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Research article

Power-series solution of compartmental epidemiological models

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Abstract: In this work, power-series solutions of compartmental epidemiological models are used to provide alternate methods to solve the corresponding systems of nonlinear differential equations. A simple and classical SIR compartmental model is considered to reveal clearly the idea of our approach. Moreover, a SAIRP compartmental model is also analyzed by using the same methodology, previously applied to the COVID-19 pandemic. Numerical experiments are performed to show the accuracy of this approach.

Keywords: compartmental epidemiological model; power-series solution; COVID-19; numerical approximation

1. Introduction

Compartmental models have been intensively used to analyze and predict the evolution of diseases and pandemic. The models may vary from the simplest and very classical SIR model to more complex proposals [1]. The idea of these compartmental models is to divide the total population N into several epidemiological classes. In the simplest case, the population is divided in Susceptible, Infected and

Recovered, giving rise to the SIR model analyzed by Kermack and McKendrick [2]. It is also possible to consider one extra class of Exposed individuals, if there exists incubation period during which individuals have been infected but are not yet infectious. Hence, the SEIR model is considered.

Several numerical methods and schemes can be used to find approximate solutions to the system of nonlinear ordinary differential equations. Recently [3] a semi-analytical solution of the model, both in the SIR and SEIR cases. Some improvements have been presented in [4].

These ideas have been used, e.g., to anticipate the number of necessary resources at intensive care units during the COVID-19 pandemic [5], by using compartmental mathematical models [6, 7], which also consider fractional derivatives as (for example) in [8–11]. Many other works have considered compartmental models to analyze the spread of the pandemic of COVID-19. In [12] the authors considered a general adapted time-window based on a SIR prediction model, changing the infection and recovery rates as the pandemic spreads. There are many other works based on the SIR model as [13, 14], in SEIR models [15–17], or using statistical tools as [18–21], and references therein. Compartmental models have been used to analyze other diseases as, for example, in [22] where the authors introduce a novel hybrid compartmental model of the dengue transmission process with memory and relapse between host-to-vector and vice versa.

In this work we explore another approach to solve mathematical compartmental models, based on the power-series expansion of the solution of the differential system. In doing so, two different approaches are considered for the SIR model, one based on the initial system and another based on the approach given in [3,4], giving rise to the same power-series solution. We use Mathematica [23] to perform some numerical simulations. Finally, we apply the main ideas to a recent compartmental model used to analyze the spread of the COVID-19 pandemic along with some numerical computations.

The structure of the paper is as follows. In section 2, we recall the SIR model as well as the semi-analytical solution of the model. In section 3, the power-series solution is used to obtain a representation of the solution of the SIR model, including numerical experiments. Another differential equation is used in section 4 to generate a power-series solution, giving rise to numerical results. Finally, in section 5, the power-series solution of a SAIRP model of COVID-19 is analyzed.

2. SIR model

Consider a simple compartmental model of Susceptible, Infected, Recovered (SIR) type

$$\begin{cases} S'(t) = -\beta S(t)I(t), \\ I'(t) = \beta S(t)I(t) - \gamma I(t), \\ R'(t) = \gamma I(t), \end{cases} \tag{2.1}$$

representing the evolution of the epidemic. Here, as usual, we have considered three compartments for the population, namely: $S(t)$ denotes the population of susceptible individuals to an infectious, but not deadly, disease at time t; $I(t)$ stands for the population of infected invididuals at time t; and $R(t)$ is used to represent the recovered individuals at time t. We have denoted with $\beta > 0$ the rate at which an infected person infects a susceptible person, and $\gamma > 0$ the rate at which infected people recover from the disease.

The differential system (2.1) can be numerically solved for any given initial conditions $S(0)$, $I(0)$ and $R(0)$, as well values of the parameters β and γ. Let us solve the system of differential equations

(2.1) for the following values of the parameters and values of the initial conditions

$$\beta = 1/2, \quad \gamma = 1/3, \quad S_0 = 0.99, \quad I_0 = 1/100, \quad R_0 = 0. \tag{2.2}$$

In doing so, we use Mathematica [23] which is using Newton's method in order to produce Figure 1.

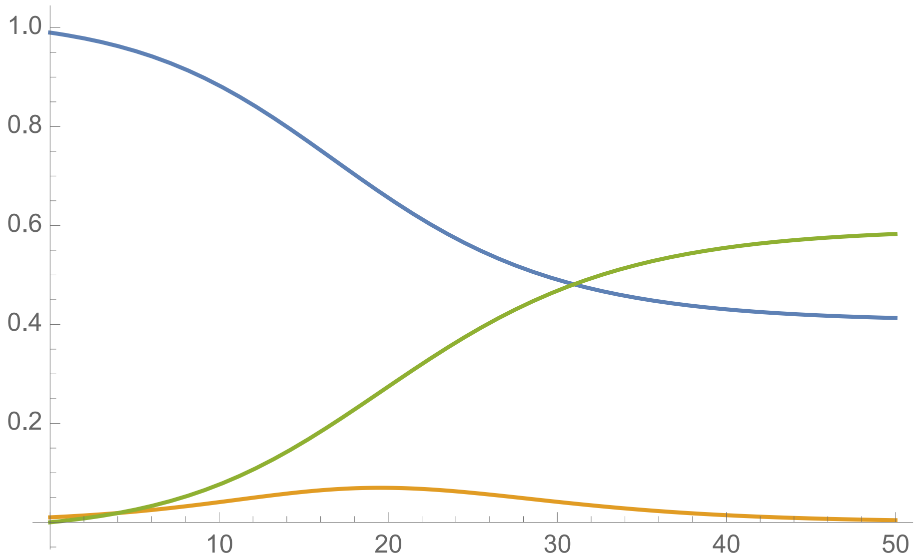

Figure 1. Numerical solution of model (2.1) for the parameters and initial conditions given in (2.2), by using Mathematica [23] by Newton's method. The evolution of $S(t)$ is represented in blue color; $I(t)$ in orange color, and $R(t)$ in green color.

