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Research article

Deterministic and stochastic dynamics of a modified Leslie-Gower prey-predator system with simplified Holling-type IV scheme

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Abstract: In this paper, a prey-predator model with modified Leslie-Gower and simplified Holling-type IV functional responses is proposed to study the dynamic behaviors. For the deterministic system, we analyze the permanence of the system and the stability of the positive equilibrium point. For the stochastic system, we not only prove the existence and uniqueness of global positive solution, but also discuss the persistence in mean and extinction of the populations. In addition, we find that stochastic system has an ergodic stationary distribution under some parameter constraints. Finally, our theoretical results are verified by numerical simulations.

Keywords: permanence; positive equilibrium point; persistence in mean; extinction; ergodic stationary distribution

1. Introduction

In the 1960s, mite pests broke out on fruit trees in Washington state of the United States, resulting in a large area of production reduction. Mite pests can not only cause the leaves of fruit trees to turn green and fall off, but also cause the trees to weaken, affecting the flowering and fruiting of the next year. Mite pests have a high reproductive rate, one female adult mite can reproduce more than 10,000 larvae within a month under suitable conditions. At the same time, mite pests are easy to develop resistance, so it is difficult to control them. Washington state is a major quality apple producing area in the United States, and its cultivation technology and pest control level are in the leading position in the United States. In 1969, Dr. S. C. Hoyt proposed a comprehensive pest management strategy combining chemical control with biological control [1]. The main pest of fruit trees, Tetranychus mcdanieli was controlled by protecting and utilizing its natural enemy, Typhlodromus occidentalis Nesbitt. In 1988, D. J. Wollkind et al. [2] studied a mite predator-prey model with Leslie term and Holling type function, and discussed the dependence between the sub-stability of the system and temperature. The form of the model is as follows:

$$\begin{cases} \dot{x}(t) = ax(t)\left(1 - \dfrac{x(t)}{K}\right) - \dfrac{kx(t)y(t)}{r_1 + x(t)}, \\ \dot{y}(t) = cy(t)\left(1 - \dfrac{y(t)}{r_2 x(t)}\right), \end{cases} \tag{1.1}$$

where $x(0) = x_0$, $y(0) = y_0$. $x(t)$ and $y(t)$ stand for the population density of prey and predator at time t, respectively. a and c are the corresponding intrinsic growth rates of the two populations. k is the maximum per capita consumption rate of prey caused by the predator, and r_1 denotes the density of prey required to reach half of this rate. K and r_2 represent the environmental carrying capacity of the prey and the conversion factor of prey into predator, respectively. $a, c, k, r_1, r_2, K > 0$. The model (1.1) adopts Holling-type II functional response, which is a common response of predatory mites. In 1997, J.B. Collings [3] studied the effects of different functional responses on the dynamic behavior of mite predator-prey interaction models. For Holling-type I, II, and III functional responses, the models have similar dynamic properties in a given temperature range, including bifurcation and stability. If Holling-type IV is used to replace Holling-type II functional response in the system (1.1), the model still has similar properties when the interaction level of prey interference is low. However, when the level of interference is higher, there will be significant differences, such as the emergence of bistability and attractors. Holling-type IV functional response [4] is the following rational nonmonotonic function:

$$h(x) = \frac{kx}{ax^2 + bx + c},$$

also known as Monod-Haldane function. The parameter a measures the level of prey interference with predation. The function reflects that the unit predation rate of predator reaches the maximum with the increase of prey density, and then decreases when it exceeds the critical value. Holling-type IV functional response is mainly used to describe the inhibition effect of microbial population, the group defense of prey population, and the aggregation effect of biological population, such as a fish school that is several miles long may make it difficult for predators to attack.

It is pointed out in [5] that when the density of apple tree pests (such as T. mcdanieli) is high, they will produce a kind of webbing which is known to interfere with predators by decreasing their walking speed and reducing their chances of contacting the prey, so as to effectively reduce the probability of being preyed upon. Sometimes predatory mites even starve in front of the prey. It is reasonable to choose Holling-type IV functional response if we assume that the prey mite and webbing density are directly related. This paper discusses the population dynamics of prey mite interference with predation, which is of great significance to the control of fruit pests.

In 1981, W. Sokol et al. [6] adopted the simplified Holling-type IV functional response function $h(x) = \frac{kx}{r_1 + x^2}$ to model and claimed that it was better consistent with the experimental data. Similarly, [7, 8] used this response function to get the following mites predator-prey model:

$$\begin{cases} \dot{x}(t) = ax(t)\left(1 - \dfrac{x(t)}{K}\right) - \dfrac{kx(t)y(t)}{r_1 + x^2(t)}, \\ \dot{y}(t) = cy(t)\left(1 - \dfrac{y(t)}{r_2 x(t)}\right). \end{cases} \tag{1.2}$$

In the second equation of model (1.2), $\frac{y(t)}{r_2 x(t)}$ is called the Leslie-Gower term [9, 10], which reflects the inverse relationship between the decrease of the predator's number and the per capita availability of their favorite prey. However, due to the influence of environment and season on population growth and reproduction, the number of prey may decrease sharply in a certain period of time. In order to survive, the predator will catch and feed on other prey to get enough food for growth. Considering that the number of prey is insufficient, reference [11] has obtained a modified Leslie-Gower term by adding a positive number to the denominator of the Leslie-Gower term: $\frac{y(t)}{r_2+x(t)}$, where r_2 expresses the quantity of available food different from the predator's favorite prey. By coupling the modified Leslie-Gower term with the system (1.2) to get the mites model as follows [12]:

$$\begin{cases} \dot{x}(t) = x(t)\left(a - bx(t) - \dfrac{ky(t)}{r_1 + x^2(t)}\right), \\ \dot{y}(t) = y(t)\left(c - \dfrac{py(t)}{r_2 + x(t)}\right), \end{cases} \tag{1.3}$$

where b and p signify the competition strength in the prey individuals and the maximum per capita reduction rate of the predator, respectively. $a, b, c, k, r_1, r_2, p > 0$.

