}{f_2}$ *, then* y_1 *becomes extinct and* y_2 *is persistent in the mean, i.e.* $\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = h_2\bar{b}_2/f_2$*, a.s.*

(ii) If $\frac{\bar{b}_1}{c_2} > \frac{\bar{b}_2}{f_2}$*, then* $\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \bar{b}_1/a - c_2\bar{b}_2/af_2$,
$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2+y_1(s)}ds = \bar{b}_2/f_2$*, a.s.*

(VII) If $\bar{b}_1 > 0, \bar{b}_2 < 0$ *and* $\bar{b}_3 > 0$*, then* y_2 *becomes extinct, and*

(i) If $\frac{\bar{b}_1}{c_3} < \frac{\bar{b}_3}{f_3}$*, then* y_1 *becomes extinct and* y_3 *is persistent in the mean, i.e.* $\lim_{t\to+\infty} t^{-1}\int_0^t y_3(s)ds = h_3\bar{b}_3/f_3$*, a.s.*

(ii) If $\frac{\bar{b}_1}{c_3} > \frac{\bar{b}_3}{f_3}$*, then* $\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \bar{b}_1/a - c_3\bar{b}_3/af_3$,
$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_3(s)}{h_3+y_1(s)}ds = \bar{b}_3/f_3$*, a.s.*

(VIII) If $\bar{b}_1 > 0, \bar{b}_2 > 0$ *and* $\bar{b}_3 > 0$*, then,*

(i) If $\bar{b}_1 < \frac{c_2}{f_2}\bar{b}_2 + \frac{c_3}{f_3}\bar{b}_3$*, then* y_1 *becomes extinct and* y_2 *and* y_3 *are persistent in the mean , i.e.* $\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = h_2\bar{b}_2/f_2, \lim_{t\to+\infty} t^{-1}\int_0^t y_3(s)ds = h_3\bar{b}_3/f_3$*, a.s.*

(ii) If $\bar{b}_1 > \frac{c_2}{f_2}\bar{b}_2 + \frac{c_3}{f_3}\bar{b}_3$*, then* $\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \bar{b}_1/a - c_2\bar{b}_2/af_2 - c_3\bar{b}_3/af_3$, $\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2+y_1(s)}ds = \bar{b}_2/f_2, \lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_3(s)}{h_3+y_1(s)}ds = \bar{b}_3/f_3$*, a.s.*

Proof. Applying Itô′s formula to model (1.3) gives

$$
\begin{aligned}
d\ln y_1(t) &= (b_1(t) + (\tilde{r}_{10} - r_{10})e^{-\alpha_1 t} - ay_1(t) - \frac{c_2 y_2(t)}{h_2 + y_1(t)} - \frac{c_3 y_3(t)}{h_3 + y_1(t)})dt \\
&\quad + \sigma_1(t)dB_1(t), \\
d\ln y_2(t) &= (b_2(t) + (\tilde{r}_{20} - r_{20})e^{-\alpha_2 t} - \frac{f_2 y_2(t)}{h_2 + y_1(t)})dt + \sigma_2(t)dB_2(t), \\
d\ln y_3(t) &= (b_3(t) + (\tilde{r}_{30} - r_{30})e^{-\alpha_3 t} - \frac{f_3 y_3(t)}{h_3 + y_1(t)})dt + \sigma_3(t)dB_3(t).
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
\ln y_1(t) - \ln y_1(0) = &\int_0^t b_1(s)ds - \frac{\tilde{r}_{10} - r_{10}}{\alpha_1}(e^{-\alpha_1 t} - 1) - a\int_0^t y_1(s)ds \\
&- c_2\int_0^t \frac{y_2(s)}{h_2 + y_1(s)}ds - c_3\int_0^t \frac{y_3(s)}{h_3 + y_1(s)}ds \\
&+ \int_0^t \sigma_1(s)dB_1(s),
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
\ln y_2(t) - \ln y_2(0) = &\int_0^t b_2(s)ds - \frac{\tilde{r}_{20} - r_{20}}{\alpha_2}(e^{-\alpha_2 t} - 1) \\
&- f_2\int_0^t \frac{y_2(s)}{h_2 + y_1(s)}ds + \int_0^t \sigma_2(s)dB_2(s),
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\ln y_3(t) - \ln y_3(0) = &\int_0^t b_3(s)ds - \frac{\tilde{r}_{30} - r_{30}}{\alpha_3}(e^{-\alpha_3 t} - 1) \\
&- f_3\int_0^t \frac{y_3(s)}{h_3 + y_1(s)}ds + \int_0^t \sigma_3(s)dB_3(s).
\end{aligned}
\tag{2.21}
$$

First, let us prove (I), we derive from (2.19) that, for sufficiently large t,

$$
t^{-1}\ln\frac{y_1(t)}{y_1(0)} \le \bar{b}_1 + \varepsilon + t^{-1}\int_0^t \sigma_1(s)dB_1(s) - \frac{\tilde{r}_{10} - r_{10}}{t\alpha_1}(e^{-\alpha_1 t} - 1). \tag{2.22}
$$

Note that $\lim_{t\to+\infty} t^{-1}\int_0^t \sigma_1(s)dB_1(s) = 0$ and $\bar{b}_1 + \varepsilon < 0$, thus $\lim_{t\to+\infty} y_1(t) = 0$, a.s. In the same way, $\bar{b}_2 < 0$ means that $\lim_{t\to+\infty} y_2(t) = 0$, a.s., $\bar{b}_3 < 0$ means that $\lim_{t\to+\infty} y_3(t) = 0$, a.s.

(II). Since $\bar{b}_1 < 0$, $\bar{b}_3 < 0$, we then deduce from (I) that $\lim_{t\to+\infty} y_1(t) = 0, \lim_{t\to+\infty} y_3(t) = 0$, a.s. Then for sufficiently large t,

$$
\ln y_2(t) \le (\bar{b}_2 + 2\varepsilon)t - \frac{f_2}{h_2 + \varepsilon}\int_0^t y_2(s)ds + \int_0^t \sigma_2(s)dB_2(s), \tag{2.23}
$$

$$
\ln y_2(t) \ge (\bar{b}_2 - 2\varepsilon)t - \frac{f_2}{h_2 - \varepsilon}\int_0^t y_2(s)ds + \int_0^t \sigma_2(s)dB_2(s). \tag{2.24}
$$

Making use of Lemma 3 to (2.23) and (2.24) respectively, we obtain

$$\begin{aligned}\limsup_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds &\le \tfrac{(h_2+\varepsilon)(\bar{b}_2+2\varepsilon)}{f_2},\\ \liminf_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds &\ge \tfrac{(h_2-\varepsilon)(\bar{b}_2-2\varepsilon)}{f_2}.\end{aligned}$$

