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Research article

Initial boundary value problem for fractional p-Laplacian Kirchhoff type equations with logarithmic nonlinearity

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Abstract: In this paper, we study the initial boundary value problem for a class of fractional p-Laplacian Kirchhoff type diffusion equations with logarithmic nonlinearity. Under suitable assumptions, we obtain the extinction property and accurate decay estimates of solutions by virtue of the logarithmic Sobolev inequality. Moreover, we discuss the blow-up property and global boundedness of solutions.

Keywords: Kirchhoff type; p-Laplacian; fractional; extinction; decay estimate

1. Introduction

In this paper, we study the extinction and the blow-up for the following fractional p-Laplacian Kirchhoff type equations with logarithmic nonlinearity.

$$\begin{cases} u_t + M(\|u\|^p)(-\Delta)_p^s u = \lambda |u|^{r-2} u \ln |u| - \beta |u|^{q-2} u, & \text{in } \Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where

$$\|u\| = \Big(\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \Big)^{\frac{1}{p}},$$

$Q = \mathbb{R}^{2N} \backslash (C\Omega \times C\Omega)$, $C\Omega = \mathbb{R}^N \backslash \Omega$, $\Omega \subset \mathbb{R}^N$ $(N > 2s)$ is a bounded domain with Lipschitz boundary, $s \in (0,1)$, $1 < p < 2$, $1 < q \le 2$, $r > 1$, λ, $\beta > 0$, $(-\Delta)_p^s$ is the fractional p-Laplacian operator and satisfies

$$(-\Delta)_p^s u(x) = 2 \lim_{\gamma \to 0^+} \int_{\mathbb{R}^N \backslash B_\gamma(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy,$$

where $u(x) \in C^\infty$ and $u(x)$ has compact support in Ω, $B_\gamma(x) \subset \mathbb{R}^N$ is the ball with center x and radius γ. $u_0(x) \in L^\infty(\Omega) \cap W_0^{s,p}(\Omega)$ is a nonzero non-negative function, where $L^\infty(\Omega)$ and $W_0^{s,p}(\Omega)$ are Lebesgue space and fractional Sobolev space respectively, which will be given in section 2. $M(\cdot)$ is a Kirchhoff function with the following assumptions

(M_1) $0 < s < 1, M(\tau) := a + b\theta\tau^{\theta-1}$ for $\tau \in \mathbb{R}_0^+ := [0, +\infty)$ ($a > 0, b \geq 0$ are two constants), $\theta \geq 1$;

(M_2) $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \backslash \{0\}$ is continuous and there exits $m_0 \geq 0$ such $M(\tau) \geq m_0$ for all $\tau \geq 0$.

It is worth pointing out that the interest in studying problems like (1.1) relies not only on mathematical purposes, but also on their significance in real models. For example, in the study of biological populations, we can use $u(x, t)$ to represent the density of the population at x at time t, the term $(-\Delta)_p^s u$ represents the diffusion of density, $\mu|u|^{q-2}u$ represents the internal source and $\lambda|u|^{r-2}u \ln|u|$ denotes external influencing factors. For more practical applications of problems like (1.1), please refer to the studies [1–3].

Compared with integer-order equations, it is very difficult to study the problem (1.1), which contains both non-local terms (including fractional p-Laplacian operators and Kirchhoff functions) and logarithmic nonlinearity. For the fractional order theory, we refer the readers to the studies [4–6]. In [7, 8], the authors use Sobolev space and Nehari manifold to study the existence of solutions for fractional equations. In [9, 10], the solutions for fractional equations are discussed by virtue of Nehari manifold and fibrillation diagram. By using different methods from above, the properties of the solutions for such partial differential equations are considered by the method of variational principle and topological theory in the the literature [11–13]. Moreover, the authors prefer to use potential well theory, Galerkin approximation and Nehari manifold method to prove the existence of solutions, decay estimation and blow-up, we refer the reader to the literature [14–16].

Existence, extinction and blow-up of solutions are three important topics which regard parabolic problems; in particular, the study of extinction properties has made great progress in recent years. In [17], Liu considered the following initial boundary value problem for the fractional p-Laplacian equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \beta u^q = \lambda u^r, \quad x \in \Omega, \quad t > 0, \tag{1.2}$$

where $1 < p < 2$, $q \leq 1$ and $r, \lambda, \beta > 0$. By employing the differential inequality and comparison principle, they obtained the extinction and the non-extinction of weak solutions. In [18], Sarra Toualbia et al. considered the following initial boundary value problem of a nonlocal heat equations with logarithmic nonlinearity

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p-2}u \log|u| - \fint_\Omega |u|^{p-2}u \log|u| dx, \ x \in \Omega, \ t > 0, \tag{1.3}$$

where $p \in (2, +\infty)$. By using the logarithmic Sobolev inequality and potential well method, they obtained decay, blow-up and non-extinction of solutions. In [19], Xiang and Yang studied the first initial boundary value problem of the following fractional p-Kirchhoff type

$$u_t + M([u]_{s,p}^p)(-\Delta)_p^s u = \lambda|u|^{r-2}u - \mu|u|^{q-2}u, \quad (x, t) \in \Omega \times (0, \infty), \tag{1.4}$$

where $M : [0, \infty) \to (0, \infty)$ is a continuous function, $0 < s < 1 < p < 2$, $1 < q \leq 2$, $r > 1$, $\lambda, \mu > 0$. Under suitable assumptions, they proved the extinction and non-extinction of solutions and perfected the Gagliardo-Nirenberg inequality. For more information on the extinction properties of the solution, please refer to the studies [20–23].

Inspired by the above works, we overcome the research difficulties of logarithmic nonlinearity, p-Laplace operator and Kirchhoff coefficients in problem (1.1) based on the potential well theory, Nehari manifold and differential inequality methods, we give the extinction and the blow-up properties of solutions. In addition, we give the global boundedness of the solution by appropriate assumptions. To the best of our knowledge, it is the first result in the literature to investigate the extinction and blow-up of solutions for fractional p-Laplacian Kirchhoff type with logarithmic nonlinearity.

In order to introduce our main results, we first give some related definitions and sets.

Definition 1.1(Weak solution). A function $u(x,t)$ is said to be a weak solution of problem (1.1), if $(x,t) \in \Omega \times [0,T)$, $u \in L^p(0,T;W_0^{s,p}(\Omega)) \cap C(0,T;L^2(\Omega))$, $u_t \in L^2(0,T;L^2(\Omega))$, $u(x,0) = u_0(x) \in W_0^{s,p}(\Omega)$, for all $v \in W_0^{s,p}(\Omega)$, $t \in (0,T)$, the following equation holds

$$\int_\Omega u_t v dx + M(\|u\|^p)\langle u,v\rangle = \lambda \int_\Omega v|u|^{r-2}u \ln|u| dx - \beta \int_\Omega v|u|^{q-2}u dx,$$

where

$$\langle u,v\rangle = \iint_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} dxdy.$$

Definition 1.2(Extinction of solutions). Let $u(t)$ be a weak solution of problem (1.1). We call $u(t)$ an extinction if there exists $T>0$ such that $u(x,t)>0$ for all $t \in (t,T)$ and $u(x,t) \equiv 0$ for all $t \in [T,+\infty)$.

