



Research article**Dynamic analysis and optimal control of an ecological snowdrift game-coupled delayed information dissemination model considering information beneficiaries and retention behavior****Xinwen Zhang¹, Rongyu Zhang¹ and Xuechao Zhang^{1,2,*}**¹ School of Economics and Law, University of Science and Technology Liaoning, Anshan, Liaoning, China² School of Business Administration, Liaoning Technical University, Huludao, Liaoning, China*** Correspondence:** Email: zhangxuechao@ustl.edu.cn.

Abstract: Online information spreading depends on individual judgment, strategy choice, and network structure. Here, we used the eco-evolutionary snowdrift game of Lv et al. and assigned the benefit b , cost c , and environmental pressure Q interpretations suited to effective information dissemination. The selected attracting ecological densities of cooperators and defectors were mapped to the exit rates of spreaders and information holders; in a multistable region, this selection may depend on the game parameters and initial state. This provided an *IHSCR* delay model with information-ignorant individuals, beneficiaries, spreaders, holders, and information-immune individuals. A survival-consistent delayed transition formulation preserved nonnegativity and yielded $R_0(\tau) = e^{-\mu\tau}R_0(0)$, with the critical delay $\tau_p = \mu^{-1} \ln R_0(0)$ when $R_0(0) > 1$. The information-free equilibrium was globally asymptotically stable for $R_0(\tau) \leq 1$, and no delay-induced imaginary-axis crossing occurred for the positive equilibrium on its existence interval. We also formulated a joint optimal-control problem for the spreader–holder transition channels. A larger cooperative ecological density raised λ_C and shortened the recurrent active cycle; the resulting decrease in R_0 does not imply a negative social effect of cooperation. Numerical simulations illustrated the threshold and showed that, in the tested settings, longer delays weaken finite-horizon control performance. Joint control gives the largest net objective among the tested strategies, while sensitivity tests across six scenarios support the dominance of u_1 under the stated control bounds and cost weights.

Keywords: information dissemination; information beneficiaries; temporary information holding; eco-evolutionary snowdrift game; survival-consistent time-delay system; persistence threshold; optimal control

Mathematics Subject Classification: 34D20, 34K18, 49J15, 91A22

1. Introduction

Online social networks have made information easier to produce, obtain, and share. Knowledge, policies, and scientific results can reach many users in a short time and affect how people think and act [1–3]. Yet, information spreading is not a simple contact–forwarding process. After receiving information, a person may share it, hold it for a time, or leave the process. This choice depends on the value of the information, the cost of sharing, and the expected benefit [4, 5]. A useful model should therefore include state transitions and individual decisions [6, 7].

Information spreading and epidemic transmission involve contact and state change. For this reason, the Daley–Kendall and Maki–Thompson models have often been used as starting points for information-spreading models [8, 9]. Later, researchers added delay and network structure [10, 11], as well as hesitation, forgetting, and behavioral differences [12–14]. Zhang et al. introduced information beneficiaries in an IHSR model. Their results suggest that the beneficiary state links information reception with later spreading behavior [15].

Information beneficiaries do not always become spreaders at once. Some hold information because they compare benefits, value an information advantage, care about privacy, or lack the ability to share it. They may later move between the spreader and holder states as incentives and information value change. Adding a holder state therefore gives a more realistic path from receiving useful information to spreading it, holding it, or leaving the process [12, 13, 15].

People often verify information and compare its value before they decide to share or hold it. This takes time and creates a behavioral decision delay. Earlier work shows that a delay can change stability and may produce periodic oscillations [10, 11, 14]. For a delayed compartmental transition, however, the delayed flux must preserve the nonnegative state space and maintain a consistent population interpretation. Optimal control is also useful when spreading goals must be balanced against intervention cost. Kang et al. studied optimal control with cross-transmission and information variation [16]. Other researchers added truth spreaders and optimized the control time [17]. Moreover, Wang et al. used educational intervention to limit rumors [18]. These examples show that the chosen control channels determine the objective functional and the adjoint system. Furthermore, networked-dynamics studies compare intervention performance across interaction settings, behavioral mechanisms, network structures, and parameter perturbations [19–21]. These comparisons motivate the additional sensitivity and convergence tests reported here.

Many spreading models treat sharing, holding, and quitting as fixed transition rates. They say little about how these rates may come from strategy choices shaped by benefit, cost, and the surrounding environment. Information-sharing behavior has a clear social-dilemma and strategic-interaction dimension [22–24]. Related studies examine competition among multiple information streams [25–27], while heterogeneous adoption behavior is studied in Ref. [28]. Spreaders pay the cost of checking, organizing, and sharing information, while free riders may benefit without paying that cost. For useful information, it is reasonable to assume $b > c > 0$, and one person's cooperation can still produce a positive outcome. This matches the snowdrift payoff structure, where unilateral cooperation is better than mutual defection [29, 30].

Lv et al. added the same-strategy encounter probability m and the environmental pressure Q to an eco-evolutionary snowdrift game [29]. They linked cooperation and defection directly to recovery channels. We keep their game structure but give b , c , and Q new meanings for effective information

spreading and change the coupling. The game is solved first to select an attracting equilibrium (x_G^*, y_G^*) according to the game parameters and initial state. These values then set the exit rates $\lambda_C = \omega_C x_G^*$ and $\lambda_S = \omega_D y_G^*$. In this way, a cooperative environment shortens long-term information holding, while a defection-dominated environment crowds out existing spreaders. In a multistable parameter region, the downstream spreading result is therefore conditional on the basin containing the reported initial game state.

Several questions remain open. Most spreading models do not derive their transition parameters from cooperation–defection dynamics driven by benefit, cost, encounter tendency, and environmental pressure. Beneficiary-based models also give little attention to information holding after benefit is obtained or to the two-way movement between spreaders and holders. Control researchers mainly focus on rumor suppression. Fewer studies ask how to promote useful information when game conditions, holding behavior, and decision delay are present at the same time. Existing game–spreading models also tend to insert strategy variables directly into recovery or exit rates. This can blur the difference between an ecological strategy density and a spreading transition rate. Moreover, equilibrium selection in a multistable game and the admissibility of a positive-delay transition mechanism need to be made explicit before the downstream threshold, stability, and control conclusions are interpreted.

Here, we build an *IHSCR* delay model and couple it with the eco-evolutionary snowdrift game. Ignorant individuals first become beneficiaries after contact with spreaders. Beneficiaries then enter one of two decision-completion channels, and the surviving members of each cohort complete the transition to spreading or holding after a fixed delay; the two active states can also change into each other. The game affects the spreading model through two exit rates. A larger x_G^* raises λ_C and shortens the holder state. Its direct negative effect on R_0 means only that active-state regeneration lasts for less time after information is submitted or shared; it does not mean that cooperation is harmful. The final effect depends on x_G^* and y_G^* and on the relative strength of the two exit channels. All such conclusions are stated for the selected attracting game equilibrium.

The paper makes four main contributions.

- (1) We assign b , c , and Q interpretations suited to useful information spreading and study the boundary equilibria and invasion conditions of the ecological snowdrift game [29]. We further give explicit coexistence candidates, a Jacobian-based local stability classification, an equilibrium-selection rule, and reproducible saddle-node and basin-separation calculations.
- (2) We add an information-holder state and two-way spreader–holder transitions. The selected attracting game densities x_G^* and y_G^* are mapped to λ_C and λ_S instead of being inserted directly as compartment sizes. The positive delay is formulated through survival-consistent decision-completion cohorts, preserving nonnegativity and an augmented population balance.
- (3) We derive R_0 and the two equilibria, study their stability at $\tau = 0$, and establish the positive-delay threshold $R_0(\tau) = e^{-\mu\tau} R_0(0)$ and the persistence critical delay τ_p . The information-free equilibrium is globally asymptotically stable when $R_0(\tau) \leq 1$, and the positive equilibrium has no delay-induced imaginary-axis crossing while it exists.
- (4) We control only the spreading layer. The control u_1 promotes $C \rightarrow S$, while u_2 limits $S \rightarrow C$. This keeps the game parameters as structural conditions and balances spreading benefit against control cost. Delay scans, game-to-spreading coupling-strength and behavioral-transition perturbations, separate-channel comparisons, and time-grid refinement are used to test robustness.

The rest of the paper is arranged as follows: In Section 2, we give the game model, the delayed

IHSCR model, and the one-way coupling between them; in Section 3, we study the game equilibria and invasion conditions, local stability, and equilibrium selection; in Section 4, we analyze R_0 , equilibrium stability, and the delay-dependent persistence threshold; in Section 5, we give the optimal-control problem; in Section 6, we report numerical results and robustness checks; and in Section 7, we conclude the paper.

2. Model construction

We first adapt the eco-evolutionary snowdrift game of Lv et al. to useful information spreading [29]. We then extend the beneficiary model in [15] by adding information holders, a behavioral decision delay, and two-way transitions between spreaders and holders. The two parts are linked through a one-way coupling.