We now recall the semi-analytical solution of the model (2.1). We follow the technique of [3] where a semi-analytical solution of the SIR model (2.1) has been given. Let us recall the method, combined with some improvements given in [4].

First of all we note that $S(t) + I(t) + R(t)$ is constant for any $t \geq 0$ since

$$S'(t) + I'(t) + R'(t) = 0.$$

We take such a constant equal to $N > 0$. Consider as initial conditions $S(0) \geq 0$, $I(0) \geq 0$ and $R(0) \geq 0$. Observe that $S(t)$ is decreasing since $S'(t) \leq 0$ and $R(t)$ is increasing since $R'(t) \geq 0$. Let us compute the derivative of the first equation of (2.1) to obtain

$$S''(t) = -\beta[S'(t)I(t) + S(t)I'(t)] \tag{2.3}$$

which implies

$$I'(t) = -\frac{1}{\beta S(t)}\left[S''(t) + \beta S'(t)I(t)\right]. \tag{2.4}$$

If we substitute $I(t)$ by using again the first equation of the model (2.1) it yields

$$I'(t) = -\frac{1}{\beta S(t)}\left[S''(t) + \beta S'(t)\frac{S'(t)}{-\beta S(t)}\right] = -\frac{1}{\beta}\left[\frac{S''(t)}{S(t)} - \left(\frac{S'(t)}{S(t)}\right)^2\right]. \tag{2.5}$$

Then, from the first equation of (2.1), (2.5) and the second equation of (2.1) we have

$$-\frac{1}{\beta}\left[\frac{S''(t)}{S(t)}-\left(\frac{S'(t)}{S(t)}\right)^2\right]=\beta S(t)\left(\frac{S'(t)}{-\beta S(t)}\right)-\gamma\left(\frac{S'(t)}{-\beta S(t)}\right), \tag{2.6}$$

or equivalently

$$\frac{S''(t)}{S(t)}-\left(\frac{S'(t)}{S(t)}\right)^2+\gamma\frac{S'(t)}{S(t)}-\beta S'(t)=0. \tag{2.7}$$

Let us consider the change of variables [4]

$$S'(t)=\frac{1}{\phi(t)}. \tag{2.8}$$

Since

$$\phi'=\frac{d\phi}{dt}=\frac{d\phi}{dS}\frac{dS}{dt}=\frac{d\phi}{dS}\frac{1}{\phi},$$

we have

$$S''(t)=-\frac{\phi'}{\phi^2}=-\frac{1}{\phi^3}\frac{d\phi}{dS}.$$

Thus, (2.7) can be rewritten as the following Bernoulli ordinary differential equation

$$\frac{d\phi}{dS}+\frac{\phi}{S}+(\beta S-\gamma)S^2=0.$$

Applying the usual substitution $u=1/\phi$, that is S', the latter Bernoulli differential equation can be solved as

$$\phi(S)=\frac{1}{S(\beta S-\gamma\ln S+c)}.$$

In order to determine the constant c, we have

$$\phi(S_0)=\frac{1}{S'(0)}=\frac{1}{S_0(\beta S_0-\gamma\ln S_0+c)}$$

so that

$$\beta S_0-\gamma\ln S_0+c=\frac{S'(0)}{S(0)}=-\beta I(0),$$

which implies

$$c=-\beta I_0+\beta S_0-\gamma\ln S_0.$$

Hence,

$$\phi(S)=\frac{1}{S(\beta S-\gamma\ln S+c)}=\frac{1}{S(\beta(S-I_0+S_0)-\gamma\ln\frac{S}{S_0})}.$$

Noting that

$$\frac{1}{S'(t)}=\phi(S(t))=\frac{1}{S(t)(\beta(S(t)-I_0+S_0)-\gamma\ln\frac{S(t)}{S_0})},$$

we obtain

$$\int_{t_0}^{t}1dt=\int_{t_0}^{t}\frac{S'(t)}{S(t)(\beta(S(t)-I_0+S_0)-\gamma\ln\frac{S(t)}{S_0})}dt,$$

and

$$t - t_0 = \int_{S_0}^{S} \frac{ds}{s(\beta(s - I_0 + S_0) - \gamma \ln \frac{s}{S_0})}.$$

For $t_0 = 0$,

$$t = \int_{S_0}^{S} \frac{ds}{s(\beta(s - I_0 + S_0) - \gamma \ln \frac{s}{S_0})}.$$

The peak of $I(t)$ occurs when $I' = 0$, i.e. $S = \gamma/\beta$, and then by using the first equation of (2.1) the time when the number of infected population attains its peak is given by the following integral

$$t_{\text{peak}} = \int_{S_0}^{\gamma/\beta} \frac{ds}{s(\beta(s - I_0 + S_0) - \gamma \ln \frac{s}{S_0})}.$$

The time when I attains its maximum is of crucial importance as we all know with the recent COVID-19 pandemic.