Nevertheless, biological populations are inevitably affected by many environmental factors, such as temperature, sunlight, humidity and so on [13]. These disturbances are random, little and independent, and white noise can be used to express the influence of these disturbances. Many scholars have studied the dynamic behavior of stochastic system by introducing environmental white noise into deterministic model [14–21]. This paper assumes that the intrinsic growth rates a and c of system (1.3) are disturbed by environmental white noise: $a \to a + \alpha\dot{B}_1(t)$, $c \to c + \beta\dot{B}_2(t)$. The following stochastic system is obtained:

$$\begin{cases} dx(t) = x(t)\left(a - bx(t) - \dfrac{ky(t)}{r_1 + x^2(t)}\right)dt + \alpha x(t)dB_1(t), \\ dy(t) = y(t)\left(c - \dfrac{py(t)}{r_2 + x(t)}\right)dt + \beta y(t)dB_2(t), \end{cases} \tag{1.4}$$

where $B_1(t)$ and $B_2(t)$ are independent Brownian motion, α^2 and β^2 represent the intensity of white noise.

In this paper, we will compare the deterministic system (1.3) with the stochastic system (1.4), focusing on the long-time dynamics of the populations. In Section 2, we first discuss the permanence of deterministic system (1.3), and then obtain the stability conditions of positive equilibrium by using Routh-Hurwitz criterion and constructing an appropriate Lyapunov function. For the stochastic system (1.4), we demonstrate the existence and uniqueness of the global positive solution. Then the conditions for the persistence in mean and extinction of populations are discussed. And the parameter constraint criterion for the existence of the ergodic stationary distribution is found, which will be presented in Section 3. In Section 4, numerical simulations are used to verify our theoretical results.

2. Deterministic system

2.1. Permanence

Definition 2.1. *[22] If there exists* $0 < M < N$ *such that for any solution* $(x(t), y(t)) \in \mathbb{R}_+^2$ *of the system* (1.3)*, there is*

$$M \leq min\left\{\lim_{t\to+\infty} \inf x(t), \lim_{t\to+\infty} \inf y(t)\right\} \leq max\left\{\lim_{t\to+\infty} \sup x(t), \lim_{t\to+\infty} \sup y(t)\right\} \leq N,$$

then the system (1.3) *is permanent.*

Theorem 2.1. *Let* $(x(t), y(t)) \in \mathbb{R}_+^2$ *be the solution of the system* (1.3) *satisfying the initial condition* $(x(0), y(0)) \in R_+^2$, *then*

$$\frac{1}{b}\left[a - \frac{ck\left(r_2 b + a\right)}{pr_1 b}\right] \leq \lim_{t\to+\infty} \inf x(t) \leq \lim_{t\to+\infty} \sup x(t) \leq \frac{a}{b},$$
$$\frac{cr_2}{p} \leq \lim_{t\to+\infty} \inf y(t) \leq \lim_{t\to+\infty} \sup y(t) \leq \frac{c}{p}\left(r_2 + \frac{a}{b}\right).$$

The system (1.3) *is permanent if* $apr_1b > ck\left(r_2b + a\right)$.

Proof. Derived from system (1.3)

$$\frac{dx}{dt} \leq x(a - bx),$$
$$\frac{dy}{dt} \geq y\left(c - \frac{py}{r_2}\right).$$

On the basis of Lemma 2.3 in [23], we have

$$\lim_{t\to+\infty} \sup x(t) \leq \frac{a}{b}, \tag{2.1}$$

$$\lim_{t\to+\infty} \inf y(t) \geq \frac{cr_2}{p}. \tag{2.2}$$

Therefore, for any $\varepsilon_1 > 0$, there exists $T_1 > 0$; when $t \geq T_1$, there is $x \leq \frac{a}{b} + \varepsilon_1$. So when $t \geq T_1$, we can get

$$\frac{dy}{dt} \leq y\left(c - \frac{py}{r_2 + \frac{a}{b} + \varepsilon_1}\right).$$

In accordance with Lemma 2.3 in [23]

$$\lim_{t\to+\infty} \sup y(t) \leq \frac{c\left(r_2 + \frac{a}{b} + \varepsilon_1\right)}{p},$$

when $\varepsilon_1 \to 0$, we obtain that

$$\lim_{t\to+\infty} \sup y(t) \leq \frac{c\left(r_2 + \frac{a}{b}\right)}{p}. \tag{2.3}$$

From (2.3), for any ε_2, there exists $T_2 > 0$; when $t \geq T_2$, there is $y \leq \frac{c\left(r_2 + \frac{a}{b}\right)}{p} + \varepsilon_2$. Hence, when $t \geq T_2$

$$\frac{dx}{dt} \geq x\left(a - bx - \frac{ky}{r_1}\right) \geq x\left[a - bx - \frac{k\left(cr_2b + ca + bp\varepsilon_2\right)}{pr_1 b}\right].$$

According to Lemma 2.3 in [23]

$$\lim_{t\to+\infty} \inf x(t) \geq \frac{1}{b}\left[a - \frac{k\left(cr_2b + ca + pb\varepsilon_2\right)}{pr_1 b}\right],$$

when $\varepsilon_2 \to 0$, we can see that

$$\lim_{t\to+\infty} \inf x(t) \geq \frac{1}{b}\left[a - \frac{ck\left(r_2 b + a\right)}{pr_1 b}\right]. \tag{2.4}$$

Consequently, we have known that the system (1.3) is permanent if $apr_1b > ck\left(r_2b + a\right)$. □

2.2. Equilibrium points analysis

In this section, we analyze the types of equilibrium points of the system(1.3). $E_1 = (0,0)$ is the trivial equilibrium point. $E_2 = \left(\frac{a}{b}, 0\right)$ and $E_3 = \left(0, \frac{cr_2}{p}\right)$ are two boundary equilibrium points. The internal equilibrium point $E^* = (x^*, y^*)$ is the intersection point of two isoclinic lines $\frac{ky}{r_1+x^2} = a - bx$ and $y = \frac{c(r_2+x)}{p}$ in the interior of the first quadrant. According to the equations

$$\begin{cases} y = \dfrac{1}{k}(a-bx)(x^2+r_1), \\ y = \dfrac{c}{p}(x+r_2), \end{cases}$$

we can obtain

$$H(x) = bx^3 - ax^2 + \left(br_1 + \frac{kc}{p}\right)x - \left(ar_1 - \frac{kcr_2}{p}\right) = 0. \tag{2.5}$$

The positive equilibrium points of the system (1.3) depend on the existence of the positive roots of equation (2.5). According to Ref. [12], it is concluded that:

(i) If $D = ar_1 - \frac{kcr_2}{p} > 0$, equation (2.5) may have one, two or three positive roots in different parameter ranges.

(ii) If $D = ar_1 - \frac{kcr_2}{p} = 0$, equation (2.5) may have one or two positive roots or no positive real roots in different parameter ranges.