2.1. Eco-evolutionary snowdrift game model

The snowdrift game describes a choice made when people share a common benefit but cooperation has a cost [29, 30]. Here, cooperation means paying the cost of checking, organizing, and sharing information. It also reduces long-term holding of useful information. Defection means avoiding this cost while having a chance to benefit from the sharing of others. To distinguish the cooperative strategy from the information-holder compartment $C(t)$, the two game strategies are denoted by C (cooperation) and $\mathcal{D}$ (defection) below.

Let b be the benefit from effective information spreading and let c be the total cost of checking, organizing, and sharing information, with $b > c > 0$. If both individuals cooperate, they divide the cost and each receives $b - c/2$. If only one cooperates, the cooperator receives $b - c$ and the defector receives b . Mutual defection gives no sharing benefit. Table 1 gives the payoff matrix.

Table 1. The payoff matrix of the snowdrift game.

Strategy	Cooperation C	Defection $\mathcal{D}$
Cooperation C	$b - \frac{c}{2}$	$b - c$
Defection $\mathcal{D}$	b	0

Let $x(t)$ and $y(t)$ be the ecological densities of cooperators and defectors. They evolve under an environmental-capacity constraint, so we do not require $x(t) + y(t) = 1$. To describe same-strategy encounters [30], an individual meets the same strategy with probability m and a randomly selected individual with probability $1 - m$, where $0 \leq m < 1$. If the random contact belongs to neither strategy group, no game occurs and no payoff is produced.

The average payoffs are

$$f_C = \left(b - \frac{c}{2}\right)[m + (1 - m)x] + (b - c)(1 - m)y, \quad (2.1)$$

$$f_{\mathcal{D}} = b(1 - m)x. \quad (2.2)$$

Let $r_0 > 0$ be the baseline proliferation rate without game payoff, and let $\mu_G > 0$ be the removal rate of the strategy groups. Adding the payoff to the baseline rate gives

$$b_C = r_0 + f_C = r_0 + \left(b - \frac{c}{2}\right)[m + (1 - m)x] + (b - c)(1 - m)y, \quad (2.3)$$

$$b_D = r_0 + f_D = r_0 + b(1 - m)x. \quad (2.4)$$

We also introduce an external environmental pressure Q , where $0 \leq Q < 1$. It summarizes limits caused by platform supervision, competition for attention, and the wider information environment. The remaining environmental capacity is $1 - x - y - Q$. The game model is

$$\begin{cases} \frac{dx}{dt} = b_C x(1 - x - y - Q) - \mu_G x, \\ \frac{dy}{dt} = b_D y(1 - x - y - Q) - \mu_G y. \end{cases} \quad (2.5)$$

Its feasible state region is

$$\Gamma_G = \{(x, y) \in \mathbb{R}_+^2 \mid x + y \leq 1 - Q\}.$$

The variables and parameters of the eco-evolutionary snowdrift game are listed in Table 2.

Table 2. Variables and parameters of the eco-evolutionary snowdrift game.

Symbol	Description
$x(t)$	Ecological density of the cooperative population at time t
$y(t)$	Ecological density of the defector population at time t
b	Benefit generated by effective information dissemination
c	Total cost of verifying, organizing, and sharing information
m	Probability of encountering individuals using the same strategy
r_0	Baseline proliferation rate without game payoffs
μ_G	Removal rate of strategy groups
Q	External environmental pressure
ω_D	Game-to-spreading coupling coefficient mapping the selected equilibrium defector density y_G^* to the effective exit rate λ_S of spreaders (time^{-1})
ω_C	Game-to-spreading coupling coefficient mapping the selected equilibrium cooperator density x_G^* to the effective exit rate λ_C of information holders (time^{-1})

2.2. IHSCR time-delay information dissemination model

Consider an open online social system whose population size changes over time. Because a positive conversion delay creates two pending decision-completion cohorts, the augmented total population at time t is

$$\widetilde{N}(t) = I(t) + H(t) + P_S(t) + P_C(t) + S(t) + C(t) + R(t),$$

where $I(t)$, $H(t)$, $S(t)$, $C(t)$, and $R(t)$ denote information-ignorant individuals, information beneficiaries, information spreaders, information holders, and information-immune individuals, respectively. The five focal groups are defined as follows:

- (1) Information-ignorant individuals $I(t)$: Individuals who have not contacted the target information but may accept it.

- (2) Information beneficiaries $H(t)$: Individuals who obtain cognition, knowledge, or actual benefits after contacting the information and are undergoing behavioral judgment before entering either decision-completion channel.
- (3) Information spreaders $S(t)$: Individuals who choose to continuously disseminate the target information after benefiting from it.
- (4) Information holders $C(t)$: Individuals who choose to temporarily hold the information after benefiting from it, without continuously spreading it.
- (5) Information-immune individuals $R(t)$: Individuals who no longer participate as active nodes in the current dissemination cycle after information-value attenuation, attention transfer, forgetting, or completion of the current sharing process.

Based on this classification, we make the following assumptions.

- (1) **Population mobility.** During each unit of time, B new members enter the system through the information-ignorant compartment. Individuals in each compartment leave the system at rate μ due to going offline, migration, or other reasons.
- (2) **Effective information contact.** When ignorant individuals have effective contact with spreaders in the network, they become information beneficiaries at the incidence rate $\alpha\langle k\rangle I(t)S(t)$, where α is the effective information-contact transmission coefficient per network connection, and $\langle k\rangle$ is the average degree of the complex network [2, 6, 7]. Similar network-based transmission formulations appear in Refs. [10, 11].
- (3) **Delayed decision-making of beneficiaries.** Information beneficiaries decide whether to spread or hold the information only after verification, benefit comparison, and behavioral trade-off [15]. At time t , beneficiaries enter a spreader-oriented decision-completion channel and a holder-oriented decision-completion channel at the current rates $\zeta H(t)$ and $\xi H(t)$, respectively. Completion of either conversion requires a fixed delay $\tau \geq 0$. During the pending period, individuals may leave the online system at rate μ ; hence, the fraction surviving until completion is

$$q_\tau = e^{-\mu\tau}. \quad (2.6)$$

The completed inflows at time t are therefore $q_\tau\zeta H(t-\tau)$ and $q_\tau\xi H(t-\tau)$.

- (4) **Spreading-temporary-holding bidirectional transition.** When the spreading benefit decreases or the information-monopoly advantage increases, spreaders become information holders at rate δ ; when sharing incentives, additional benefits, or information value change, information holders return to the spreader state at rate σ .
- (5) **Information-value attenuation.** Spreaders and information holders enter the information-immune state at rates ε and θ , respectively, due to information-value attenuation, attention transfer, or forgetting.
- (6) **Strategy-dependent exit.** Let the selected attracting cooperative ecological density and the selected attracting ecological density of defectors generated by the eco-evolutionary snowdrift game be x_G^* and y_G^* , respectively. Define

$$\lambda_S = \omega_D y_G^*, \quad \lambda_C = \omega_C x_G^*. \quad (2.7)$$

Here, λ_S denotes the effective exit rate of spreaders induced by a defection-dominated environment, whereas λ_C denotes the additional completion or resolution rate from the

information-holder state under a cooperative environment. The defection-dominated environment crowds out existing spreaders, while the cooperative environment reduces long-term information retention. After completing a one-off information use, verification, submission or sharing, some holders no longer remain active in the current dissemination cycle. The rate λ_C is not a direct $C \rightarrow S$ transition; σ remains the holder-to-spreader rate. Moreover, R_0 measures regeneration of the recurrent active dissemination cycle and is not a social-welfare index.

For population accounting, define the auxiliary pending cohorts

$$P_S(t) = \int_{t-\tau}^t e^{-\mu(t-s)} \zeta H(s) ds, \quad P_C(t) = \int_{t-\tau}^t e^{-\mu(t-s)} \xi H(s) ds. \quad (2.8)$$

They contain beneficiaries who have entered one of the two decision-completion channels but have not completed the delayed transition. They are determined by the history of H and need not be integrated as independent state equations. Their derivatives are

$$\begin{aligned} \dot{P}_S(t) &= \zeta H(t) - q_\tau \zeta H(t - \tau) - \mu P_S(t), \\ \dot{P}_C(t) &= \xi H(t) - q_\tau \xi H(t - \tau) - \mu P_C(t). \end{aligned} \quad (2.9)$$

Figure 1 shows the delayed *IHSCR* process based on these states and transitions.