On the other hand, from the first and third equations of (2.1) we can eliminate $I(t)$ so that

$$R'(t) = -\frac{\gamma}{\beta}\left(\frac{S'(t)}{S(t)}\right). \tag{2.9}$$

Integrate the latter equation to obtain

$$S(t) = \alpha_0 \exp\left(-\frac{\beta}{\gamma}R(t)\right), \tag{2.10}$$

where α_0 is a positive constant. If we substitute $t = 0$ in (2.10), we have

$$S_0 = \alpha_0 \exp\left(-\frac{\beta}{\gamma}R_0\right), \quad S(0) = S_0, \quad R(0) = R_0,$$

or equivalently

$$\alpha_0 = S_0 \exp\left(\frac{\beta}{\gamma}R_0\right). \tag{2.11}$$

Let us now compute the derivate with respect to t of (2.10),

$$S'(t) = -\frac{\alpha_0\beta}{\gamma} \exp\left(-\frac{\beta}{\gamma}R(t)\right)R'(t), \tag{2.12}$$

and the derivative with respect to t of (2.9) yields

$$R''(t) = -\frac{\gamma}{\beta}\left(\frac{S''(t)S(t) - (S'(t))^2}{(S(t))^2}\right). \tag{2.13}$$

If we substitute (2.9), (2.12) and (2.13), into (2.7) we obtain a second order differential equation for $R(t)$

$$R''(t) = S_0\beta \exp\left(-\frac{\beta}{\gamma}R(t)\right)R'(t) - \gamma R'(t). \tag{2.14}$$

Let us now introduce

$$u(t) = \exp\left(-\frac{\beta}{\gamma}R(t)\right), \tag{2.15}$$

which at $t = 0$ gives

$$u(0) := u_0 = \exp\left(-\frac{\beta}{\gamma}R_0\right). \tag{2.16}$$

If we substitute (2.15) into (2.14) we obtain

$$u(t)u''(t) - [u'(t)]^2 + [\gamma - S_0\beta u(t)]u(t)u'(t) = 0. \tag{2.17}$$

If one is able to solve this nonlinear ordinary differential equation with the initial condition (2.9), then

$$S(t) = S_0 u(t). \tag{2.18}$$

From

$$R'(t) = \gamma I(t) = \frac{\gamma}{\beta}\frac{S'(t)}{S(t)}$$

it yields

$$R(t) = -\frac{\gamma}{\beta}\ln u(t), \tag{2.19}$$

and finally

$$I(t) = N - S_0 u(t) + \frac{\gamma}{\beta}\ln u(t), \quad I(0) = I_0. \tag{2.20}$$

3. Direct power-series solution

Let us assume that (2.1) has solution of the form

$$S(t) = \sum_{n=0}^{\infty} s_n y^n, \quad I(t) = \sum_{n=0}^{\infty} i_n t^n, \quad R(t) = \sum_{n=0}^{\infty} r_n t^n, \tag{3.1}$$

where $s_0 = S(0)$, $i_0 = I_0$, and $r_0 = R_0$. Then,

$$S'(t) = \sum_{n=0}^{\infty} s_{n+1}(n+1)t^n, \quad I'(t) = \sum_{n=0}^{\infty} i_{n+1}(n+1)t^n, \quad R'(t) = \sum_{n=0}^{\infty} r_{n+1}(n+1)t^n. \tag{3.2}$$

Therefore, from (2.1) we have the following recurrence relations for the coefficients

$$s_{n+1} = -\frac{\beta}{n+1}\sum_{k=0}^{n} s_k i_{n-k}, \quad i_{n+1} = \frac{1}{n+1}\left[\beta\sum_{k=0}^{n} s_k i_{n-k} - \gamma i_n\right], \quad r_{n+1} = \frac{\gamma i_n}{n+1}. \tag{3.3}$$

Then, we obtain the following first coefficients

$$\begin{aligned} s_1 &= -\beta i_0 s_0, \\ i_1 &= i_0(\beta s_0 - \gamma), \\ r_1 &= \gamma i_0, \end{aligned}$$

$$s_2 = \frac{1}{2}\beta i_0 s_0 \left(\beta i_0 - \beta s_0 + \gamma\right),$$
$$i_2 = \frac{1}{2} i_0 \left(\beta s_0 \left(-\beta i_0 + \beta s_0 - 2\gamma\right) + \gamma^2\right),$$
$$r_2 = -\frac{1}{2}\gamma i_0 \left(\gamma - \beta s_0\right),$$
$$s_3 = -\frac{1}{6}\beta i_0 s_0 \left(\beta^2 i_0^2 + \beta i_0 \left(3\gamma - 4\beta s_0\right) + \left(\gamma - \beta s_0\right)^2\right),$$
$$i_3 = \frac{1}{6} i_0 \left(\beta s_0 \left(\beta \left(4 i_0 \left(\gamma - \beta s_0\right) + s_0 \left(\beta s_0 - 3\gamma\right) + \beta i_0^2\right) + 3\gamma^2\right) - \gamma^3\right),$$
$$r_3 = \frac{1}{6}\gamma i_0 \left(\beta s_0 \left(-\beta i_0 + \beta s_0 - 2\gamma\right) + \gamma^2\right).$$

In Figures 2–4 a comparison between Mathematica's solution and power-series solution is presented, considering the same initial conditions as well as the same values of the parameters given in (2.2).

