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Research article

Stability and Hopf bifurcation for a delayed diffusive competition model with saturation effect

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Abstract: This paper presents an investigation on the dynamics of a delayed diffusive competition model with saturation effect. We first perform the stability analysis of the positive equilibrium and the existence of Hopf bifurcations. It is shown that the positive equilibrium is asymptotically stable under some conditions, and that there exists a critical value of delay, when the delay increases across it, the positive equilibrium loses its stability and a spatially homogeneous or inhomogeneous periodic solution emerges from the positive equilibrium. Then, we derive the formulas for the determination of the direction of Hopf bifurcation and the properties of the bifurcating periodic solutions. Finally, some numerical simulations are performed to illustrate the obtained results.

Keywords: competitive model; time delay; diffusion; stability; Hopf bifurcation

1. Introduction

The competition model is an important part in the study of population dynamics and has been discussed widely [1–8]. The classical two-dimension Lotka-Volterra competition model is constructed based on the law of mass action, that is, the competitive capacity of one species is assumed to be proportional to the number or density of its competitive species. Though the classical Lotka-Volterra competition model and its various extensions have been extensively studied in the literature [9–16], it has a deficiency that the competitive capacity of one species may increase and go to infinity with the increase of its competitors density.

Let $u = u(t)$ and $v = v(t)$ be respectively the densities of two competitive species. The following model with saturation competitive capacity has properly overcome the above deficiency of the classical

Lotka-Volterra competition model:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u\left(a - bu - \dfrac{cv}{1+v}\right), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(d - ev - \dfrac{fu}{1+u}\right), \end{cases} \tag{1}$$

where a and d are the two species' respective intrinsic growth rates; b, e and c, f are their respective intra- and inter-specific interaction coefficients. The parameters here are all assumed to be positive. We refer the readers to references [17–19] for some related extension works.

Due to the ubiquitousness of spatial diffusion in the real world, the spatiotemporal dynamics of the reaction-diffusion equation have recently been increasingly discussed by many authors [20, 22–26]. For example, one may focus on the pattern formation of biological tissues to reveal the mechanism behind the forms [26]. In addition, in order to better understand the interaction between prey and the predators, some diffusive population models are discussed to illustrate the influence of diffusion and the way in which different individuals interact (cooperation, competition, herd behaviors etc.) [20, 22–25]. To observe how the diffusion affects the dynamical behaviour of model (1), in a recent paper [21], we proposed and analyzed a diffusive competition model with saturated interaction terms, and showed that cross-diffusion is of great importance in the spatial pattern formation. Besides, time delay is another important component in constructing population models. Researchers investigated the effects of both diffusion and time delay, and have observed some interesting phenomena, such as Turing pattern [27–29], stability [30, 31] and the occurrence of Hopf bifurcation induced by time delay [32–34].

Motivated by the above mentioned biological facts and the observed nonlinear phenomena, we propose the following diffusive competition model with delay:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - d_1\Delta u(x,t) = u(x,t)\left(a - bu(t-\tau) - \dfrac{cv(t-\tau)}{1+v(t-\tau)}\right), \\ \dfrac{\partial v(x,t)}{\partial t} - d_2\Delta v(x,t) = v(x,t)\left(d - ev(t-\tau) - \dfrac{fu(t-\tau)}{1+u(t-\tau)}\right), \\ u_x(0,t) = u_x(\pi,t) = v_x(0,t) = v_x(\pi,t) = 0, \ \ t \geq 0, \\ u(x,t) = \phi_1(x,t) \geq 0, v(x,t) = \phi_2(x,t) \geq 0, \ \ (x,t) \in [0,\pi] \times [-\tau, 0], \end{cases} \tag{2}$$

where d_1 and d_2 are respectively the diffusion constants for the two species, and Δ the usual Laplacian operator. The other symbols play similar roles as in model (1). The main objective of this paper is to study the effect of time delay on population dynamics of model (2). The conclusions show that delay may destabilize the stable positive equilibrium, resulting the spatially homogeneous or nonhomogeneous Hopf bifurcations.

The organization of this paper is as follows. In Section 2, by analyzing the characteristic equation, the stability of positive equilibrium and the existence of Hopf bifurcations are studied. In Section 3, we obtain the formulae that determine the direction and stability of the spatial Hopf bifurcation by applying the normal form theory and the center manifold theorem. In Section 4, some numerical simulations are given to demonstrate the theoretical results, and finally, some conclusion and discussion are presented in Section 5.

2. Stability of the positive equilibrium and existence of Hopf bifurcations

We have shown in [21] that when $[(a+b)d-af][(e+d)a-cd]>0$, model (1) has a unique positive constant equilibrium $E_* = (u_*, v_*)$, which is global asymptotically stable provided

$$(H):\quad (a+b)d > af,\ (e+d)a > cd,$$

otherwise it is unstable. Note that model (1) has the same constant equilibrium as that in model (2). Now we discuss the stability of E_* and the existence of Hopf bifurcation for system (2). Under the hypothesis (H), we shift the equilibrium E_* to the origin by letting $\tilde{u}(x,t) = u(x,t) - u_*$, $\tilde{v}(x,t) = v(x,t) - v_*$, which yields $u(x,t) = u_* + \tilde{u}(x,t)$, $v(x,t) = v_* + \tilde{v}(x,t)$. Therefore, we obtain the following system (dropping off the tildes)