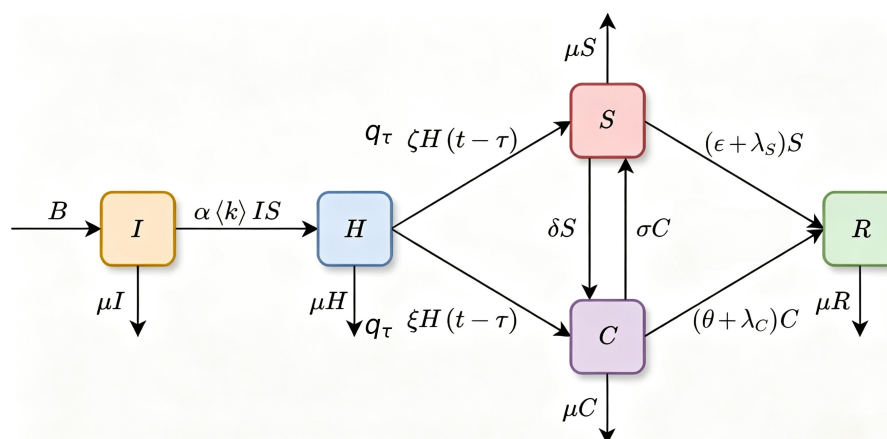


Figure 1. Flow diagram of the *IHSCR* time-delay information dissemination model. The two delayed inflows include the survival factor $q_\tau = e^{-\mu\tau}$.

In a complex network, the node degree k denotes the number of network connections possessed by an individual, and the degree distribution $P(k)$ denotes the probability that a randomly selected node has k connections. The average degree of the network is defined as

$$\langle k \rangle = \sum_{k=1}^n k P(k). \quad (2.10)$$

The average degree gives the mean number of connections per node. A larger $\langle k \rangle$ gives ignorant individuals more chances to contact spreaders. We use a mean-field model and do not divide each compartment by degree. The contact term is therefore $\alpha \langle k \rangle I(t) S(t)$. For shorter notation, set $\kappa = \alpha \langle k \rangle$.

These transition rules give the delayed *IHSCR* model

$$\begin{cases} \frac{dI(t)}{dt} = B - \kappa I(t)S(t) - \mu I(t), \\ \frac{dH(t)}{dt} = \kappa I(t)S(t) - (\zeta + \xi + \mu)H(t), \\ \frac{dS(t)}{dt} = q_\tau \zeta H(t - \tau) + \sigma C(t) - (\delta + \varepsilon + \mu + \lambda_S)S(t), \\ \frac{dC(t)}{dt} = q_\tau \xi H(t - \tau) + \delta S(t) - (\sigma + \theta + \mu + \lambda_C)C(t), \\ \frac{dR(t)}{dt} = (\varepsilon + \lambda_S)S(t) + (\theta + \lambda_C)C(t) - \mu R(t). \end{cases} \quad (2.11)$$

Together with Eq (2.9), model (2.11) satisfies the augmented population balance

$$\frac{d\tilde{N}(t)}{dt} = B - \mu\tilde{N}(t). \quad (2.12)$$

The model variables and parameters are listed in Table 3.

Table 3. Variables and parameters of the *IHSCR* time-delay information dissemination model.

Symbol	Meaning
$I(t)$	Density of information-ignorant individuals at time t
$H(t)$	Density of information beneficiaries who have not entered a decision-completion channel at time t
$S(t)$	Density of information spreaders at time t
$C(t)$	Density of information holders at time t
$R(t)$	Density of information-immune individuals at time t
$P_S(t), P_C(t)$	Densities of the auxiliary pending spreader-oriented and holder-oriented completion cohorts
B	Rescaled inflow rate of new individuals into the system
α	Effective contact transmission coefficient of a single connection
$\langle k \rangle$	Average degree of the complex network
ζ	Rate at which beneficiaries enter the spreader-oriented decision-completion channel
ξ	Rate at which beneficiaries enter the holder-oriented decision-completion channel
δ	Rate at which spreaders become holders
σ	Rate at which holders return to the spreader state
ε	Transition rate of spreaders into the information-immune state due to information-value attenuation
θ	Transition rate of holders into the information-immune state due to information-value attenuation
μ	Rate at which individuals leave the system
τ	Behavioral decision-completion delay of beneficiaries
q_τ	Survival fraction during the decision-completion period
λ_S	Effective exit rate of spreaders induced by the defection-dominated environment
λ_C	Effective completion or resolution rate of holders induced by the cooperative environment
ω_D	Game-to-spreading coupling coefficient mapping the selected equilibrium defector density y_G^* to the effective exit rate λ_S of spreaders (time^{-1})
ω_C	Game-to-spreading coupling coefficient mapping the selected equilibrium cooperator density x_G^* to the effective exit rate λ_C of information holders (time^{-1})

Unless stated otherwise, the rate parameters B , α , ζ , ξ , δ , σ , ε , θ , and μ , together with $\langle k \rangle$, are positive. The coupling-induced exit rates satisfy $\lambda_S, \lambda_C \geq 0$, and the decision delay satisfies $\tau \geq 0$. The focal and pending state variables are rescaled densities; $\langle k \rangle$ and q_τ are dimensionless; B has units of density per unit time; α has units of inverse density per unit time; τ has units of time; and ζ , ξ , δ , σ , ε , θ , μ , λ_S , λ_C , ω_D , and ω_C have units of inverse time. The coefficients ω_D and ω_C are illustrative structural scaling coefficients rather than empirically calibrated quantities.

The initial history conditions of model (2.11) satisfy

$$\begin{cases} I(\vartheta) = \phi_1(\vartheta) \geq 0, \\ H(\vartheta) = \phi_2(\vartheta) \geq 0, \\ S(\vartheta) = \phi_3(\vartheta) \geq 0, \\ C(\vartheta) = \phi_4(\vartheta) \geq 0, \\ R(\vartheta) = \phi_5(\vartheta) \geq 0, \end{cases} \quad \vartheta \in [-\tau, 0], \quad (2.13)$$

where $\phi_i \in C([-\tau, 0], \mathbb{R}_+)$, $i = 1, 2, \dots, 5$. In particular, when $\vartheta = 0$, we have

$$I(0) = I_0 \geq 0, \quad H(0) = H_0 \geq 0, \quad S(0) = S_0 \geq 0, \quad C(0) = C_0 \geq 0, \quad R(0) = R_{\text{ini}} \geq 0.$$

The corresponding initial pending-cohort values are fixed by the same complete history:

$$P_S(0) = \int_{-\tau}^0 e^{\mu s} \zeta \phi_2(s) ds, \quad P_C(0) = \int_{-\tau}^0 e^{\mu s} \xi \phi_2(s) ds. \quad (2.14)$$

2.3. Coupling mechanism between the eco-evolutionary game and the information dissemination model

Inspired by Lv et al. [29], we use a two-stage, one-way coupling. For given b , c , m , Q , r_0 , μ_G , $x(0)$, and $y(0)$, the game system (2.5) is solved first. Its selected attracting densities x_G^* and y_G^* are then mapped to the two exit rates in Eq (2.7).

For clarity, let

$$p_G = (b, c, m, Q, r_0, \mu_G), \quad z_0 = (x_0, y_0) \in \Gamma_G,$$

and denote the solution semiflow of the game system by $\Phi_G(t; z_0, p_G)$. For an initial state whose trajectory converges to a locally asymptotically stable feasible equilibrium, define

$$E_G^{\text{sel}}(p_G, z_0) = \lim_{t \rightarrow \infty} \Phi_G(t; z_0, p_G) = (x_G^*, y_G^*). \quad (2.15)$$

Thus, x_G^* and y_G^* do not denote an equilibrium assumed to be globally unique. In a multistable parameter region, the selected branch depends on the basin containing z_0 . An initial state on a saddle, its stable manifold, or another exceptional invariant set is not assigned an attracting branch by default.

The quantities x_G^* and y_G^* are dimensionless ecological densities. Hence, ω_D and ω_C have units of inverse time, and λ_S and λ_C are transition rates. The game variables $x(t)$ and $y(t)$ are strategy densities under normalized capacity. The spreading variables are rescaled group densities [15]. Equation (2.7) therefore does not equate the sizes of the two populations. It uses only the selected attracting game densities to set spreading transition rates.

This coupling mechanism is

$$(p_G, z_0) \longrightarrow E_G^{\text{sel}}(p_G, z_0) \longrightarrow (\lambda_S, \lambda_C) \longrightarrow (I, H, S, C, R). \quad (2.16)$$

The game strategies do not directly decide whether a beneficiary first becomes a spreader or a holder. A defection-dominated environment raises λ_S and removes existing spreaders more quickly. A cooperative environment raises λ_C and reduces long-term holding. Here, λ_C describes holders who leave the active cycle after completing a one-off information use, verification, submitting, or sharing information. It is not a direct $C \rightarrow S$ transition. The coupling is therefore based on strategy-dependent exit rates, not on real-time two-way feedback between the two systems. Accordingly, a negative derivative of R_0 with respect to λ_C indicates a shorter recurrent active-cycle duration, not a negative social effect of cooperation. Every downstream spreading conclusion is conditional on the reported game parameters, initial state, and selected basin.

