MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **3**, Number **2**, April **2006**

http://www.mbejournal.org/

pp. **385–387**

REMARK ON THE PAPER BY RAO AND KAKEHASHI (2005)

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(Communicated by Patrick Nelson)

The paper by Rao and Kakehashi [1] uses the Weibull distribution (given by equation (3) in the paper) and a truncated version of it (given by equations (4) and (5)) to model HIV/AIDS data in India. After a careful reading, I have found that all of the results presented in the appendix are incorrect (starting with equation (11) itself). Instead of pointing out all of the specific errors, I have chosen to give the *correct* formulas (and a brief outline of their derivation) for the rth moment of the truncated Weibull distribution.

The probability density function (pdf) of the truncated Weibull distribution is given by

$$f(y) = \begin{cases} \dfrac{\beta}{\alpha}\left(\dfrac{y}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\dfrac{y}{\alpha}\right)^{\beta}\right\}, \\ \quad \text{if } 0 < y < t, \\ \dfrac{\beta}{\alpha}\exp\left\{-\dfrac{\beta}{\alpha}\left(\dfrac{t}{\alpha}\right)^{\beta-1}(t-y)-\left(\dfrac{y}{\alpha}\right)^{\beta}\right\}\left\{\left(\dfrac{t}{\alpha}\right)^{\beta-1}+\left(\dfrac{y}{\alpha}\right)^{\beta-1}\right\}, \\ \quad \text{if } y \geq t \end{cases} \tag{1}$$

for $\alpha > 0$, $\beta > 0$ and $t > 0$ (see equation (5) in the paper by Rao and Kakehashi [1]). The corresponding rth moment can be expressed as

$$\begin{aligned} T_r &= \int_0^t y^r \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{\beta-1}\exp\left\{-\left(\frac{y}{\alpha}\right)^{\beta}\right\} dy \\ &\quad + \int_t^\infty y^r \frac{\beta}{\alpha}\exp\left\{-\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}(t-y)-\left(\frac{y}{\alpha}\right)^{\beta}\right\}\left(\frac{t}{\alpha}\right)^{\beta-1} dy \\ &\quad + \int_t^\infty y^r \frac{\beta}{\alpha}\exp\left\{-\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}(t-y)-\left(\frac{y}{\alpha}\right)^{\beta}\right\}\left(\frac{y}{\alpha}\right)^{\beta-1} dy \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{2}$$

where I_1, I_2 and I_3 denote the three integrals. Using the substitution $z = (y/\alpha)^\beta$ and the incomplete gamma function defined by

$$\gamma(a, x) = \int_0^x t^{a-1}\exp(-t)\,dt,$$

2000 *Mathematics Subject Classification.* 92A15, 62A10.

Key words and phrases. Incubation time distribution, Moments, Truncated Weibull distribution.

one can calculate I_1 as

$$I_1 = \alpha^r \gamma \left(\frac{r}{\beta} + 1, \left(\frac{t}{\alpha} \right)^{\beta} \right). \tag{3}$$

Note that (3) holds for any $\beta > 0$. However, such general expressions for I_2 and I_3 are difficult to find but one derive closed forms for the special cases $\beta = 1$ and $\beta = 2$ (as correctly pointed out by Rao and Kakehashi [1]). If $\beta = 1$, using the substitution $z = 2y/\alpha$ and the complementary incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} \exp(-t)\, dt,$$

one can calculate I_2 and I_3 as

$$I_2 = \alpha^r 2^{-r-1} \exp\left(\frac{t}{\alpha} \right) \Gamma \left(r + 1, \frac{2t}{\alpha} \right) \tag{4}$$

and

$$I_3 = \alpha^r 2^{-r-1} \exp\left(\frac{t}{\alpha} \right) \Gamma \left(r + 1, \frac{2t}{\alpha} \right). \tag{5}$$

If $\beta = 2$, one can write

$$\begin{aligned} I_2 &= \frac{2t}{\alpha^2} \exp\left(\frac{2t^2}{\alpha^2} \right) \int_t^{\infty} y^r \exp\left(-\frac{2ty}{\alpha^2} - \frac{y^2}{\alpha^2} \right) dy \\ &= \frac{2t}{\alpha^2} \exp\left(\frac{2t^2}{\alpha^2} \right) \int_t^{\infty} (y - t + t)^r \exp\left(-\frac{2ty}{\alpha^2} - \frac{y^2}{\alpha^2} \right) dy \\ &= \frac{2t}{\alpha^2} \exp\left(\frac{2t^2}{\alpha^2} \right) \sum_{k=0}^{r} \binom{r}{k} t^{r-k} \int_t^{\infty} (y - t)^k \exp\left(-\frac{2ty}{\alpha^2} - \frac{y^2}{\alpha^2} \right) dy \\ &= tr! \exp\left(\frac{t^2}{\alpha^2} \right) \sum_{k=0}^{r} \frac{t^{r-k} \alpha^{k-1}}{2^{(k-1)/2} (r-k)!} D_{-k-1} \left(\frac{2\sqrt{2}t}{\alpha} \right), \end{aligned} \tag{6}$$

where we have assumed that r is a positive integer (as usually the case). The last step of the above argument follows from equation (2.3.15.1) in Prudnikov et al. [2] and $D_p(\cdot)$ denotes the parabolic cylinder function defined by

$$D_p(x) = \frac{\exp\left(-x^2/4\right)}{\Gamma(-p)} \int_0^{\infty} \exp\left\{ -\left(tx + t^2/2\right) \right\} t^{-(p+1)} dt.$$

A similar argument shows that

$$I_3 = (r+1)! \exp\left(\frac{t^2}{\alpha^2} \right) \sum_{k=0}^{r+1} \frac{t^{r+1-k} \alpha^{k-1}}{2^{(k-1)/2} (r+1-k)!} D_{-k-1} \left(\frac{2\sqrt{2}t}{\alpha} \right). \tag{7}$$

Now, substituting (3), (4) and (5) into (2), the rth moment of the truncated Weibull distribution for $\beta = 1$ takes the form

$$T_r = \alpha^r \gamma \left(r + 1, \frac{t}{\alpha} \right) + 2\alpha^r 2^{-r-1} \exp\left(\frac{t}{\alpha} \right) \Gamma \left(r + 1, \frac{2t}{\alpha} \right).$$

Substituting (3), (6) and (7) into (2), the rth moment for $\beta = 2$ becomes

$$\begin{aligned} T_r &= \alpha^r \gamma\left(\frac{r}{2}+1, \frac{t^2}{\alpha^2}\right) + tr! \exp\left(\frac{t^2}{\alpha^2}\right) \sum_{k=0}^{r} \frac{t^{r-k}\alpha^{k-1}}{2^{(k-1)/2}(r-k)!} D_{-k-1}\left(\frac{2\sqrt{2}t}{\alpha}\right) \\ &\quad + (r+1)! \exp\left(\frac{t^2}{\alpha^2}\right) \sum_{k=0}^{r+1} \frac{t^{r+1-k}\alpha^{k-1}}{2^{(k-1)/2}(r+1-k)!} D_{-k-1}\left(\frac{2\sqrt{2}t}{\alpha}\right). \end{aligned}$$

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Received on January 6, 2006. Accepted on January 17, 2006.

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