Therefore, from the arbitrariness of ε, we can derive that $\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = h_2\bar{b}_2/f_2$, a.s.
The proof of (III) and (IV) is similar to that of (II), thus is omitted here.
Now let us prove (V). According to $\bar{b}_1 < 0$, we have $\lim_{t\to+\infty} y_1(t) = 0$. The proof is similar to (II), so here is omitted.
(VI). First of all, $\bar{b}_3 < 0$ implies that $\lim_{t\to+\infty} y_3(t) = 0$, a.s.
(i) Compute that (2.19) $\times f_2$ − (2.20) $\times c_2$, then we get

$$\begin{aligned}f_2 t^{-1}\ln\frac{y_1(t)}{y_1(0)} =& c_2 t^{-1}\ln\frac{y_2(t)}{y_2(0)} + t^{-1}f_2\int_0^t b_1(s)ds - t^{-1}c_2\int_0^t b_2(s)ds\\ &-\frac{\tilde{r}_{10}-r_{10}}{\alpha_1 t}(e^{-\alpha_1 t}-1)f_2 + \frac{\tilde{r}_{20}-r_{20}}{\alpha_2 t}(e^{-\alpha_2 t}-1)c_2\\ &-af_2t^{-1}\int_0^t y_1(s)ds - f_2c_3t^{-1}\int_0^t \frac{y_3(s)}{h_3+y_1(s)}ds\\ &+f_2t^{-1}\int_0^t\sigma_1(s)dB_1(s) - c_2t^{-1}\int_0^t\sigma_2(s)dB_2(s)\\ \le& c_2 t^{-1}\ln\frac{y_2(t)}{y_2(0)} + t^{-1}f_2\int_0^t b_1(s)ds - t^{-1}c_2\int_0^t b_2(s)ds\\ &-\frac{\tilde{r}_{10}-r_{10}}{\alpha_1 t}(e^{-\alpha_1 t}-1)f_2 + \frac{\tilde{r}_{20}-r_{20}}{\alpha_2 t}(e^{-\alpha_2 t}-1)c_2\\ &-af_2t^{-1}\int_0^t y_1(s)ds + f_2t^{-1}\int_0^t\sigma_1(s)dB_1(s)\\ &-c_2t^{-1}\int_0^t\sigma_2(s)dB_2(s).\end{aligned}\tag{2.25}$$

In view of Lemma 4, for any $\varepsilon > 0$, there exists a $T > 0$ such that for $t > T$, we can see that

$$\begin{aligned}&c_2t^{-1}\ln\frac{y_2(t)}{y_2(0)} \le \varepsilon/4,\\ &f_2t^{-1}\ln y_1(0) \le \varepsilon/4,\\ &\frac{\tilde{r}_{20}-r_{20}}{\alpha_2 t}(e^{-\alpha_2 t}-1)c_2 - \frac{\tilde{r}_{10}-r_{10}}{\alpha_1 t}(e^{-\alpha_1 t}-1)f_2 \le \varepsilon/4,\\ &f_2t^{-1}\int_0^t\sigma_1(s)dB_1(s) - c_2t^{-1}\int_0^t\sigma_2(s)dB_2(s) \le \varepsilon/4,\\ &t^{-1}f_2\int_0^t b_1(s)ds - t^{-1}c_2\int_0^t b_2(s)ds \le f_2\bar{b}_1 - c_2\bar{b}_2 + f_2\varepsilon + c_2\varepsilon.\end{aligned}$$

As a consequence, for $t > T$,

$$t^{-1}f_2\ln y_1(t) \le (f_2+c_2+1)\varepsilon + f_2\bar{b}_1 - c_2\bar{b}_2.\tag{2.26}$$

Let ε be sufficiently small such that $0 < \varepsilon < \frac{c_2\bar{b}_2 - \bar{b}_1 f_2}{f_2 + c_2 + 1}$. Consequently, $\lim_{t\to+\infty} y_1(t) = 0$. The proof of $\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2\bar{b}_2}{f_2}$ is similar to that of (II) and here is omitted.

(ii) It follows from (2.20) that

$$\begin{aligned} t^{-1}\ln y_2(t) - t^{-1}\ln y_2(0) = t^{-1}\int_0^t b_2(s)ds - \frac{\tilde{r}_{20} - r_{20}}{\alpha_2 t}(e^{-\alpha_2 t} - 1) - f_2 t^{-1}\int_0^t \frac{y_2(s)}{h_2 + y_1(s)}ds \\ + t^{-1}\int_0^t \sigma_2(s)dB_2(s). \end{aligned} \tag{2.27}$$

Making use of Lemma 4 and $\lim_{t\to+\infty} t^{-1}\int_0^t \sigma_2(s)dB_2(s) = 0$, we can obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2 + y_1(s)}ds = \frac{\bar{b}_2}{f_2}. \tag{2.28}$$

Then, for any $\varepsilon > 0$, there is $T > 0$ such that for $t > T$,

$$\begin{aligned} -\frac{c_2\bar{b}_2}{f_2} - \varepsilon &\le -\frac{\tilde{r}_{10} - r_{10}}{\alpha_1 t}(e^{-\alpha_1 t} - 1) - c_2 t^{-1}\int_0^t \frac{y_2(s)}{h_2 + y_1(s)}ds + t^{-1}\ln y_1(0) - c_3 t^{-1}\int_0^t \frac{y_3(s)}{h_3 + y_1(s)}ds \\ &\le -\frac{c_2\bar{b}_2}{f_2} + \varepsilon. \end{aligned} \tag{2.29}$$

Substitute (2.29) into (2.19), and we can get that, for $t \ge T$,

$$\begin{aligned} t^{-1}\ln y_1(t) &\ge \bar{b}_1 - \frac{c_2\bar{b}_2}{f_2} - 2\varepsilon - at^{-1}\int_0^t y_1(s)ds + t^{-1}\int_0^t \sigma_1(s)dB_1(s), \\ t^{-1}\ln y_1(t) &\le \bar{b}_1 - \frac{c_2\bar{b}_2}{f_2} + 2\varepsilon - at^{-1}\int_0^t y_1(s)ds + t^{-1}\int_0^t \sigma_1(s)dB_1(s). \end{aligned}$$

Let ε be sufficiently small such that $0 < \varepsilon < (\bar{b}_1 - c_2\bar{b}_2/f_2)/2$. Thanks to Lemma 3, we have

$$\begin{aligned} \frac{\bar{b}_1}{a} - \frac{c_2\bar{b}_2}{af_2} - \frac{2\varepsilon}{a} &\le \liminf_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds \le \limsup_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds \\ &\le \frac{\bar{b}_1}{a} - \frac{c_2\bar{b}_2}{af_2} + \frac{2\varepsilon}{a}. \end{aligned}$$

We then derive from the arbitrariness of ε that $\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \bar{b}_1/a - c_2\bar{b}_2/(af_2)$.
The proof of (VII) is similar to that of (VI), so it is omitted here.
Now let us proof (VIII).