(iii) If $D = ar_1 - \frac{kcr_2}{p} < 0$, equation (2.5) may have one or two positive roots or no positive real roots in different parameter ranges.

Therefore, the model (1.3) can have at most three positive equilibrium points, which may be saddle point, node, focus or center under different conditions. By means of numerical simulations, the authors revealed that the system had abundant dynamic properties including noninfinitesimal limit cycle and bifurcation (for details, please refer to Ref. [12]).

In conclusion, if $D = ar_1 - \frac{kcr_2}{p} > 0$, the model (1.3) has at least one positive equilibrium point. Next, we focus on the stability of the positive equilibrium point $E^* = (x^*, y^*)$.

For the convenience of discussion, the following assumptions are given:

(H_1): $a > \frac{ckr_2}{pr_1}$;

(H_2): $a < \frac{ckr_2}{pr_1}$;

(H_3): $\frac{2kx^*y^*}{\left(r_1+x^{*2}\right)^2} < \min\left\{b + \frac{c}{x^*}, b + \frac{ck}{p\left(r_1+x^{*2}\right)}\right\}$;

(H_4): $-b + \frac{ky^*(x^*+\frac{a}{b})}{r_1\left(r_1+x^{*2}\right)} < -\frac{y^*(r_2+\frac{a}{b})}{4p(r_2+x^*)}\left(\frac{p^2}{r_2{}^2} + \frac{k^2}{r_1{}^2}\right)$.

Based on the analysis of the types of equilibrium points in [12], we can know that: $E_1 = (0,0)$ is an unstable node; $E_2 = \left(\frac{a}{b}, 0\right)$ is a saddle point; $E_3 = \left(0, \frac{cr_2}{p}\right)$ is a saddle point if the condition (H_1) holds; $E_3 = \left(0, \frac{cr_2}{p}\right)$ is a stable node if the condition (H_2) holds. We begin to analyze the stability of the positive equilibrium point $E^* = (x^*, y^*)$ in the following content.

Theorem 2.2. *If conditions (H_1) and (H_3) hold, thus $E^* = (x^*, y^*)$ is locally asymptotically stable. Furthermore, if condition (H_4) holds, then $E^* = (x^*, y^*)$ is globally asymptotically stable.*

Proof. The Jacobian matrix at $E^* = (x^*, y^*)$ is

$$J_{(x^*,y^*)} = \begin{pmatrix} -bx^* + \dfrac{2kx^{*2}y^*}{\left(r_1 + x^{*2}\right)^2} & -\dfrac{kx^*}{r_1 + x^{*2}} \\ \dfrac{c^2}{p} & -c \end{pmatrix}.$$

The characteristic equation of $J_{(x^*,y^*)}$ is

$$\lambda^2 + \left(c + bx^* - \frac{2kx^{*2}y^*}{\left(r_1 + x^{*2}\right)^2}\right)\lambda + \left(bx^* - \frac{2kx^{*2}y^*}{\left(r_1 + x^{*2}\right)^2}\right)c + \frac{c^2}{p}\cdot\frac{kx^*}{r_1 + x^{*2}} = 0.$$

By (H_3) we can obtain that $c + bx^* - \frac{2kx^{*2}y^*}{\left(r_1+x^{*2}\right)^2} > 0$, $\left(bx^* - \frac{2kx^{*2}y^*}{\left(r_1+x^{*2}\right)^2}\right) + \frac{c}{p}\cdot\frac{kx^*}{r_1+x^{*2}} > 0$. Thus, according to the Routh-Hurwitz criterion, $E^* = (x^*, y^*)$ is locally asymptotically stable.

Next, we study the global asymptotic stability of E^*, as long as it is globally attractive. Define

$$V(x, y) = \int_{x^*}^{x} \frac{\xi - x^*}{\xi} d\xi + h \int_{y^*}^{y} \frac{\eta - y^*}{\eta} d\eta,$$

where h is a positive number, as defined in the following text. We can see that $V(x, y)$ is a non-negative function, $V(x, y) = 0$ if and only if $(x, y) = (x^*, y^*)$.

$$\begin{aligned} \frac{dV}{dt} &= \frac{x - x^*}{x}\cdot x\left(a - bx - \frac{ky}{r_1 + x^2}\right) + h\cdot\frac{y - y^*}{y}\cdot y\left(c - \frac{py}{r_2 + x}\right) \\ &= (x - x^*)\left(a - bx - \frac{ky}{r_1 + x^2} - a + bx^* + \frac{ky^*}{r_1 + x^{*2}}\right) + h(y - y^*)\left(c - \frac{py}{r_2 + x} - c + \frac{py^*}{r_2 + x^*}\right) \\ &= -b(x - x^*)^2 + k(x - x^*)\frac{y^*(x + x^*)(x - x^*) - \left(r_1 + x^{*2}\right)(y - y^*)}{\left(r_1 + x^{*2}\right)(r_1 + x^2)} \\ &\quad + ph(y - y^*)\frac{y^*(x - x^*) - (r_2 + x^*)(y - y^*)}{(r_2 + x^*)(r_2 + x)}. \end{aligned}$$

Choose $h = \dfrac{r_2 + x^*}{y^*}$, hence

$$\begin{aligned} \frac{dV}{dt} =& (x - x^*)^2\left[-b + \frac{ky^*(x + x^*)}{\left(r_1 + x^{*2}\right)(r_1 + x^2)}\right] + (x - x^*)(y - y^*)\left(\frac{p}{r_2 + x} - \frac{k}{r_1 + x^2}\right) \\ &- \frac{p(r_2 + x^*)(y - y^*)^2}{y^*(r_2 + x)}. \end{aligned}$$

The above equation can be written as

$$\frac{dV}{dt} = -(x - x^*, y - y^*)\begin{pmatrix} -A & -B \\ -B & C \end{pmatrix}\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix},$$

where

$$A = -b + \frac{ky^*(x+x^*)}{\left(r_1 + x^{*2}\right)\left(r_1 + x^2\right)},$$

$$B = \frac{1}{2}\left(\frac{p}{r_2 + x} - \frac{k}{r_1 + x^2}\right),$$

$$C = \frac{p(r_2 + x^*)}{y^*(r_2 + x)}.$$

Then $\frac{dV}{dt} < 0$ if the above matrix is a positive definite matrix(see Ref. [11]). According to the judgment of positive definite matrix, we only need to prove that its leading principal minors are positive. That is to say:

$(i) : -A > 0$,

$(ii) : -AC - B^2 > 0$.

