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DELAY DIFFERENTIAL EQUATIONS VIA THE MATRIX LAMBERT W FUNCTION AND BIFURCATION ANALYSIS: APPLICATION TO MACHINE TOOL CHATTER

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Abstract. In a turning process modeled using delay differential equations (DDEs), we investigate the stability of the regenerative machine tool chatter problem. An approach using the matrix Lambert W function for the analytical solution to systems of delay differential equations is applied to this problem and compared with the result obtained using a bifurcation analysis. The Lambert W function, known to be useful for solving scalar first-order DDEs, has recently been extended to a matrix Lambert W function approach to solve systems of DDEs. The essential advantages of the matrix Lambert W approach are not only the similarity to the concept of the state transition matrix in linear ordinary differential equations, enabling its use for general classes of linear delay differential equations, but also the observation that we need only the principal branch among an infinite number of roots to determine the stability of a system of DDEs. The bifurcation method combined with Sturm sequences provides an algorithm for determining the stability of DDEs without restrictive geometric analysis. With this approach, one can obtain the critical values of delay, which determine the stability of a system and hence the preferred operating spindle speed without chatter. We apply both the matrix Lambert W function and the bifurcation analysis approach to the problem of chatter stability in turning, and compare the results obtained to existing methods. The two new approaches show excellent accuracy and certain other advantages, when compared to traditional graphical, computational and approximate methods.

1. Introduction.

Machine tool chatter, which can be modeled as a time-delayed system, is one of the major constraints that limit the productivity of the turning process. Chatter is the self-excited vibration that is caused by the interaction between the machine structure and the cutting process dynamics. The interaction

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between the tool-workpiece structure and the cutting process dynamics can be described as a closed-loop system. If this system becomes unstable, chatter occurs and leads to deteriorated surface finish, dimensional inaccuracy in the machined part, and unexpected damage to the machine tool, including breakage. Following the introduction of the classical chatter theories introduced by Tobias [1] and Tlusty [2] in the 1960s, various models were developed to predict the onset of chatter. Tobias et al. [1] developed a graphical method and an algebraic method to determine the onset of instability of a system with multiple degrees of freedom (DOF). Merit presented a theory to calculate the stability boundary by plotting the harmonic solutions of the system's characteristic equation, assuming that there was no dynamics in the cutting process, and also proposed a simple asymptotic borderline to assure chatter-free performance at all spindle speeds [3]. Opitz and Bernardi [4] developed a general closed loop representation of the cutting system dynamics for turning and milling processes. The machine structural dynamics was generally expressed in terms of transfer matrices, while the cutting process was limited by two assumptions: (1) direction of the dynamic cutting force is fixed during cutting, and (2) the effects of feed and cutting speed are neglected. These assumptions were later removed by Minis et al. [5], who described the system stability in terms of a characteristic equation and then applied the Nyquist stability criterion to determine the stability of the system. Chen et al. [6] introduced a computational method that avoids lengthy algebraic (symbolic) manipulations in solving the characteristic equation. In [6], the characteristic equation was numerically formulated as an equation in a single unknown but well bounded variable. The stability criteria for time-delay systems were analytically derived by Stépán et al. [7] [8], by Kuang [9], and using the Hopf Bifurcation Theorem [10]-[12]. Recently, Olgac and Sipahi developed the "Cluster Treatment of Characteristic Roots" examining one infinite cluster of roots at a time for stability of delay systems has been developed to enable the determination of the complete stability regions of delay [13], and also applied to machining chatter [14]. In this paper, an approach to solve the chatter equation using the matrix Lambert W function is presented. By applying the matrix Lambert W function to the chatter equation, we can solve systems of DDEs in the time domain and check the stability of the system. With this method one can obtain ranges of preferred operating spindle speed that do not cause chatter. The form of the solution obtained is analogous to the general solution form for ordinary differential equations (ODEs), and the concept of the state transition matrix in ODEs can be generalized to DDEs with the presented method. The result is compared with results obtained using a bifurcation analysis method with Sturm sequences. This method provides a useful algorithm for determining the stability of systems of DDEs without restrictive geometric analysis.

2. **The chatter equation in the turning process.** In the turning process, a cylindrical workpiece rotates with constant angular velocity, and the tool generates a surface as material is removed. Any vibration of the tool is reflected in this surface, which means that the cutting force depends on the position of the tool edge for the current revolution as well as the previous one. Delay differential equations, thus, have been widely used as models for regenerative machine tool vibration. The model of tool vibration, assuming a 1-DOF orthogonal cutting depicted in Figure 1, can be expressed as [11]