(i) Compute that (2.19) $\times f_2 f_3$ − (2.20) $\times f_3 c_2$ − (2.21) $\times f_2 c_3$,

$$f_2 f_3 t^{-1}\ln\frac{y_1(t)}{y_1(0)} = f_3 c_2 t^{-1}\ln\frac{y_2(t)}{y_2(0)} + f_2 c_3 t^{-1}\ln\frac{y_3(t)}{y_3(0)}$$

$$
\begin{aligned}
&+f_2f_3t^{-1}\int_0^t b_1(s)ds - t^{-1}c_2f_3\int_0^t b_2(s)ds\\
&-c_3f_2t^{-1}\int_0^t b_3(s)ds - f_2f_3\frac{\tilde{r}_{10}-r_{10}}{\alpha_1 t}(e^{-\alpha_1 t}-1)\\
&+c_2f_3\frac{\tilde{r}_{20}-r_{20}}{\alpha_2 t}(e^{-\alpha_2 t}-1) + c_3f_2\frac{\tilde{r}_{30}-r_{30}}{\alpha_3 t}(e^{-\alpha_3 t}-1)\\
&-af_2f_3t^{-1}\int_0^t y_1(s)ds + f_2f_3t^{-1}\int_0^t \sigma_1(s)dB_1(s)\\
&-c_2f_3t^{-1}\int_0^t \sigma_2(s)dB_2(s) - c_3f_2t^{-1}\int_0^t \sigma_3(s)dB_3(s).
\end{aligned}
$$

In view of Lemma 4, for any $\varepsilon > 0$, there exists a $T > 0$ such that for $t > T$, we can see that

$$
\begin{aligned}
&f_3c_2t^{-1}\ln\frac{y_2(t)}{y_2(0)} + f_2c_3t^{-1}\ln\frac{y_3(t)}{y_3(0)} \le \varepsilon/4,\\
&f_2f_3t^{-1}\ln y_1(0) \le \varepsilon/4,\\
&-f_2f_3\frac{\tilde{r}_{10}-r_{10}}{\alpha_1 t}(e^{-\alpha_1 t}-1) + c_2f_3\frac{\tilde{r}_{20}-r_{20}}{\alpha_2 t}(e^{-\alpha_2 t}-1) - c_3f_2\frac{\tilde{r}_{30}-r_{30}}{\alpha_1 t}(e^{-\alpha_1 t}-1) \le \varepsilon/4,\\
&f_2f_3t^{-1}\int_0^t \sigma_1(s)dB_1(s) - c_2f_3t^{-1}\int_0^t \sigma_2(s)dB_2(s) - c_3f_2t^{-1}\int_0^t \sigma_3(s)dB_3(s) \le \varepsilon/4,\\
&t^{-1}f_2f_3\int_0^t b_1(s)ds - t^{-1}f_3c_2\int_0^t b_2(s)ds - t^{-1}c_3f_2\int_0^t b_3(s)ds\\
&\quad \le f_2f_3\bar{b}_1 - c_2f_3\bar{b}_2 - c_3f_2\bar{b}_3 + (f_2f_3 + c_2f_3 + c_3f_2)\varepsilon.
\end{aligned}
$$

As a consequence, for $t > T$,

$$
t^{-1}f_2f_3\ln y_1(t) \le (1 + f_2f_3 + c_2f_3 + c_3f_2)\varepsilon + f_2f_3\bar{b}_1 - c_2f_3\bar{b}_2 - c_3f_2\bar{b}_3. \tag{2.30}
$$

Let ε be sufficiently small such that $0 < \varepsilon < \frac{c_3f_2\bar{b}_3+c_2f_3\bar{b}_2-f_2f_3\bar{b}_1}{1+f_2f_3+c_2f_3+c_3f_2}$, we have $\lim_{t\to+\infty} y_1(t) = 0$. According to the proof of (II), we derive that

$$
\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2\bar{b}_2}{f_2}.
$$

Similar to the proof of (II), we can obtain

$$
\lim_{t\to+\infty} t^{-1}\int_0^t y_3(s)ds = \frac{h_3\bar{b}_3}{f_3}.
$$

(ii) It follows from (2.20), (2.21), Lemma 4 and $\lim_{t\to+\infty} t^{-1}\int_0^t \sigma_i(s)dB_i(s) = 0, i = 1, 2, 3$ that

$$
\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2+y_1(s)}ds = \frac{\bar{b}_2}{f_2}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_3(s)}{h_3+y_1(s)}ds = \frac{\bar{b}_3}{f_3}.
$$

Then, for any $\varepsilon > 0$, there is $T > 0$ such that for $t > T$,

$$\begin{aligned}-\frac{c_2\bar{b}_2}{f_2}-\frac{c_3\bar{b}_3}{f_3}-\varepsilon \le & -\frac{\tilde{r}_{10}-r_{10}}{\alpha_1 t}(e^{-\alpha_1 t}-1)-c_2t^{-1}\int_0^t\frac{y_2(s)}{h_2+y_1(s)}ds\\ & -c_3t^{-1}\int_0^t\frac{y_3(s)}{h_3+y_1(s)}ds+t^{-1}\ln y_1(0)\\ \le & -\frac{c_2\bar{b}_2}{f_2}-\frac{c_3\bar{b}_3}{f_3}+\varepsilon.\end{aligned} \tag{2.31}$$

Substitute (2.31) into (2.19), we can get that, for $t \ge T$,

$$t^{-1}\ln y_1(t) \ge \bar{b}_1-\frac{c_2\bar{b}_2}{f_2}-\frac{c_3\bar{b}_3}{f_3}-2\varepsilon-at^{-1}\int_0^t y_1(t)ds+t^{-1}\int_0^t\sigma_1(s)dB_1(s),$$
$$t^{-1}\ln y_1(t) \le \bar{b}_1-\frac{c_2\bar{b}_2}{f_2}-\frac{c_3\bar{b}_3}{f_3}+2\varepsilon-at^{-1}\int_0^t y_1(t)ds+t^{-1}\int_0^t\sigma_1(s)dB_1(s).$$

Let ε be sufficiently small such that $0 < \varepsilon < (\bar{b}_1 - c_2\bar{b}_2/f_2 - c_3\bar{b}_3/f_3)/2$. Thanks to Lemma 3, we have

$$\begin{aligned}\frac{\bar{b}_1}{a}-\frac{c_2\bar{b}_2}{af_2}-\frac{c_3\bar{b}_3}{af_3}-\frac{2\varepsilon}{a} \le & \liminf_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds \le \limsup_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds\\ \le & \frac{\bar{b}_1}{a}-\frac{c_2\bar{b}_2}{af_2}-\frac{c_3\bar{b}_3}{af_3}+\frac{2\varepsilon}{a}.\end{aligned}$$

We then derive from the arbitrariness of ε that $\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \bar{b}_1/a - c_2\bar{b}_2/(af_2) - c_3\bar{b}_3/(af_3)$.

The proof of Theorem 1 is completed.

3. Numerical simulations

Now we use the Milstein method offered in [36] to verify the theoretical results numerically (here we only provide the functions of ξ_i^2 since the functions of α_i can be proffered analogously).