3. Analysis of the ecological snowdrift game model

We now study the equilibria and invasion conditions of the game system (2.5) [29]. We also state how the stable ecological densities used in the coupling are selected. For shorter notation, let

$$A = b - \frac{c}{2}, \quad L = 1 - Q. \quad (3.1)$$

Because $b > c > 0$ and $0 \leq Q < 1$, A and L are positive.

3.1. Boundary equilibria

The equilibria of the eco-evolutionary snowdrift game system satisfy

$$x[b_C(1 - x - y - Q) - \mu_G] = 0, \quad y[b_D(1 - x - y - Q) - \mu_G] = 0. \quad (3.2)$$

Proposition 1. System (2.5) always has the empty-strategy equilibrium

$$E_G^0 = (0, 0). \quad (3.3)$$

When $\mu_G < r_0(1 - Q)$, the system has the pure defection equilibrium

$$E_G^{\mathcal{D}} = \left(0, 1 - Q - \frac{\mu_G}{r_0}\right). \quad (3.4)$$

Additionally, when the following quadratic equation has real roots in the feasible interval $(0, 1 - Q)$, the system has pure cooperation equilibria $E_G^{C\pm} = (x_{\pm}, 0)$:

$$A(1 - m)x^2 - [A(1 - m)L - Am - r_0]x + \mu_G - (r_0 + Am)L = 0, \quad (3.5)$$

where

$$x_{\pm} = \frac{A[(1 - m)L - m] - r_0 \pm \sqrt{\Delta}}{2A(1 - m)}, \quad (3.6)$$

$$\Delta = \{A[(1 - m)L + m] + r_0\}^2 - 4A(1 - m)\mu_G. \quad (3.7)$$

Therefore, a necessary condition for the existence of pure cooperation equilibria is $\Delta \geq 0$, and the roots must satisfy $0 < x_{\pm} < L$.

Proof. Setting $x = y = 0$ directly yields E_G^0 . Setting $x = 0$ and $y > 0$, the second equilibrium equation gives $r_0(L - y) - \mu_G = 0$, which leads to Eq (3.4); its ecological feasibility requires $1 - Q - \mu_G/r_0 > 0$. Setting $y = 0$ and $x > 0$, substituting $b_C = r_0 + A[m + (1 - m)x]$ into the first equilibrium equation and rearranging yields Eq (3.5). Applying the quadratic root formula then gives Eqs (3.6) and (3.7). The proof is complete. $\square$

If $x > 0$ and $y > 0$, the two strategies have the same equilibrium proliferation condition. Any positive solution located in the interior of Γ_G and satisfying

$$Am - \frac{c}{2}(1 - m)x + (b - c)(1 - m)y = 0, \quad (3.8)$$

$$[r_0 + b(1 - m)x](L - x - y) = \mu_G \quad (3.9)$$

constitutes a cooperation–defection coexistence equilibrium state.

For the parameter range $0 \leq m < 1$, the coexistence equilibrium can also be written explicitly. Let $d = b - c$, $u = 1 - m$, $\rho = A/d$, and $K_0 = L + Am/(du)$. Equation (3.8) gives

$$y = \frac{c}{2d}x - \frac{Am}{du}. \quad (3.10)$$

Substitution into Eq (3.9) yields

$$a_G x^2 + b_G x + c_G = 0, \quad (3.11)$$

where

$$a_G = bu\rho, \quad b_G = r_0\rho - buK_0, \quad c_G = \mu_G - r_0K_0. \quad (3.12)$$

If $\Delta_G = b_G^2 - 4a_Gc_G \geq 0$, all interior candidates are

$$x_{G,\pm} = \frac{-b_G \pm \sqrt{\Delta_G}}{2a_G}, \quad y_{G,\pm} = \frac{c}{2d}x_{G,\pm} - \frac{Am}{du}. \quad (3.13)$$

Only candidates satisfying $x_{G,\pm} > 0$, $y_{G,\pm} > 0$, and $x_{G,\pm} + y_{G,\pm} < L$ are retained. The endpoint $m = 1$, if considered, is checked directly from Eqs (3.8) and (3.9).

3.2. Strategy invasion conditions

An invasion condition asks whether a rare strategy can grow when the other strategy dominates. We check the sign of its per-capita growth rate near the relevant boundary equilibrium [29, 30].

Theorem 2. When the pure defection equilibrium E_G^D exists, the cooperation strategy can invade the pure defection population. The equivalent cooperation invasion condition is

$$\mu_G < r_0 \left[L + \frac{Am}{(b - c)(1 - m)} \right]. \quad (3.14)$$

Proof. Near E_G^D , the per-capita density growth rate of the cooperation strategy is

$$\frac{1}{x} \frac{dx}{dt} \Big|_{E_G^D} = \frac{\mu_G}{r_0} \left[Am + (b - c)(1 - m) \left(L - \frac{\mu_G}{r_0} \right) \right]. \quad (3.15)$$

The positivity of Eq (3.15) is equivalent to Eq (3.14). On the other hand, the existence of E_G^D requires $\mu_G < r_0L$, and the right-hand side of Eq (3.14) is greater than r_0L , so Eq (3.14) holds automatically. Therefore, as long as the pure defection equilibrium is ecologically meaningful, the rare cooperation strategy can obtain a positive growth rate. The proof is complete. $\square$

The condition also shows how cooperation gains a foothold. A larger m or a larger net benefit A helps a rare cooperator, while a larger checking and sharing cost c weakens this advantage [30].

Theorem 3. Let $E_G^C = (x_c^e, 0)$ be an ecologically meaningful pure cooperation equilibrium. The defection strategy can invade the pure cooperation population if and only if

$$x_c^e > \frac{(2b - c)m}{c(1 - m)}. \quad (3.16)$$

Proof. At the pure cooperation equilibrium, the cooperation strategy satisfies

$$b_C(x_c^e, 0)(L - x_c^e) = \mu_G. \quad (3.17)$$

The per-capita density growth rate of the defection strategy is

$$\left. \frac{1}{y} \frac{dy}{dt} \right|_{E_G^C} = (L - x_c^e) [b_D(x_c^e) - b_C(x_c^e, 0)]. \quad (3.18)$$

Since E_G^C lies in the interior of the feasible region, $L - x_c^e > 0$. Further computation gives

$$b_D(x_c^e) - b_C(x_c^e, 0) = \frac{c}{2}(1 - m)x_c^e - Am. \quad (3.19)$$

Therefore, Eq (3.18) is positive if and only if

$$\frac{c}{2}(1 - m)x_c^e > Am, \quad (3.20)$$

which, after rearrangement, yields Eq (3.16). The proof is complete. $\square$

Equation (3.16) gives the opposite effect. When the cooperative density is high, free riders receive more benefit from others' sharing and defection may invade. This helps explain why cooperation and defection can coexist in the snowdrift game [29].

3.3. Determination of the stable strategy state

For given b, c, m, Q, r_0, μ_G , and initial states $x(0) = x_0, y(0) = y_0$, system (2.5) is solved numerically. Let $\epsilon_0 > 0$ be the convergence tolerance. When

$$\left| \frac{dx}{dt} \right| < \epsilon_0, \quad \left| \frac{dy}{dt} \right| < \epsilon_0, \quad (3.21)$$

and these conditions persist over a prescribed stability-checking interval, the state at the terminal time t_{end} is defined as the stable ecological densities of cooperators and defectors:

$$x_G^* = x(t_{\text{end}}), \quad y_G^* = y(t_{\text{end}}). \quad (3.22)$$

The feasible set Γ_G is positively invariant under system (2.5). In this paper, x_G^* and y_G^* denote a parameter- and initial-condition-dependent attracting ecological state; when multiple feasible equilibria coexist, the state used in Eq (2.7) is identified numerically from trajectory convergence under the prescribed initial condition.

To classify the analytically enumerated equilibria, write $\dot{x} = xF_C(x, y)$ and $\dot{y} = yF_D(x, y)$, where the strategy labels C and D avoid confusion with the holder compartment $C(t)$. The Jacobian is

$$J_G(x, y) = \begin{pmatrix} F_C + x[A(1-m)(L-x-y) - b_C] & x[(b-c)(1-m)(L-x-y) - b_C] \\ y[b(1-m)(L-x-y) - b_D] & F_D - yb_D \end{pmatrix}. \quad (3.23)$$

At an interior equilibrium, $F_C = F_D = 0$. Hence, $\text{tr } J_G < 0$ and $\det J_G > 0$ imply local asymptotic stability, whereas $\det J_G < 0$ identifies a saddle. For $(b, c, m, Q, r_0, \mu_G) = (1, 0.4, 0, 0, 0.1, 0.05)$, the unique feasible coexistence equilibrium is $E_G^* \approx (0.70331865, 0.23443955)$, with eigenvalues -0.70954283 and -0.00875516 . At $Q = 0.58$, $E_G^+ \approx (0.18204026, 0.06068009)$ is stable, while $E_G^- \approx (0.03295974, 0.01098658)$ is a saddle.