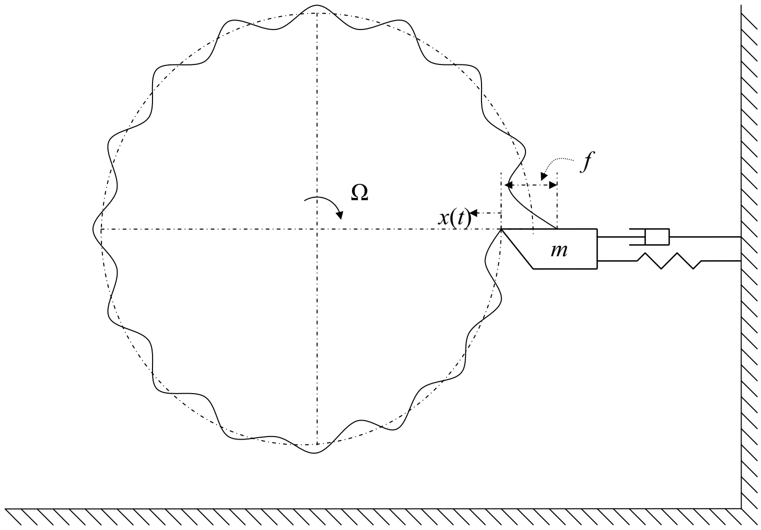

FIGURE 1. 1 DOF orthogonal cutting model

$$\begin{aligned} &\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \left(\omega_n^2 + \frac{k_c}{m}\right)x(t) - \frac{k_c}{m}x(t-T) \\ &= \frac{k_c}{8f_0 m}\left((x(t) - x(t-T))^2 - \frac{5}{12f_0}(x(t) - x(t-T))^3\right), \end{aligned} \tag{1}$$

where $x(t)$ is the general coordinate of tool edge position and the delay $T = 2\pi/\Omega$ is the time period for one revolution, with ω being the angular velocity of the rotating workpiece. The coefficient k_c is the cutting coefficient derived from a stationary cutting force model as an empirical function of the parameters such as the chip width, the chip thickness f (nominally f_0 at steady-state), and the cutting speed. The natural angular frequency of the undamped free oscillating system ω_n, and ζ is the relative damping factor. Note that the zero value of the general coordinate $x(t)$ of the tool edge position is selected such that the x component of the cutting force is in balance with the stiffness when the chip thickness f is at the nominal value f_0 [11].

To linearize (1), define $x_1 \equiv x$ and $x_2 \equiv \dot{x}$, and rewrite the equation in first-order form as

$$\begin{aligned} \dot{x}_1 &= x_2(t), \\ \dot{x}_2 &= -2\zeta\omega_n x_2(t) - \left(\omega_n^2 + \frac{k_c}{m}\right)x_1(t) - \frac{k_c}{m}x_1(t-T) \\ &+ \frac{k_c}{8f_0 m}\left((x_1(t) - x_1(t-T))^2 - \frac{5}{12f_0}(x_1(t) - x_1(t-T))^3\right). \end{aligned} \tag{2}$$

At equilibrium, the condition, $\dot{x}_1(t) = \dot{x}_2(t) = 0$, is satisfied; that is,

$$\begin{aligned} 0 &= x_2(t), \\ 0 &= -2\zeta\omega_n x_2(t) - \left(\omega_n^2 + \frac{k_c}{m}\right)x_1(t) - \frac{k_c}{m}x_1(t-T) \\ &+ \frac{k_c}{8f_0 m}\left((x_1(t) - x_1(t-T))^2 - \frac{5}{12f_0}(x_1(t) - x_1(t-T))^3\right). \end{aligned} \tag{3}$$

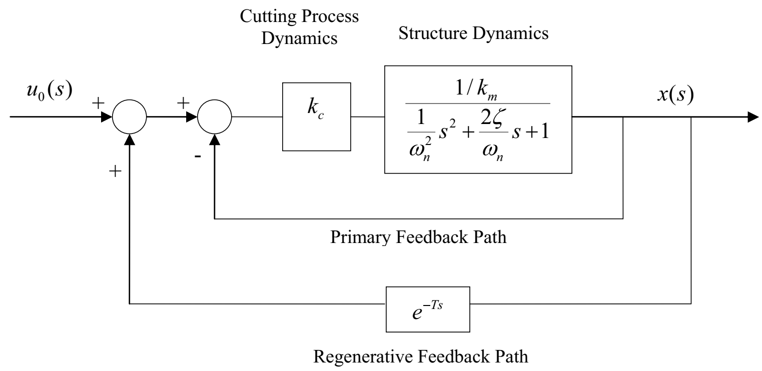

FIGURE 2. Block diagram of chatter loop [3]. Two feedback paths exist: a negative feedback of position (primary path) and a positive feedback of delayed position (regenerative path). Chatter occurs when this closed loop system becomes unstable.

and if no vibration from previous processing is left, then $x_1(t) = x_1(t-T) = 0$. Therefore, we can conclude that one of the equilibrium points is

$$\bar{x}_1(t) = \bar{x}_1(t-T) = \bar{x}_2(t) = 0, \tag{4}$$

which means that at this equilibrium point, the tool edge is in the zero position as defined previously. Linearizing (2) using a Jacobian matrix evaluated at the equilibrium point gives

$$\left\{ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ -\left(\omega_n^2 + \dfrac{k_c}{m}\right) & -2\zeta\omega_n \end{array} \right] \left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} + \left[\begin{array}{cc} 0 & 0 \\ \dfrac{k_c}{m} & 0 \end{array} \right] \left\{ \begin{array}{c} x_1(t-T) \\ x_2(t-T) \end{array} \right\}. \tag{5}$$

Equivalently, (5) can be written as

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \left(\omega_n^2 + \frac{k_c}{m}\right)x(t) - \frac{k_c}{m}x(t-T) = 0 \tag{6}$$

or in the form, [6]

$$\frac{1}{\omega_n^2}\ddot{x}(t) + \frac{2\zeta}{\omega_n}\dot{x}(t) + x(t) = -\frac{k_c}{k_m}\left(x(t) - x(t-T)\right), \tag{7}$$

where k_m is structural stiffness (N/m) and $m\omega_n^2 \equiv k_m$.