In Figure 1–4, choose $r_{10} = 0.5$, $r_{20} = 0.5$, $r_{30} = 0.3$, $\tilde{r}_{10} = 0.3$, $\tilde{r}_{20} = 0.25$, $\tilde{r}_{30} = 0.2$, $r_{11} = r_{21} = r_{31} = 0.9$, $a = 0.25$, $c_2 = 0.36$, $c_3 = 0.4$, $f_2 = 0.5$, $f_3 = 0.47$, $h_2 = h_3 = 1$, $\alpha_1 = 0.31$, $\alpha_2 = 0.35$, $\alpha_3 = 0.42$, $\gamma = 2$, $k_1 = 0.2$, $k_2 = 0.26$ $k_3 = 0.21$, $l_1 = 0.8$, $l_2 = 0.7$, $l_3 = 0.7$, $h = 0.9$, $q = 0.5$. The only difference between conditions of Figure1–4 is that the values of ξ_1^2, ξ_2^2 and ξ_3^2 are different.

(I) In Figure 1, we choose $\xi_1^2 = 0.55$, $\xi_2^2 = 0.9$ and $\xi_3^2 = 0.5$. Then $\bar{b}_1 = -0.006 < 0$, $\bar{b}_2 = -0.2357 < 0$, $\bar{b}_3 = -0.0726 < 0$. According to (I) in Theorem 1, y_1, y_2 and y_3 die out.
Figure 1 confirms these.

(II) In Figure 2, we choose $\xi_1^2 = 0.55$, $\xi_2^2 = 0.16$ and $\xi_3^2 = 0.5$. Then $\bar{b}_1 = -0.006 < 0$, $\bar{b}_2 = 0.2929 > 0$, $\bar{b}_3 = -0.0726 < 0$. On the basis of (II) in Theorem 1, y_1 and y_3 die out and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2\bar{b}_2}{f_2} = 0.5857.$$

See Figure 2.

(III) In Figure 3, we choose $\xi_1^2 = 0.15$, $\xi_2^2 = 0.9$ and $\xi_3^2 = 0.5$. Then $\bar{b}_1 = 0.2915 > 0$, $\bar{b}_2 = -0.2357 < 0$, $\bar{b}_3 = -0.0726 < 0$. On the basis of (IV) in Theorem 1, y_2 and y_3 die out and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{\bar{b}_1}{a} = 1.2661.$$

See Figure 3.

(IV) In Figure 4, we choose $\xi_1^2 = 0.55$, $\xi_2^2 = 0.16$ and $\xi_3^2 = 0.09$. Then $\bar{b}_1 = -0.006 < 0$, $\bar{b}_2 = \bar{b}_2 = 0.2929 > 0$, $\bar{b}_3 = 0.1714 > 0$. On the basis of (V) in Theorem 1, y_1 dies out and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2\bar{b}_2}{f_2} = 0.5857,$$
$$\lim_{t\to+\infty} t^{-1}\int_0^t y_3(s)ds = \frac{h_3\bar{b}_3}{f_3} = 0.3647.$$

Figure 4 confirms these.

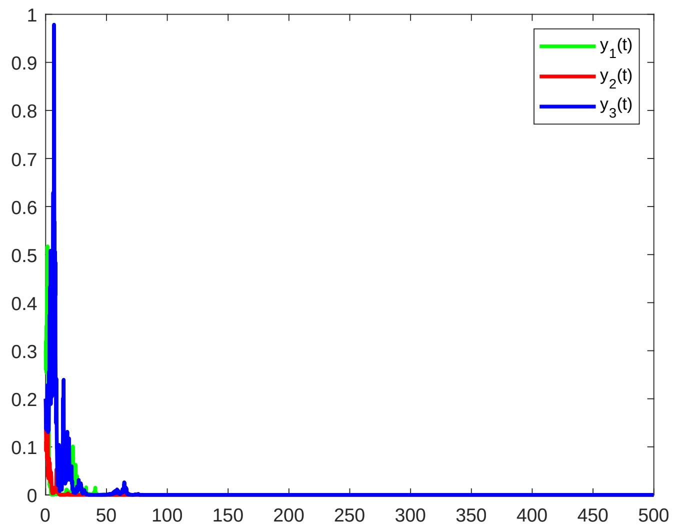

Figure 1. Solutions of system (1.4) for $r_{10} = 0.5, r_{20} = 0.5, r_{30} = 0.3, \tilde{r}_{10} = 0.3, \tilde{r}_{20} = 0.25, \tilde{r}_{30} = 0.2, r_{11} = r_{21} = r_{31} = 0.9, a = 0.25, c_2 = 0.36, c_3 = 0.4, f_2 = 0.5, f_3 = 0.47, h_2 = h_3 = 1, \alpha_1 = 0.31, \alpha_2 = 0.35, \alpha_3 = 0.42, \gamma = 2, k_1 = 0.2, k_2 = 0.26, k_3 = 0.21, l_1 = 0.8, l_2 = 0.7, l_3 = 0.7, h = 0.9, q = 0.5$, and $\xi_1^2 = 0.55, \xi_2^2 = 0.9, \xi_3^2 = 0.5$.

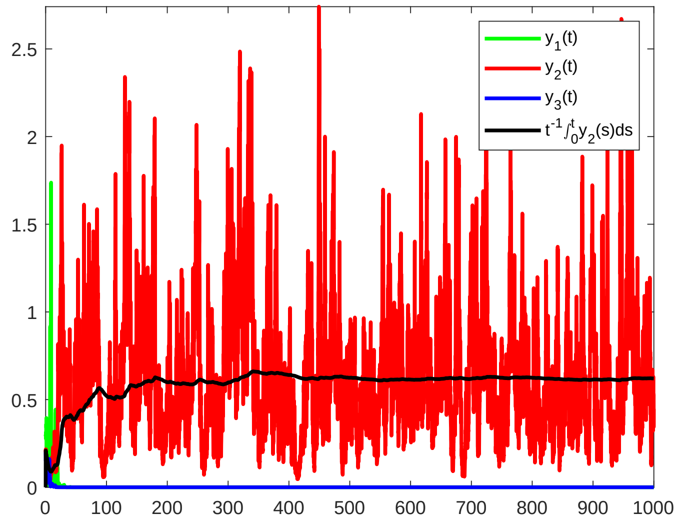

Figure 2. Solutions of system (1.4) for $r_{10} = 0.5, r_{20} = 0.5, r_{30} = 0.3, \tilde{r}_{10} = 0.3, \tilde{r}_{20} = 0.25, \tilde{r}_{30} = 0.2, r_{11} = r_{21} = r_{31} = 0.9, a = 0.25, c_2 = 0.36, c_3 = 0.4, f_2 = 0.5, f_3 = 0.47, h_2 = h_3 = 1, \alpha_1 = 0.31, \alpha_2 = 0.35, \alpha_3 = 0.42, \gamma = 2, k_1 = 0.2, k_2 = 0.26, k_3 = 0.21, l_1 = 0.8, l_2 = 0.7, l_3 = 0.7, h = 0.9, q = 0.5$, and $\xi_1^2 = 0.55, \xi_2^2 = 0.16, \xi_3^2 = 0.5$.