All feasible equilibria are first enumerated from the boundary formulas and Eq (3.13). In the numerical selection rule, $\epsilon_0 = 10^{-9}$, the final stability-checking window has length 100, and $t_{\text{end}} \leq 5000$. A trajectory is assigned to one attracting equilibrium E only when Eq (3.21) and $\|(x(t), y(t)) - E\|_\infty < 10^{-7}$ hold at every checking node throughout that final window; otherwise, it is retained as unresolved. Thus, the selected game equilibrium is explicitly a function of the parameter vector and initial state, rather than an equilibrium assumed to be globally unique.

For the branch computation under the baseline payoff parameters, the transcritical and saddle-node thresholds are $Q_T = 0.5$ and $Q_{SN} = 17/15 - 2\sqrt{15}/15 \approx 0.6169$. At $Q = 0.58$, the separatrix is obtained by backward integration from E_G^- along its stable eigenvector using DOP853 with relative tolerance 10^{-10} , absolute tolerance 10^{-12} , and maximum step 0.25. The interior basin grid contains 3081 points with spacing 0.00525; halving the RK4 step from 0.25 to 0.125 changes no attractor label, and 40 selected labels are independently confirmed with DOP853. These checks establish reproducibility at the stated numerical resolution, not an exact analytic location of the separatrix.

4. Dynamical analysis of the IHSCR time-delay information dissemination model

To simplify subsequent expressions, denote

$$m_1 = \zeta + \xi + \mu, \quad m_2 = \delta + \varepsilon + \mu + \lambda_S, \quad m_3 = \sigma + \theta + \mu + \lambda_C, \quad (4.1)$$

and let

$$D = m_2 m_3 - \delta \sigma. \quad (4.2)$$

From the nonnegativity of the model parameters and $\mu > 0$, we have

$$D = \delta(\theta + \mu + \lambda_C) + \sigma(\varepsilon + \mu + \lambda_S) + (\varepsilon + \mu + \lambda_S)(\theta + \mu + \lambda_C) > 0. \quad (4.3)$$

4.1. Nonnegativity and boundedness of solutions

Lemma 4. For every continuous nonnegative initial history $\Phi \in C([-\tau, 0], \mathbb{R}_+^5)$, model (2.11) has a unique global solution that remains nonnegative. Moreover, with the two auxiliary decision-completion

cohorts defined in Section 2, the augmented system size

$$\widetilde{N}(t) = I(t) + H(t) + P_S(t) + P_C(t) + S(t) + C(t) + R(t) \quad (4.4)$$

satisfies

$$\frac{d\widetilde{N}(t)}{dt} = B - \mu\widetilde{N}(t). \quad (4.5)$$

Therefore,

$$\widetilde{N}(t) = \left(\widetilde{N}(0) - \frac{B}{\mu} \right) e^{-\mu t} + \frac{B}{\mu}, \quad (4.6)$$

and every component is ultimately bounded.

Proof. The right-hand side of model (2.11) is locally Lipschitz on the history space. On the coordinate boundaries,

$$\begin{aligned} I = 0 &\Rightarrow I' = B > 0, & H = 0 &\Rightarrow H' = \kappa IS \geq 0, \\ S = 0 &\Rightarrow S' = q_\tau \zeta H(t - \tau) + \sigma C \geq 0, & C = 0 &\Rightarrow C' = q_\tau \xi H(t - \tau) + \delta S \geq 0, \\ R = 0 &\Rightarrow R' = (\varepsilon + \lambda_S)S + (\theta + \lambda_C)C \geq 0. \end{aligned}$$

Hence, the vector field is quasipositive, and the nonnegative history space is positively invariant. The integral definitions of P_S and P_C give $P_S(t), P_C(t) \geq 0$. Adding model (2.11) to the two auxiliary cohort equations makes the current channel-entry fluxes and the surviving delayed completion fluxes cancel, yielding Eq (4.5). Solving this first-order linear equation gives Eq (4.6). The uniform bound excludes finite-time blow-up, so the local solution extends globally. $\square$

4.2. Basic reproduction number

The basic reproduction number R_0 is the mean number of new active information participants produced by one spreader when the information is rare. We use the next-generation matrix method [15, 17, 31] and take the active subsystem

$$X = (H, S, C)^T. \quad (4.7)$$

In the threshold computation, we first set $\tau = 0$; then the active subsystem is

$$\frac{dX}{dt} = \mathcal{F}(X) - \mathcal{V}(X), \quad (4.8)$$

where

$$\mathcal{F}(X) = \begin{pmatrix} \kappa IS \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V}(X) = \begin{pmatrix} m_1 H \\ m_2 S - \zeta H - \sigma C \\ m_3 C - \xi H - \delta S \end{pmatrix}. \quad (4.9)$$

At the information-free equilibrium, $I_0 = B/\mu$. Taking the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ for (H, S, C) gives

$$F = \begin{pmatrix} 0 & \frac{\kappa B}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} m_1 & 0 & 0 \\ -\zeta & m_2 & -\sigma \\ -\xi & -\delta & m_3 \end{pmatrix}. \quad (4.10)$$

Since $D > 0$, the matrix V is invertible, and

$$V^{-1} = \begin{pmatrix} \frac{1}{m_1} & 0 & 0 \\ \frac{m_3\zeta + \sigma\xi}{m_1D} & \frac{m_3}{D} & \frac{\sigma}{D} \\ \frac{\delta\zeta + m_2\xi}{m_1D} & \frac{\delta}{D} & \frac{m_2}{D} \end{pmatrix}. \quad (4.11)$$

The zero-delay reproduction number is

$$R_0(0) = \rho(FV^{-1}) = \frac{\kappa B(m_3\zeta + \sigma\xi)}{\mu m_1(m_2m_3 - \delta\sigma)}. \quad (4.12)$$

Substituting $\kappa = \alpha\langle k \rangle$ gives

$$R_0(0) = \frac{\alpha\langle k \rangle B [\zeta(\sigma + \theta + \mu + \lambda_C) + \sigma\xi]}{\mu(\zeta + \xi + \mu) [(\delta + \varepsilon + \mu + \lambda_S)(\sigma + \theta + \mu + \lambda_C) - \delta\sigma]}. \quad (4.13)$$

In Eq (4.12), $m_3\zeta$ comes from the direct path $H \rightarrow S$, while $\sigma\xi$ comes from the indirect path $H \rightarrow C \rightarrow S$. Holders who later return to spreading therefore also help reproduce active states.

For positive delay, a beneficiary enters the spreader-oriented and holder-oriented completion channels with probabilities ζ/m_1 and ξ/m_1 , respectively, and survives the fixed completion period with probability $q_\tau = e^{-\mu\tau}$. Once the individual reaches the (S, C) subsystem, the expected cumulative time spent in the spreader state is m_3/D when entering S and σ/D when entering C . Hence, the next-generation contribution is multiplied by q_τ , and the positive-delay threshold is

$$R_0(\tau) = \frac{q_\tau \kappa B(m_3\zeta + \sigma\xi)}{\mu m_1 D} = e^{-\mu\tau} R_0(0). \quad (4.14)$$

Proposition 5. With the other parameters fixed, $R_0(\tau)$ satisfies

$$\frac{\partial R_0(\tau)}{\partial \tau} = -\mu R_0(\tau) < 0, \quad (4.15)$$

$$\frac{\partial R_0(\tau)}{\partial \lambda_S} = -\frac{q_\tau \kappa B m_3(m_3\zeta + \sigma\xi)}{\mu m_1 D^2} < 0, \quad (4.16)$$

$$\frac{\partial R_0(\tau)}{\partial \lambda_C} = -\frac{q_\tau \kappa B \sigma(\delta\zeta + m_2\xi)}{\mu m_1 D^2} < 0, \quad (4.17)$$

and

$$\frac{\partial R_0(\tau)}{\partial \zeta} = \frac{q_\tau \kappa B [\xi(m_3 - \sigma) + m_3\mu]}{\mu D m_1^2} > 0. \quad (4.18)$$

Proof. Taking the partial derivatives of Eq (4.14) and using $\partial m_2/\partial \lambda_S = 1$, $\partial m_3/\partial \lambda_C = 1$, and $q_\tau = e^{-\mu\tau}$ gives Eqs (4.15)–(4.18). The proof is complete. $\square$

Proposition 5 gives two exit effects. A larger λ_S removes spreaders more quickly and reduces persistent spreading. A larger λ_C shortens the holder state after information has been submitted or shared. Thus, $\partial R_0/\partial \lambda_C < 0$ reflects a shorter active period, not a harmful effect of cooperation. The total effect of b , c , m , and Q depends on x_G^* and y_G^* and on the strength of the two exit channels. Here, R_0 measures regeneration of the recurrent active spreading cycle; it is not a social-welfare measure.