Define the following two functionals on $W_0^{s,p}(\Omega)$

$$E(u) = \frac{1}{p}a\|u\|^p + \frac{1}{p}b\|u\|^{\theta p} - \lambda\frac{1}{r}\int_\Omega |u|^r \ln|u| dx + \lambda\frac{1}{r^2}\int_\Omega |u|^r dx + \beta\frac{1}{q}\int_\Omega |u|^q dx, \tag{1.5}$$

$$I(u) = a\|u\|^p + \theta b\|u\|^{\theta p} - \lambda\int_\Omega |u|^r \ln|u| dx + \beta\|u\|_q^q. \tag{1.6}$$

Let

$$Z = \left\{u \in L^p(0,T;W_0^{s,p}(\Omega)) \cap C(0,T;L^2(\Omega)), u_t \in L^2(0,T;L^2(\Omega))\right\}.$$

Remark 1 Since $u \in Z$, $1<p<2$, $M(\cdot)$ is a continuous function and

$$\int_\Omega |u|^r \ln|u| dx \le \frac{1}{\sigma}\|u\|_{r+\sigma}^{r+\sigma} \le \frac{1}{\sigma}C_{r+\sigma}^{r+\sigma}\|u\|^{r+\sigma},$$

where $0<\sigma<p_s^*-r$, then we can claim that $E(u)$ and $I(u)$ are well-defined in $W_0^{s,p}(\Omega)$. Further, by arguing essentially as in [24], one can prove the that

$$u \mapsto \int_\Omega |u|^r \ln|u| dx$$

is continuous from $W_0^{s,p}(\Omega)$ to $\mathbb{R}$. It follows that $E(u)$ and $I(u)$ are continuous.

Define some sets as follows

$$W := \{u \in W_0^{s,p}(\Omega) \mid I(u)>0,\ E(u)<h\} \cup \{0\}, \tag{1.7}$$

$$V := \{u \in W_0^{s,p}(\Omega) \mid I(u)<0,\ E(u)<h\}, \tag{1.8}$$

the mountain pass level

$$h := \inf_{u\in\mathcal{N}} E(u), \tag{1.9}$$

the Nehari manifold

$$\mathcal{N} := \{u \in W_0^{s,p}(\Omega)\backslash\{0\} \mid I(u) = 0\}. \tag{1.10}$$

Moreover, we define

$$\mathcal{N}_+ := \{u \in W_0^{s,p}(\Omega) \mid I(u) > 0\}, \tag{1.11}$$

$$\mathcal{N}_- := \{u \in W_0^{s,p}(\Omega) \mid I(u) < 0\}. \tag{1.12}$$

Let λ_1 be the first eigenvalue of the problem

$$(-\Delta)_p^s u = \lambda|u|^{p-2}u \text{ in } \Omega, \quad u|_{\mathbb{R}^N\backslash\Omega} = 0, \tag{1.13}$$

and $\phi(x) > 0$ a.e. in Ω be the eigenfunction corresponding to the eigenvalue $\lambda_1 > 0$, $\phi(x) \in L^\infty(\Omega)$ and $\|\phi\|_{L^\infty(\Omega)} \leq 1$.

First, we give some results satisfying $I(u_0) > 0$ and $q = 2$.

Theorem 1.1 Assume that $I(u_0) > 0$, $r = p$ and $q = 2$. Let m_0 be as in assumption (M_2), and let

$$l := \frac{2N - (s+N)p}{sp}, \; P_1 := \frac{m_0}{\lambda_1 L(p,\Omega) + \ln(R)}, \; P_2 := \frac{\lambda_1 m_0 l p^{p-1}}{[\lambda_1 L(p,\Omega) + \ln(R)](p+l-1)^{p-1}},$$

where $L(p,\Omega)$ and R are given in Lemma 2.1 and Lemma 2.5. Then, there exist positive constants C_1, C_2, T_1 and T_2 such that

(i) If $\lambda < \lambda_1 P_1$, then the weak solution of (1.1) vanishes in the sense of $\|\cdot\|_2$ as $t \to +\infty$.

(ii) If $2N/(N+2s) < p < 2$ and $\lambda < \lambda_1 P_1$ or $1 < p \leq 2N/(N+2s)$ and $\lambda < P_2$, then the nonnegative solutions of (1.1) vanish in finite time, and

$$\begin{cases} \|u\|_2 \leq \left[\left(\|u_0\|_2^{2-p} + \frac{C_1}{\beta}\right)e^{(p-2)\beta t} - \frac{C_1}{\beta}\right]^{\frac{1}{2-p}}, & t \in [0, T_1), \\ \|u\|_2 \equiv 0, & t \in [T_1, \infty), \end{cases}$$

for $2N/(N+2s) < p < 2$, and

$$\begin{cases} \|u\|_{l+1} \leq \left[\left(\|u_0\|_{l+1}^{2-p} + \frac{C_2}{\beta}\right)e^{(p-2)\beta t} - \frac{C_2}{\beta}\right]^{\frac{1}{2-p}}, & t \in [0, T_2), \\ \|u\|_{l+1} \equiv 0, & t \in [T_2, \infty), \end{cases}$$

for $1 < p < 2N/(N+2s)$.

Theorem 1.2 Assume that $I(u_0) > 0$, $0 < \sigma \leq p_s^* - r$, $r > p$ and $q = 2$. Let m_0 be as in assumption (M_2), and let

$$P_3 := \max\Big\{\frac{m_0}{L(r,\Omega)R^{r-p} + \varepsilon\Phi}, \frac{\beta}{\Phi\varepsilon^{\frac{r(\vartheta_1-1)}{p-r(1-\vartheta_1)}}}\Big\}, \; P_4 := \max\Big\{\frac{m_0 l p^p}{\varepsilon_1(p+l-1)^p}, \beta\varepsilon_1^{\frac{r_2(1-\vartheta_2)}{p-r_2(1-\vartheta_2)}}\Big\},$$

where

$$\Phi = \ln(R)|\Omega|^{\frac{s_1-r}{s_1}} C_{p_s^*}^{r(1-\vartheta_1)},\ p_s^* = \frac{Np}{N-sp},\ s_1 = 2\vartheta_1 + p_s^*(1-\vartheta_1), \vartheta_1 = \frac{2(r-p)}{r(2-p)},\ r_2 = \frac{p(l+r+\sigma-1)}{l+p-1},$$

$C_{p_s^*}$ is the best constant of embedding from $W_0^{s,p}(\Omega)$ to $L^{p_s^*}(\Omega)$, $L(p,\Omega)$ and R are given in Lemma 2.1 and Lemma 2.5, and $\varepsilon, \varepsilon_1 > 0$ are two constants. Then, there exist positive constants C_4, C_5, C_6, C_7, T_3 and T_4 such that the non-negative weak solution of problem (1.1) vanishes in finite time and

$$\begin{cases} \|u\|_2 \le \left[\left(\|u_0\|_{l+1}^{2-p} + \frac{C_5}{C_4}\right)e^{(p-2)C_4 t} - \frac{C_5}{C_4}\right]^{\frac{1}{2-p}}, & t \in [0, T_3), \\ \|u\|_2 \equiv 0, & t \in [T_3, \infty), \end{cases}$$

for $2N/(N+2s) \le p < 2$, $\lambda < P_3$, and

$$\begin{cases} \|u\|_{l+1} \le \left[\left(\|u_0\|_{l+1}^{2-p} + \frac{C_6}{C_7}\right)e^{(p-2)C_7 t} - \frac{C_6}{C_7}\right]^{\frac{1}{2-p}}, & t \in [0, T_4), \\ \|u\|_{l+1} \equiv 0, & t \in [T_4, \infty), \end{cases}$$

for $1 < p < 2N/(N+2s)$, $\lambda < P_4$.