Figure 2 shows the block diagram of the chatter loop. In the diagram, two feedback paths exist: a negative feedback of position (primary path) and a positive feedback of delayed position (regenerative path). The $u_0(s)$ is the nominal depth of cut initially set to zero [3]. Chatter occurs when this closed loop system becomes unstable. The stability of the linearized model in (7) can be used to determine the conditions for the onset of chatter; however, the linearized equations do not capture the amplitude limiting nonlinearities associated with the chatter vibrations. Although comparison with experimental data is not provided in this paper, similar models have been extensively studied and validated in prior works [1]-[8].

3. **Solving DDEs using the Lambert W function.** The linearized chatter equation (7) can be expressed in state space form as

$$\dot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-T) = 0. \tag{8}$$

Defining $\mathbf{x} = \{x \quad \dot{x}\}^T$, where ^{T} indicates transpose, equation (7) can be expressed as

$$\mathbf{A} = -\begin{bmatrix} 0 & 1 \\ -\left(1+\dfrac{k_c}{k_m}\right)\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad \mathbf{A_d} = -\begin{bmatrix} 0 & 0 \\ \dfrac{k_c}{k_m}\omega_n^2 & 0 \end{bmatrix}. \tag{9}$$

$\mathbf{A}$ and $\mathbf{A_d}$ are the linearized coefficient matrices of the process model and are functions of the machine-tool and workpiece structural parameters such as natural frequency, damping, and stiffness. The analytical method to solve scalar DDEs, and systems of DDEs as in (8) using the matrix Lambert W function was introduced by Asl and Ulsoy but is exact only in the case where the matrices $\mathbf{A}$ and $\mathbf{A_d}$ commute [15]. In [16], the matrix Lambert W function approach is extended to obtain the solution of general systems of DDEs in matrix-vector form. Here we briefly summarize the results.

First we assume a solution form for (8) as

$$\mathbf{x}(t) = e^{\mathbf{S}t}\mathbf{x}_0, \tag{10}$$

where $\mathbf{S}$ is $n \times n$ matrix. In the usual case, the characteristic equation for (8) is obtained from the equation by looking for nontrivial solution of the form $e^{st}\mathbf{C}$ where s is a scalar variable and $\mathbf{C}$ is constant [17]. However, such an approach can neither lead to any interesting result nor help in deriving a solution to systems of DDEs in (8). Alternatively, one could assume the form of (10) to derive the solution to systems of DDEs in (8) using the matrix Lambert W function. Substitution it into (8) yields

$$\mathbf{S}e^{\mathbf{S}t}\mathbf{x}_0 + \mathbf{A}e^{\mathbf{S}t}\mathbf{x}_0 + \mathbf{A_d}e^{\mathbf{S}(t-T)}\mathbf{x}_0 = \mathbf{0}, \tag{11}$$

and using the property of the exponential

$$e^{\mathbf{S}(t-T)} = e^{\mathbf{S}(-T+t)} = e^{\mathbf{S}(-T)}e^{\mathbf{S}t} \tag{12}$$

we can rewrite as

$$\begin{aligned} &\mathbf{S}e^{\mathbf{S}t}\mathbf{x}_0 + \mathbf{A}e^{\mathbf{S}t}\mathbf{x}_0 + \mathbf{A_d}e^{-\mathbf{S}T}e^{\mathbf{S}t}\mathbf{x}_0 \\ &= (\mathbf{S} + \mathbf{A} + \mathbf{A_d}e^{-\mathbf{S}T})e^{\mathbf{S}t}\mathbf{x}_0 \\ &= \mathbf{0}. \end{aligned} \tag{13}$$

Because the matrix $\mathbf{S}$ is a inherent characteristic of a system and independent of initial condition, we can conclude that for equation (13) to be satisfied for any arbitrary initial condition, $\mathbf{x}_0$, and every time, t, we must have

$$\mathbf{S} + \mathbf{A} + \mathbf{A_d}e^{-\mathbf{S}T} = \mathbf{0}. \tag{14}$$

In the special case that $\mathbf{A_d} = \mathbf{0}$, the delay term in (8) disappears, (8) becomes ODE, and (14) is

$$\mathbf{S} + \mathbf{A} = \mathbf{0} \Longleftrightarrow \mathbf{S} = -\mathbf{A}. \tag{15}$$

Then, substitution into (10) yields

$$\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{x}_0 \tag{16}$$

This is the typical solution to ODE in terms of the matrix exponential. Multiply $Te^{\mathbf{S}T}e^{\mathbf{A}T}$ on both sides of (14) and rearrange to obtain,

$$T(\mathbf{S} + \mathbf{A})e^{\mathbf{S}T}e^{\mathbf{A}T} = -\mathbf{A_d}Te^{\mathbf{A}T}. \tag{17}$$