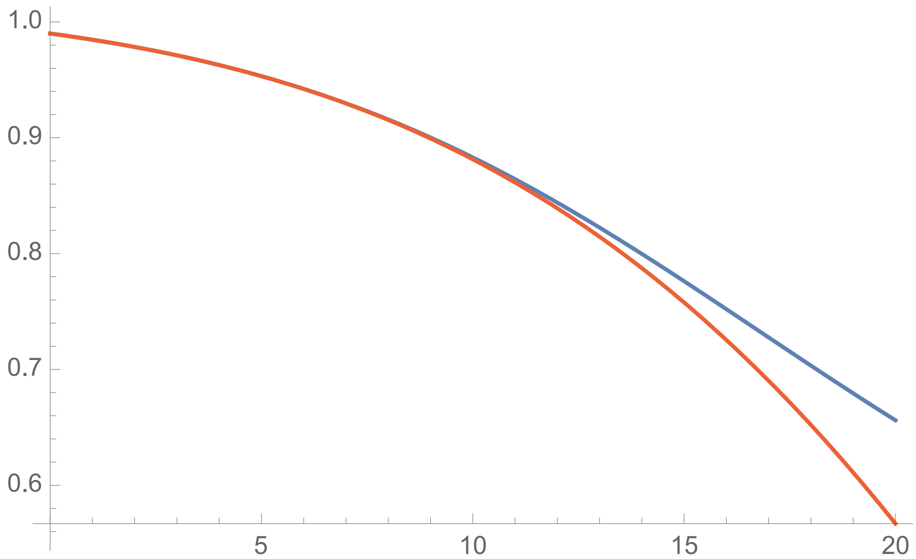

Figure 2. Comparison between the numerical solution of (2.1) for parameters and initial conditions given in (2.2), by using Mathematica which is using Newton's method and the power series (3.1). The evolution of $S(t)$ given by solving the system of differential equations is represented in blue color; the solution given by the power series truncating at degree 5 in red color.

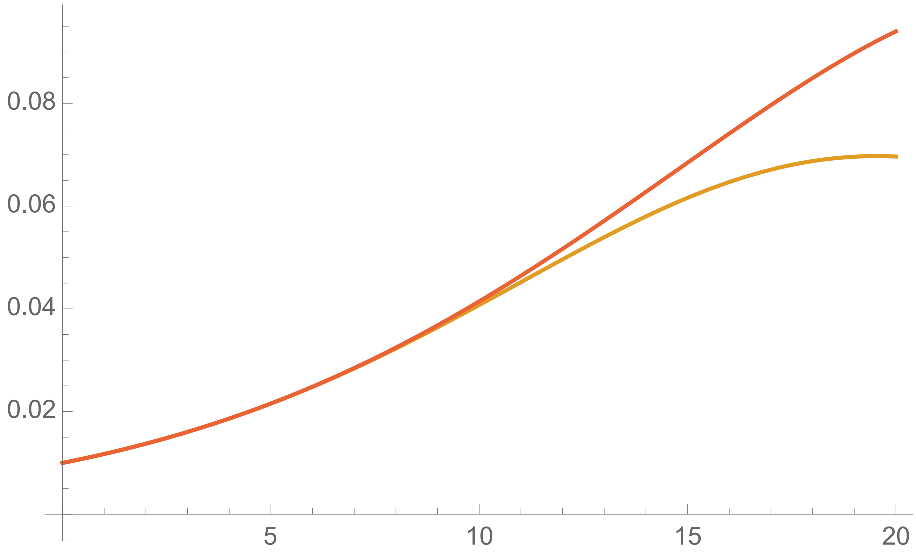

Figure 3. Comparison between the numerical solution of (2.1) for parameters and initial conditions given in (2.2), by using Mathematica which is using Newton's method and the power series (3.1). The evolution of $I(t)$ given by solving the system of differential equations is represented in orange color; the solution given by the power series truncating at degree 5 in red color.

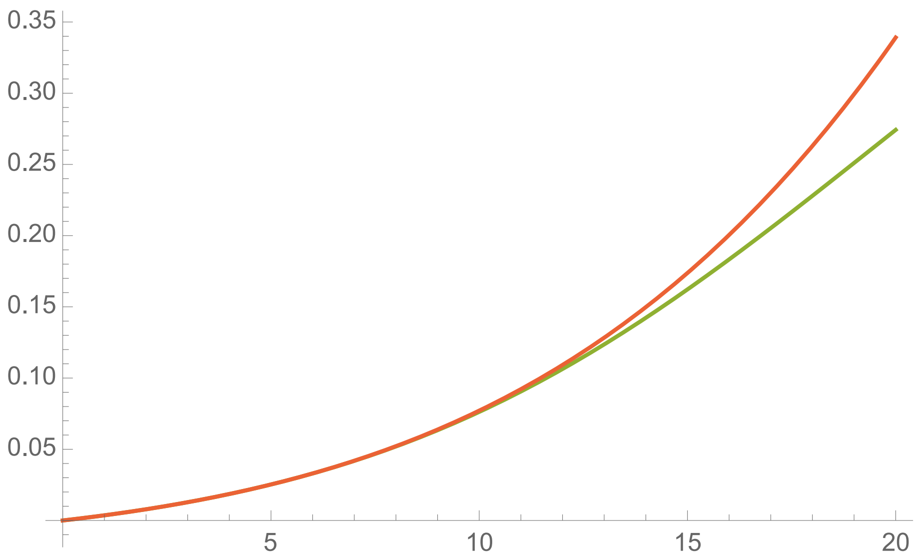

Figure 4. Comparison between the numerical solution of (2.1) for parameters and initial conditions given in (2.2), by using Mathematica which is using Newton's method and the power series (3.1). The evolution of $R(t)$ given by solving the system of differential equations is represented in green color; the solution given by the power series truncating at degree 5 in red color.