Secondly, we give some results satisfying $I(u_0) > 0$ and $q < 2$.

Theorem 1.3 Assume that $I(u_0) > 0$, $p = r$, $l_1 > l \ge 1$ and $1 < q < 2$. Let m_0 be as in assumption (M_2), and let

$$P_5 := \frac{\lambda_1 m_0 l_1 p^{p-1}}{[\lambda_1 L(p,\Omega) + \ln(R)](p+l_1-1)^{p-1}},$$

where $L(p,\Omega)$ and R are given in Lemma 2.1 and Lemma 2.5. If $2N/(N+2s) < p < 2$ with $\lambda < \lambda_1 P_1$ or $1 < p \le 2N/(N+2s)$ with $\lambda < P_5$, then the non-negative weak solution of problem (1.1) vanishes in finite time for any non-negative initial data.

Theorem 1.4 Assume that $I(u_0) > 0$, $l_1 > l \ge 1$, $r \le 2$ and $1 < q < 2$. Let m_0 be as in assumption (M_2), and let

$$P_6 := \max\left\{\frac{m_0}{L(r,\Omega)R^{r-p}}, \frac{\beta\varepsilon_3^{\frac{(1-\vartheta_4)r_4}{p-(1-\vartheta_4)r_4}} C_{p_s^*}^{(\vartheta_4-1)r_4} R^{r_4-r}}{\ln(R)|\Omega|^{\frac{2-r}{2}}}\right\},\ P_7 := \frac{1}{C}\beta\varepsilon_2^{\frac{(\vartheta_3-1)r_3}{(1-\vartheta_3)-(p+l_1-1)}} C_{p_s^*}^{\frac{(\vartheta_3-1)pr_3}{p+l_1-1}},$$

$$\vartheta_3 = \frac{[(q+l_1-1)p_s^* - (l_1+1)p](p+l_1-1)}{[(q+l_1-1)p_s^* - (p+l_1-1)p](l_1+1)},\ \vartheta_4 = \frac{(p_s^*-2)p}{(p_s^*-q)2},\ C = \frac{1}{e\sigma}|\Omega|^{1-\frac{r_5}{l_1+1}} C_{l_1+1}^{r_5} R^{r_5-r_3},$$

$$r_3 = \frac{(q+l_1-1)(p+l_1-1)}{\vartheta_3(p+l_1-1) + (q+l_1-1)(1-\vartheta_3)},\ r_4 = \frac{qp}{q(1-\vartheta_4)+\vartheta_4 p},\ r_5 = l_1 + r + \sigma - 1,$$

and ε_2, $\varepsilon_3 > 0$ are two constants. If $2N/(N+2s) < p < 2$ with $\lambda < P_6$ or $1 < p \le 2N/(N+2s)$ with $\lambda < P_7$, then the non-negative weak solution of problem (1.1) vanishes in finite time for any non-negative initial data.

Finally, we discuss the global boundedness and blow up of weak solutions.

Theorem 1.5 Let $u(x,t)$ be the weak solution of problem (1.1).

(i) If $E(u_0) < 0$, $r = p > q$ and $\theta = 1$, then the weak solution $u(x,t)$ blows up at $+\infty$;

(ii) If $0 < E(u_0) \le h$ and $I(u_0) \ge 0$, then the weak solution $u(x,t)$ is globally bounded.

The rest of the paper is organized as follows. In Section 2, we give some related spaces and lemmas. In Section 3, we give the proof process for the main results of problem (1.1).

2. Preliminaries

In order to facilitate the proof of the main results, we start this section by introducing some symbols and Lemmas that will be used throughout the paper.

In this section, we assume that $0 < s < 1 < p < 2$ and $\Omega \in \mathbb{R}^N$ $(N > 2s)$ is a bounded domain with Lipschitz boundary. We denote by $\|u\|_i$ $(i \geq 1)$ the norm of Lebesgue space $L^i(\Omega)$. Let $W^{s,p}(\Omega)$ be the linear space of Lebesgue measurable functions u from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in $W^{s,p}(\Omega)$ belongs to $L^p(\Omega)$ and

$$\iint_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy < \infty,$$

where $Q = \mathbb{R}^{2N} \backslash (C\Omega \times C\Omega)$, $C\Omega = \mathbb{R}^N \backslash \Omega$. The space $W^{s,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(\|u\|_p^p + \iint_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy\Big)^{\frac{1}{p}}.$$

We further give a closed linear subspace

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\Omega) | u(x) = 0 \;\text{ a.e. in } \mathbb{R}^N \backslash \Omega\}.$$

As shown in [19], it can be concluded that

$$\|u\| = \Big(\iint_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy\Big)^{\frac{1}{p}}$$

is an equivalent norm of $W_0^{s,p}(\Omega)$.

Next we give the necessary Lemmas.

Lemma 2.1 (**[25]**) Let $u \in W_0^{s,p}(\Omega) \backslash \{0\}$. Then

$$\int_\Omega |u|^p \ln |u| dx \leq L(p, \Omega)\|u\|^p + \ln(\|u\|) \int_\Omega |u|^p dx,$$

where $L(p, \Omega) := \frac{|\Omega|}{ep} + \frac{1}{e(p_s^* - p)} C_{p_s^*}^{p_s^*}$, $C_{p_s^*}$ is the best constant of embedding from $W_0^{s,p}(\Omega)$ to $L^{p_s^*}(\Omega)$.

Lemma 2.2 (**[26]**) Let $y(t)$ be a non-negative absolutely continuous function on $[T_0, +\infty)$ satisfying

$$\frac{dy}{dt} + \alpha y^k + \beta y \leq 0, \quad t \geq 0, \quad y(0) \geq 0,$$

where $\alpha, \beta > 0$ are constants and $k \in (0, 1)$. Then

$$\begin{cases} y(t) \leq \Big[(y^{1-k}(T_0) + \frac{\alpha}{\beta}) e^{(k-1)\beta(t-T_0)} - \frac{\alpha}{\beta}\Big]^{\frac{1}{1-k}}, & t \in [T_0, T_*), \\ y(t) \equiv 0, & t \in [T_*, +\infty), \end{cases}$$

where $T_* = \frac{1}{(1-k)\beta} \ln\big(1 + \frac{\beta}{\alpha} y^{1-k}(T_0)\big)$.