4.3. Existence of equilibria

Setting the right-hand sides of all equations of model (2.11) to zero, and noting that at an equilibrium state $H(t - \tau) = H(t) = H^*$, we have the following result:

Theorem 6. Model (2.11) always has the information-free equilibrium

$$E_0 = \left(\frac{B}{\mu}, 0, 0, 0, 0 \right). \quad (4.19)$$

When $R_0(\tau) > 1$, the model has a unique information-spreading equilibrium

$$E^*(\tau) = (I^*, H^*, S^*, C^*, R^*), \quad (4.20)$$

where

$$I^* = \frac{B}{\mu R_0(\tau)}, \quad (4.21)$$

$$H^* = \frac{B(R_0(\tau) - 1)}{m_1 R_0(\tau)}, \quad (4.22)$$

$$S^* = \frac{\mu(R_0(\tau) - 1)}{\kappa}, \quad (4.23)$$

$$C^* = \frac{\mu(\delta\zeta + m_2\xi)(R_0(\tau) - 1)}{\kappa(m_3\zeta + \sigma\xi)}, \quad (4.24)$$

$$R^* = \frac{(\varepsilon + \lambda_S)S^* + (\theta + \lambda_C)C^*}{\mu}. \quad (4.25)$$

Proof. Setting $H = S = C = R = 0$, the equation $I' = B - \mu I = 0$ yields Eq (4.19). For the positive equilibrium, let

$$\Gamma = m_3\zeta + \sigma\xi, \quad \Lambda = \delta\zeta + m_2\xi. \quad (4.26)$$

From the equilibrium equations of S and C , we obtain

$$S^* = \frac{q_\tau \Gamma}{D} H^*, \quad C^* = \frac{q_\tau \Lambda}{D} H^*. \quad (4.27)$$

Combining with the equilibrium equation of H gives $I^* = m_1 D / (q_\tau \kappa \Gamma) = B / [\mu R_0(\tau)]$. Then, from the equilibrium equation of I , Eqs (4.22)–(4.25) are obtained successively. Since $D > 0$, $\Gamma > 0$, and $\Lambda > 0$, all the above components are positive if and only if $R_0(\tau) > 1$. The proof is complete. $\square$

If $R_0(0) > 1$, the positive equilibrium reaches the persistence threshold at

$$\tau_p = \frac{1}{\mu} \ln R_0(0). \quad (4.28)$$

Thus, $E^*(\tau)$ exists for $0 \leq \tau < \tau_p$ and approaches E_0 as $\tau \uparrow \tau_p$. Moreover,

$$\frac{d\tau_p}{d\zeta} = \frac{\xi(m_3 - \sigma) + m_3\mu}{\mu m_1 \Gamma} > 0. \quad (4.29)$$

The term persistence threshold is used here for positive-equilibrium existence and the extinction threshold; a uniform-persistence theorem for all initial histories is not asserted.

4.4. Local stability of the equilibria at zero delay

Since $R(t)$ receives only inflows from the other compartments and does not feed back into the first four equations, it corresponds to one characteristic root $-\mu < 0$ in the linearized system. This is enough to study the four-dimensional subsystem (I, H, S, C) .

4.4.1. Information-free equilibrium

When $\tau = 0$, the Jacobian matrix of the model at E_0 is

$$J(E_0) = \begin{pmatrix} -\mu & 0 & -\frac{\kappa B}{\mu} & 0 \\ 0 & -m_1 & \frac{\kappa B}{\mu} & 0 \\ 0 & \zeta & -m_2 & \sigma \\ 0 & \xi & \delta & -m_3 \end{pmatrix}. \quad (4.30)$$

Apart from the characteristic root $-\mu$, the remaining characteristic roots satisfy

$$f(\lambda) = (\lambda + m_1) [(\lambda + m_2)(\lambda + m_3) - \delta\sigma] - \frac{\kappa B}{\mu} [\zeta(\lambda + m_3) + \sigma\xi] = 0. \quad (4.31)$$

Expanding Eq (4.31) gives

$$f(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0, \quad (4.32)$$

where

$$A_2 = m_1 + m_2 + m_3, \quad (4.33)$$

$$A_1 = m_1(m_2 + m_3) + D - \frac{\kappa B}{\mu}\zeta, \quad (4.34)$$

$$A_0 = m_1D - \frac{\kappa B}{\mu}\Gamma = m_1D[1 - R_0(0)]. \quad (4.35)$$

Theorem 7. When $R_0(0) < 1$, the information-free equilibrium E_0 is locally asymptotically stable at $\tau = 0$; when $R_0(0) > 1$, E_0 is unstable.

Proof. The matrix V is a nonsingular M-matrix. According to the next-generation matrix theory, $\rho(FV^{-1}) = R_0(0) < 1$ is equivalent to $V - F$ being a nonsingular M-matrix, and hence all characteristic roots of the active subsystem $F - V$ have negative real parts. Combining this with the independent characteristic root $-\mu$, we conclude that E_0 is locally asymptotically stable.

When $R_0(0) > 1$, Eq (4.35) implies $A_0 < 0$. Since $f(0) = A_0 < 0$ and $f(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$, equation $f(\lambda) = 0$ has at least one positive real root, so E_0 is unstable. The proof is complete. $\square$

Theorem 8 (Global stability of the information-free equilibrium at $\tau = 0$). Assume $\tau = 0$ and $R_0(0) < 1$. Then the information-free equilibrium

$$E_0 = \left(\frac{B}{\mu}, 0, 0, 0, 0 \right)$$

is globally asymptotically stable in the nonnegative feasible region.

Proof. When $\tau = 0$, model (2.11) becomes an ordinary differential equation system. On the coordinate boundaries,

$$\begin{aligned} I = 0 &\Rightarrow I' = B > 0, & H = 0 &\Rightarrow H' = \kappa IS \geq 0, \\ S = 0 &\Rightarrow S' = \zeta H + \sigma C \geq 0, & C = 0 &\Rightarrow C' = \xi H + \delta S \geq 0, \\ R = 0 &\Rightarrow R' = (\varepsilon + \lambda_S)S + (\theta + \lambda_C)C \geq 0. \end{aligned}$$

Hence, the nonnegative region is positively invariant. Lemma 4 further gives boundedness.

Let

$$I_0 = \frac{B}{\mu}, \quad D = m_2 m_3 - \delta \sigma, \quad \Gamma = m_3 \zeta + \sigma \xi,$$

and define

$$V = I - I_0 - I_0 \ln \frac{I}{I_0} + H + \frac{m_1 m_3}{\Gamma} S + \frac{m_1 \sigma}{\Gamma} C.$$

Since $u - 1 - \ln u \geq 0$ for $u > 0$, we have $V \geq 0$. Along solutions,

$$\begin{aligned} \dot{V} &= \frac{I - I_0}{I} (B - \kappa IS - \mu I) + (\kappa IS - m_1 H) \\ &\quad + \frac{m_1 m_3}{\Gamma} (\zeta H + \sigma C - m_2 S) + \frac{m_1 \sigma}{\Gamma} (\xi H + \delta S - m_3 C). \end{aligned}$$

Using $B = \mu I_0$, $\Gamma = m_3 \zeta + \sigma \xi$, and $D = m_2 m_3 - \delta \sigma$, we obtain

$$\dot{V} = -\frac{\mu(I - I_0)^2}{I} + \left(\kappa I_0 - \frac{m_1 D}{\Gamma} \right) S.$$

Because

$$R_0(0) = \frac{\kappa I_0 \Gamma}{m_1 D},$$

it follows that

$$\dot{V} = -\frac{\mu(I - I_0)^2}{I} - \frac{m_1 D}{\Gamma} [1 - R_0(0)] S \leq 0.$$

For $R_0(0) < 1$, equality requires $I = I_0$ and $S = 0$. To remain in this set, one must have $S' = \zeta H + \sigma C = 0$. Since $\zeta > 0$, $\sigma > 0$ and the states are nonnegative, this forces $H = C = 0$. Thus, the active subsystem converges to $(I_0, 0, 0, 0)$. On this limiting set, $R' = -\mu R$, so $R(t) \rightarrow 0$. By LaSalle's invariance principle, every feasible solution converges to E_0 . Therefore, E_0 is globally asymptotically stable.