In the general case, when the matrices $\mathbf{A}$ and $\mathbf{A_d}$ don't commute, neither do $\mathbf{S}$ and $\mathbf{A}$ [16]; thus

$$T(\mathbf{S}+\mathbf{A})e^{\mathbf{S}T}e^{\mathbf{A}T} \neq T(\mathbf{S}+\mathbf{A})e^{(\mathbf{S}+\mathbf{A})T}. \tag{18}$$

Consequently, to adjust the inequality in (18) and to take advantage of the property of the matrix Lambert W function defined by

$$\mathbf{W}(\mathbf{H})e^{\mathbf{W}(\mathbf{H})} = \mathbf{H}, \tag{19}$$

we introduce an unknown matrix $\mathbf{Q}$ so that satisfies,

$$T(\mathbf{S}+\mathbf{A})e^{(\mathbf{S}+\mathbf{A})T} = -\mathbf{A_d}T\mathbf{Q}. \tag{20}$$

Comparing (19) and (20) we note that,

$$(\mathbf{S}+\mathbf{A})T = \mathbf{W}(-\mathbf{A_d}T\mathbf{Q}). \tag{21}$$

Then from (21), solving for $\mathbf{S}$ gives

$$\mathbf{S} = \frac{1}{T}\mathbf{W}(-\mathbf{A_d}T\mathbf{Q}) - \mathbf{A}. \tag{22}$$

Substituting (22) into (17) yields the following condition, which can be used to solve for the unknown matrix $\mathbf{Q}$:

$$\mathbf{W}(-\mathbf{A}_dT\mathbf{Q})e^{\mathbf{W}(-\mathbf{A_d}T\mathbf{Q})-\mathbf{A}T} = -\mathbf{A_d}T. \tag{23}$$

In the many examples we have studied, (23) always has a unique solution $\mathbf{Q}_k$ for each branch, k. The solution is obtained numerically, for a variety of initial conditions, using the 'fsolve' function in Matlab. The matrix Lambert W function defined in (19) contains an infinite number of branches [18]. Corresponding to each branch, k $(= -\infty, \cdots, -1, 0, 1, \cdots, \infty)$, of the Lambert W function, for $\mathbf{H}_k = -\mathbf{A_d}T\mathbf{Q}_k$, we compute the eigenvalues $\hat{\lambda}_{ki}$, $i = 1, 2$, of $\mathbf{H}_k$ and the corresponding eigenvector matrix $\mathbf{V}_k$. Hence, the matrix Lambert W function is

$$\mathbf{W}_k(\mathbf{H}_k) = \mathbf{V}_k \begin{bmatrix} W_k(\hat{\lambda}_{k1}) & 0 \\ 0 & W_k(\hat{\lambda}_{k2}) \end{bmatrix} \mathbf{V}_k^{-1}. \tag{24}$$

Finally, $\mathbf{S}_k$ is computed corresponding to $\mathbf{W}_k$ from (22) and summated to be the solution to the systems of DDEs (8) as

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k \tag{25}$$

where the $\mathbf{C}_k$ is a 2×1 coefficient matrix computed from a given preshape function $\mathbf{x}(t) = \mathbf{g}(t)$, which is initial state of DDEs (8), for $t \in [-T, 0]$ [16].

Each branch of the Lambert W function can be computed analytically as shown in [18], and one of the merits of the matrix Lambert W function approach is that one can compute all of the branches of the function using commands already embedded in the various commercial software packages, such as Matlab, Maple, and Mathematica.

4. **Stability by the Lambert W method.** Here we apply the matrix Lambert W function to the chatter problem introduced in section 2. Assume the unknown $\mathbf{Q}$ in (23) as

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}. \tag{26}$$

Then with (9) and (26), the argument of the Lambert W function, "$-\mathbf{A_d}T\mathbf{Q}$" is

$$-\mathbf{A_d}T\mathbf{Q} = \begin{bmatrix} 0 & 0 \\ q_{11}\dfrac{k_c}{k_m}\omega_n^2 T & q_{12}\dfrac{k_c}{k_m}\omega_n^2 T \end{bmatrix}. \tag{27}$$

Hence, the eigenvalue matrix and the eigenvector matrix for $-\mathbf{A_d}T\mathbf{Q}$ are

$$\mathbf{d} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} q_{12}\dfrac{k_c}{k_m}\omega_n^2 T & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 0 & -\dfrac{q_{12}}{q_{11}} \\ 1 & 1 \end{bmatrix}. \tag{28}$$

As seen in (28), one of the eigenvalues is zero. This point makes the chatter equation unusual, because of the following property of the Lambert function [18]:

$$W_k(0) = \begin{cases} 0 & \text{when } k = 0 \\ -\infty & \text{when } k \neq 0 \end{cases} \tag{29}$$

Because of this property, in contrast to the typical case where identical branches ($k_1 = k_2$) are used in (24) [16], here it is necessary to use hybrid branches ($k_1 \neq k_2$) of the matrix Lambert W function defined as

$$\mathbf{W}_{k_1,k_2}(-\mathbf{A_d}T\mathbf{Q}) = \mathbf{V} \begin{bmatrix} W_{k_1}(q_{12}\dfrac{k_c}{k_m}\omega_n^2 T) & 0 \\ 0 & W_{k_2}(0) \end{bmatrix} \mathbf{V}^{-1}. \tag{30}$$

By setting $k_2 = 0$ and varying only k_1 from $-\infty$ to ∞, we can solve (23) to get $\mathbf{Q}_{k_1,0}$; then using (22), we determine the transition matrices of the system (8). The results for gain $(k_c/k_m) = 0.25$, spindle speed $(1/T) = 50$, $\omega_n = 150(\text{sec}^{-2})$, and $\zeta = 0.05$, are in Table 1. As seen in Table 1, even though k_1 varies, we observe that the eigenvalues for $k_1 = k_2 = 0$ repeat, which is caused by the fact that one of the branches (k_2) is always zero. The eigenvalues in Table 1 are displayed in the complex plane in Figure 3.