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - d_1\Delta u(x,t) = (u(x,t)+u_*)\left(a - b\left(u\left(x,t-\tau\right)+u_*\right) - \dfrac{c\left(v\left(x,t-\tau\right)+v_*\right)}{1+v\left(x,t-\tau\right)+v_*}\right), \\ \dfrac{\partial v(x,t)}{\partial t} - d_2\Delta v(x,t) = (v(x,t)+v_*)\left(d - e\left(v\left(x,t-\tau\right)+v_*\right) - \dfrac{f\left(u\left(x,t-\tau\right)+u_*\right)}{1+u\left(x,t-\tau\right)+u_*}\right), \\ u_x(0,t) = u_x(\pi,t) = v_x(0,t) = v_x(\pi,t) = 0, t \geq 0, \\ u(x,t) = \phi_1\left(x,t\right) \geq 0, v\left(x,t\right) = \phi_2\left(x,t\right) \geq 0, (x,t) \in [0,\pi]\times[-\tau,0]. \end{cases} \tag{3}$$

Then the linearized system corresponds to model (3) at E_* is

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - d_1\Delta u(x,t) = a_{11}u(x,t-\tau) + a_{12}v(x,t-\tau), \\ \dfrac{\partial v(x,t)}{\partial t} - d_2\Delta v(x,t) = a_{21}u(x,t-\tau) + a_{22}v(x,t-\tau). \end{cases} \tag{4}$$

where

$$a_{11} = -bu_*,\quad a_{12} = -\frac{cu_*}{(1+v_*)^2},\quad a_{21} = -ev_*,\quad a_{22} = -\frac{fv_*}{(1+u_*)^2}. \tag{5}$$

Let $U = (u,v)^T$, $U_t\left(\theta\right) = U\left(t+\theta\right) = \varphi\left(\theta\right) = \left(\varphi_1,\varphi_2\right)^T, \theta \in [-\tau,0]$ and $D\Delta = \begin{pmatrix} d_1\Delta & 0 \\ 0 & d_2\Delta \end{pmatrix}$. We can rewrite model (4) as an abstract functional differential equation in $C\left([-\tau,0],X\right)$:

$$\frac{\partial U}{\partial t} - D\Delta U = L\left(U_t\right), \tag{6}$$

where $X = \{u, v \in W^{2,2}\left(0, l\pi\right) : u_x = v_x = 0 \text{ for } x = 0, l\pi\}$, the linear operator $L : C\left([-\tau,0], X\right) \to X$ is given by

$$L\left(\varphi\right) = \begin{pmatrix} a_{11}\varphi_1\left(-\tau\right) & a_{12}\varphi_2\left(-\tau\right) \\ a_{21}\varphi_1\left(-\tau\right) & a_{22}\varphi_2\left(-\tau\right) \end{pmatrix}$$

with characteristic equation being

$$\lambda y - D\Delta y - L\left(ye^{\lambda\cdot}\right) = 0,\quad y \in dom\left(\Delta\right)\backslash\{0\}, \tag{7}$$

where $dom(\Delta) \subset X$ and $(e^{\lambda \cdot}y)(\theta) = e^{\lambda\theta}y$. Notice that the linear operator Δ on X has the eigenvalues $-k^2$ $(k \geq 0)$. It follows from [35] that λ is a characteristic value of Eq (7) provided that λ satisfies

$$det\left[\begin{pmatrix} \lambda - a_{11}e^{-\lambda\tau} + k^2 & -a_{12}e^{-\lambda\tau} \\ -a_{21}e^{-\lambda\tau} & \lambda - a_{22}e^{-\lambda\tau} + k^2 \end{pmatrix}\right] = 0,$$

that is

$$\left(\lambda + k^2\right)^2 - (a_{11} + a_{22})\left(\lambda + k^2\right)e^{-\lambda\tau} + (a_{11}a_{22} - a_{12}a_{21})\, e^{-2\lambda\tau} = 0. \tag{8}$$

Under condition (H), we have

$$T = -(a_{11} + a_{22}) > 0,$$
$$D = a_{11}a_{22} - a_{12}a_{21} > 0.$$

Let $z = \left(\lambda + k^2\right)e^{\lambda\tau}$, Eq (8) will be transformed into the following form:

$$z^2 + Tz + D = 0. \tag{9}$$

Because $T^2 - 4D = (a_{11} - a_{22})^2 + 4a_{12}a_{21} > 0$, Eq (9) has two negative real roots:

$$z_1 = \frac{-T - \sqrt{T^2 - 4D}}{2}, z_2 = \frac{-T + \sqrt{T^2 - 4D}}{2}$$

and $z_1 < z_2 < 0$. Eq (8) is equivalent to

$$\left(\lambda + k^2\right)e^{\lambda\tau} = z_n, \quad (n = 1, 2)\,. \tag{10}$$

When $\tau = 0$, Eq (8) has two negative real eigenvalues $\lambda_1 = -k^2 + z_1, \lambda_2 = -k^2 + z_2$ and $\lambda_1 < \lambda_2 < 0$ for any k. We can have the following lemma:

Lemma 2.1. *Suppose that condition (H) holds. Then all the eigenvalues of characteristic Eq (8) with $\tau = 0$ have negative real parts for all $k \geq 0$.*