In fact, if (H_4) holds, then $A < 0$ is obviously true.

And

$$\begin{aligned} AC + B^2 &= \left[-b + \frac{ky^*(x+x^*)}{\left(r_1 + x^{*2}\right)\left(r_1 + x^2\right)}\right] \cdot \frac{p(r_2 + x^*)}{y^*(r_2 + x)} + \frac{1}{4}\left(\frac{p}{r_2 + x} - \frac{k}{r_1 + x^2}\right)^2 \\ &< \left[-b + \frac{ky^*\left(\frac{a}{b} + x^*\right)}{r_1\left(r_1 + x^{*2}\right)}\right] \cdot \frac{p(r_2 + x^*)}{y^*(r_2 + \frac{a}{b})} + \frac{1}{4}\left(\frac{p^2}{{r_2}^2} + \frac{k^2}{{r_1}^2}\right). \end{aligned}$$

From the condition (H_4), $AC + B^2 < 0$ clearly. Hence, $\frac{dV}{dt} < 0$ for any point except the equilibrium point $E^* = (x^*, y^*)$, and $\frac{dV}{dt}|(x^*, y^*) = 0$. In the end, we can conclude that E^* is globally asymptotically stable. □

The global stability of the positive equilibrium point in the deterministic system means that under certain conditions, the two populations will coexist with each other over time and the population density will be stable near the positive equilibrium point.

3. Stochastic system

In this section, we introduce environmental white noise into the deterministic system (1.3) to obtain the stochastic system (1.4). Next, we consider the influence of white noise on the dynamic behavior of the system (1.4).

3.1. The existence and uniqueness of global positive solution

We consider whether system (1.4) has a unique global positive solution for any given initial value $(x(0), y(0)) \in \mathbb{R}_+^2$. Generally, we will first prove that there exists a unique local positive solution, and then show that the solution is also global.

Theorem 3.1. *For any given initial value* $(x(0), y(0)) \in \mathbb{R}_+^2$ *, the stochastic system* (1.4) *has unique global positive solution* $(x(t), y(t)) \in \mathbb{R}_+^2$ *for all* $t \geq 0$ *and exists in* $\mathbb{R}_+^2$ *with probability* 1.

Proof. The equation coefficients of the stochastic system (1.4) neither satisfy the linear growth condition, nor the local Lipschitz condition [24]. Let $u(t) = \ln x(t), v(t) = \ln y(t)$, then the system (1.4) is transformed into the following equivalent system:

$$\begin{cases} du(t) = \left(a - \dfrac{\alpha^2}{2} - b\, e^{u(t)} - \dfrac{k\, e^{v(t)}}{r_1 + e^{2u(t)}}\right) dt + \alpha dB_1(t), \\ dv(t) = \left(c - \dfrac{\beta^2}{2} - \dfrac{p\, e^{v(t)}}{r_2 + e^{u(t)}}\right) dt + \beta dB_2(t). \end{cases} \tag{3.1}$$

Obviously, the coefficients of equation (3.1) satisfy the local Lipschitz condition, then (3.1) exists a unique local solution $(u(t), v(t)), t \in [0, \tau_e)$. τ_e is the explosion time which means that $\lim_{t\to\tau_e} |u(t)| = \infty$ or $\lim_{t\to\tau_e} |v(t)| = \infty$. Since the (3.1) is equivalent to the (1.4), system (1.4) has a unique local solution. On the basis of Itô's formula, it can be known that $(x(t), y(t)) = \left(e^{u(t)}, e^{v(t)}\right)$ is the only local solution satisfying the initial value $(x(0), y(0)) \in \mathbb{R}_+^2$ for the stochastic system (1.4). In order to prove the solution is global, we only need to prove $\tau_e = +\infty$.

The following proof process is similar to [25], we can refer to the detailed proof of it. In the proof process, the most momentous thing is to construct a suitable Lyapunov function.

We define a C^2-function $V(x, y) : \mathbb{R}_+^2 \to \overline{\mathbb{R}}_+$,

$$V(x, y) = x - \log x + y - \log y.$$

Its nonnegativity can be determined by $z - \log z \geq 1$ for any $z > 0$. In the light of Itô's formula to get

$$\begin{aligned} LV(x, y) &= (x - 1)\left(a - bx - \frac{ky}{r_1 + x^2}\right) + (y - 1)\left(c - \frac{py}{r_2 + x}\right) + \frac{\alpha^2}{2} + \frac{\beta^2}{2} \\ &\leq (a + b)\, x + (c + \frac{k}{r_1} + \frac{p}{r_2})y + (-a - c + \frac{\alpha^2}{2} + \frac{\beta^2}{2}). \end{aligned}$$

Define two positive constants

$$h_1 = (a + b),\ h_2 = (c + \frac{k}{r_1} + \frac{p}{r_2}).$$

Then

$$LV(x, y) \leq h_1 x + h_2 y + (-a - c + \frac{\alpha^2}{2} + \frac{\beta^2}{2}).$$

By the inequality $x \leq 2(x - 1 - \log x) + \log 4,\ y \leq 2(y - 1 - \log y) + \log 4$,

$$\begin{aligned} LV(x, y) &\leq 2h_1(x - 1 - \log x) + 2h_2(y - 1 - \log y) + h_1 \log 4 + h_2 \log 4 + (-a - c + \frac{\alpha^2}{2} + \frac{\beta^2}{2}) \\ &\leq 2(h_1 + h_2)V(x, y) + (h_1 + h_2) \log 4 + (-a - c + \frac{\alpha^2}{2} + \frac{\beta^2}{2}) \\ &\leq h_3(1 + V(x, y)), \end{aligned} \tag{3.2}$$

where $h_3 = \max\{2(h_1 + h_2), (h_1 + h_2) \log 4 + (-a - c + \frac{\alpha^2}{2} + \frac{\beta^2}{2})\}$. From (3.2), we have

$$\begin{aligned} dV(x, y) &= LV(x, y)\, dt + \alpha\, (x - 1)\, dB_1(t) + \beta\, (y - 1)\, dB_2(t) \\ &\leq h_3(1 + V(x, y))dt + \alpha\, (x - 1)\, dB_1(t) + \beta\, (y - 1)\, dB_2(t). \end{aligned}$$

Integrating both sides from 0 to $\tau_k \wedge T$ (τ_k is a stopping time and $T > 0$ is a constant, as defined in Ref. [25]) and taking the expectation

$$\begin{aligned} EV(x(\tau_k \wedge T), y(\tau_k \wedge T)) &\le V(x(0), y(0)) + E\int_0^{\tau_k \wedge T} h_3(1 + V(x(t), y(t)))dt \\ &\le V(x(0), y(0)) + h_3 T + h_3 E\int_0^{\tau_k \wedge T} V(x(t), y(t))dt \\ &\le V(x(0), y(0)) + h_3 T + h_3 \int_0^{T} EV(x(\tau_k \wedge t), y(\tau_k \wedge t))dt. \end{aligned}$$

By Gronwall inequality

$$E\left(V\left(x(\tau_k \wedge T), y(\tau_k \wedge T)\right)\right) \le \left(V\left(x(0), y(0)\right) + h_3 T\right) e^{h_3 T}.$$

The following proof is the same as [25], so we omit it here. □

3.2. Extinction and persistence in mean

In this section, we demonstrate that population extinction and persistence in mean will occur under some parameter constraints.