Lemma 2.3 (**[27]**) Suppose that $\beta^* \geq 0$, $N > sp \geq 1$, and $1 \leq r \leq q \leq (\beta^* + 1)p_s^*$, then for u such that $|u|^{\beta^*} u \in W_0^{s,p}(\Omega)$, we have

$$\|u\|_q \leq C_{p_s^*}^{\frac{1-\vartheta}{\beta^*+1}} \|u\|_r^\vartheta \||u|^{\beta^*} u\|^{\frac{1-\vartheta}{\beta^*+1}},$$

with $\vartheta = \frac{[(\beta^*+1)p_s^*-q]r}{[(\beta^*+1)p_s^*-r]q}$, where $C_{p_s^*}$ is the embedding constant for $W_0^{s,p}(\Omega) \hookrightarrow L^{p_s^*}(\Omega)$.
Lemma 2.4 (**[19]**) Let $1 < p < \infty$ and $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. Define

$$G(t) = \int_0^t g'(\tau)^{\frac{1}{p}} d\tau, \quad t \in \mathbb{R}$$

then

$$|a-b|^{p-2}(a-b)(g(a)-g(b)) \geq |G(a)-G(b)|^p \quad \text{for all } a, b \in \mathbb{R}.$$

Lemma 2.5 Assume that (M_1) holds. Let $u \in W_0^{s,p}(\Omega)\backslash\{0\}$, $0 < \sigma \leq p_s^* - r$. We have
(i) if $0 < \|u\| \leq R$, then $I(u) > 0$;
(ii) if $I(u) \leq 0$, then $\|u\| > R$,
where

$$R = \Big(\frac{a\sigma}{\lambda C_{r+\sigma}^{r+\sigma}}\Big)^{\frac{1}{r+\sigma-p}},$$

$C_{r+\sigma}$ is the embedding constant for $W_0^{s,p}(\Omega) \hookrightarrow L^{r+\sigma}(\Omega)$.
Proof. Since $u \in W_0^{s,p}(\Omega)\backslash\{0\}$, and

$$\sigma \ln |u(x)| < |u(x)|^\sigma \quad \text{for } a.e. x \in \Omega.$$

Then by the definition of $I(u)$, we obtain

$$\begin{aligned} I(u) =& a\|u\|^p + \theta b\|u\|^{\theta p} + \beta\|u\|_q^q - \lambda \int_\Omega |u|^r \ln |u| dx \\ >& a\|u\|^p + \theta b\|u\|^{\theta p} + \beta\|u\|_q^q - \lambda \frac{1}{\sigma}\|u\|_{r+\sigma}^{r+\sigma} \\ \geq& a\|u\|^p - \lambda \frac{1}{\sigma}\|u\|_{r+\sigma}^{r+\sigma} \\ \geq& (a - \lambda \frac{1}{\sigma} C_{r+\sigma}^{r+\sigma}\|u\|^{r+\sigma-p})\|u\|^p, \end{aligned}$$

where $C_{r+\sigma}$ is the embedding constant for $W_0^{s,p}(\Omega) \hookrightarrow L^{r+\sigma}(\Omega)$.
We can get

$$I(u) > (a - \lambda \frac{1}{\sigma} C_{r+\sigma}^{r+\sigma}\|u\|^{r+\sigma-p})\|u\|^p. \tag{2.1}$$

If $0 < \|u\| \leq R$, then it follows from the definition of R that

$$a - \lambda \frac{1}{\sigma} C_{r+\sigma}^{r+\sigma}\|u\|^{r+\sigma-p} \geq 0,$$

thus (i) holds.
If $I(u) \leq 0$, by (2.1), we have

$$a - \lambda \frac{1}{\sigma} C_{r+\sigma}^{r+\sigma}\|u\|^{r+\sigma-p} < 0,$$

thus (ii) holds. □

Lemmas 2.6 is similar to [28, Lemmas 9], so we ignore its proof.
Lemma 2.6 (**[28]**) Assume that $E(u_0) \leq h$, then the sets $\mathcal{N}_+$ and $\mathcal{N}_-$ are both invariant for $u(t)$, i.e, if $u_0 \in \mathcal{N}_-$ (resp. $u_0 \in \mathcal{N}_+$), then $u(t) \in \mathcal{N}_-$ (resp. $u(t) \in \mathcal{N}_+$) for all $t \in [0, T)$.
Lemma 2.7 (**[29]**) Let α be positive. Then

$$t^p \ln(t) \leq \frac{1}{e\alpha} t^{p+\alpha}, \quad \text{for all } p, t > 0.$$

3. Proof of main results

In this section, we prove that the main results of problem (1.1).

Proof of Theorem 1.1

(1) Taking $v = u$ in Definition 1.1, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + M(\|u\|^p)\|u\|^p = \lambda\int_\Omega |u|^p \ln|u|dx - \beta\int_\Omega u^2 dx. \tag{3.1}$$

By Lemma 2.1, Lemma 2.5 and Lemma 2.6, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + (m_0 - \lambda L(p,\Omega) - \frac{\lambda}{\lambda_1}\ln(R))\|u\|^p + \beta\int_\Omega u^2 dx \le 0.$$

Since $\lambda < \lambda_1 P_1$, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \beta\int_\Omega u^2 dx \le 0.$$

thus

$$\|u(\cdot,t)\|_2^2 \le \|u_0\|_2^2 e^{-2\beta t}.$$

which implies that $\|u(\cdot,t)\|_2 \to 0$ as $t \to +\infty$.

(2)We consider first the case $2N/(N+2s) < p < 2$ with $\lambda < \lambda_1 P_1$. By (3.1) and Lemma 2.1, Lemma 2.5 and Lemma 2.6, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \Big(m_0 - \frac{\lambda}{\lambda_1}(\lambda_1 L(p,\Omega) + \ln(R))\Big)\|u\|^p + \beta\|u\|_2^2 \le 0. \tag{3.2}$$

Using Hölder's inequality and the fractional Sobolev embedding theorem, we have

$$\|u\|_2 \le |\Omega|^{\frac{1}{2}-\frac{N-sp}{Np}}\|u\|_{\frac{Np}{N-sp}} \le C_{p_s^*}|\Omega|^{\frac{1}{2}-\frac{N-sp}{Np}}\|u\|, \tag{3.3}$$

where $C_{p_s^*} > 0$ is the embedding constant. By (3.2), (3.3) and $\lambda < \lambda_1 P_1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + C_1\|u\|_2^p + \beta\|u\|_2^2 \le 0, \tag{3.4}$$

where

$$C_1 = C_{p_s^*}^{-p}|\Omega|^{\frac{N-sp}{N}-\frac{p}{2}}\Big(m_0 - \frac{\lambda}{\lambda_1}(\lambda_1 L(p,\Omega) + \ln(R))\Big) > 0. \tag{3.5}$$