Now we examine whether there is a positive constant τ^* such that $\mathrm{Re}\left(\lambda_n\left(\tau^*\right)\right) = 0$. Let $\lambda = \pm\omega i$, $\omega > 0$ be a pair of roots of Eq (10), then

$$\left(i\omega + k^2\right)e^{i\omega\tau} = z_n, \quad (n = 1, 2)\,, \tag{11}$$

separating the real and imaginary parts, we obtain

$$\begin{cases} k^2\cos\omega\tau - \omega\sin\omega\tau = \mathrm{Re}\left(\lambda_n\right) = z_n, \\ \omega\cos\omega\tau + k^2\sin\omega\tau = \mathrm{Im}\left(\lambda_n\right) = 0, \end{cases}$$

which lead to

$$\cos\omega\tau = \frac{k^2 z_n}{k^4 + \omega^2}, \quad \sin\omega\tau = \frac{-\omega z_n}{k^4 + \omega^2}.$$

Since $\cos^2\omega\tau + \sin^2\omega\tau = 1$, the above equation is equivalent to

$$k^4 + \omega^2 - {z_n}^2 = 0, \tag{12}$$

then

$$\omega = \sqrt{z_n^2 - k^4}. \tag{13}$$

If $k = 0$, we can obtain that $\omega_n = -z_n$, cos $\omega\tau$=0, sin $\omega\tau = 0$, which implies that

$$\tau = \frac{1}{\omega}\left(\frac{\pi}{2} + 2j\pi\right),\ j = 0, 1, 2, \cdots .$$

Let

$$\tau_j^n = -\frac{1}{z_n}\left(\frac{\pi}{2} + 2j\pi\right),\ n = 1, 2; j = 0, 1, 2, \cdots . \tag{14}$$

Thus, when $k = 0$ and $\tau = \tau_j^n$, Eq (8) has a pair of pure imaginary roots $\pm i\omega_n$.

If $k \neq 0$, *i.e.* $k \geq 1$. We have the following three cases.

Case 1: $-1 < z_1 < z_2 < 0$.

In this case, since $z_n^2 < 1 \leq k^4$, no positive ω satisfies the Eq (10). That is to say that all the eigenvalues of characteristic Eq (8) have negative real parts for all $k \geq 1$. Noting that $z_1 < z_2 < 0$, therefore, $\tau_j^1 < \tau_j^2$ and $\tau_j^n < \tau_{j+1}^n$. We have

$$\tau_0^1 = \min\left\{\tau_j^n\right\},\ n = 1, 2; j = 0, 1, 2, \cdots .$$

We have the following conclusion.

Lemma 2.2. *Let* $\tau_0 = \tau_0^1$*, Then the following conclusions hold:*

(1) For $0 \leq \tau < \tau_0$*, all roots of Eq (8) have negative real parts;*

(2) For $\tau = \tau_0$*, we only have a pair of pure imaginary roots* $\pm iz_1$ *for Eq (8) that corresponds to* $k = 0$*, and all other roots have negative real parts;*

(3) For $\tau = \tau_j^n$ $(n = 1, 2; j = 0, 1, 2, \cdots)$*, we only have a pair of pure imaginary roots* $\pm iz_n$ *for each* $n = 1, 2$*corresponding to* $k = 0$*, and all other roots have negative real parts.*

Next, we will examine whether the transversality condition holds. Suppose $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ denotes the eigenvalue of Eq (10) near $\tau = \tau_j^n$ satisfying $\alpha(\tau_j^n) = 0$, $\omega(\tau_j^n) = \omega_n$ $(n = 1, 2; j = 0, 1, 2, \cdots)$. Differentiating Eq (10) with respect to τ, we obtain

$$\frac{d\lambda}{d\tau}e^{\lambda\tau} + \lambda\left(\lambda e^{\lambda\tau} + \tau e^{\lambda\tau}\frac{d\lambda}{d\tau}\right) + k^2\left(\lambda e^{\lambda\tau} + \tau e^{\lambda\tau}\frac{d\lambda}{d\tau}\right) = 0.$$

Thus, by Eq (10), we have

$$\begin{aligned}
\mathrm{Re}\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_j^n}\right) &= \mathrm{Re}\left(-\frac{1}{i\omega_n\left(i\omega_n + k^2\right)} - \frac{\tau_j^n}{i\omega_n}\right)\\
&= \mathrm{Re}\left(i\frac{\tau_j^n}{\omega_n} + \frac{\omega_n^2 + ik^2\omega_n}{\omega_n^4 + k^4\omega_n^2}\right)\\
&= \frac{1}{\omega_n^2 + k^4} > 0.
\end{aligned}$$

Thus, we have

Lemma 2.3. *Suppose that condition* (H) *is satisfied. Then*

$$\left(\frac{d\alpha}{d\tau}\right)\Bigg|_{\tau=\tau_j^n} > 0$$

for $n = 1, 2; j = 0, 1, 2, \cdots$.

Combining Lemmas 2.2 and 2.3, we can have the following result.

Theorem 2.4. *Suppose that condition* (H) *is satisfied, then for system (2) we have*

(1) if $\tau \in [0, \tau_0)$, *then the equilibrium* E_* *of model (2) is asymptotically stable;*

(2) if $\tau > \tau_0$, *then the equilibrium* E_* *of model (2) is unstable;*

(3) Hopf bifurcations occur at the equilibrium of (2) when $\tau = \tau_j^n (n = 1, 2; j = 0, 1, 2, \cdots)$, *and these Hopf bifurcations are all spatially homogeneous.*

Case 2: $z_1 < -1 < z_2 < 0$.