If $I(0) = 0$, then $I'(0) = B > 0$, so $I(t) > 0$ for every $t > 0$. The preceding Lyapunov argument can therefore be applied from any positive time and extended to the initial boundary by continuity. $\square$

4.4.2. Information-spreading equilibrium

Let

$$\chi = \mu + \kappa S^* = \mu R_0(0), \quad \varpi = \kappa I^*, \quad a = \zeta + \xi. \quad (4.36)$$

At E^* , the characteristic equation for $\tau = 0$ is

$$(\lambda + \chi)(\lambda + m_1) [(\lambda + m_2)(\lambda + m_3) - \delta \sigma] - \varpi(\lambda + \mu) [\zeta(\lambda + m_3) + \sigma \xi] = 0. \quad (4.37)$$

Expanding it as

$$\lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0 = 0, \quad (4.38)$$

where

$$B_3 = \chi + m_1 + m_2 + m_3, \quad (4.39)$$

$$B_2 = \chi m_1 + (\chi + m_1)(m_2 + m_3) + D - \varpi\zeta, \quad (4.40)$$

$$B_1 = (\chi + m_1)D + \chi m_1(m_2 + m_3) - \varpi(\Gamma + \mu\zeta), \quad (4.41)$$

$$B_0 = \chi m_1 D - \varpi\mu\Gamma. \quad (4.42)$$

From the equilibrium relation $\varpi\Gamma = m_1 D$, we obtain

$$B_0 = (\chi - \mu)m_1 D = \kappa S^* m_1 D > 0. \quad (4.43)$$

Theorem 9. When $R_0(0) > 1$ and

$$B_i > 0, \quad i = 0, 1, 2, 3, \quad (4.44)$$

$$B_3 B_2 > B_1, \quad (4.45)$$

$$B_3 B_2 B_1 > B_1^2 + B_3^2 B_0, \quad (4.46)$$

the information-spreading equilibrium E^* is locally asymptotically stable at $\tau = 0$.

Proof. Equations (4.44)–(4.46) are the Routh–Hurwitz conditions for the quartic characteristic polynomial (4.38). Therefore, all its characteristic roots have negative real parts. Combining this with the characteristic root $-\mu$ corresponding to the R compartment, we conclude that the information-spreading equilibrium E^* is locally asymptotically stable at $\tau = 0$. The proof is complete. $\square$

$R_0(0) > 1$ guarantees only the existence of the information-spreading equilibrium and does not by itself guarantee its local stability; the stability also depends on whether the bidirectional transitions between spreaders and information holders, the strategy-dependent exit rates, and the other parameters satisfy Eqs (4.44)–(4.46).

4.5. Delay-dependent persistence threshold and stability

We now examine the corrected positive-delay model on the existence interval of the information-spreading equilibrium. Define

$$G(\lambda) = (\lambda + m_2)(\lambda + m_3) - \delta\sigma, \quad L(\lambda) = \zeta(\lambda + m_3) + \sigma\xi. \quad (4.47)$$

Theorem 10. For every fixed $\tau \geq 0$, if $R_0(\tau) \leq 1$, the information-free equilibrium E_0 is globally asymptotically stable in the nonnegative history space. If $R_0(\tau) > 1$, E_0 is unstable.

Proof. Let $I_0 = B/\mu$ and define the Lyapunov functional

$$\mathcal{V}(t) = I - I_0 - I_0 \ln \frac{I}{I_0} + H + \frac{m_1 m_3}{q_\tau \Gamma} S + \frac{m_1 \sigma}{q_\tau \Gamma} C + m_1 \int_{t-\tau}^t H(s) ds. \quad (4.48)$$

For $I(t) > 0$, differentiation along solutions gives

$$\dot{V} = -\frac{\mu(I - I_0)^2}{I} - \frac{m_1 D}{q_\tau \Gamma} [1 - R_0(\tau)] S \leq 0. \quad (4.49)$$

If $R_0(\tau) < 1$, equality requires $I = I_0$ and $S = 0$. At $R_0(\tau) = 1$, it first requires $I = I_0$; invariance of this equality in the I equation again forces $S = 0$. In either case, maintaining $S = 0$ forces $H = C = 0$, and then $R(t) \rightarrow 0$. Bounded solutions have precompact history orbits after a positive time, so the invariance principle yields convergence to E_0 . The functional controls deviations of I, H, S, C , while R is a stable scalar equation driven by S and C ; this also gives stability. The case $I(0) = 0$ follows from $I'(0) = B > 0$ and continuity.

For instability, the active characteristic equation at E_0 is

$$(\lambda + m_1)G(\lambda) - q_\tau \frac{\kappa B}{\mu} L(\lambda) e^{-\lambda\tau} = 0. \quad (4.50)$$

At $\lambda = 0$, its left-hand side equals $m_1 D[1 - R_0(\tau)]$. If $R_0(\tau) > 1$, this value is negative, whereas the expression tends to $+\infty$ on the positive real axis. Hence a positive real characteristic root exists and E_0 is unstable. $\square$

At the positive equilibrium, let $\chi = \mu + \kappa S^* = \mu R_0(\tau)$ and $\varpi = \kappa I^* = m_1 D/(q_\tau \Gamma)$. After removal of the stable characteristic factor $\lambda + \mu$, the characteristic equation is

$$(\lambda + \chi)(\lambda + m_1)G(\lambda) - \frac{m_1 D}{\Gamma} (\lambda + \mu)L(\lambda)e^{-\lambda\tau} = 0. \quad (4.51)$$

Theorem 11. Assume $R_0(\tau) > 1$. Equation (4.51) has no nonzero purely imaginary root. Consequently, if $E^*(0)$ is locally asymptotically stable, then $E^*(\tau)$ remains locally asymptotically stable throughout $0 \leq \tau < \tau_p$; no delay-induced oscillatory stability switch occurs while the positive equilibrium exists.

Proof. Suppose $\lambda = i\omega$, $\omega > 0$, satisfies Eq (4.51). Taking moduli gives

$$1 = \frac{\sqrt{\omega^2 + \chi^2}}{\sqrt{\omega^2 + \mu^2}} \frac{\sqrt{\omega^2 + m_1^2}}{m_1} \frac{\Gamma |G(i\omega)|}{D |L(i\omega)|}. \quad (4.52)$$

Since $R_0(\tau) > 1$ and $\chi = \mu R_0(\tau) > \mu$, the first and second factors exceed one. Moreover,

$$|G(i\omega)|^2 = D^2 + (m_2^2 + m_3^2 + 2\delta\sigma)\omega^2 + \omega^4, \quad |L(i\omega)|^2 = \Gamma^2 + \zeta^2\omega^2.$$

Because $\Gamma m_2 - \zeta D = \sigma(\xi m_2 + \delta\zeta) > 0$, it follows that $\Gamma |G(i\omega)|/[D |L(i\omega)|] > 1$. Thus, the right-hand side of Eq (4.52) is strictly greater than one, which is a contradiction. At $\lambda = 0$, the left-hand side of Eq (4.51) equals $m_1 D(\chi - \mu) > 0$. Hence, no characteristic root crosses the imaginary axis on $0 \leq \tau < \tau_p$, and stability at zero delay is preserved throughout this interval. $\square$

This spectral result excludes a delay-induced Hopf root crossing on the positive-equilibrium existence interval. It does not establish global attraction of the positive equilibrium or exclude every possible nonlinear periodic orbit. The threshold τ_p separates positive-equilibrium existence from the globally attracting information-free regime.

5. Optimal control analysis

The game parameters b , c , and Q describe structural conditions and are not easy to change continuously over a short control period. We therefore keep the stable game outcome x_G^*, y_G^* fixed during one control horizon. The controls act only on the two $S \rightleftharpoons C$ transition channels in the spreading layer. The game layer sets the environment, and the spreading layer carries the time-dependent control.

5.1. Control variables and the controlled system

Let $T > 0$ be the terminal control time. The following two time-varying control variables are introduced:

- (1) Positive guidance $u_1(t)$ for the transition from holders to spreaders. This control represents lowering the sharing threshold of information holders through measures such as sharing incentives, convenient interaction channels, information verification tools, and collaborative feedback. As an additional transition rate, $u_1(t)$ is added to the $C \rightarrow S$ channel and satisfies

$$0 \leq u_1(t) \leq u_{1\max}. \quad (5.1)$$

Accordingly, the total transition rate from holders to spreaders is adjusted from σ to $\sigma + u_1(t)$.

- (2) Reverse suppression $u_2(t)$ for the transition from spreaders to holders. This control represents restraining spreaders from stopping diffusion and entering the holder state by stabilizing spreading benefits, reducing spreading uncertainty, and providing continuous feedback. $u_2(t)$ is a dimensionless suppression proportion and satisfies

$$0 \leq u_2(t) \leq u_{2\max} \leq 1. \quad (5.2)$$

Accordingly, the transition rate from spreaders to holders is adjusted from δ to $(1 - u_2(t))\delta$.

The admissible control set is defined as

$$\mathcal{U} = \{(u_1, u_2) : u_i \in L^\infty(0, T), 0 \leq u_i(t) \leq u_{i\max}, i = 1, 2\}. \quad (5.3)$$

Here, u_1 is an additional transition rate with unit time^{-1} , while u_2 is a dimensionless suppression proportion. The upper bounds $u_{1\max}$ and $u_{2\max}$ describe illustrative intervention capacities rather than empirically calibrated limits.