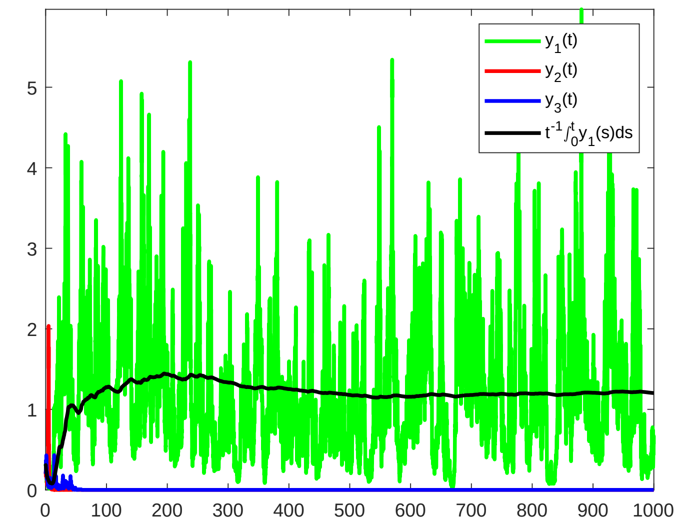

Figure 3. Solutions of system (1.4) for $r_{10} = 0.5, r_{20} = 0.5, r_{30} = 0.3, \tilde{r}_{10} = 0.3, \tilde{r}_{20} = 0.25, \tilde{r}_{30} = 0.2, r_{11} = r_{21} = r_{31} = 0.9, a = 0.25, c_2 = 0.36, c_3 = 0.4, f_2 = 0.5, f_3 = 0.47, h_2 = h_3 = 1, \alpha_1 = 0.31, \alpha_2 = 0.35, \alpha_3 = 0.42, \gamma = 2, k_1 = 0.2, k_2 = 0.26, k_3 = 0.21, l_1 = 0.8, l_2 = 0.7, l_3 = 0.7, h = 0.9, q = 0.5$, and $\xi_1^2 = 0.15, \xi_2^2 = 0.9, \xi_3^2 = 0.5$.

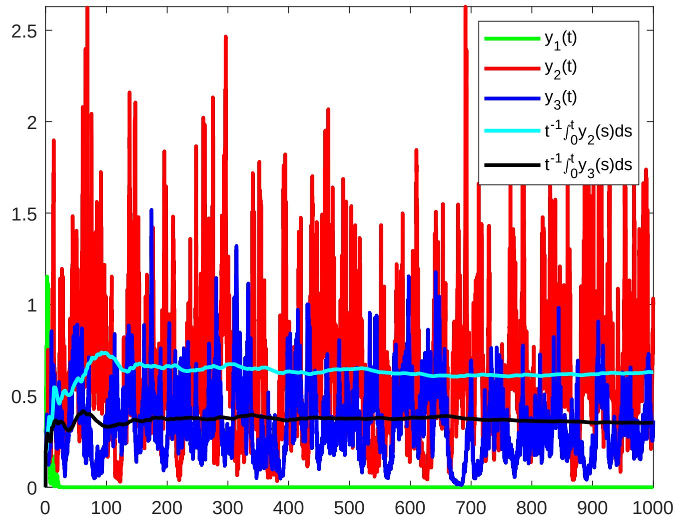

Figure 4. Solutions of system (1.4) for $r_{10} = 0.5, r_{20} = 0.5, r_{30} = 0.3, \tilde{r}_{10} = 0.3, \tilde{r}_{20} = 0.25, \tilde{r}_{30} = 0.2, r_{11} = r_{21} = r_{31} = 0.9, a = 0.25, c_2 = 0.36, c_3 = 0.4, f_2 = 0.5, f_3 = 0.47, h_2 = h_3 = 1, \alpha_1 = 0.31, \alpha_2 = 0.35, \alpha_3 = 0.42, \gamma = 2, k_1 = 0.2, k_2 = 0.26, k_3 = 0.21, l_1 = 0.8, l_2 = 0.7, l_3 = 0.7, h = 0.9, q = 0.5$, and $\xi_1^2 = 0.55, \xi_2^2 = 0.16, \xi_3^2 = 0.09$.

In Figures 5 and 6, we choose $r_{10} = 0.6$, $r_{20} = 0.8$, $r_{30} = 0.3$, $\tilde{r}_{10} = 0.3$, $\tilde{r}_{20} = 0.25$, $\tilde{r}_{30} = 0.2$, $r_{11} = r_{21} = r_{31} = 0.9$, $a = 0.25$, $c_2 = 0.36$, $c_3 = 0.4$, $f_2 = 0.5$, $f_3 = 0.47$, $h_2 = h_3 = 1$, $\alpha_1 = 0.81$, $\alpha_2 = 0.65$, $\alpha_3 = 0.42$, $\gamma = 2$, $k_1 = 0.2$, $k_2 = 0.23$ $k_3 = 0.45$, $l_1 = 0.4$, $l_2 = 0.8$, $l_3 = 0.7$, $h = 0.9$, $q = 0.5$. The only difference between conditions of Figures 5 and 6 is that the values of ξ_1^2, ξ_2^2 and ξ_3^2 are different.

(V) In Figure 5, we choose $\xi_1^2 = 0.5$, $\xi_2^2 = 0.06$ and $\xi_3^2 = 0.5$. Then $\bar{b}_1 = 0.3207 > 0$, $\bar{b}_2 = 0.7050 > 0$, $\bar{b}_3 = -0.1583 < 0$ and $\bar{b}_1 < c_2\bar{b}_2/f_2 = 0.5076$. On the basis of (VI) in Theorem 1, y_1 and y_3 die out and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2\bar{b}_2}{f_2} = 1.4101.$$

See Figure 5.

(VI) In Figure 6, we choose $\xi_1^2 = 0.5$, $\xi_2^2 = 0.86$ and $\xi_3^2 = 0.5$. Then $\bar{b}_1 = 0.3207 > 0$, $\bar{b}_2 = 0.3974 > 0$, $\bar{b}_3 = -0.1583 < 0$ and $\bar{b}_1 > c_2\bar{b}_2/f_2 = 0.2861$. On the basis of (VI) in Theorem 1, y_3 dies out and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{\bar{b}_1}{a} - \frac{c_2\bar{b}_2}{af_2} = 0.1383,$$
$$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2 + y_1(s)}ds = \frac{\bar{b}_2}{f_2} = 0.7947.$$

See Figure 6.