Setting $y(t) = \|u(\cdot,t)\|_2^2$, $y(0) = \|u_0(\cdot)\|_2^2$, by Lemma 2.2, we obtain

$$\begin{cases} \|u\|_2 \le \Big[\Big(\|u_0\|_2^{2-p} + \frac{C_1}{\beta}\Big)e^{(p-2)\beta t} - \frac{C_1}{\beta}\Big]^{\frac{1}{2-p}}, & t \in [0, T_1), \\ \|u\|_2 \equiv 0, & t \in [T_1, \infty), \end{cases}$$

where

$$T_1 = \frac{1}{(2-p)\beta}\ln\Big(1 + \frac{\beta}{C_1}\|u_0\|_2^{2-p}\Big). \tag{3.6}$$

Next, we consider the case $1 < p \leq 2N/(N+2s)$ and $\lambda < \lambda_1 P_2$. Taking $v = u^l$ in Definition 1.1, where $l = \frac{2N-(s+N)p}{sp} \geq 1$, by Lemma 2.1, Lemma 2.4, Lemma 2.5 and Lemma 2.6, we obtain

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + G\|u^{\frac{p+l-1}{p}}\|^p + \beta\|u\|_{l+1}^{l+1} \leq 0, \tag{3.7}$$

where $G = \Big(\frac{m_0 l p^p}{(p+l-1)^p} - \frac{\lambda p L(p,\Omega)}{p+l-1} - \frac{\lambda p \ln(R)}{\lambda_1(p+l-1)}\Big)$. By the very choice of l and the fractional Sobolev embedding theorem, we have

$$\|u\|_{l+1}^{\frac{p+l-1}{p}} = \Big(\int_\Omega u^{\frac{p+l-1}{p}\cdot\frac{Np}{N-sp}}dx\Big)^{\frac{N-sp}{Np}} \leq C_{p_s^*}\|u^{\frac{p+l-1}{p}}\|. \tag{3.8}$$

Hence,

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + C_2\|u\|_{l+1}^{p+l-1} + \beta\|u\|_{l+1}^{l+1} \leq 0, \tag{3.9}$$

where

$$C_2 = C_{p_s^*}^{-p}\Big(\frac{m_0 l p^p}{(p+l-1)^p} - \frac{\lambda p L(p,\Omega)}{p+l-1} - \frac{\lambda p \ln(R)}{\lambda_1(p+l-1)}\Big), \tag{3.10}$$

since $\lambda < P_2$, then $C_2 > 0$. Setting $y(t) = \|u(\cdot,t)\|_{l+1}$, $y(0) = \|u_0(\cdot)\|_{l+1}$, by Lemma 2.2, we obtain

$$\begin{cases} \|u\|_{l+1} \leq \Big[\Big(\|u_0\|_{l+1}^{2-p} + \frac{C_2}{\beta}\Big)e^{(p-2)\beta t} - \frac{C_2}{\beta}\Big]^{\frac{1}{2-p}}, & t \in [0, T_2), \\ \|u\|_{l+1} \equiv 0, & t \in [T_2, \infty), \end{cases}$$

where

$$T_2 = \frac{1}{(2-p)\beta}\ln\big(1 + \frac{\beta}{C_2}\|u_0\|_{l+1}^{2-p}\big). \tag{3.11}$$

The proof is completed. □

Proof of Theorem 1.2

We consider first the case $p < r < 2$ and $2N/(N+2s) < p < 2$. Taking $v = u$ in Definition 1.1, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + M(\|u\|^p)\|u\|^p = \int_\Omega |u|^r \ln|u| dx - \beta\|u\|_2^2. \tag{3.12}$$

By Lemma 2.1, Lemma 2.5 and Lemma 2.6, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + M(\|u\|^p)\|u\|^p \leq \lambda L(r,\Omega)\|u\|^r + \lambda\ln(R)\|u\|_r^r - \beta\|u\|_2^2. \tag{3.13}$$

Using Hölder's inequality and the interpolation inequality, the fractional Sobolev embedding theorem and Young inequality, we can easily obtain (see [19])

$$\|u\|_r^r \leq |\Omega|^{\frac{s_1-r}{s_1}}\|u\|_{s_1}^r \leq |\Omega|^{\frac{s_1-r}{s_1}}C_{p_s^*}^{r(1-\vartheta_1)}(\varepsilon\|u\|^p + \varepsilon^{\frac{r(\vartheta_1-1)}{p-r(1-\vartheta_1)}}\|u\|_2^2), \tag{3.14}$$

where $s_1 > r$, $\vartheta_1 \in (0,1)$, $\varepsilon > 0$ and

$$s_1 = 2\vartheta_1 + p_s^*(1-\vartheta_1), \ \ \vartheta_1 = \frac{2(r-p)}{r(2-p)}.$$

By (3.13) and (3.14), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + C_3\|u\|^p + C_4\|u\|_2^2 \leq 0, \tag{3.15}$$

where

$$C_3 = m_0 - \lambda L(r,\Omega)R^{r-p} - \lambda\varepsilon\ln(R)|\Omega|^{\frac{s_1-r}{s_1}}C_{p_s^*}^{r(1-\vartheta_1)}, \tag{3.16}$$

$$C_4 = \beta - \lambda\ln(R)|\Omega|^{\frac{s_1-r}{s_1}}C_{p_s^*}^{r(1-\vartheta_1)}\varepsilon^{\frac{r(\vartheta_1-1)}{p-r(1-\vartheta_1)}}. \tag{3.17}$$

Since $2N/(N+2s) < p < 2$ and $\lambda < P_3$, by the fractional embedding theorem and (3.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + C_5\|u\|_2^p + C_4\|u\|_2^2 \leq 0, \tag{3.18}$$

and $C_3 > 0$, $C_4 > 0$, where

$$C_5 = C_3C_{p_s^*}^{-p}|\Omega|^{\frac{N-sp}{N}-\frac{p}{2}}. \tag{3.19}$$

Similar to the Theorem 1.1, one can prove the that

$$\begin{cases}\|u\|_2 \leq \left[\left(\|u_0\|_{l+1}^{2-p} + \frac{C_5}{C_4}\right)e^{(p-2)C_4t} - \frac{C_5}{C_4}\right]^{\frac{1}{2-p}}, & t\in[0,T_3),\\ \|u\|_2 \equiv 0, & t\in[T_3,\infty),\end{cases}$$

where

$$T_3 = \frac{1}{(2-p)C_4}\ln\left(1 + \frac{C_4}{C_5}\|u_0\|_2^{2-p}\right). \tag{3.20}$$