According to $\omega_k = \sqrt{{z_n}^2 - k^4}$, there exists an positive integer N_0 such that

$$\begin{cases} {z_1}^2 - k^4 > 0, & k < N_0, \\ {z_1}^2 - k^4 < 0, & k \geq N_0. \end{cases}$$

For any positive integer $k \in [1, N_0]$, Eq (13) exists a positive root $\omega_{k1} = \sqrt{{z_1}^2 - k^4}$. Arguing similarly as in Theorem 2.1, as τ passes through the critical values

$$\tau_{kj}^1 = \frac{1}{\omega_{k1}}[\arccos\left(\frac{k^2}{z_1}\right) + 2j\pi], \quad k = 0, 1, 2, \cdots, N_0, \quad j = 0, 1, 2, \cdots,$$

system (2) will undergo a spatially inhomogeneous Hopf bifurcation at equilibrium point E_*.

Case 3: $z_1 < z_2 < -1$

Define $N_n, n = 1, 2$ by

$$\begin{cases} {z_n}^2 - k^4 > 0, & k < N_n, \\ {z_n}^2 - k^4 < 0, & k \geq N_n. \end{cases}$$

For any positive $k \in [1, N_n]$, the Eq (13) exists a positive root $\omega_{kn} = \sqrt{{z_n}^2 - k^4}$. As τ passes through the critical values

$$\tau_{kj}^n = \frac{1}{\omega_{kn}}[\arccos\left(\frac{k^2}{z_n}\right) + 2j\pi], \quad n = 1, 2, \quad k = 0, 1, 2, \cdots, N_0, \quad j = 0, 1, 2, \cdots,$$

model (2) will similarly undergo a spatially inhomogeneous Hopf bifurcation at equilibrium point E_*.

To summarize, we have the following result.

Theorem 2.5. *Suppose condition* (H) *holds, then we have for model (2) that*

(1) *if* $-1 < z_1 < z_2 < 0$, *a spatially homogeneous Hopf bifurcation occurs at equilibrium* E_* *as* τ *passes though* $\tau = \tau_j^n$*;*

(2) *if* $z_1 < -1 < z_2 < 0$, *a spatially homogeneous Hopf bifurcation occurs at equilibrium* E_* *as* τ *passes though* $\tau = \tau_{0j}^1$ $(j = 0, 1, 2, \cdots)$ *and a spatially inhomogeneous Hopf bifurcation occurs at* E_* *as* τ *passes though* $\tau = \tau_{kj}^1$ $(k = 1, 2, \cdots, N_0; j = 0, 1, 2, \cdots)$*;*

(3) *if* $z_1 < z_2 < -1$, *a spatially homogeneous Hopf bifurcation occus at equilibrium point* E_* *as* τ *passes though* $\tau = \tau_{0j}^n$ $(n = 1, 2; j = 0, 1, 2, \cdots)$ *and a spatially inhomogeneous Hopf bifurcation occurs at* E_* *as* τ *passes though* $\tau = \tau_{kj}^n$ $(n = 1, 2; k = 1, 2, \cdots, N_0; j = 0, 1, 2, \cdots)$.

3. Direction and stability of the spatial Hopf bifurcation

In this section, using the normal form theory and center manifold theorem [35, 36], we investigate the direction of Hopf bifurcation and the properties of the bifurcating periodic solutions at the critical value τ_{kj}^n. Throughout this section, we always assume that condition (H) holds, and denote τ_{kj}^n by $\tilde{\tau}$ for fixed $n \in 1, 2$, $k \in 0, 1, 2, \cdots$ and $j \in 0, 1, 2, \cdots$. Then at $\tau = \tilde{\tau}$, $\pm i\omega_0$ are the corresponding purely imaginary roots of the characteristic Eq (8). Letting $t \to \frac{t}{\tau}$ and $\tau = \tilde{\tau} + \mu$, then model (3) is equivalently transformed into the following equation in the phase space $C = C([-1, 0], X)$:

$$\frac{\partial U}{\partial t} = L(U_t) + \tilde{\tau} D\Delta U + \mathrm{G}(U_t, \mu), \tag{15}$$

where

$$L(\varphi) = \tilde{\tau}\begin{pmatrix} a_{11}\varphi(-1) & a_{12}\varphi(-1) \\ a_{21}\varphi(-1) & a_{22}\varphi(-1) \end{pmatrix}, \quad G(\varphi, \mu) = \mu L(\varphi) + \varphi D \vartriangle U + F(\varphi),$$

$$F(\varphi) = \begin{pmatrix} \frac{f_{20}^{(1)}}{2}\varphi_1(0)\varphi_1(-1) + f_{11}^{(1)}\varphi_1(0)\varphi_2(-1) + \frac{f_{02}^{(1)}}{2}\varphi_2^2(-1) + \cdots \\ \frac{f_{20}^{(2)}}{2}\varphi_1{}^2(-1) + f_{11}^{(2)}\varphi_1(-1)\varphi_2(0) + \frac{f_{02}^{(2)}}{2}\varphi_2(0)\varphi_2(-1) + \cdots \end{pmatrix},$$

$$f_{ij}^{(n)} = \frac{\partial^{i+j} f^{(n)}}{\partial u^i \partial v^j}\,(n = 1, 2).$$

The linear system of Eq (15) is

$$\frac{\partial U}{\partial t} = L(U_t) + \tilde{\tau} D\Delta U. \tag{16}$$

By the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \mu)$, $\theta \in [-1, 0]$ with its elements being of bounded variation such that

$$L(\varphi) + D\Delta\varphi(0) = \int_{-1}^{0} d\eta(\theta,\ \mu)\varphi(\theta), \quad \varphi(\theta) \in C. \tag{17}$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{pmatrix} -k^2 & 0 \\ 0 & -k^2 \end{pmatrix}\delta(\theta) + \tilde{\tau}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\delta(\theta + 1),$$

where $\delta(\theta)$ is the Dirac function.