Figure 3 shows that the eigenvalues obtained using the principal branch ($k_1 = k_2 = 0$) are closest to the imaginary axis and determine the stability of the system. Therefore,

$$\text{Re}\{\text{eigenvalues for } k_1 = k_2 = 0\} \geq \text{Re}\{\text{all other eigenvalues}\} \tag{31}$$

For the scalar DDE case, it has been proven that the root obtained using the principal branch always determines stability [19], and such a proof can readily be extended to systems of DDEs where $\mathbf{A}$ and $\mathbf{A_d}$ commute. However, such a proof is not available in the case of general matrix-vector DDEs. Nevertheless, we have observed the same behavior in all the examples we have considered. That is, the eigenvalues of $\mathbf{S}_{0,0}$, obtained using the principal branch for both of k_1 and k_2, are closest to the imaginary axis, and their real parts are negative. Furthermore, using additional branches to calculate the eigenvalues always yields eigenvalues whose real parts are further to the left in the s-plane. Thus, we conclude that the system is stable.

The responses, with the transition matrices in Table 1, are illustrated in Figure 4 and compared with the response using a numerical integration with nonlinear

TABLE 1. Results of calculation for the chatter equation

	$\mathbf{S}_{k_1,k_2}$	Eigenvalues of $\mathbf{S}_{k_1,k_2}$
$k_1 = k_2 = 0$	$\begin{bmatrix} 0 & 1 \\ -33083 & -0.24 \end{bmatrix}$	$\begin{cases} -0.12 + 181.88i \\ -0.12 - 181.88i \end{cases}$
$k_1 = -1$ & $k_2 = 0$	$\begin{bmatrix} 0 & 1 \\ -77988 + 32093i & -177 - 247i \end{bmatrix}$	$\begin{cases} -0.12 + 181.88i \\ -176.73 - 428.66i \end{cases}$
	$\begin{bmatrix} 0 & 1 \\ -11 - 1663i & -92 - 182i \end{bmatrix}$	$\begin{cases} -91.61 \\ -0.12 - 181.88i \end{cases}$
$k_1 = 1$ & $k_2 = 0$	$\begin{bmatrix} 0 & 1 \\ -77988 - 32093i & -177 + 247i \end{bmatrix}$	$\begin{cases} -0.12 - 181.88i \\ -176.73 + 428.66i \end{cases}$
	$\begin{bmatrix} 0 & 1 \\ -11 + 1663i & -92 + 182i \end{bmatrix}$	$\begin{cases} -91.61 \\ -0.12 + 181.88i \end{cases}$
$k_1 = -2$ & $k_2 = 0$	$\begin{bmatrix} 0 & 1 \\ -137360 + 42340i & -230 - 570i \end{bmatrix}$	$\begin{cases} -0.12 + 181.88i \\ -233.30 - 755.05i \end{cases}$
	$\begin{bmatrix} 0 & 1 \\ 77945 - 31297i & -177 - 611i \end{bmatrix}$	$\begin{cases} -0.12 - 181.88i \\ -176.73 - 428.66i \end{cases}$
$k_1 = 2$ & $k_2 = 0$	$\begin{bmatrix} 0 & 1 \\ -137360 - 42340i & -230 + 570i \end{bmatrix}$	$\begin{cases} -0.12 - 181.88i \\ -233.30 + 755.05i \end{cases}$
	$\begin{bmatrix} 0 & 1 \\ 77945 + 31297i & -177 + 611i \end{bmatrix}$	$\begin{cases} -0.12 + 181.88i \\ -176.73 + 428.66i \end{cases}$
$\vdots$	$\vdots$	$\vdots$

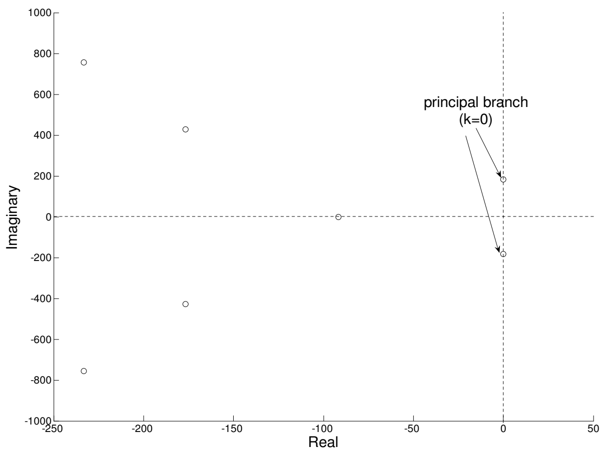

FIGURE 3. Eigenvalues in Table 1 in the complex plane. The eigenvalues obtained using the principal branch ($k = 0$) are dominant and determine the stability of the system.