First of all, we give condition (H_5): $k^2 + 4p(r_2 k - r_1 p) \le 0$, then there is:

Theorem 3.2. *For the stochastic system* (1.4)*, the following conclusions hold:*

(i) If $a < \frac{\alpha^2}{2}$, $c < \frac{\beta^2}{2}$, both the predator and the prey are extinct.

(ii) If $a > \frac{\alpha^2}{2}$, $c < \frac{\beta^2}{2}$, the prey $x(t)$ is persistent in mean and the predator $y(t)$ is extinct.

(iii) If $a < \frac{\alpha^2}{2}$, $c > \frac{\beta^2}{2}$, the prey $x(t)$ is extinct and the predator $y(t)$ is persistent in mean .

(iv) If $a - \frac{\alpha^2}{2} > c - \frac{\beta^2}{2} > 0$ and the condition (H_5) is satisfied, both the prey $x(t)$ and the predator $y(t)$ are persistent in mean.

Proof. According to Itô's formula

$$d\log x(t) = \left(a - bx(t) - \frac{ky(t)}{r_1 + x^2(t)} - \frac{\alpha^2}{2}\right)dt + \alpha dB_1(t),$$

$$d\log y(t) = \left(c - \frac{py(t)}{r_2 + x(t)} - \frac{\beta^2}{2}\right)dt + \beta dB_2(t).$$

Integrating both sides from 0 to t and dividing t, respectively

$$\frac{\log x(t)}{t} = \frac{\log x(0)}{t} + \left(a - \frac{\alpha^2}{2}\right) - \frac{b}{t}\int_0^t x(s)ds - \frac{1}{t}\int_0^t \frac{ky(s)}{r_1 + x^2(s)}ds + \frac{\alpha}{t}\int_0^t dB_1(s), \tag{3.3}$$

$$\frac{\log y(t)}{t} = \frac{\log y(0)}{t} + \left(c - \frac{\beta^2}{2}\right) - \frac{1}{t}\int_0^t \frac{py(s)}{r_2 + x(s)}ds + \frac{\beta}{t}\int_0^t dB_2(s). \tag{3.4}$$

(i) Applying the strong law of large number for local martingales, it can be seen from (3.3) and (3.4)

$$\lim_{t\to+\infty} \frac{\log x(t)}{t} \le a - \frac{\alpha^2}{2} < 0,$$

$$\lim_{t\to+\infty}\frac{\log y(t)}{t}\leq c-\frac{\beta^2}{2}<0.$$

Consequently

$$\lim_{t\to+\infty}x(t)=0,\ \lim_{t\to+\infty}y(t)=0.$$

Then, the prey $x(t)$ and the predator $y(t)$ are both extinct.

(ii) From (3.4), we have $\lim\limits_{t\to+\infty}y(t)=0$. In this case, the limit system of (1.4) is

$$dx(t)=x(t)\left(a-bx(t)\right)dt+\alpha x(t)dB_1(t).$$

Therefore, on the basis of Lemma A.1 in [26]

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t x(s)ds=\frac{a-\frac{\alpha^2}{2}}{b}>0,$$

which means that the prey $x(t)$ is persistent in mean.

(iii) According to (3.3), we can obviously get $\lim\limits_{t\to+\infty}x(t)=0$. Moreover, the limit system of (1.4) is

$$dy(t)=y(t)\left(c-\frac{p}{r_2}y(t)\right)dt+\beta y(t)dB_2(t).$$

Similarly

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t y(s)ds=\frac{r_2(c-\frac{\beta^2}{2})}{p}>0.$$

It signifies that the predator $y(t)$ is persistent in mean.

(iv) In the light of (3.4)

$$\frac{\log y(t)}{t}\geq\frac{\log y(0)}{t}+\left(c-\frac{\beta^2}{2}\right)-\frac{1}{t}\int_0^t\frac{py(s)}{r_2}ds+\frac{\beta}{t}\int_0^t dB_2(s).$$

Thereupon, in view of Lemma 2 in [27]

$$\lim_{t\to+\infty}\inf\frac{1}{t}\int_0^t y(s)ds\geq\frac{r_2\left(c-\frac{\beta^2}{2}\right)}{p}.$$

In the same way, combining (3.4) and Lemma 2 of [28], we can obtain

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t\frac{py(s)}{r_2+x(s)}ds=c-\frac{\beta^2}{2}.$$

According to the condition (H_5), there is $\dfrac{k}{r_1+x^2}\leq\dfrac{p}{r_2+x}$, then combining (3.3) to get

$$\begin{aligned}\frac{\log x(t)}{t}&=\frac{\log x(0)}{t}+\left(a-\frac{\alpha^2}{2}\right)-\frac{b}{t}\int_0^t x(s)ds-\frac{1}{t}\int_0^t\frac{ky(s)}{r_1+x^2(s)}ds+\frac{\alpha}{t}\int_0^t dB_1(s)\\&\geq\frac{\log x(0)}{t}+\left(a-\frac{\alpha^2}{2}\right)-\frac{b}{t}\int_0^t x(s)ds-\frac{1}{t}\int_0^t\frac{py(s)}{r_2+x(s)}ds+\frac{\alpha}{t}\int_0^t dB_1(s).\end{aligned}$$

Similarly, we get

$$\lim_{t\to+\infty}\inf\frac{1}{t}\int_0^t x(s)ds\geq\frac{a-\frac{\alpha^2}{2}-\left(c-\frac{\beta^2}{2}\right)}{b}.$$

As a result, both the prey $x(t)$ and the predator $y(t)$ are persistent in mean. □

3.3. Ergodic stationary distribution

We can testify the stability of the deterministic system by analyzing the local and global stability of the equilibrium points. However, there are no equilibrium points in the stochastic system, so we can not use the equilibrium points theory to certify the stability. Therefore, we utilize the stationary distribution to depict the stochastic weak stability. And ergodicity means that the system is persistent [25]. Under some conditions, we prove that the stochastic system (1.4) has an ergodic stationary distribution, which will be given in the following.