For $\varphi \in C^1\left([-1,0], \mathbb{R}^2\right)$, define

$$A(\mu)\varphi(\theta) = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} d\eta(\theta)\varphi(\theta), & \theta = 0, \end{cases} \tag{18}$$

and

$$R\phi(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ f(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (15) is equivalent to

$$\dot{x}_t = A(\mu)x_t + Rx_t. \tag{19}$$

For $\psi \in C^1\left([0,1], \mathbb{R}^2\right)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1), \\ \displaystyle\int_{-1}^{0} d\eta(-s)\psi(s), & s = 0. \end{cases} \tag{20}$$

Here, A and A^* are a pair of adjoint operators under the following bilinear inner product

$$\begin{aligned} \langle \psi(s), \varphi(\theta) \rangle &= \bar{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{s=0}^{\theta} \bar{\psi}(s-\theta)d\eta(\theta)\varphi(s)ds \\ &= \bar{\psi}(0)\varphi(0) - \tilde{\tau}\int_{-1}^{0} \psi(s+1)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\varphi(s)ds. \end{aligned}$$

According to the discussions in previous section, we know that $\pm i\omega$ are eigenvalues of A, and therefore they are also eigenvalues of A^*. Suppose that $q(\theta) = (1,\alpha)^T e^{i\omega_0\tilde{\tau}\theta}$ is the eigenvector of A corresponding to $i\omega_0\tilde{\tau}$, and $q^*(s) = D(1,\beta)e^{i\omega_0\tilde{\tau}s}$ is the eigenvector of A^* corresponding to $-i\omega_0\tilde{\tau}$. From $Aq(\theta) = i\omega_0\tilde{\tau}q(\theta)$, $A^*q^*(s) = -i\omega_0\tilde{\tau}q^*(s)$ and $\langle q^*(s), q(\theta)\rangle = 1$, we can easily obtain

$$\alpha = -\frac{a_{11}e^{-i\omega_0\tilde{\tau}} - k^2 - i\omega_0}{a_{12}e^{-i\omega_0\tilde{\tau}}}, \quad \beta = -\frac{a_{11}e^{i\omega_0\tilde{\tau}} - k^2 + i\omega_0}{a_{21}e^{i\omega_0\tilde{\tau}}},$$

$$D = \left[1 + \bar{\alpha}\beta + \tilde{\tau}\left(a_{11} + \bar{\alpha}a_{12} + \beta a_{21} + \bar{\alpha}\beta a_{22}\right)e^{i\omega_0\tilde{\tau}}\right]^{-1}.$$

Next, we use the same notation as in Hassard et al. [36]. We first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of Eq (15) when $\mu = 0$. Define

$$z(t) = < q^*, u_t >, \quad W(t,\theta) = u_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta). \tag{21}$$

On the center manifold C_0,we have $W(t,\theta) = W(z,\bar{z},\theta) = \left(W^{(1)}, W^{(2)}\right)^T$ with

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{22}$$

where z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Note that W is real if u_t is real, we only consider real solutions. For solution $u_t \in C_0$ of Eq (15), since $\mu = 0$, we have

$$\begin{aligned}\dot{z} &= i\omega_0\tilde{\tau}_0 z(t) + \overline{q}^*(0) f(0, u_t)\\ &= i\omega_0\tilde{\tau}_0 z(t) + \overline{q}^*(0) f(0, zq(\theta) + \bar{z}\bar{q}(\theta) + W(z, \bar{z}, \theta))\\ &= i\omega_0\tilde{\tau}_0 z(t) + \overline{q}^*(0) f_0,\end{aligned}$$

where

$$f_0 = f_{z^2}\frac{z^2}{2} + f_{z\bar{z}}z\bar{z} + f_{\bar{z}^2}\frac{\bar{z}}{2} + f_{z^2\bar{z}}\frac{z^2\bar{z}}{2} + \cdots. \tag{23}$$

Then, we rewrite this equation as

$$\begin{aligned}\dot{z}(t) &= i\omega_0\tilde{\tau}_0 z(t) + g(z, \bar{z})\\ &= i\omega_0\tilde{\tau}_0 z(t) + g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots,\end{aligned} \tag{24}$$

Notice that $u_t(\theta) = W(t, \theta) + z(t) q(\theta) + \bar{z}(t)\bar{q}(\theta)$, $q(\theta) = (1, \alpha)^T e^{i\omega_0\tilde{\tau}\theta}$ and $q^*(s) = D(1, \beta) e^{-i\omega_0\tilde{\tau}s}$. Through simple calculations, we can compare the coefficients of like terms, z^2, $z\bar{z}$, $\bar{z}^2$, $z^2\bar{z}$ in Eq (24) with that in Eq (23), and obtain