When $1 < p \leq 2N/(N+2s)$, $p < r \leq 2$ and $\lambda < P_4$. Taking $v = u^l$ ($l = \frac{2N-sp-Np}{sp} \geq 1$) in Definition 1.1, by Lemma 2.4 and Lemma 2.7, we obtain

$$\frac{1}{l+1}\frac{d}{dt}\|u\|_{l+1}^{l+1} + \frac{m_0lp^p}{(p+l-1)^p}\|u^{\frac{l+p-1}{p}}\|^p \leq \lambda\frac{1}{e\sigma}\|u\|_{l+r+\sigma-1}^{l+r+\sigma-1} - \beta\|u\|_{l+1}^{l+1} \tag{3.21}$$

further, we have

$$\frac{1}{l+1}\frac{d}{dt}\|u^{\frac{l+p-1}{p}}\|_{r_1}^{r_1} + \frac{m_0lp^p}{(p+l-1)^p}\|u^{\frac{l+p-1}{p}}\|^p \leq \lambda\frac{1}{e\sigma}\|u^{\frac{l+p-1}{p}}\|_{r_2}^{r_2} - \beta\|u^{\frac{l+p-1}{p}}\|_{r_1}^{r_1} \tag{3.22}$$

where $r_1 = \frac{p(l+1)}{l+p-1}$, $r_2 = \frac{p(l+r+\sigma-1)}{l+p-1}$. Note that, since $r_1 < p_s^*$, by the Hölder's inequality and the fractional Sobolev embedding theorem , we have

$$\|u^{\frac{l+p-1}{p}}\|_{r_1}^p \leq |\Omega|^{\frac{(p_s^*-r_1)p}{r_1p_s^*}}\|u^{\frac{l+p-1}{p}}\|_{p_s^*}^p \leq |\Omega|^{\frac{(p_s^*-r_1)p}{r_1p_s^*}}C_{p_s^*}^p\|u^{\frac{l+p-1}{p}}\|^p. \tag{3.23}$$

Using the same discussion as above, one can conclude that

$$\|u^{\frac{l+p-1}{p}}\|_{r_2}^{r_2} \leq |\Omega|^{\frac{s_2-r_2}{s_2}}C_{p_s^*}^{r_2(1-\vartheta_2)}\left(\varepsilon_1\|u^{\frac{l+p-1}{p}}\|^p + \varepsilon_1^{\frac{r_2(\vartheta_2-1)}{p-r_2(1-\vartheta_2)}}\|u^{\frac{l+p-1}{p}}\|_{r_1}^{r_1}\right), \tag{3.24}$$

where $s_2 > r_2$, $\vartheta_2 \in (0,1)$, $\varepsilon_1 > 0$ and

$$s_2 = r_1\vartheta_2 + p_s^*(1-\vartheta_2),\quad \vartheta_2 = \frac{r_1(r_2-p)}{r_2(r_1-p)}.$$

Combining (3.22)-(3.24) and $\lambda < P_4$, we obtain

$$\frac{1}{l+1}\frac{d}{dt}\|u^{\frac{l+p-1}{p}}\|_{r_1}^{r_1} + C_6\|u^{\frac{l+p-1}{p}}\|_{r_1}^{p} + C_7\|u^{\frac{l+p-1}{p}}\|_{r_1}^{r_1} \le 0, \tag{3.25}$$

and $C_6 > 0$, $C_7 > 0$, where

$$C_6 = \Big(\frac{m_0 l p^p}{(l+p-1)^p} - \lambda\varepsilon_1 \frac{1}{e\sigma} C_{p_s^*}^{r_2(1-\vartheta_2)} |\Omega|^{\frac{s_2-r_2}{s_2}}\Big)|\Omega|^{\frac{(r_1-p_s^*)p}{r_1 p_s^*}} C_{p_s^*}^{-p}, \tag{3.26}$$

$$C_7 = \beta - \lambda\frac{1}{e\sigma}|\Omega|^{\frac{s_2-r_2}{s_2}} C_{p_s^*}^{r_2(1-\vartheta_2)} \varepsilon_1^{\frac{r_2(\vartheta_2-1)}{p-r_2(1-\vartheta_2)}}. \tag{3.27}$$

Using Lemma 2.2 and a direct calculation, we have

$$\begin{cases} \|u\|_{l+1} \le \Big[\Big(\|u_0\|_{l+1}^{2-p} + \frac{C_6}{C_7}\Big)e^{(p-2)C_7 t} - \frac{C_6}{C_7}\Big]^{\frac{1}{2-p}}, & t \in [0, T_4), \\ \|u\|_{l+1} \equiv 0, & t \in [T_4, \infty), \end{cases}$$

where

$$T_4 = \frac{1}{(2-p)C_7}\ln\Big(\frac{C_7}{C_6}\|u_0\|_{l+1}^{2-p} + 1\Big). \tag{3.28}$$

The proof is completed. □

Proof of Theorem 1.3

We consider first the case $2N/(N+2s) < p < 2$ with $\lambda < \lambda_1 P_1$. Taking $v = u$ in Definition 1.1, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + M(\|u\|^p)\|u\|^p = \lambda\int_\Omega |u|^p \ln|u| dx - \beta\int_\Omega u^q dx. \tag{3.29}$$

Note that $\|u\|^p \ge \lambda_1\|u\|_p^p$ and by Lemma 2.1, Lemma 2.5 and Lemma 2.6, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \Big(m_0 - \frac{\lambda}{\lambda_1}(\lambda_1 L(p,\Omega) + \ln(R))\Big)\|u\|^p + \beta\|u\|_q^q \le 0. \tag{3.30}$$

By (3.3) and $\beta > 0$ and $\lambda < \lambda_1 P_1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + C_8\|u\|_2^p \le 0, \tag{3.31}$$

where

$$C_8 = C_{p_s^*}^{-p}|\Omega|^{\frac{2(N-sp)-Np}{2N}}\Big(m_0 - \frac{\lambda}{\lambda_1}(\lambda_1 L(p,\Omega) + \ln(R))\Big) > 0. \tag{3.32}$$

By a direct calculation, we obtain

$$\begin{cases} \|u\|_2 \le \Big[\Big(\|u_0\|_2^{2-p} + C_8(p-2)t\Big]^{\frac{1}{2-p}}, & t \in [0, T_5), \\ \|u\|_2 \equiv 0, & t \in [T_5, \infty), \end{cases}$$

where

$$T_5 = \frac{1}{(2-p)C_8}\|u_0\|_2^{2-p}. \tag{3.33}$$

Next, we consider the case $1 < p \le 2N(N+2s)$ with $\lambda < P_2$. Multiplying (1.1) by u^{l_1} ($l_1 > l \ge 1$) and integrating, by Lemma 2.1, Lemma 2.4, Lemma 2.5 and Lemma 2.6, we obtain

$$\frac{1}{l_1+1}\frac{d}{dt}\|u\|_{l_1+1}^{l_1+1} + G_1\|u^{\frac{p+l_1-1}{p}}\|^p + \beta\|u\|_{q+l_1-1}^{q+l_1-1} \le 0, \tag{3.34}$$

where $G_1 = \left(\frac{m_0 l_1 p^p}{(p+l_1-1)^p} - \frac{\lambda p L(p,\Omega)}{p+l_1-1} - \frac{\lambda p\ln(R)}{\lambda_1(p+l_1-1)}\right)$. By Lemma 2.3 , we obtain

$$\|u\|_{l_1+1} \le C_{p_s^*}^{\frac{(1-\vartheta_3)p}{p+l_1-1}}\|u^{\frac{p+l_1-1}{p}}\|^{\frac{(1-\vartheta_3)p}{p+l_1-1}}\|u\|_{q+l_1-1}^{\vartheta_3}, \tag{3.35}$$

where

$$\vartheta_3 = \frac{[(q+l_1-1)p_s^* - (l_1+1)p](p+l_1-1)}{[(q+l_1-1)p_s^* - (p+l_1-1)p](l_1+1)}.$$