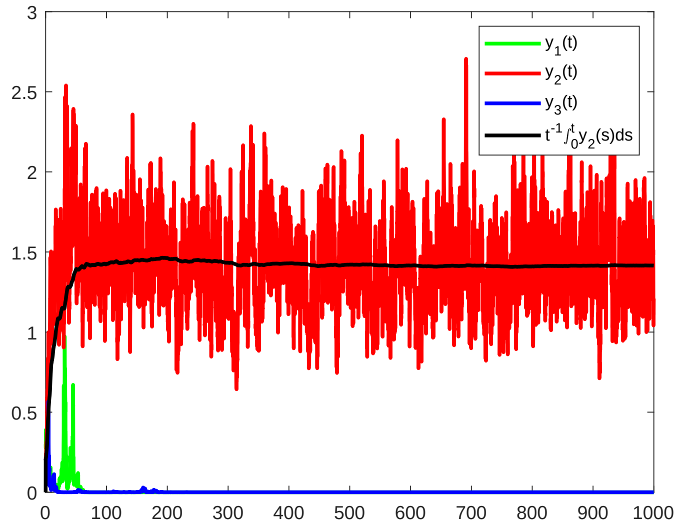

Figure 5. Solutions of system (1.4) for $r_{10} = 0.6, r_{20} = 0.8, r_{30} = 0.3, \tilde{r}_{10} = 0.3, \tilde{r}_{20} = 0.25, \tilde{r}_{30} = 0.2, r_{11} = r_{21} = r_{31} = 0.9, a = 0.25, c_2 = 0.36, c_3 = 0.4, f_2 = 0.5, f_3 = 0.47, h_2 = h_3 = 1, \alpha_1 = 0.81, \alpha_2 = 0.65, \alpha_3 = 0.42, \gamma = 2, k_1 = 0.2, k_2 = 0.23, k_3 = 0.45, l_1 = 0.4, l_2 = 0.8, l_3 = 0.7, h = 0.9, q = 0.5$, and $\xi_1^2 = 0.5, \xi_2^2 = 0.06, \xi_3^2 = 0.5$.

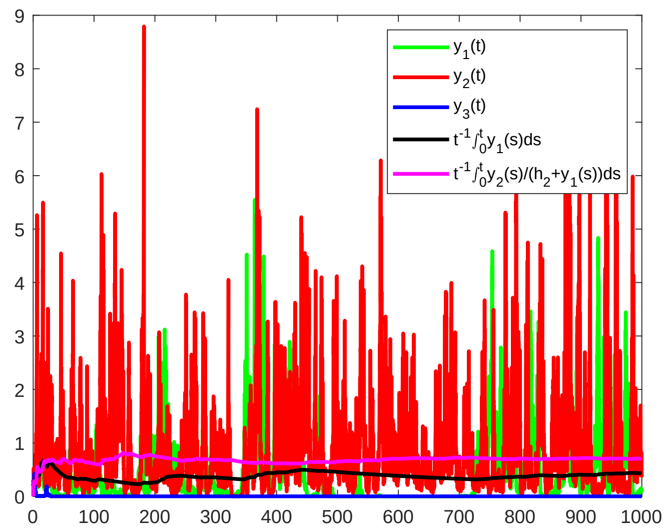

Figure 6. Solutions of system (1.4) for $r_{10} = 0.6, r_{20} = 0.8, r_{30} = 0.3, \tilde{r}_{10} = 0.3, \tilde{r}_{20} = 0.25, \tilde{r}_{30} = 0.2, r_{11} = r_{21} = r_{31} = 0.9, a = 0.25, c_2 = 0.36, c_3 = 0.4, f_2 = 0.5, f_3 = 0.47, h_2 = h_3 = 1, \alpha_1 = 0.81, \alpha_2 = 0.65, \alpha_3 = 0.42, \gamma = 2, k_1 = 0.2, k_2 = 0.23, k_3 = 0.45, l_1 = 0.4, l_2 = 0.8, l_3 = 0.7, h = 0.9, q = 0.5$, and $\xi_1^2 = 0.5, \xi_2^2 = 0.86, \xi_3^2 = 0.5$.

In Figures 7 and 8, we choose $r_{10} = 0.6$, $r_{20} = 0.6$, $r_{30} = 0.4$, $\tilde{r}_{10} = 0.3$, $\tilde{r}_{20} = 0.15$, $\tilde{r}_{30} = 0.3$,

$r_{11} = r_{21} = r_{31} = 0.1$, $a = 0.25$, $c_2 = 0.36$, $c_3 = 0.4$, $f_2 = 0.75$, $f_3 = 0.77$, $h_2 = h_3 = 1$, $\alpha_1 = 0.61$, $\alpha_2 = 0.85$, $\alpha_3 = 0.89$, $\gamma = 2$, $k_1 = 0.32$, $k_2 = 0.05$ $k_3 = 0.25$, $l_1 = 0.9$, $l_2 = 0.7$, $l_3 = 0.8$, $h = 0.9$, $q = 0.6$. The only difference between conditions of Figures 7 and 8 is that the values of ξ_1^2, ξ_2^2 and ξ_3^2 are different.

(VII) In Figure 7, we choose $\xi_1^2 = 0.5$, $\xi_2^2 = 0.006$ and $\xi_3^2 = 0.009$. Then $\bar{b}_1 = 0.3832 > 0$, $\bar{b}_2 = 0.5959 > 0$, $\bar{b}_3 = 0.3871 > 0$ and $\bar{b}_1 < c_2\bar{b}_2/f_2 + c_3\bar{b}_3/f_3 = 0.4871$. On the basis of (VIII) in Theorem 1, y_1 dies out and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2\bar{b}_2}{f_2} = 0.7945,$$
$$\lim_{t\to+\infty} t^{-1}\int_0^t y_3(s)ds = \frac{h_3\bar{b}_3}{f_3} = 0.5027.$$

See Figure 7.

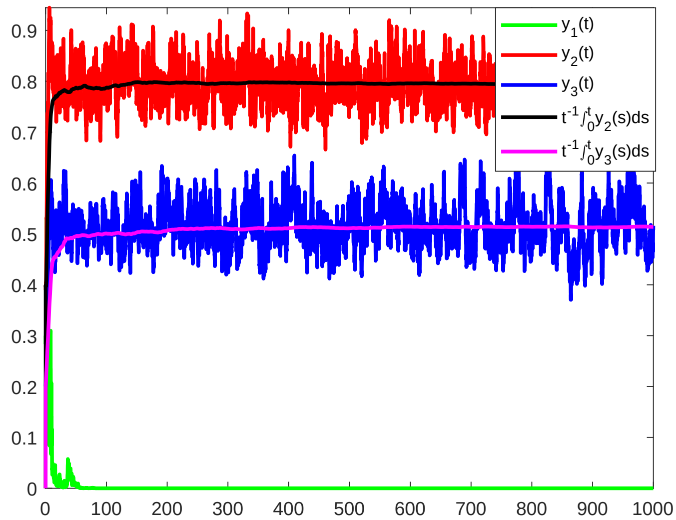

Figure 7. Solutions of system (1.4) for $r_{10} = 0.6, r_{20} = 0.6, r_{30} = 0.4, \tilde{r}_{10} = 0.3, \tilde{r}_{20} = 0.15, \tilde{r}_{30} = 0.3, r_{11} = r_{21} = r_{31} = 0.1, a = 0.25, c_2 = 0.36, c_3 = 0.4, f_2 = 0.75, f_3 = 0.77, h_2 = h_3 = 1, \alpha_1 = 0.61, \alpha_2 = 0.85, \alpha_3 = 0.89, \gamma = 2, k_1 = 0.32, k_2 = 0.05, k_3 = 0.25, l_1 = 0.9, l_2 = 0.7, l_3 = 0.8, h = 0.9, q = 0.6$, and $\xi_1^2 = 0.5, \xi_2^2 = 0.006, \xi_3^2 = 0.009$.