$$\begin{aligned}g_{20} &= \bar{D}\tilde{\tau}e^{-i\omega_0\tilde{\tau}}\left[\sigma_1 + \bar{\beta}\sigma_2\right],\\ g_{11} &= \bar{D}\tilde{\tau}e^{-i\omega_0\tilde{\tau}}\left[\frac{f_{20}^{(1)}}{2} + f_{11}^{(1)}\alpha + \frac{f_{02}^{(1)}}{2}\alpha\bar{\alpha}e^{i\omega_0\tilde{\tau}} + \bar{\beta}\left(\frac{f_{20}^{(2)}}{2}e^{i\omega_0\tilde{\tau}} + f_{11}^{(2)}\bar{\alpha} + \frac{f_{02}^{(2)}}{2}\alpha\bar{\alpha}\right)\right]\\ &\quad + \bar{D}\tilde{\tau}e^{i\omega_0\tilde{\tau}}\left[\frac{f_{20}^{(1)}}{2} + f_{11}^{(1)}\bar{\alpha} + \frac{f_{02}^{(1)}}{2}\bar{\alpha}\alpha e^{-i\omega_0\tilde{\tau}} + \bar{\beta}\left(\frac{f_{20}^{(2)}}{2}e^{-i\omega_0\tilde{\tau}} + f_{11}^{(2)}\alpha + \frac{f_{02}^{(2)}}{2}\bar{\alpha}\alpha\right)\right],\\ g_{02} &= \bar{D}\tilde{\tau}e^{i\omega_0\tilde{\tau}}\left[\bar{\sigma}_1 + \bar{\beta}\bar{\sigma}_2\right],\\ g_{21} &= \bar{D}\tilde{\tau}\left(w_1 + \bar{\beta}w_2\right),\end{aligned}$$

where

$$\begin{aligned}\sigma_1 &= f_{20}^{(1)} + 2f_{11}^{(1)}\alpha + f_{02}^{(1)}\alpha^2 e^{-i\omega_0\tilde{\tau}},\\ \sigma_2 &= f_{20}^{(2)}e^{-i\omega_0\tilde{\tau}} + 2f_{11}^{(2)}\alpha + f_{02}^{(2)}\alpha^2,\end{aligned}$$

$$\begin{aligned}w_1 =& f_{20}^{(1)}\left(\frac{W_{20}^{(1)}(-1)}{2} + \frac{e^{i\omega_0\tilde{\tau}}W_{20}^{(1)}(0)}{2} + W_{11}^{(1)}(-1) + e^{-i\omega_0\tilde{\tau}}W_{11}^{(1)}(0)\right)\\ &+ 2f_{11}^{(1)}\left(\frac{W_{20}^{(2)}(-1)}{2} + \frac{\bar{\alpha}e^{i\omega_0\tilde{\tau}}W_{20}^{(1)}(0)}{2} + W_{11}^{(2)}(-1) + \alpha e^{-i\omega_0\tilde{\tau}}W_{11}^{(1)}(0)\right)\\ &+ f_{02}^{(1)}\left(\bar{\alpha}e^{i\omega_0\tilde{\tau}}W_{20}^{(2)}(-1) + 2\bar{\alpha}e^{-i\omega_0\tilde{\tau}}W_{11}^{(2)}(-1)\right),\\ w_2 =& f_{20}^{(2)}\left(e^{i\omega_0\tilde{\tau}}W_{20}^{(1)}(-1) + 2e^{-i\omega_0\tilde{\tau}}W_{11}^{(1)}(-1)\right)\\ &+ 2f_{11}^{(2)}\left(\frac{\bar{\alpha}W_{20}^{(1)}(-1)}{2} + \frac{e^{i\omega_0\tilde{\tau}}W_{20}^{(2)}(0)}{2} + \alpha W_{11}^{(1)}(-1) + e^{-i\omega_0\tilde{\tau}}W_{11}^{(2)}(0)\right)\\ &+ f_{02}^{(2)}\left(\frac{\bar{\alpha}W_{20}^{(2)}(-1)}{2} + \frac{\bar{a}e^{i\omega_0\tilde{\tau}}W_{20}^{(2)}(0)}{2} + \alpha W_{11}^{(2)}(-1) + \alpha e^{-i\omega_0\tilde{\tau}}W_{11}^{(2)}(0)\right),\end{aligned}$$

We have already determined g_{20}, g_{11} and g_{02}. Next we compute $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}.

We can easily get from Eq (21) that

$$\dot{W}(t,\theta) = \dot{x}_t(\theta) - \dot{z}q(\theta) - \dot{\bar{z}}\bar{q}(\theta).$$

From Eqs (18), (19) and (24), notice that $Azq(\theta) = i\omega_0\tilde{\tau}zq(\theta)$. We have

$$\dot{W} = \begin{cases} AW - gq(\theta) - \bar{g}\bar{q}(\theta), & \theta \in [-1,0), \\ AW - gq(\theta) - \bar{g}\bar{q}(\theta) + f_0, & \theta = 0. \end{cases} \tag{25}$$

On the other hand, in the center manifold C_0, $W(z,\bar{z})$ satisfies

$$\begin{aligned} \dot{W} &= W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}} \\ &= [W_{20}(\theta)z + W_{11}(\theta)\bar{z}] \cdot [i\omega_0\tilde{\tau}z + g(z,\bar{z})] \\ &\quad + [W_{11}(\theta)z + W_{02}(\theta)\bar{z}] \cdot [-i\omega_0\tilde{\tau}\bar{z} + \bar{g}(z,\bar{z})] \\ &= i\omega_0\tilde{\tau}W_{20}(\theta)z^2 - i\omega_0\tilde{\tau}W_{02}(\theta)\bar{z}^2 + o\left(|(z,\bar{z})|^2\right). \end{aligned} \tag{26}$$

From Eqs (22) and (24), comparing the coefficients of Eq (26) with Eq (25) about z^2 and $z\bar{z}$, we obtain the following two equations

$$2i\omega_0\tilde{\tau}W_{20}(\theta) - AW_{20}(\theta) = \begin{cases} -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), & \theta \in [-1,0), \\ -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) + f_{z^2}, & \theta = 0, \end{cases} \tag{27}$$

and

$$-AW_{11}(\theta) = \begin{cases} -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta), & \theta \in [-1,0), \\ -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) + f_{z\bar{z}}, & \theta = 0. \end{cases} \tag{28}$$