By the choice of l_1, we have $0 < \vartheta_3 < 1$. Hence, using the Young inequality, for every $r_3 > 0$ and $\varepsilon_2 > 0$, we obtain

$$\|u\|_{l_1+1}^{r_3} \le C_{p_s^*}^{\frac{(1-\vartheta_3)pr_3}{p+l_1-1}}\left(\varepsilon_2\|u^{\frac{p+l_1-1}{p}}\|^p + \varepsilon_2^{\frac{(1-\vartheta_3)r_3}{(1-\vartheta_3)r_3-(p+l_1-1)}}\|u\|_{q+l_1-1}^{\frac{\vartheta_3 r_3(p+l_1-1)}{(p+l_1-1)-(1-\vartheta_3)r_3}}\right). \tag{3.36}$$

We now choose

$$r_3 = \frac{(q+l_1-1)(p+l_1-1)}{\vartheta_3(p+l_1-1)+(q+l_1-1)(1-\vartheta_3)},$$

and we notice that $\frac{\vartheta_3 r_3(p+l_1-1)}{(p+l_1-1)-(1-\vartheta_3)r_3} = q+l_1-1$. That means

$$\|u\|_{l_1+1}^{r_3} \le C_{p_s^*}^{\frac{(1-\vartheta_3)pr_3}{p+l_1-1}}\left(\varepsilon_2\|u^{\frac{p+l_1-1}{p}}\|^p + \varepsilon_2^{\frac{(1-\vartheta_3)r_3}{(1-\vartheta_3)r_3-(p+l_1-1)}}\|u\|_{q+l_1-1}^{q+l_1-1}\right). \tag{3.37}$$

We choose

$$\varepsilon_2 = \Big[\frac{1}{\beta}\Big(\frac{m_0 l_1 p^p}{(p+l_1-1)^p} - \frac{\lambda pL(p,\Omega)}{p+l_1-1} - \frac{\lambda p\ln(R)}{\lambda_1(p+l_1-1)}\Big)\Big]^{\frac{(1-\vartheta_3)r_3-(p+l_1-1)}{-(p+l_1-1)}}.$$

By (3.34) and (3.37) and $\lambda < P_5$, we obtain

$$\frac{1}{l_1+1}\frac{d}{dt}\|u\|_{l_1+1}^{l_1+1} + C_9\|u\|_{l_1+1}^{r_3} \le 0,$$

where

$$C_9 = C_{p_s^*}^{\frac{(\vartheta_3-1)pr_3}{p+l_1-1}}\beta\Big[\frac{1}{\beta}\Big(\frac{m_0 l_1 p^p}{(p+l_1-1)^p} - \frac{\lambda pL(p,\Omega)}{p+l_1-1} - \frac{\lambda p\ln(R)}{\lambda_1(p+l_1-1)}\Big)\Big]^{\frac{(\vartheta_3-1)r_3}{-(p+l_1-1)}},$$

and $C_9 > 0$, which implies that

$$\begin{cases}\|u\|_{l_1+1} \le \left[\|u_0\|_{l_1+1}^{l_1+1-r_3} + (r_3-l_1-1)C_9 t\right]^{\frac{1}{l_1+1-r_3}}, & t\in[0,T_6),\\ \|u\|_{l_1+1} \equiv 0, & t\in[T_6,\infty),\end{cases}$$

where

$$T_6 = \frac{1}{(l_1+1-r_3)C_9}\|u_0\|_{l_1+1}^{l_1+1-r_3}.$$

The proof is completed. □

Proof of Theorem 1.4

We consider first the case $r<2$. When $2N/(N+2)<p<2$, multiplying (1.1) by u, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + M(\|u\|^p)\|u\|^p = \lambda\int_\Omega |u|^r\ln|u|dx - \beta\int_\Omega u^q dx. \tag{3.38}$$

Similar to the Theorem 1.2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + (m_0-\lambda L(r,\Omega)R^{r-p})\|u\|^p \le \lambda\ln(R)|\Omega|^{\frac{2-r}{2}}\|u\|_2^r - \beta\|u\|_q^q. \tag{3.39}$$

Taking $\beta^*=0$ in Lemma 2.3, we have

$$\|u\|_2 \le \|u\|_{p_s^*}^{(1-\vartheta_4)}\|u\|_q^{\vartheta_4} \le C_{p_S^*}^{(1-\vartheta_4)}\|u\|^{(1-\vartheta_4)}\|u\|_q^{\vartheta_4}, \tag{3.40}$$

where $\vartheta_4=\frac{(p_s^*-2)p}{(p_s^*-q)2}$. Then, taking into account that $\vartheta_4\in(0,1)$, we can apply Young's inequality: for every $r_4>0$ and $\varepsilon_3>0$, we have

$$\|u\|_2^{r_4} \le C_{p_s^*}^{(1-\vartheta_4)r_4}\Big(\varepsilon_3\|u\|^p + \varepsilon_3^{\frac{(\vartheta_4-1)r_4}{p-(1-\vartheta_4)r_4}}\|u\|_q^{\frac{\vartheta_4 r_4 p}{p-r_4(1-\vartheta_4)}}\Big). \tag{3.41}$$

Taking

$$r_4=\frac{qp}{q(1-\vartheta_4)+\vartheta_4 p},$$

then $\frac{\vartheta_4 r_4 p}{p-k_2(1-\vartheta_4)}=q$. By (3.39), (3.41), Lemma 2.5 and $\lambda<P_6$, and let $\varepsilon_3=\Big(\frac{m_0-\lambda L(r,\Omega)R^{r-p}}{\beta}\Big)^{\frac{p-(1-\vartheta_4)r_4}{p}}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + C_{10}\|u\|_2^r \le 0, \tag{3.42}$$

where

$$C_{10}=\beta\varepsilon_3^{\frac{(1-\vartheta_4)r_4}{p-(1-\vartheta_4)r_4}}C_{p_s^*}^{(\vartheta_4-1)r_4}R^{r_4-r} - \lambda\ln(R)|\Omega|^{\frac{2-r}{2}}>0,$$

which implies that our result holds.