equation (1) and linearized one (7). Note that this is for the linearized equation given by (8). As seen in Figure 4, because there are infinite numbers of transition matrices for DDEs with varying branches, as more transition matrices are utilized, the response approaches the numerically obtained response. If we observe

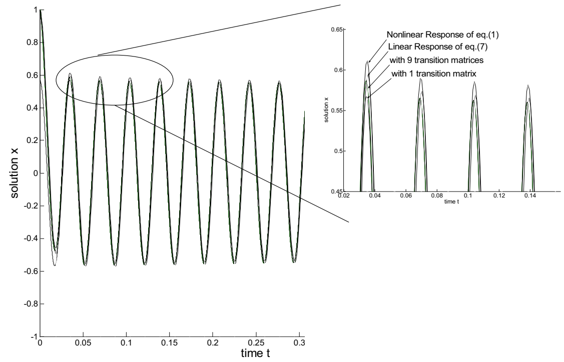

FIGURE 4. Responses for the chatter equation in (8). With more branches used, the results show better agreement.

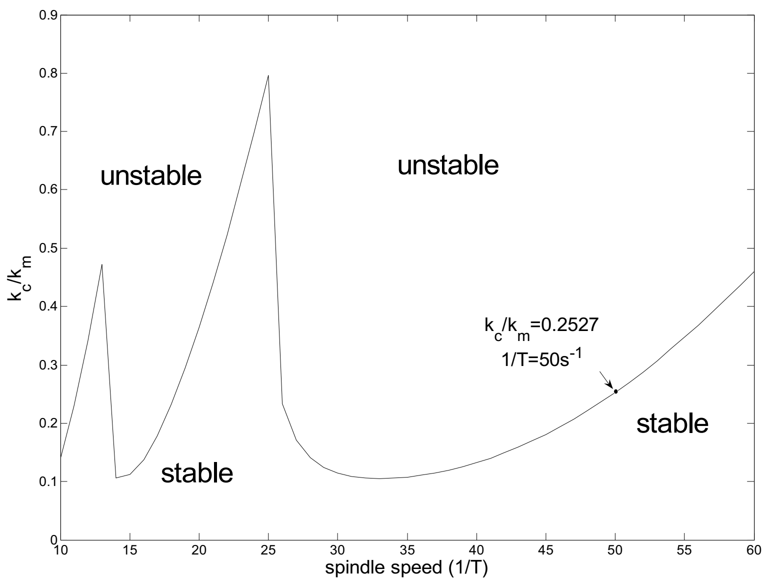

FIGURE 5. Stability lobes for the chatter equation

the roots obtained using the principal branch, we can find the critical point when the roots cross the imaginary axis. For example, when spindle speed $(1/T) = 50$, $\omega_n = 150(\sec^{-2})$ and $\zeta = 0.05$, the critical ratio of gains (k_c/k_m) is 0.2527. This value agrees with the result obtained by the Lyapunov method [20], the Nyquist criterion and the computational method of [6]. The stability lobes by this method are depicted in Figure 5 with respect to the spindle speed (rps, revolution per second).

In obtaining the result shown in the Figure 5, we note that the roots obtained using the principal branch always determine stability. One of the advantages of using the matrix Lambert function over other methods appears to be the observation that the stability of the system can be obtained from only the principal branch among an infinite number of roots. The main advantage of this method is that solution (25) in terms of the matrix Lambert function is similar to that of ODEs. Hence, the concept of the state transition matrix in ODEs can be generalized to DDEs using the matrix Lambert function. This suggests that some analyses used in systems of ODEs, based upon the concept of the state transition matrix, can potentially be extended to systems of DDEs. For example, the approach presented based on the matrix Lambert function, may be useful in controller design via eigenvalue assignment for systems of DDEs. Similarly, concepts of observability, controllability with their Gramians, and state estimator design may be tractable and is being studied by the authors. The analytical approach using the matrix Lambert function for "time-varying" DDEs based on Floquet theory is also being currently investigated.

5. **Bifurcation analysis** [21]**.** Recently, Forde and Nelson [21] developed a bifurcation analysis combined with Sturm sequences, for determining the stability of delay differential equations. The method simplifies the task of determining the necessary and sufficient conditions for the roots of a quasi-polynomial to have negative real parts, and was applied to a biological system [21]. For the chatter problem considered here, the bifurcation analysis presented in [21] also provides a useful algorithm for determining stability.