Let us prove that the solution of the SIR model is analytical. First of all, we consider

$$S(t) = s_0 + \sum_{n=1}^{\infty} s_n t^n = s_0 + \sum_{n=0}^{\infty} s_{n+1} t^{n+1} = s_0 + t \sum_{n=0}^{\infty} \left(-\frac{\beta}{n+1} \sum_{k=0}^{n} s_k i_{n-k} \right) t^n.$$

We have

$$\left| -\frac{\beta}{n+1} \sum_{k=0}^{n} s_k i_{n-k} \right| \leq \frac{\beta}{n+1} \sum_{k=0}^{\infty} |s_k|\,|i_{n-k}|.$$

Thus, we consider the following power series:

$$\tilde{S}(t) = \sum_{n=0}^{\infty} |s_n| t^n, \quad \tilde{I}(t) = \sum_{n=0}^{\infty} |i_n| t^n, \quad \tilde{R}(t) = \sum_{n=0}^{\infty} |r_n| t^n,$$

$$\tilde{S}(t) = s_0 + t\beta \tilde{S}(t)\tilde{I}(t), \quad \tilde{I}(t) = i_0 + t\beta \tilde{S}(t)\tilde{I}(t) + \gamma \tilde{I}(t), \quad \tilde{R}(t) = r_0 + \gamma \tilde{I}(t).$$

Define

$$\Phi(t, x, y, z) = (x - s_0 - t\beta xy, y - i_0 - t\beta xy - t\gamma y, z - r_0 - t\gamma y),$$

so that

$$J\Phi(t, x, y, z) = \begin{vmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_1}{\partial z} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_2}{\partial z} \\ \frac{\partial \phi_3}{\partial x} & \frac{\partial \phi_3}{\partial y} & \frac{\partial \phi_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 - t\beta y & -t\beta x & 0 \\ -t\beta y & 1 - t\beta x - t\gamma & 0 \\ 0 & -t\gamma & 1 \end{vmatrix}$$

At $t = 0$ the determinant equals to 1. By the inverse function theorem, in a neighborhood of $t = 0$, $\Phi(t, x, y, z)$ has a unique solution $(x(t), y(t), z(t)) = (\tilde{S}(t), \tilde{I}(t), \tilde{R}(t))$. Since Φ is analytical, so is $\tilde{S}(t), \tilde{I}(t), \tilde{R}(t)$ in a neighborhood of $t = 0$, that is, the radius of convergence of the following power series:

$$\sum_{n=0}^{\infty} |s_n| t^n, \sum_{n=0}^{\infty} |i_n| t^n, \sum_{n=0}^{\infty} |r_n| t^n,$$

are positive. Therefore, the radii of convergence of the power series of $S(t)$, $I(t)$, and $R(t)$ are positive and therefore they are analytical functions.

Remark 1. We could repeat the analysis done by using the power series at any point t_0 in the form

$$S(t) = \sum_{n=0}^{\infty} s_n (t - t_0)^n, \quad I(t) = \sum_{n=0}^{\infty} i_n (t - t_0)^n, R(t) = \sum_{n=0}^{\infty} r_n (t - t_0)^n.$$

In doing so, the same recurrences (3.3) appear for the coefficients s_n, i_n and r_n. Therefore, it is possible to use, for example, the evaluation of the truncated series expansion at a certain point t_k to build another truncated series expansion at that point. As an example, for the initial conditions and parameters given in (2.2) we obtain

$$S_1(t) = \frac{8257447t^5}{864000000000000} - \frac{266717t^4}{720000000000} - \frac{42097t^3}{2400000000} - \frac{1551t^2}{4000000} - \frac{99t}{20000} + \frac{99}{100},$$

$$I_1(t) = -\frac{229421789t^5}{23328000000000000} + \frac{20221t^4}{4860000000000} + \frac{284819t^3}{64800000000} + \frac{4259t^2}{36000000} + \frac{97t}{60000} + \frac{1}{100},$$
$$R_1(t) = \frac{20221t^5}{72900000000000} + \frac{284819t^4}{777600000000} + \frac{4259t^3}{324000000} + \frac{97t^2}{360000} + \frac{t}{300},$$

so that, for example, at $t = 5$ we obtain the following approximations:

$$S(5) \approx S_1(5) = \frac{263530240567}{276480000000} = 0.9531620390878183,$$
$$I(5) \approx I_1(5) = \frac{160961399971}{7464960000000} = 0.021562258869571974,$$
$$R(5) \approx R_1(5) = \frac{2358526309}{93312000000} = 0.02527570204260974.$$

Repeating the process we obtain the following approximation as a truncated power series at $t = 5$, by using exactly the same recurrence relations (3.3)

$$S_2(t) = 6.564434306730253 * 10^{-8}(t-5)^5 + 5.042256514874686 * 10^{-7}(t-5)^4 - 0.0000186083(t-5)^3 - 0.000680624(t-5)^2 - 0.0102762(t-5) + 0.953162,$$
$$I_2(t) = -3.101608855459257 * 10^{-8}(t-5)^5 - 5.194238176906494 * 10^{-7}(t-5)^4 + 1.8237799443817078 * 10^{-7}(t-5)^3 + 0.000165833(t-5)^2 + 0.00308874(t-5) + 0.0215623,$$
$$R_2(t) = -3.462825451270996 * 10^{-8}(t-5)^5 + 1.5198166203180895 * 10^{-8}(t-5)^4 + 0.0000184259(t-5)^3 + 0.000514791(t-5)^2 + 0.00718742(t-5) + 0.0252757.$$

The evaluations at $t = 10$ are

$$S_2(10) = 0.8829598639697187,$$
$$I_2(10) = 0.04075304367445933,$$
$$R_2(10) = 0.07628709235582196.$$

Then, we can repeat again the same idea to obtain another approximation for $t \in [10, 15]$, and continue the same idea.