When $\theta \in [-1,0)$, we have from Eq (18)

$$\frac{dW_{20}(\theta)}{d\theta} = 2i\omega_0\tilde{\tau}W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \tag{29}$$

Therefore,

$$W_{20}(\theta) = \frac{ig_{20}q(0)e^{i\omega_0\tilde{\tau}\theta}}{\omega_0\tilde{\tau}} + \frac{i\bar{g}_{02}\bar{q}(0)e^{-i\omega_0\tilde{\tau}\theta}}{3\omega_0\tilde{\tau}} + E_1e^{2i\omega_0\tilde{\tau}\theta}, \tag{30}$$

where E_1 is constant vector. When $\theta = 0$, we have from Eq (18)

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tilde{\tau}W_{20}(0) + g_{20}q(0) + \bar{g}_{02}\bar{q}(0) - f_{z^2}. \tag{31}$$

Substituting Eq (30) into Eq (31), we have

$$\begin{aligned} &\int_{-1}^{0} d\eta.\left[\frac{ig_{20}q(0)}{\omega_0\tilde{\tau}}e^{i\omega_0\tilde{\tau}\theta} + \frac{i\bar{g}_{02}\bar{q}(0)e^{-i\omega_0\tilde{\tau}\theta}}{3\omega_0\tilde{\tau}} + E_1e^{2i\omega_0\tilde{\tau}\theta}\right] \\ &= -g_{20}q(0) + \frac{1}{3}\bar{g}_{02}\bar{q}(0) + 2i\omega_0\tilde{\tau}E_1 - f_z^2. \end{aligned}$$

Therefore,

$$E_1 = \left(2i\omega_0\tilde{\tau}I - \int_{-1}^{0} d\eta(\theta) \cdot e^{2i\omega_0\tilde{\tau}\theta}\right)^{-1} f_{z^2},$$

where

$$f_{z^2} = \begin{bmatrix} f_{20}^{(1)}e^{-i\omega_0\tilde{\tau}} + 2f_{11}^{(1)}\alpha e^{-i\omega_0\tilde{\tau}} + f_{02}^{(1)}\alpha^2 e^{-2i\omega_0\tilde{\tau}} \\ f_{20}^{(2)}e^{-2i\omega_0\tilde{\tau}} + 2f_{11}^{(2)}\alpha e^{-i\omega_0\tilde{\tau}} + f_{02}^{(2)}\alpha^2 e^{-i\omega_0\tilde{\tau}} \end{bmatrix}.$$

Similar to the discussion of solving W_{20}, we have

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tilde{\tau}}q(0)e^{i\omega_0\tilde{\tau}\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tilde{\tau}\theta} + E_2,$$

and

$$E_2 = -\left[\int_{-1}^{0} d\eta(\theta)\right]^{-1} f_{z\bar{z}},$$

where

$$f_{z\bar{z}} = \begin{pmatrix} \frac{f_{20}^{(1)}}{2} \cdot \left(e^{i\omega_0\tilde{\tau}} + e^{-i\omega_0\tilde{\tau}}\right) + f_{11}^{(1)} \cdot \left(\bar{\alpha}e^{i\omega_0\tilde{\tau}} + \alpha e^{-i\omega_0\tilde{\tau}}\right) + \frac{f_{02}^{(1)}}{2} \cdot (2\alpha\bar{\alpha}) \\ \frac{f_{20}^{(2)}}{2} \cdot 2 + f_{11}^{(2)} \cdot \left(\alpha e^{i\omega_0\tilde{\tau}} + \bar{\alpha}e^{-i\omega_0\tilde{\tau}}\right) + \frac{f_{02}^{(2)}}{2}\left(\alpha\bar{\alpha}e^{i\omega_0\tilde{\tau}} + \bar{\alpha}\alpha e^{-i\omega_0\tilde{\tau}}\right) \end{pmatrix}.$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$. Furthermore, we can determine g_{21}. According to the notation from [36], we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tilde{\tau}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tilde{\tau})\}}, \quad \beta_2 = 2\,\mathrm{Re}(c_1(0)), \\ T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\,\mathrm{Im}\left\{\lambda'(\tilde{\tau})\right\}}{\omega_0\tilde{\tau}}. \end{aligned}$$

From the conclusion [36], we have the main results in this section:

Theorem 3.1. *Assume that model (2) undergoes a Hopf bifurcation when delay τ crosses through the τ_{kj}^n, then the following statements are true:*

(1) μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical),

(2) β_2 determines the stability of the bifurcating periodic solution: if $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solution is stable (unstable),

(3) T_2 determines the period of the bifurcating periodic solution: if $T_2 > 0$ ($T_0 < 0$), then the period increases (decreases).

4. Numerical simulations

To support the theoretical results obtained in this paper, we illustrate some numerical simulations. We choose the parameters as $a = 1.5$, $b = 1.1$, $c = 1.1$, $d = 1.1$, $e = 1.15$, $f = 0.5$, $d_1 = d_2 = 1$.It is easy to check that condition (H) holds, which indicates that there exists a unique positive equilibrium and it is globally asymptotically stable.