(VIII) In Figure 8, we choose $\xi_1^2 = 0.0002$, $\xi_2^2 = 0.006$ and $\xi_3^2 = 0.009$. Then $\bar{b}_1 = 0.5881 > 0$, $\bar{b}_2 = 0.5959 > 0$, $\bar{b}_3 = 0.3871 > 0$ and $\bar{b}_1 > c_2\bar{b}_2/f_2 + c_3\bar{b}_3/f_3 = 0.4871$. On the basis of (VIII) in Theorem 1, we have

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{\bar{b}_1}{a} - \frac{c_2\bar{b}_2}{af_2} - \frac{c_3\bar{b}_3}{af_3} = 0.4040,$$
$$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2 + y_1(s)}ds = \frac{\bar{b}_2}{f_2} = 0.7945,$$

$$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_3(s)}{h_3+y_1(s)}ds = \frac{\bar{b}_3}{f_3} = 0.5027.$$

See Figure 8.

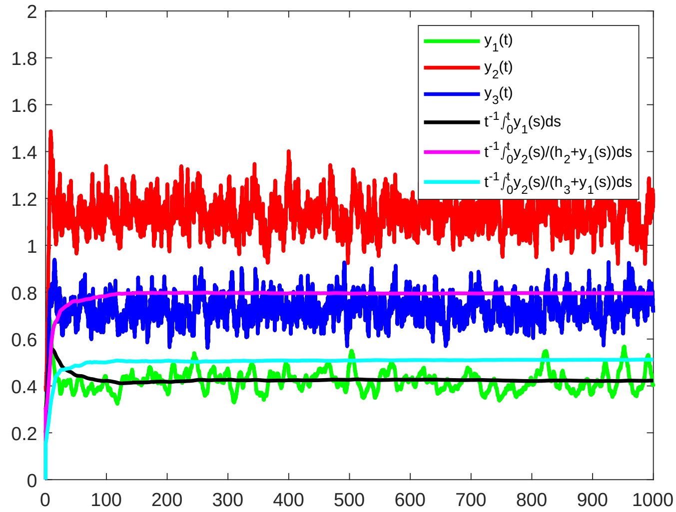

Figure 8. Solutions of system (1.4) for $r_{10} = 0.6, r_{20} = 0.6, r_{30} = 0.4, \tilde{r}_{10} = 0.3, \tilde{r}_{20} = 0.15, \tilde{r}_{30} = 0.3, r_{11} = r_{21} = r_{31} = 0.1, a = 0.25, c_2 = 0.36, c_3 = 0.4, f_2 = 0.75, f_3 = 0.77, h_2 = h_3 = 1, \alpha_1 = 0.61, \alpha_2 = 0.85, \alpha_3 = 0.89, \gamma = 2, k_1 = 0.32, k_2 = 0.05, k_3 = 0.25, l_1 = 0.9, l_2 = 0.7, l_3 = 0.8, h = 0.9, q = 0.6$, and $\xi_1^2 = 0.0002, \xi_2^2 = 0.006, \xi_3^2 = 0.009$.

In Figures 9 and 10, we choose $r_{10} = 0.2$, $r_{20} = 0.1$, $r_{30} = 0.2$, $\tilde{r}_{10} = 0.1$, $\tilde{r}_{20} = 0.05$, $\tilde{r}_{30} = 0.13$, $r_{11} = 2$, $r_{21} = 1$, $r_{31} = 2.6$, $a = 0.5$, $c_2 = 0.2$, $c_3 = 0.2$, $f_2 = 0.95$, $f_3 = 0.97$, $h_2 = h_3 = 1$, $\alpha_1 = 0.81$, $\alpha_2 = 0.86$, $\alpha_3 = 0.8$, $k_1 = 0.11$, $k_2 = 0.12$ $k_3 = 0.125$, $l_1 = 0.6$, $l_2 = 0.7$, $l_3 = 0.7$, $h = 0.9$, $q = 0.12$, $\xi_1^2 = 0.02$, $\xi_2^2 = 0.006$, $\xi_3^2 = 0.009$. The only difference between conditions of Figures 9 and 10 is that the values of γ are different.

(IX) In Figure 9, we choose $\gamma = 4$. Then $\bar{b}_1 = 0.1816 > 0$, $\bar{b}_2 = 0.0925 > 0$, $\bar{b}_3 = 0.1817 > 0$ and $\bar{b}_1 > c_2\bar{b}_2/f_2 + c_3\bar{b}_3/f_3 = 0.0569$. On the basis of (VIII) in Theorem 1, we have

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{\bar{b}_1}{a} - \frac{c_2\bar{b}_2}{af_2} - \frac{c_3\bar{b}_3}{af_3} = 0.2493,$$

$$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2+y_1(s)}ds = \frac{\bar{b}_2}{f_2} = 0.0974,$$

$$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_3(s)}{h_3+y_1(s)}ds = \frac{\bar{b}_3}{f_3} = 0.1873.$$

See Figure 9.

In Figure 10, choose $\gamma = 0.23$. Then $\bar{b}_1 = -0.0187 < 0$, $\bar{b}_2 = -0.0011 < 0$ and $\bar{b}_3 = -0.0720 < 0$. On the basis of (I) in Theorem 1, y_1, y_2 and y_3 die out.

See Figure 10.

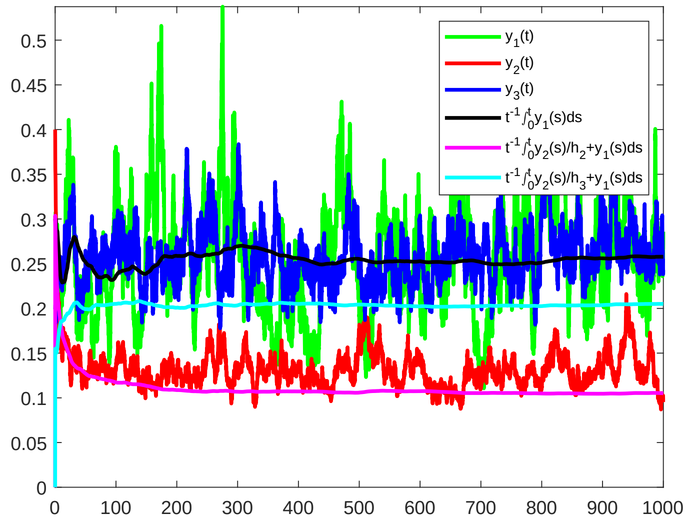

Figure 9. Solutions of system (1.4) for $r_{10} = 0.2, r_{20} = 0.1, r_{30} = 0.2, \tilde{r}_{10} = 0.1, \tilde{r}_{20} = 0.05, \tilde{r}_{30} = 0.13, r_{11} = 2, r_{21} = 1, r_{31} = 2.6, a = 0.5, c_2 = 0.2, c_3 = 0.2, f_2 = 0.95, f_3 = 0.97, h_2 = h_3 = 1, \alpha_1 = 0.81, \alpha_2 = 0.86, \alpha_3 = 0.8, k_1 = 0.11, k_2 = 0.12, k_3 = 0.125, l_1 = 0.6, l_2 = 0.7, l_3 = 0.7, h = 0.9, q = 0.12, \xi_1^2 = 0.02, \xi_2^2 = 0.006, \xi_3^2 = 0.009$ and $\gamma = 4$.