In Figure 5 we have repeated this idea 10 times for the initial values and parameters given in (2.2), representing as well the solution given by Mathematica [23]. Since both graphs do exactly coincide, we have tiny shifted the approximation.

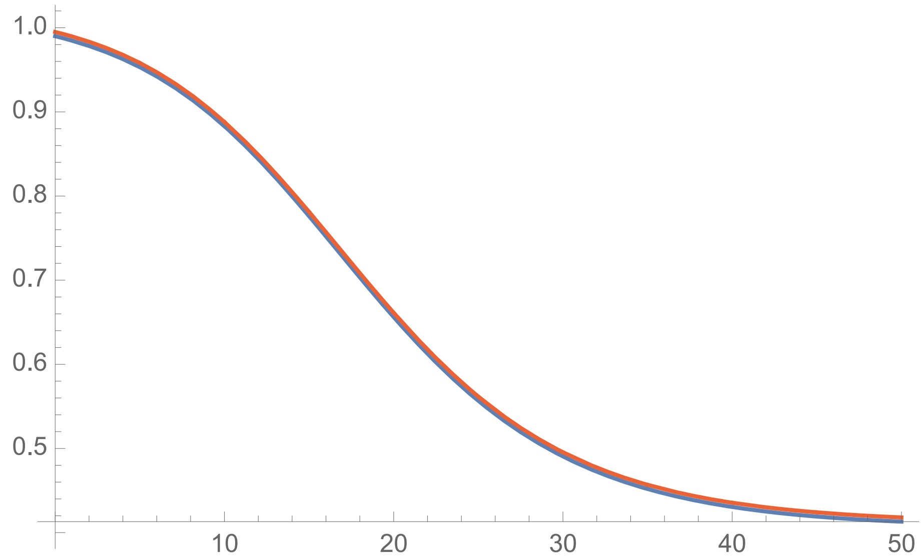

Figure 5. Repeated use of the power series to produce 10 polynomials of degree 5 that approximate $S(t)$ (red color), compared with the numerical solution of $S(t)$ in (2.1) for parameters and initial conditions given in (2.2), by using Mathematica (blue color). Since both graphs do exactly coincide, we have tiny shifted the approximation.

4. Power-series solution for the susceptibles $S(t)$

Let us recall the approach of [3] and [4]. From (2.7) we have

$$S(t)S''(t) - [S'(t)]^2 - \beta[S(t)]^2 S'(t) + \gamma S(t)S'(t) = 0. \tag{4.1}$$

Let us assume that there exist a power-series expansion of the form

$$S(t) = \sum_{n=0}^{\infty} s_n t^n, \quad s_0 = S(0), \tag{4.2}$$

in which we know s_1 since $S'(0) = -\beta s_0 i_0$. From the power-series expansion we also have

$$S'(t) = \sum_{n=0}^{\infty} \overbrace{S_{n+1}(n+1)}^{u_n} t^n, \quad S''(t) = \sum_{n=0}^{\infty} \overbrace{S_{n+2}(n+2)(n+1)}^{v_n} t^n,$$

$$S(t)S''(t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} s_k v_{n-k}\right] t^n, \quad [S'(t)]^2 = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} u_k u_{n-k}\right] t^n,$$

$$S(t)S'(t) = \sum_{n=0}^{\infty} \overbrace{\left[\sum_{k=0}^{n} s_k u_{n-k}\right]}^{w_n} t^n, \quad [S(t)]^2 S'(t) = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} s_k w_{n-k}\right] t^n.$$

Therefore, we obtain the following recurrence relation for the coefficients

$$\sum_{k=0}^{n} s_k v_{n-k} - \sum_{k=0}^{n} u_k u_{n-k} - \beta \sum_{k=0}^{n} s_k w_{n-k} + \gamma \sum_{k=0}^{n} s_k u_{n-k} = 0 \tag{4.3}$$

with the initial conditions $s_0 = S(0)$ and $s_1 = -\beta S(0)I(0) = -\beta s_0 i_0$. The above expression can be also written as

$$\begin{aligned} s_{n+2} = -\frac{1}{s_0(n+2)(n+1)} \Bigg\{ & \sum_{k=1}^{n} [s_k v_{n-k}] - \sum_{k=0}^{n} [s_{k+1} s_{n-k+1}(k+1)(n-k+1)] \\ & -\beta \sum_{k=0}^{n} \left[s_k w_{n-k} \sum_{p=0}^{n-k} s_p s_{n-k-p+1}(n-k-p+1) \right] + \gamma \sum_{k=0}^{n} [s_k s_{n-k+1}(n-k+1)] \Bigg\}. \end{aligned} \tag{4.4}$$

The first coefficients are hence given by

$$\begin{aligned} s_2 &= \frac{1}{2}\beta i_0 s_0 \left[\beta i_0 - \beta s_0 + \gamma\right], \\ s_3 &= -\frac{1}{6}\beta i_0 s_0 \left[\beta^2 i_0^2 + \beta i_0 (3\gamma - 4\beta s_0) + (\gamma - \beta s_0)^2\right], \\ s_4 &= \frac{1}{24}\beta i_0 s_0 \left[\beta^2 i_0^2 (6\gamma - 11\beta s_0) + \beta^3 i_0^3 + \beta i_0 (7\gamma - 11\beta s_0)(\gamma - \beta s_0) + (\gamma - \beta s_0)^3\right]. \end{aligned}$$

In Figure 6 we compare the numerical solution of (2.1) for $\beta = 1/2$, $\gamma = 1/3$, with the initial conditions $S_0 = 1$, $I_0 = 1/100$, and $R_0 = 0$, by using Mathematica which is using Newton's method and the power series (4.2).