In the case that the rank of $\mathbf{A_d}$ in (8) is one, the characteristic equation of (7) can be written in the form

$$P(\lambda, T) \equiv P_1(\lambda) + P_2(\lambda)e^{\lambda T} \tag{32}$$

where

$$P_1(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2\left(1 + \frac{k_c}{k_m}\right), \quad P_2(\lambda) = -\omega_n^2\frac{k_c}{k_m}. \tag{33}$$

If we begin by looking for purely imaginary roots, $i\nu$, $\nu \in \Re$, of (32),

$$P_1(i\nu) + P_2(i\nu)e^{i\nu T} = 0 \tag{34}$$

we can then separate the polynomial into its real and imaginary parts and write the exponential in terms of trigonometric functions to get

$$R_1(\nu) + iQ_1(\nu) + (R_2(\nu) + iQ_2(\nu))(\cos(\nu T) - i\sin(\nu T)) = 0, \tag{35}$$

where

$$R_1 = \omega_n^2\left(1 + \frac{k_c}{k_m}\right), \quad Q_1 = 2\zeta\omega_n, \quad R_2 = -\omega_n^2\frac{k_c}{k_m}, \quad Q_2 = 0. \tag{36}$$

For (35) to hold, both the real and imaginary parts must be zero, so we get the pair of equations

$$\begin{aligned} R_1(\nu) + R_2(\nu)\cos(\nu T) + Q_2(\nu)\sin(\nu T) = 0, \\ Q_1(\nu) - R_2(\nu)\sin(\nu T) + Q_2(\nu)\cos(\nu T) = 0. \end{aligned} \tag{37}$$

Squaring each equation and summing the results yields

$$R_1(\nu)^2 + Q_1(\nu)^2 = R_2(\nu)^2 + Q_2(\nu)^2. \tag{38}$$

Note that (38) is a polynomial equation where the trigonometric terms have disappeared and the delay, T, has been eliminated. By defining a new variable $\mu = \nu^2$, (38) can be written in terms of μ as

$$\sigma(\mu) = \mu^2 + 2\omega_n^2 \left\{ 2\zeta^2 - \frac{k_c}{k_m} - 1 \right\} \mu + \omega_n^4 \left(1 + 2\frac{k_c}{k_m} \right) = 0, \tag{39}$$

where σ is a polynomial in μ. Note that we are only interested in $\nu \in \Re$. Thus, if all of the real roots of σ are negative, there can be no simultaneous solution ν^* of (37). Conversely, if there is a positive real root to (39), then there is a delay T^* corresponding to $\nu^* = \pm\sqrt{\mu^*}$ which solves both equations in (37). There are many approaches one might take to determine whether a polynomial has any positive real roots. For second-order characteristic polynomials, there is always the quadratic formula. For third- and fourth-order polynomials, there are also explicit algorithms. One approach to showing that no bifurcation exists is to apply the Routh-Hurwitz condition. If these conditions are satisfied, then all of the roots of (39) have negative real parts, and thus none are positive and real. This condition is not sharp, however, since there remains the possibility that the polynomial (39) has a conjugate pair of roots with a positive real part and a nonzero imaginary part. For example, consider the characteristic polynomial

$$\lambda^2 + 3\lambda + 5 + \lambda e^{-\lambda T} = 0. \tag{40}$$

In the absence of delay, this becomes

$$\lambda^2 + 4\lambda + 5 = 0, \tag{41}$$

which clearly has only roots with negative real parts, and thus the steady state is stable. Explicitly, the roots are $\lambda_{1,2} = -2 \pm i$. The polynomial produced by the process we have described from (34) to (39) is

$$\mu^2 - 2\mu + 25 = 0, \tag{42}$$

whose roots are $1 \pm 2\sqrt{6}i$. This polynomial has no positive real solution, and yet fails the Routh-Hurwitz conditions. In other words, the Routh-Hurwitz conditions can guarantee the absence of a bifurcation but cannot give conditions under which a bifurcation does occur with increasing T. A simple approach to determining whether a positive real root exists is Descartes' Rule of Signs, whereby the number of sign changes in the coefficients is equal to the number of positive real roots, modulo 2. If the number of sign changes is odd, then a solution is guaranteed. If, however, the number of sign changes is even, the rule cannot distinguish between, for example, 2 roots and 0 roots.

A more general approach to this problem uses Sturm sequences [21]. Suppose that a polynomial σ as in (39) has no repeated roots. Then σ_0 is relatively prime. Let $\sigma = \sigma_0$ and $\sigma' = \sigma_1$. We obtain the following sequence of equations by the division algorithm

$$\begin{aligned} \sigma_0 &= a_0\sigma_1 - \sigma_2, \\ \sigma_1 &= a_1\sigma_2 - \sigma_3, \\ &\vdots \\ \sigma_{s-2} &= a_{s-2}\sigma_{s-1} - K, \end{aligned} \tag{43}$$

where K is some constant. The sequence of Sturm functions, σ_0, σ_1, $\sigma_2, \cdots$, σ_{s-1}, $\sigma_s(= K)$ is called a Sturm chain. We may determine the number of real roots of the polynomial in any interval in the following manner: Plug in each endpoint of the interval and obtain a sequence of signs. The number of real roots in the interval is the difference between the number of sign changes in the sequence at each endpoint. For a complete proof of the method of Sturm sequences, see [22]. For example, consider the polynomial

$$\mu^2 - \mu - 2 = 0. \tag{44}$$

Using $\sigma = \sigma_0$, $\sigma' = \sigma_1$, and (43), the corresponding Sturm chain is

$$\sigma_0 = \mu^2 - \mu - 2, \quad \sigma_1 = 2\mu - 1, \quad \sigma_2 = \frac{9}{4}. \tag{45}$$

We evaluate these in the interval $[-3, 3]$ and $[0, 3]$, and construct a table of the signs at these end points. Considering the difference in the number of sign changes

	-3	3		0	3
σ_0	$+$	$+$		$-$	$+$
σ_1	$-$	$+$		$-$	$+$
σ_2	$+$	$+$		$+$	$+$

at each endpoint, we know that there must be two real roots in the interval [-3, 3], and one in [0, 3]. Given a specified parameter set, this method gives a simple, implementable algorithm for determining whether a bifurcation occurs, without the need to run the full simulation of the system of equations for various delays.