First, we define a parameter:

$$\lambda = \left(a - \frac{\alpha^2}{2}\right) - \left(c - \frac{\beta^2}{2}\right) - \frac{p}{r_2},$$

then

Theorem 3.3. *If the parameters of the system* (1.4) *satisfy* $c - \beta^2 > 0,\ \lambda > 0$ *and condition* (H_5)*, then system* (1.4) *has unique ergodic stationary distribution.*

Proof. For the sake of proving Theorem 3.3, we simply need to confirm the two conditions in Lemma 2.1 [29]. At first, the diffusion matrix of system (1.4) is

$$A = \begin{pmatrix} \alpha^2 x^2 & 0 \\ 0 & \beta^2 y^2 \end{pmatrix}.$$

We think that U is a bounded domain, and it is apparent that there exists

$$M = \min_{(x,y)\in\overline{U}\subset\mathbb{R}_+^2} \left\{\alpha^2 x^2, \beta^2 y^2\right\} > 0,$$

such that

$$\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j = \alpha^2 x^2 \xi_1{}^2 + \beta^2 y^2 \xi_2{}^2 \geq M \| \xi \|^2 \text{ for all } (x,y) \in \overline{U}, \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Then the first condition holds, and next we verify the second condition.

Define the C^2-function $\overline{V}(x,y) : \mathbb{R}_+^2 \to \mathbb{R}_+$

$$\overline{V}(x,y) = x - 1 - \log x + \frac{1}{y} + \log y.$$

Obviously, $\overline{V}(x,y)$ has a unique stationary point $(1,1)$. It is easy to calculate that $(1,1)$ is the minimum point and the minimum value is $\overline{V}(1,1) = 1$. Define the Lyapunov function

$$\widetilde{V}(x,y) = \overline{V}(x,y) - \overline{V}(1,1) = V_1(x,y) + V_2(x,y) - 1,$$

where $V_1(x,y) = x - 1 - \log x,\ V_2(x,y) = \dfrac{1}{y} + \log y.$

Utilizing Itô's formula and condition (H_5), we have

$$\begin{aligned} LV_1(x,y) &= (x-1)\left(a - bx - \frac{ky}{r_1 + x^2}\right) + \frac{1}{2}\alpha^2 \\ &= -b\,x^2 + (a+b)x - \frac{kxy}{r_1 + x^2} + \frac{ky}{r_1 + x^2} - (a - \frac{1}{2}\alpha^2) \\ &\leq -b\,x^2 + (a+b)x - \frac{kxy}{r_1 + x^2} + \frac{py}{r_2 + x} - (a - \frac{1}{2}\alpha^2). \end{aligned}$$

$$
\begin{aligned}
LV_2(x,y) &= \left(-\frac{1}{y^2}+\frac{1}{y}\right)\cdot y\left(c-\frac{py}{r_2+x}\right)+\frac{1}{2}\beta^2 y^2\left(\frac{2}{y^3}-\frac{1}{y^2}\right)\\
&= -\frac{1}{y}(c-\beta^2)+\frac{p}{r_2+x}-\frac{py}{r_2+x}+(c-\frac{1}{2}\beta^2)\\
&\le -\frac{1}{y}(c-\beta^2)-\frac{py}{r_2+x}+(c-\frac{1}{2}\beta^2)+\frac{p}{r_2}.
\end{aligned}
$$

Then

$$
L\widetilde{V}(x,y) \le -b\,x^2 +(a+b)x-\frac{kxy}{r_1+x^2}-\frac{1}{y}(c-\beta^2)-\lambda.
$$

We choose sufficiently small $\varepsilon_1, \varepsilon_2$ such that

$$
0<\varepsilon_1<\frac{\lambda}{2\,(a+b),} \tag{3.5}
$$

$$
0<\varepsilon_2<\frac{4b(c-\beta^2)}{4b+(a+b)^2}, \tag{3.6}
$$

$$
0<{\varepsilon_1}^2<\frac{b^2}{2b+(a+b)^2}, \tag{3.7}
$$

$$
0<\varepsilon_2<\frac{k\varepsilon_1^2}{(r_1\varepsilon_1^2+1)(a+b)}. \tag{3.8}
$$

And we consider bounded open set

$$
U_{\varepsilon 1,2}=\left\{(x,y)\in\mathbb{R}_+^2|\varepsilon_1<x<\frac{1}{\varepsilon_1},\varepsilon_2<y<\frac{1}{\varepsilon_2}\right\}.
$$

Denote

$$
U_1=\left\{(x,y)\in\mathbb{R}_+^2|0<x\le\varepsilon_1\right\},\qquad U_2=\left\{(x,y)\in\mathbb{R}_+^2|0<y\le\varepsilon_2\right\},
$$

$$
U_3=\left\{(x,y)\in\mathbb{R}_+^2|x\ge\frac{1}{\varepsilon_1}\right\},\qquad U_4=\left\{(x,y)\in\mathbb{R}_+^2|\varepsilon_1<x\le\frac{1}{\varepsilon_1},y\ge\frac{1}{\varepsilon_2}\right\}.
$$

Evidently, $U^c_{\varepsilon 1,2} = U_1\cup U_2\cup U_3\cup U_4$. Now we just need to demonstrate that $L\widetilde{V}(x,y)$ is negative for any $(x,y)\in U^c_{\varepsilon 1,2}$.