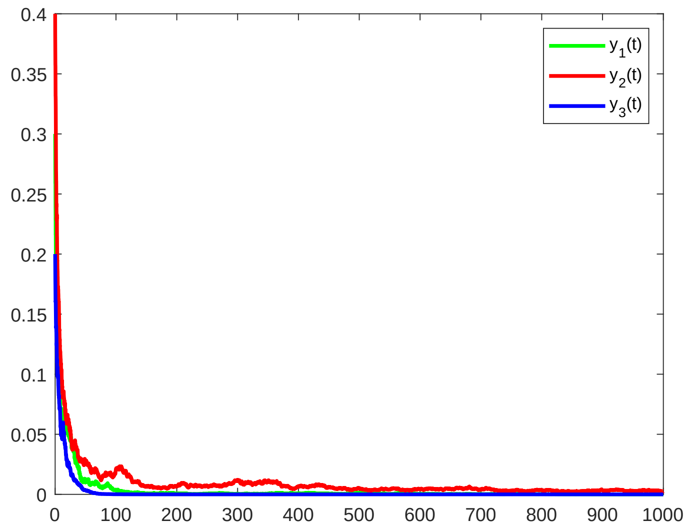

Figure 10. Solutions of system (1.4) for $r_{10} = 0.2, r_{20} = 0.1, r_{30} = 0.2, \tilde{r}_{10} = 0.1, \tilde{r}_{20} = 0.05, \tilde{r}_{30} = 0.13, r_{11} = 2, r_{21} = 1, r_{31} = 2.6, a = 0.5, c_2 = 0.2, c_3 = 0.2, f_2 = 0.95, f_3 = 0.97, h_2 = h_3 = 1, \alpha_1 = 0.81, \alpha_2 = 0.86, \alpha_3 = 0.8, k_1 = 0.11, k_2 = 0.12, k_3 = 0.125, l_1 = 0.6, l_2 = 0.7, l_3 = 0.7, h = 0.9, q = 0.4, \xi_1^2 = 0.02, \xi_2^2 = 0.006, \xi_3^2 = 0.009$ and $\gamma = 0.23$.

By comparing Figure 9 with Figure 10, one can observe that with the decrease of toxicant impulsive period γ, species tends to die out.

4. Conclusions

In this paper, we take advantage of a mean-reverting Ornstein-Uhlenbeck process to portray the random perturbations in the environment and assume that the toxicants are released in regular pulses. Based on the classical deterministic predator-prey model with modified Leslie-Gower Holling-type II schemes, we present a three-species predator prey stochastic model with modified Leslie-Gower Holling-type II schemes, and use more appropriate methods to describe random perturbations in the environment. We obtain sharp sufficient conditions for persistence in the mean and extinction for each species of model (1.4).

Theorem 1 has some interesting biological interpretations. By Theorem 1, we can see that each species is either extinct or persistent in the mean, relying on the sign of $\bar{b}_i (i = 1, 2, 3)$, $\bar{b}_1 f_2 - c_2\bar{b}_2$, $\bar{b}_1 f_3 - c_3\bar{b}_3$, and $f_2 f_3\bar{b}_1 - c_2 f_3\bar{b}_2 - c_3 f_2\bar{b}_3$.

We note that the intensity of the perturbation ξ_i^2 and the speed of reversion α_i are two key parameters in the Ornstein-Uhlenbeck process. Obviously,

$$
\begin{aligned}
&\frac{d\bar{b}_i}{d\alpha_i} > 0, \quad \frac{d(\bar{b}_1 f_2 - c_2\bar{b}_2)}{d\alpha_1} > 0, \quad \frac{d(\bar{b}_1 f_2 f_3 - c_2 f_3\bar{b}_2 - c_3 f_2\bar{b}_3)}{d\alpha_1} > 0,\\
&\frac{d(\bar{b}_1 f_3 - c_3\bar{b}_3)}{d\alpha_1} > 0, \quad \frac{d(\bar{b}_1 f_2 - c_2\bar{b}_2)}{d\alpha_2} < 0, \quad \frac{d(\bar{b}_1 f_2 f_3 - c_2 f_3\bar{b}_2 - c_3 f_2\bar{b}_3)}{d\alpha_2} < 0,\\
&\frac{d(\bar{b}_1 f_3 - c_3\bar{b}_3)}{d\alpha_3} < 0, \quad \frac{d(\bar{b}_1 f_2 f_3 - c_2 f_3\bar{b}_2 - c_3 f_2\bar{b}_3)}{d\alpha_3} < 0, \quad \frac{d\bar{b}_i}{d\xi_i^2} < 0,\\
&\frac{d(\bar{b}_1 f_2 - c_2\bar{b}_2)}{d\xi_1^2} < 0, \quad \frac{d(\bar{b}_1 f_3 - c_3\bar{b}_3)}{d\xi_1^2} < 0, \quad \frac{d(\bar{b}_1 f_2 f_3 - c_2 f_3\bar{b}_2 - c_3 f_2\bar{b}_3)}{d\xi_1^2} < 0,\\
&\frac{d(\bar{b}_1 f_2 - c_2\bar{b}_2)}{d\xi_2^2} > 0, \quad \frac{d(\bar{b}_1 f_2 f_3 - c_2 f_3\bar{b}_2 - c_3 f_2\bar{b}_3)}{d\xi_2^2} > 0,\\
&\frac{d(\bar{b}_1 f_3 - c_3\bar{b}_3)}{d\xi_3^2} > 0, \quad \frac{d(\bar{b}_1 f_2 f_3 - c_2 f_3\bar{b}_2 - c_3 f_2\bar{b}_3)}{d\xi_3^2} > 0.
\end{aligned}
$$

Therefore, with the increase of α_i (respectively, ξ_i^2), species y_i tends to be persistent (respectively, extinct), $i = 1, 2, 3$. Furthermore, with the increase of α_2 or α_3 (respectively, ξ_2^2 or ξ_3^2), the prey population y_1 tends to die out (respectively, be persistent) provided $\bar{b}_i > 0, i = 1, 2, 3$.

Some interesting topics remain to be solved. For example, it would be interesting to dissect other random noises such as the telephone noise (see [37]), the Lévy noise (see [38]) or reaction diffusion (see [39]) etc. We leave these questions for future research.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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