Equation (39) has a positive real root when gain $(k_c/k_m) = 0.2527$, $\omega_n = 150(\text{sec}^{-2})$, and $\zeta = 0.05$, we can obtain two roots μ_1^* and μ_2^*, and using $\mu^* = \nu^{*2}$ we get

$$\nu^* = \pm\sqrt{\mu_1}, \ \pm\sqrt{\mu_2} \tag{46}$$

Substituting the roots in (46) into (37) yields the critical values of delay, T^*, such as,

$$T^* = \cdots, 11.2364, \ 12.4700, \ 18.3490, \ 25.8137, \ 49.9977, \cdots \tag{47}$$

The results agree with previous results, as shown in the Figure 6, and provide an analytical method to determine the exact values of T^* that cause bifurcation. Note that the value of $1/T^*$ corresponds to the spindle speed.

The characteristic equation (32) has two roots in left-half plane (LHP) when the gain (k_c/k_m) is zero and in the region under the borderline no bifurcation occurs. Therefore, the region under the borderline is a stable one, which means chatter free. At the bifurcation point such as "A", the root crosses the imaginary axis of the complex plane (LHP to RHP (right-half plane) or RHP to LHP) if and only if [21]

$$R_1(\nu^*)R_1'(\nu^*) + Q_1(\nu^*)Q_1'(\nu^*) \neq R_2(\nu^*)R_2'(\nu^*) + Q_2(\nu^*)Q_2'(\nu^*). \tag{48}$$

The point "A" satisfies this condition; therefore, the right side is a stable region, and the left side is an unstable region. By marking the values of critical delays, T^*, and varying the ratio, (k_c/k_m), we can obtain exactly the same stability lobes as in Figure 5. With this method it is possible to obtain the critical delayed time T^* that determines the stability of a system without restrictive geometric analysis (e.g., Nyquist method) and to predict analytically the preferred operating spindle speed at which chatter does not occur. This method can also be used for less than fourth-order characteristic equations to get analytical solutions, and is also suitable for higher-order cases via numerical calculations [21].

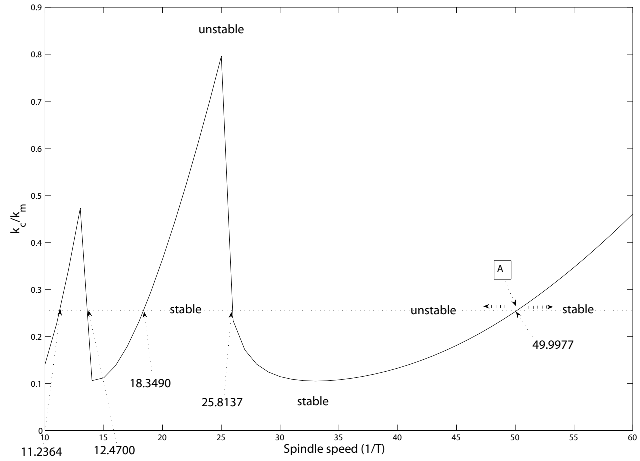

FIGURE 6. Comparison of the bifurcation analysis [21] with the Lambert method shows excellent agreement

6. **Concluding remarks.** In this paper, two new approaches for the stability analysis of machining tool chatter problems, which can be expressed as systems of linear delay differential equations, have been presented using the matrix Lambert function and a bifurcation analysis. The main advantage of the analytical approach based on the matrix Lambert function lies in the fact that one can obtain the solution to systems of linear DDEs in the time domain, and the solution has a form analogous to the state transition matrix in systems of linear ordinary differential equations. It can be applied to systems of linear DDEs of arbitrary order, and thus can be used in chatter models that include multiple structural vibration modes. Though the solution is in the form of an infinite series of modes computed with different branches, we observe that the principal branch always determines the stability of a system. Therefore, it appears that one has only to check the solution using the principal branch to determine the stability of the system. The results show excellent agreement with those obtained using traditional methods, e.g., Lyapunov [20], Nyquist, and the numerical method used in [6]. The method not only yields stability results but also can be used to obtain the free and forced response of the linearized machine tool dynamics. The results obtained with the Lambert method also are compared with those obtained using the bifurcation analysis with Sturm sequences. Compared with the Nyquist or Lyapunov methods, using the bifurcation analysis method we can determine the critical values of delay at the stability limit of the system with relatively simple calculations, avoiding restrictive geometric analysis. The bifurcation analysis method is well suited for the stability analysis of 1-DOF (second order) machining dynamics problems. With this approach, the higher DOF systems can also be analyzed in a similar way using Sturm sequences.

The matrix Lambert W function and Sturm sequence methods demonstrated here for machine tool chatter are applicable to a wide variety of systems represented by systems of linear delay differential equations. The authors are currently working on application of these methods to HIV dynamics.

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