Case1. When $(x,y)\in U_1$, in the light of (3.5), we can get

$$
\begin{aligned}
L\widetilde{V}(x,y) &\le -b\,x^2 +(a+b)x-\frac{kxy}{r_1+x^2}-\frac{1}{y}(c-\beta^2)-\lambda\\
&\le (a+b)\varepsilon_1-\lambda\\
&\le -\frac{\lambda}{2}.
\end{aligned}
$$

Case2. When $(x, y) \in U_2$, in consideration of (3.6), it is obvious that

$$\begin{aligned}
L\widetilde{V}(x, y) &\leq -b\, x^2 + (a+b)x - \frac{kxy}{r_1 + x^2} - \frac{1}{y}(c - \beta^2) - \lambda \\
&\leq -\frac{1}{y}(c - \beta^2) - b\, x^2 + (a+b)x \\
&\leq -\frac{1}{\varepsilon_2}(c - \beta^2) + \frac{(a+b)^2}{4b} \\
&\leq -1.
\end{aligned}$$

Case3. When $(x, y) \in U_3$, according to (3.7), we can see that

$$\begin{aligned}
L\widetilde{V}(x, y) &\leq -b\, x^2 + (a+b)x - \frac{kxy}{r_1 + x^2} - \frac{1}{y}(c - \beta^2) - \lambda \\
&\leq -\frac{b}{2}\, x^2 - \frac{b}{2}\, x^2 + (a+b)x \\
&\leq -\frac{b}{2\, {\varepsilon_1}^2} + \frac{(a+b)^2}{2b} \\
&\leq -1.
\end{aligned}$$

Case4. When $(x, y) \in U_4$, in view of (3.8) to have

$$\begin{aligned}
L\widetilde{V}(x, y) &\leq -b\, x^2 + (a+b)x - \frac{kxy}{r_1 + x^2} - \frac{1}{y}(c - \beta^2) - \lambda \\
&\leq (a+b)x - \frac{kxy}{r_1 + x^2} - \lambda \\
&\leq x\left[(a+b) - \frac{k\, {\varepsilon_1}^2}{\varepsilon_2(r_1\, {\varepsilon_1}^2 + 1)}\right] - \lambda \\
&\leq -\lambda.
\end{aligned}$$

Hence, $L\widetilde{V}(x, y)$ is negative for any $(x, y) \in U^c_{\varepsilon 1,2}$, the second condition is proved. Apparently, we have verified the conclusion of Theorem 3.3. □

4. Numerical simulations and conclusions

In this paper, a modified Leslie-Gower predation model is established, and three ecological issues are considered. First, the functional response of predator to prey is a rational non-monotone function. The second is that the predator is omnivore and will look for alternative food when the preferred food is scarce. Third, the intrinsic growth rates of the populations are disturbed by white noise. Coupling the environmental white noise into the differential equation for modeling is more consistent with the survival environment of the population. By comparing the deterministic system with the corresponding stochastic system, we obtain rich and interesting dynamic properties. To our knowledge, there are few studies that consider the effects of Holling-type IV functional response, generalist predators and environmental noise disturbance to model simultaneously.

First of all, we testify that the deterministic system (1.3) is permanent. Then we prove the stability of the positive equilibrium point E^* by applying the Routh-Hurwitz criterion and constructing a suitable Lyapunov function. Next, we use numerical simulations to illustrate the stability of E^*. We choose parameters:

$$a = 4, b = 0.8, k = 0.7, r_1 = 4, c = 0.2, p = 0.4, r_2 = 1.$$

By calculating we can know that (H_1) is satisfied, which means that the system (1.3) has the positive equilibrium point E^*. Through further calculation, the condition (H_3) and (H_4) are also satisfied. Then $E^* = (4.91, 2.95)$ is globally asymptotically stable. Figure 1 is obtained by numerical simulations, which describes the trend of solution with the initial value $(0.2, 0.1)$ (see Figure 1).

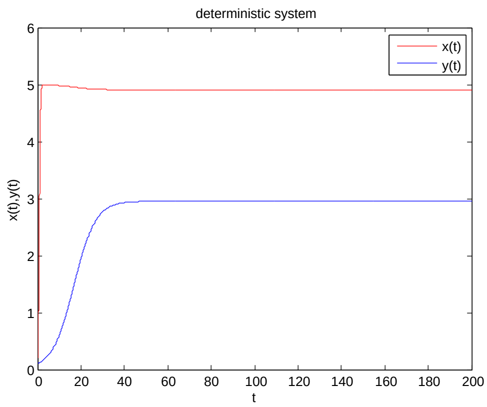

Figure 1. Solution trajectories of the deterministic system (1.3) with the given initial value.

In addition, we research the dynamic behavior of the stochastic system (1.4), which takes into account the influence of white noise. To start with, we prove the existence and uniqueness of global positive solution. Secondly, we discover the conditions that make the population persistent in mean or extinct in four cases. And we conclude that when the white noise intensity is small enough, the population is persistent in mean. Nevertheless, when the intensity of white noise increases gradually, the population will be threatened and tend to be extinct. Furthermore, it can be seen that when the white noise of the stochastic system (1.4) is small, the population densities fluctuate around the deterministic steady-state values of x^* and y^*. Next, we apply numerical simulations to validate our conclusions.

In order to satisfy the conditions of Theorem 3.2, we take $\alpha = 3.4, \beta = 1.5$; $\alpha = 0.05, \beta = 1$; $\alpha = 3.3, \beta = 0.04$; $\alpha = 0.05, \beta = 0.05$, respectively. Other parameters in model (1.4) have the same values as Figure 1. Using Matlab to get the following images, which conform to our theoretical results(see Figures 2–5).

On the other hand, there is a stationary distribution in system (1.4), which means that the two populations tend to coexist for a long time and the system is weakly stable. Similarly, the other parameters are the same as in Figure 1, we take $\alpha = 0.06$, $\beta = 0.04$ to satisfy the conditions of Theorem 3.3, and we can see from the images that the densities of the two populations are close to the positive equilibrium point of x^* and y^* (see Figures 6–8).

As for the analysis of two systems, we can know that the difference between the stochastic and deterministic systems is that the stochastic system will inevitably be disturbed by the environment. To study the stochastic system is more in line with the characteristics of population environment variability. By comparing the two systems with numerical simulations, we can clearly see that when

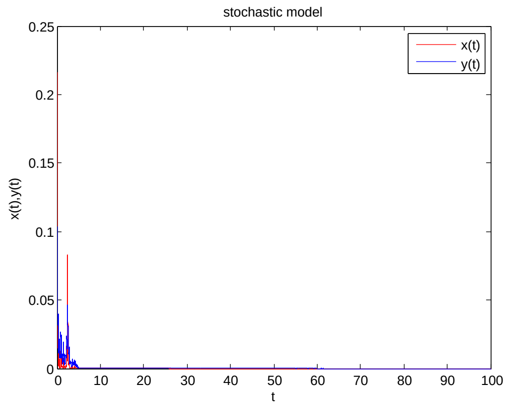

Figure 2. Numerical simulation of the stochastic system (1.4) shows that prey and predator are extinct.