By simple calculations, we can know that the positive equilibrium is $E_* = (0.9363, 0.7463)$ and $z_1 = -1.1465 < -1 < z_2 = -0.7416$. In this case, $\omega_0 = 1.1466$ and $\tau_0 = 1.365$. Based on Theorem 2.5, we know that the positive equilibrium E_* is asymptotically stable when $0 \le \tau < \tau_0$ and unstable when $\tau > \tau_0$. And a spatially homogenous periodic solution emerges from the positive equilibrium E_* when delay τ is increasing across the critical value $\tau_0 = 1.365$. We can see that if $\tau = 1.2 < \tau_0$, the positive equilibrium E_* is stable and an uniform distribution can be observed (see Figure 1). If $\tau = 1.37 > \tau_0$, the positive equilibrium E_* lose its stability and a spatially homogeneous periodic distribution can be observed (see Figure 1). For the direction and stability of spatial Hopf bifurcation, we can obtain that $\mu_2 = 0.1691, \beta_2 = -0.4445$ and $T_2 = 0.1544$ if $\tau = \tau_0$. Due to $\mu_2 > 0$, $\beta_2 < 0$, by Theorem 3.1, it is obvious that the directions of Hopf bifurcations are supercritical and the periodic solutions are stable. In addition, the period of periodic solutions increases with the increase of delay τ since $T_2 > 0$. These phenomenon also can be observed in Figure 2.

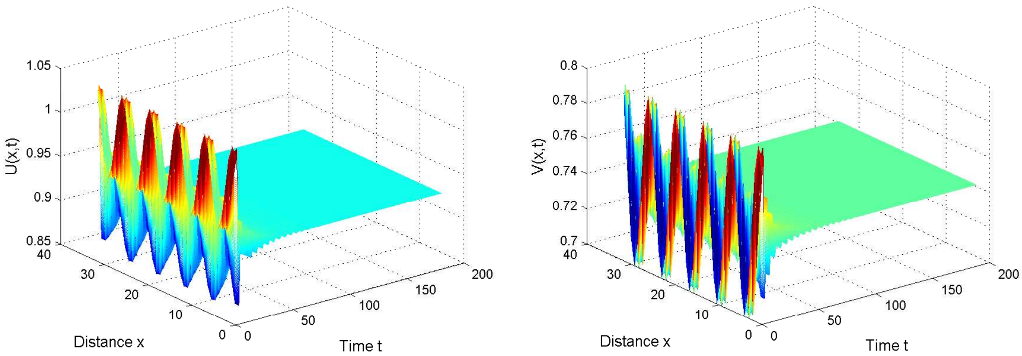

Figure 1. The stable solution of system (2) with $\tau = 1.20 < \tau_0$ and the initial conditions $u(x,t) = 1.0 + 0.05\sin(x,t), v(x,t) = 0.75 + 0.05\sin(x,t), x \in [0, 10\pi], t \in [-\tau, 0]$.

5. Conclusions and discussions

In this paper, we investigated the rich dynamics of a delayed diffusive competition model with saturation effect. It is known that the classical competition model does not have the Hopf bifurcation near the positive equilibrium E_*. However, when the delay is included, our results indicate that the delayed diffusive competition model (2) show more complex dynamics, such as the existence of the Hopf bifurcation. For example, by taking delay τ as the bifurcation parameter, stability switches phenomenon of the positive equilibrium E_* can be observed. We demonstrated that if $\tau < \tau_0$, the positive equilibrium E_* is asymptotically stable. It is unstable if delay $\tau > \tau_0$. And we further proved that model (2) can generate spatially homogeneous and nonhomogeneous Hopf bifurcations when the

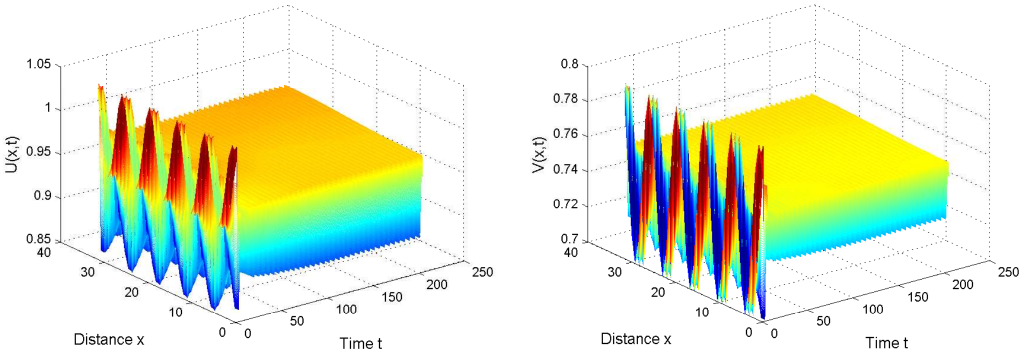

Figure 2. The spatially homogeneous periodic solution of system (2) with $\tau = 1.37 > \tau_0$ and the initial conditions $u(x,t) = 1.0 + 0.05\sin(x,t)$, $v(x,t) = 0.75 + 0.05\sin(x,t)$, $x \in [0, 10\pi]$, $t \in [-\tau, 0]$.

delay τ passes through some critical values. Besides, by using the normal form theory and center manifold theorems, we analysed the directions of the Hopf bifurcations and the stability of the bifurcating periodic solutions and show the Hopf bifurcation of system (2) at the equilibrium E_* when delay $\tau = \tau_0$ is supercritical and the periodic solution is asymptotically stable.

In fact, time delay can be found in various biological applications, such as infectious disease modeling, population modeling and the response of vegetation coverage to climate change so on. Some rich phenomenon can be induced by time delay. For example, time delay may induce the occurrence of spatiotemporal patterns [37, 38]. In this paper, we only consider the occurrence of Hopf bifurcation induced by time delay. The global dynamics of the diffusive competition model still needs further study.

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Conflict of interests

The authors declare there is no conflict of interest.

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