Moreover, we would like to emphasize that the expressions for the coefficients coincide with those given in the previous section and therefore both power series (3.1) and (4.2) do also coincide.

5. Power-series solution of a SAIRP model of COVID-19

Very recently [24] a SAIRP model has been applied to analyze the evolution of the pandemic of COVID-19 in Portugal. More precisely, the total population was divided in five compartments, namely: Susceptible, Asymptomatic, Confirmed/active infected, Recovered/removed (includes deaths by COVID-19) and Protected/prevented. The system of differential equations reads as

$$\begin{cases} \dfrac{dS}{dt} &= [-\beta(1-p)(\theta A(t) + I(t)) - \phi p]S(t) + \omega P(t), \\ \dfrac{dA}{dt} &= \beta(1-p)(\theta A(t) + I(t))S(t) - \nu A(t), \\ \dfrac{dI}{dt} &= \nu A(t) - \delta I(t), \\ \dfrac{dR}{dt} &= \delta I(t), \\ \dfrac{dP}{dt} &= \phi p S(t) - \omega P(t). \end{cases} \tag{5.1}$$

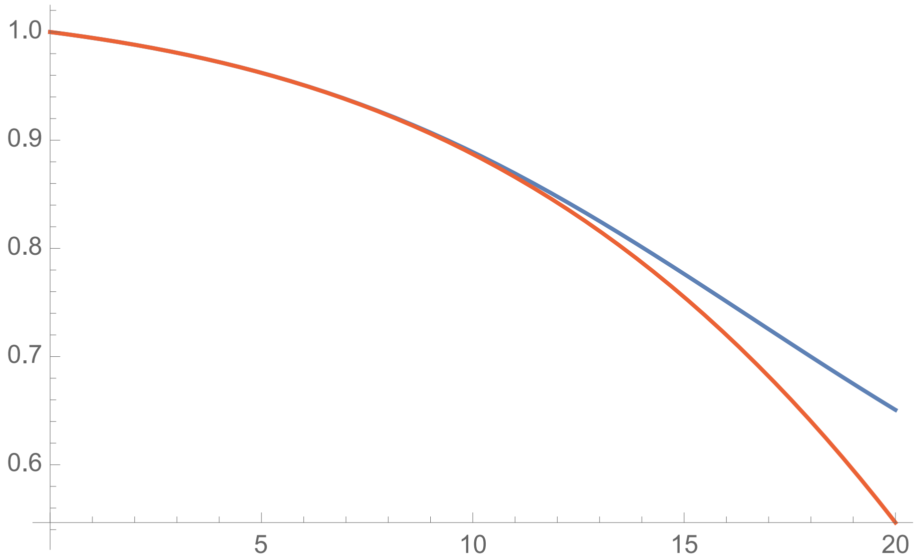

Figure 6. Comparison between the numerical solution of (2.1) for parameters and initial conditions given in (2.2), by using Mathematica which is using Newton's method and the power series (4.2). The evolution of $S(t)$ given by solving the system of differential equations is represented in blue color; the solution given by the power series truncating at degree 5 in red color.

In this system β denotes the transmission rate; θ the modification parameter: p the fraction of susceptible transferred to the protected class P; ϕ the transition rate of susceptible S to protected class P; $\omega = wm$; w the transition rate of protected P to susceptible S; m fraction of protected P transferred to susceptible S; $\nu = vq$; v the transition rate of asymptomatic A to active/confirmed infected I; q the fraction of asymptomatic A infected individuals; and δ the transition rate from active/confirmed infected I to removed/recovered R. Notice that the total population $N'(t) = S'(t)+A'(t)+I'(t)+R'(t)+P'(t) = 0$ which implies that N is constant over time. Without loss of generality it is possible to normalize the system with $N = 1$. By using that all the parameters of the model are non-negative, for a given non-negative initial condition the solution of the system is also non-negative and $S(t) + A(t) + I(t) + R(t) + P(t) = 1$. Moreover, by using this conservation law, it is possible to simplify the system to a SAIP model as

$$\begin{cases} \dfrac{dS}{dt} &= [-\beta(1-p)(\theta A(t) + I(t)) - \phi p]S(t) + \omega P(t), \\ \dfrac{dA}{dt} &= \beta(1-p)(\theta A(t) + I(t))S(t) - \nu A(t), \\ \dfrac{dI}{dt} &= \nu A(t) - \delta I(t), \\ \dfrac{dP}{dt} &= \phi p S(t) - \omega P(t). \end{cases} \tag{5.2}$$

since

$$R(t) = R(0) + \delta \int_0^1 I(s)ds.$$

To develop the method of power-series solutions, let us assume that

$$S(t) = \sum_{n=0}^{\infty} s_n y^n, \quad A(t) = \sum_{n=0}^{\infty} a_n t^n, \quad I(t) = \sum_{n=0}^{\infty} i_n t^n, \quad P(t) = \sum_{n=0}^{\infty} p_n t^n. \tag{5.3}$$

Mutatis mutandis, the following recurrence relations appear

$$\begin{cases} (n+1)s_{n+1} & = -\beta(1-p)\theta \sum_{k=0}^{n} a_k s_{n-k} - \beta(1-p) \sum_{k=0}^{n} i_k s_{n-k} - \phi p s_n + \omega p_n, \\ (n+1)a_{n+1} & = -\beta(1-p)\theta \sum_{k=0}^{n} a_k s_{n-k} + \beta(1-p) \sum_{k=0}^{n} i_k s_{n-k} + \nu a_n, \\ (n+1)i_{n+1} & = \nu a_n + \delta i_n, \\ (n+1)p_{n+1} & = \phi p s_n - \omega p_n. \end{cases} \tag{5.4}$$