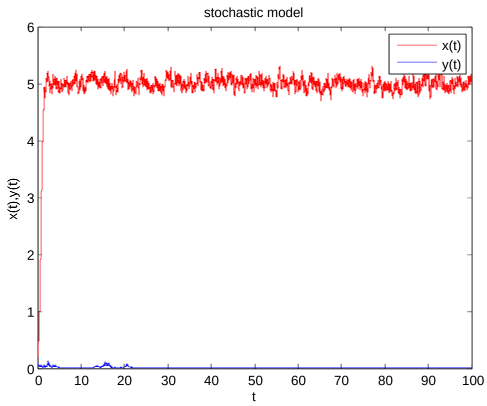

Figure 3. Numerical simulation of the stochastic system (1.4) shows that prey is persistent in mean and predator is extinct.

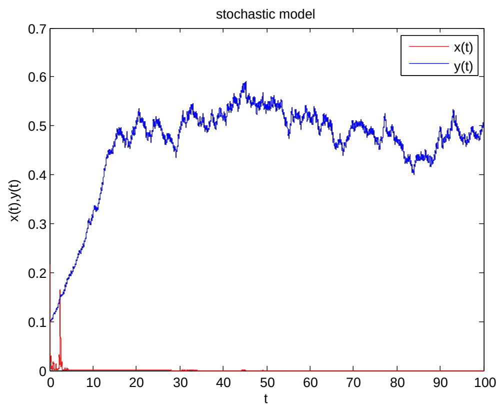

Figure 4. Numerical simulation of the stochastic system (1.4) shows that prey is extinct and predator is persistent in mean.

the white noise intensity is small, the population is persistent and fluctuates around the equilibrium state of the deterministic model. As the noise intensity increases, the population density oscillates and

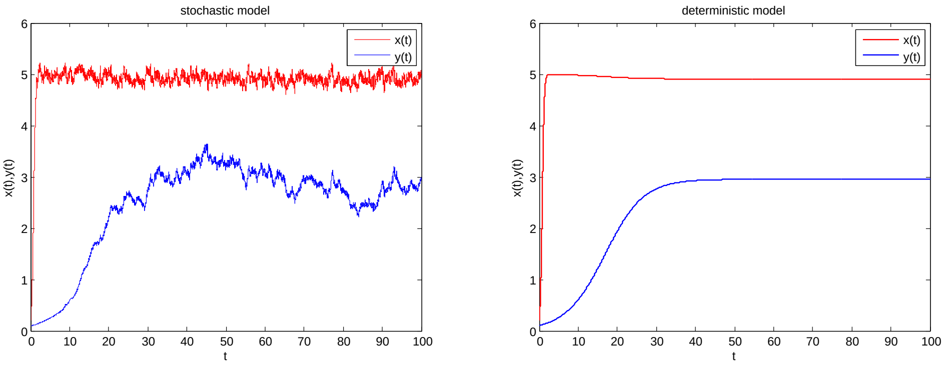

Figure 5. Numerical simulation of stochastic system (1.4) shows that the populations are both persistent in mean and the density values fluctuate around the deterministic steady-state values respectively.

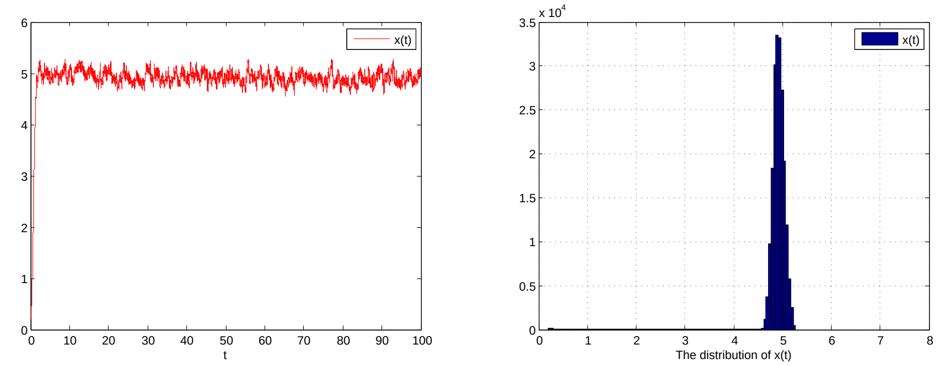

Figure 6. The path and density function distribution of prey x(t) with $\alpha = 0.06$.

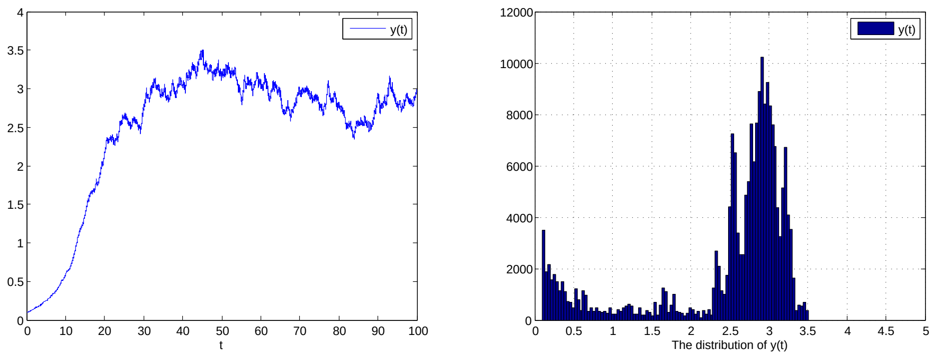

Figure 7. The path and density function distribution of predator y(t) with $\beta = 0.04$.

may eventually lead to extinction. Therefore, the existence of white noise is not conducive to the long-term survival of the population. From the point of view of pest management, white noise is a favorable factor and can be used for biological control. Obviously, the study of environmental noise disturbance is of great significance not only for integrated pest management but also for the development and utilization of biological resources.

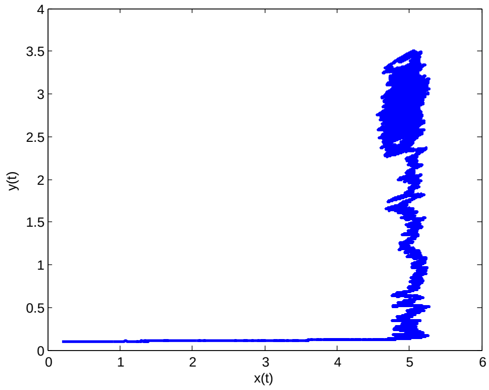

Figure 8. Sample path distribution of solution of stochastic system (1.4) in phase space.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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