When $1<p\le 2N/(N+2s)$, we multiply (1.1) by u^{l_1} ($l_1>l\ge 1$), by Lemma 2.4 and Lemma 2.7, we obtain

$$\frac{1}{l_1+1}\frac{d}{dt}\|u\|_{l_1+1}^{l_1+1} + \frac{m_0 l_1 p^p}{(p+l_1-1)^p}\|u^{\frac{l_1+p-1}{p}}\|^p \le \lambda\frac{1}{e\sigma}\|u\|_{r_5}^{r_5} - \beta\|u\|_{q+l_1-1}^{q+l_1-1}, \tag{3.43}$$

where $r_5=l_1+r+\sigma-1$. Using the Holder inequality and (3.37), and we choose

$$\varepsilon_2=\Big[\frac{m_0 l_1 p^p}{\beta(p+l_1-1)^p}\Big]^{\frac{p+l_1-1+(1-\vartheta_3)r_3}{p+l_1-1}},$$

then, we have

$$\frac{1}{l_1+1}\frac{d}{dt}\|u\|_{l_1+1}^{l_1+1} + \beta\varepsilon_2^{\frac{(\vartheta_3-1)r_3}{(1-\vartheta_3)r_3-(p+l_1-1)}}C_{p_s^*}^{\frac{(\vartheta_3-1)pr_3}{p+l_1-1}}\|u\|_{l_1+1}^{r_3} \le \lambda\frac{1}{e\sigma}|\Omega|^{1-\frac{r_5}{l_1+1}}\|u\|_{l_1+1}^{r_5}, \tag{3.44}$$

for $0<\sigma<2-r$. After performing some simple calculations, we finally obtain

$$\frac{1}{l_1+1}\frac{d}{dt}\|u\|_{l_1+1}^{l_1+1} + C_{11}\|u\|_{l_1+1}^{r_3} \le 0, \tag{3.45}$$

where

$$C_{11}=\beta\varepsilon_2^{\frac{(\vartheta_3-1)r_3}{(1-\vartheta_3)r_3-(p+l_1-1)}}C_{p_s^*}^{\frac{(\vartheta_3-1)pr_3}{p+l_1-1}}-\lambda C,\ \ C=\frac{1}{e\sigma}|\Omega|^{1-\frac{r_5}{l_1+1}}C_{l_1+1}^{r_5}R^{r_5-r_3}.$$

Note that $\lambda<P_7$, then $C_{11}>0$, which implies that

$$\begin{cases}\|u\|_{l_1+1}\leq\left[\|u_0\|_{l_1+1}^{l_1+1-r_3}+(r_3-l_1-1)C_{11}t\right]^{\frac{1}{l_1+1-r_3}}, & t\in[0,T_7),\\ \|u\|_{l_1+1}\equiv 0, & t\in[T_7,\infty),\end{cases}$$

where

$$T_7=\frac{1}{(l_1+1-r_3)C_{11}}\|u_0\|_{l_1+1}^{l_1+1-r_3}.$$

The proof is completed. □

Proof of Theorem 1.5

(i) By the definition of $E(u)$ and $I(u)$, we obtain

$$E(u)=\frac{1}{p}I(u)+\frac{1}{p}b(1-\theta)\|u\|^{\theta p}+\frac{\beta(p-q)}{qp}\|u\|_q^q+\lambda\frac{1}{p^2}\|u\|_p^p. \tag{3.46}$$

Choosing $v=u_t$ in Definition 1.1, we have

$$\int_\Omega u_tu_tdx=-M(\|u\|^p)\langle u,u_t\rangle+\lambda\int_\Omega u_t|u|^{r-2}u\ln|u|dx-\beta\int_\Omega u_t|u|^{q-2}udx. \tag{3.47}$$

Note that

$$\frac{d}{dt}E(u)=\frac{1}{p}\frac{d}{dt}(a\|u\|^p+b\|u\|^{\theta p})-\lambda\int_\Omega|u|^{r-2}uu_t\ln|u|dx+\beta\int_\Omega|u|^{q-2}uu_tdx. \tag{3.48}$$

By (3.47) and (3.48), we obtain

$$\frac{d}{dt}E(u)+\int_\Omega u_tu_tdx=0,$$

which implies that

$$E(u)=E(u_0)-\int_0^t\|u_\tau\|_2^2d\tau. \tag{3.49}$$

Setting $\Gamma(t)=\frac{1}{2}\int_\Omega|u(x,t)|^2dx$, then we have

$$\Gamma'(t)=\int_\Omega u_tudx=-I(u). \tag{3.50}$$

By (3.46) and (3.49) and (3.50), we obtain

$$\begin{aligned}\Gamma'(t)=&-pE(u)+b(1-\theta)\|u\|^{\theta p}+\frac{1}{q}\beta(p-q)\|u\|_q^q+\lambda\frac{1}{p}\|u\|_p^p\\ =&-pE(u_0)+b(1-\theta)\|u\|^{\theta p}+\frac{1}{q}\beta(p-q)\|u\|_q^q\\ &+\lambda\frac{1}{p}\|u\|_p^p+p\int_0^t\|u_\tau\|_2^2d\tau.\end{aligned}$$

Since $p > q$, then $\frac{1}{q}\mu(p-q)\|u\|_q^q > 0$, we obtain

$$\Gamma'(t) \geq -pE(u_0) > 0.$$

By a simple calculation, we get

$$\|u\|_2^2 \geq -2pE(u_0)t + 2\|u_0\|_2^2, \text{ for all } t > 0,$$

which implies that our result holds.

(ii) Here, we only prove the case of $E(u_0) < h$, and the proof of $E(u_0) = h$ is similar. Choosing $v = u$ in Definition 1.1, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + M(\|u\|^p)\|u\|^p = \int_\Omega |u|^r \ln|u| dx - \beta\|u\|_q^q, \tag{3.51}$$

namely

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + I(u) = 0. \tag{3.52}$$

Next, taking $v = u_t$ in Definition 1.1 and integrating with respect to time from 0 to t, we have

$$\int_0^t \|u_\tau\|_2^2 dx + E(u(t)) = E(u_0) < h, \text{for } t > 0. \tag{3.53}$$

We claim that $u(x,t) \in W$ for any $t > 0$. If it is false, there exists a $t_0 \in \mathbb{R}_0^+\backslash\{0\}$ such that $u(t_0) \in \partial W$, which implies

$$I(u(x,t_0)) = 0 \text{ or } E(u(x,t_0)) = h.$$

From (3.53), $E(u(t_0)) = h$ is not true. So $u(t_0) \in \mathcal{N}$, then by the definition of h in (1.9), we have $E(u(t_0)) \geq h$, which also contradicts with (3.53). Hence, $u(t_0) \in W$. By (3.52) and $u(t) \in W$ for all $t > 0$, we obtain

$$\|u\|_2^2 \leq \|u_0\|_2^2.$$

□

Remark 2 Compared with problem (1.4), we not only discuss the extinction of weak solutions of problem (1.1) with logarithmic nonlinearity, but also prove that the weak solutions are globally bounded and blow up at infinity.

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Conflict of interest

The authors declare no potential conflict of interests.

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