The first coefficients are then given by

$$\begin{aligned} s_1 &= \beta(p-1)s_0\left(a_0\theta + i_0\right) + \omega p_0 - p\phi s_0, \\ a_1 &= -a_0\left(\beta(p-1)s_0\theta + \nu\right) - \beta i_0(p-1)s_0, \\ i_1 &= a_0\nu - \delta i_0, \\ p_1 &= p\phi s_0 - \omega p_0, \end{aligned}$$

$$\begin{aligned} s_2 &= \frac{1}{2}\Big(\beta^2(p-1)^2 s_0\left(a_0\theta + i_0\right)\left(\theta\left(a_0 - s_0\right) + i_0\right) + \beta(p-1)\left(a_0 s_0(\nu - \theta(\nu + 2p\phi))\right. \\ &\quad \left. + a_0\omega p_0\theta - i_0 s_0(\delta + 2p\phi) + i_0\omega p_0\right) + (-\omega - p\phi)\left(\omega p_0 - p\phi s_0\right)\Big), \\ a_2 &= \frac{1}{2}\Big(a_0^2\left(-\beta^2\right)(p-1)^2 s_0\theta^2 + a_0\left(\beta(p-1)\theta\left(\beta(p-1)s_0\left(s_0\theta - 2i_0\right) - \omega p_0 + p\phi s_0\right)\right. \\ &\quad \left. + \beta\nu(p-1)s_0(2\theta - 1) + \nu^2\right) + \beta i_0(p-1)\left(s_0\left(-\beta(p-1)\left(i_0 - s_0\theta\right) + \delta + \nu + p\phi\right) - \omega p_0\right)\Big), \\ i_2 &= \frac{1}{2}\Big(-\nu\left(a_0\left(\beta(p-1)s_0\theta + \nu\right) + \beta i_0(p-1)s_0\right) - a_0\delta\nu + \delta^2 i_0\Big), \\ p_2 &= \frac{1}{2}\Big(-p\phi s_0\left(p\phi - \beta(p-1)\left(a_0\theta + i_0\right)\right) + \omega^2 p_0 + \omega p\phi\left(p_0 - s_0\right)\Big). \end{aligned}$$

In Figure 7 we compare the numerical solution of (2.1) for parameters and initial conditions given in [24] that is,

$$\begin{gathered} \phi = 1/12, \quad \beta = 1.492, \quad p = 0.675, \quad \omega = 1/45, \quad \delta = 1/30, \quad \theta = 1, \\ \nu = 0.15, \quad N = 10295909, \quad s_0 = 1, \quad i_0 = 2/N, \quad a_0 = (2/0.15)/N, \quad p_0 = 0, \end{gathered}$$

by using Mathematica [23] which is using Newton's method and the power series (4.2).

A similar analysis as done in the previous section could be followed to prove that the power series has a positive radius of convergence. Similarly as we showed in the previous section, since the recurrence relations (5.4) are the same if we consider power series at point t_0, we repeat the approximation in order to improve the numerical solution in the whole interval.

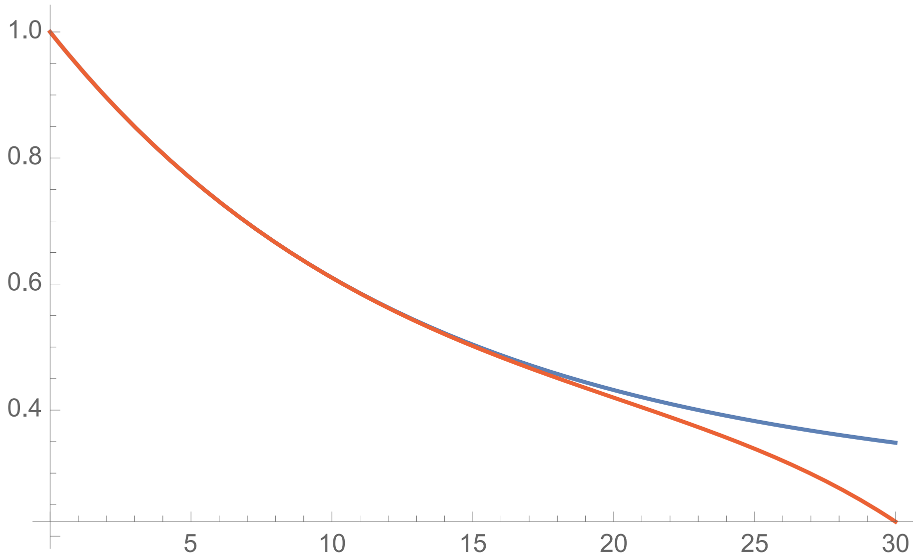

Figure 7. Comparison between the numerical solution of (5.2) for parameters and initial conditions given in [24], by using Mathematica which is using Newton's method and the power series (5.3). The evolution of $S(t)$ given by solving the system of differential equations is represented in blue color; the solution given by the power series truncating at degree 5 in red color.

6. Conclusions

Epidemiological models are gaining relevance due to the worldwide pandemic of COVID-19. In this paper we study some new aspects of the classical SIR compartmental model as well as some of its qualitative properties from the analytical point of view. The time when the infected population peaks is a relevant and critical time and the solutions can be approximated by using a development in power series. Some numerical simulations are presented using Mathematica [23] to reveal the adequacy of the approach. Finally we use the same methodology to study a SAIRP comparmental model which has been used to analyze the evolution of COVID-19 pandemic.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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