Keeping the beneficiary delay $H(t - \tau)$ and the one-way game coupling unchanged, the controlled IHSCR time-delay information dissemination system is

$$\begin{cases} \frac{dI(t)}{dt} = B - \kappa I(t)S(t) - \mu I(t), \\ \frac{dH(t)}{dt} = \kappa I(t)S(t) - (\zeta + \xi + \mu)H(t), \\ \frac{dS(t)}{dt} = q_\tau \zeta H(t - \tau) + [\sigma + u_1(t)]C(t) - [(1 - u_2(t))\delta + \varepsilon + \mu + \lambda_S]S(t), \\ \frac{dC(t)}{dt} = q_\tau \xi H(t - \tau) + (1 - u_2(t))\delta S(t) - [\sigma + u_1(t) + \theta + \mu + \lambda_C]C(t), \\ \frac{dR(t)}{dt} = (\varepsilon + \lambda_S)S(t) + (\theta + \lambda_C)C(t) - \mu R(t). \end{cases} \quad (5.4)$$

The controlled system adopts the same continuous nonnegative initial history functions as in Eq (2.13). Every admissible control pair preserves quasipositivity. Since the controls redistribute only individuals between S and C , the augmented total population, including the two auxiliary completion cohorts, still satisfies

$$\frac{d\tilde{N}(t)}{dt} = B - \mu\tilde{N}(t). \quad (5.5)$$

The controls do not change the total population size. They change only how the active population is divided between spreaders and information holders.

5.2. Objective functional and existence of the optimal control

Let $\varrho > 0$ be the weight assigned to the spreader population, and let $\nu_1 > 0$ and $\nu_2 > 0$ be the cost weights of the two controls, respectively. To enlarge the spreader scale while avoiding excessively high intervention costs, the objective functional is defined as

$$J(u_1, u_2) = \int_0^T \left[\varrho S(t) - \frac{\nu_1}{2} u_1^2(t) - \frac{\nu_2}{2} u_2^2(t) \right] dt. \quad (5.6)$$

The quadratic cost terms show that the marginal cost increases with the control intensity and make the integrand strictly concave in the control variables. If $[J] = U$, dimensional consistency gives $[\varrho] = U/(\text{density} \cdot \text{time})$, $[\nu_1] = U \cdot \text{time}$, and $[\nu_2] = U/\text{time}$. Thus, setting their illustrative numerical values equal to one does not make the two interventions physically equivalent. The optimal control problem is to find $(u_1^*, u_2^*) \in \mathcal{U}$ such that

$$J(u_1^*, u_2^*) = \max_{(u_1, u_2) \in \mathcal{U}} J(u_1, u_2). \quad (5.7)$$

Theorem 12. Under the given continuous nonnegative initial history functions, the corrected controlled system (5.4) has a unique feasible state solution for every $(u_1, u_2) \in \mathcal{U}$, and the optimal control problem (5.7) admits at least one optimal control pair $(u_1^*, u_2^*) \in \mathcal{U}$.

Proof. It is convenient to rewrite the maximization problem as the equivalent minimization problem

$$\min_{(u_1, u_2) \in \mathcal{U}} \mathcal{I}(u_1, u_2), \quad \mathcal{I}(u_1, u_2) = -J(u_1, u_2) = \int_0^T \left[\frac{\nu_1}{2} u_1^2(t) + \frac{\nu_2}{2} u_2^2(t) - \varrho S(t) \right] dt.$$

The admissible set $\mathcal{U}$ is nonempty, closed, bounded, and convex in $L^2(0, T)^2$. The controlled vector field is continuous in the state variables, locally Lipschitz on bounded feasible sets, and affine in (u_1, u_2) . Quasipositivity and the augmented population identity (5.5) give a unique global feasible trajectory and a uniform state bound on $[0, T]$ for every admissible control pair.

Let $\{(u_1^n, u_2^n)\} \subset \mathcal{U}$ be a minimizing sequence for $\mathcal{I}$. Since $\mathcal{U}$ is bounded in $L^2(0, T)^2$, a subsequence converges weakly to some $(u_1^*, u_2^*) \in \mathcal{U}$. The affine dependence of the dynamics on the controls, together with the uniform state bounds and the well-posedness, yields a further subsequence for which the associated state trajectories converge uniformly on $[0, T]$ to the state generated by (u_1^*, u_2^*) . The term $-\varrho S$ therefore converges strongly in $L^1(0, T)$, whereas the quadratic control terms are convex and weakly lower semicontinuous. Consequently,

$$\mathcal{I}(u_1^*, u_2^*) \leq \liminf_{n \rightarrow \infty} \mathcal{I}(u_1^n, u_2^n) = \inf_{(u_1, u_2) \in \mathcal{U}} \mathcal{I}(u_1, u_2).$$

Thus, (u_1^*, u_2^*) minimizes $\mathcal{I} = -J$ and hence maximizes J . Therefore, the optimal control problem admits at least one optimal control pair. The argument follows the standard compactness and lower-semicontinuity framework for optimal control problems with ordinary state systems [32, 33]. Related formulations for delayed optimal-control systems are given in Refs. [34, 35]. $\square$

5.3. Hamiltonian and the adjoint system

Let the state vector and the adjoint variables be

$$X(t) = (I(t), H(t), S(t), C(t), R(t))^T, \quad (5.8)$$

$$\Psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t), \psi_5(t))^T. \quad (5.9)$$

The Hamiltonian is defined as

$$\begin{aligned} \mathcal{H} = & \varrho S - \frac{\nu_1}{2} u_1^2 - \frac{\nu_2}{2} u_2^2 \\ & + \psi_1 (B - \kappa IS - \mu I) \\ & + \psi_2 [\kappa IS - (\zeta + \xi + \mu)H] \\ & + q_\tau \zeta \psi_3 H(t - \tau) + (\sigma + u_1) \psi_3 C - [(1 - u_2)\delta + \varepsilon + \mu + \lambda_S] \psi_3 S \\ & + q_\tau \xi \psi_4 H(t - \tau) + (1 - u_2)\delta \psi_4 S - (\sigma + u_1 + \theta + \mu + \lambda_C) \psi_4 C \\ & + (\varepsilon + \lambda_S) \psi_5 S + (\theta + \lambda_C) \psi_5 C - \mu \psi_5 R. \end{aligned} \quad (5.10)$$

The state variable $H(t)$ enters the Hamiltonian at the future time $t + \tau$ in the form of the delayed variable $H((t + \tau) - \tau)$, so its adjoint equation contains advanced terms. For a unified representation, the adjoint variables are extended as

$$\psi_i(t) = 0, \quad T < t \leq T + \tau, \quad i = 1, 2, \dots, 5. \quad (5.11)$$

Theorem 13. Let (u_1^*, u_2^*) be an optimal control pair and $X^*(t)$ the optimal state trajectory. Then there exist absolutely continuous adjoint variables $\psi_i(t)$ satisfying the transversality conditions

$$\psi_i(T) = 0, \quad i = 1, 2, \dots, 5, \quad (5.12)$$

and the following adjoint system:

$$\frac{d\psi_1}{dt} = (\kappa S^*(t) + \mu) \psi_1 - \kappa S^*(t) \psi_2, \quad (5.13)$$

$$\frac{d\psi_2}{dt} = m_1 \psi_2 - q_\tau \mathbf{1}_{[0, T-\tau]}(t) [\zeta \psi_3(t + \tau) + \xi \psi_4(t + \tau)], \quad (5.14)$$

$$\frac{d\psi_3}{dt} = -\varrho + \kappa I^*(t)(\psi_1 - \psi_2) + (1 - u_2^*)\delta(\psi_3 - \psi_4) + (\varepsilon + \mu + \lambda_S)\psi_3 - (\varepsilon + \lambda_S)\psi_5, \quad (5.15)$$

$$\frac{d\psi_4}{dt} = (\sigma + u_1^*)(\psi_4 - \psi_3) + (\theta + \mu + \lambda_C)\psi_4 - (\theta + \lambda_C)\psi_5, \quad (5.16)$$

$$\frac{d\psi_5}{dt} = \mu \psi_5, \quad (5.17)$$

where $\mathbf{1}_{[0, T-\tau]}(t)$ is the indicator function.

Proof. According to the Pontryagin maximum principle for systems with discrete time delays [34, 35], the adjoint equations include contributions from the partial derivatives of the current Hamiltonian with respect to the current states and from the partial derivatives of the future Hamiltonian with respect to the delayed states. Taking the partial derivatives of Eq (5.10) for I , H , S , C , and R , respectively, yields Eqs (5.13)–(5.17). Since the terminal state is free and the objective functional contains no terminal payoff, the transversality conditions are given by Eq (5.12). The proof is complete. $\square$

From $\psi_5(T) = 0$ and Eq (5.17), we obtain

$$\psi_5(t) \equiv 0, \quad 0 \leq t \leq T. \quad (5.18)$$

Retaining ψ_5 helps to display the complete Hamiltonian system of the five-compartment model. Because beneficiary outflow is current and the completion inflows are delayed, Eq (5.14) contains no future $\psi_2(t + \tau)$ term; the future ψ_3 and ψ_4 contributions are multiplied by q_τ . When $\tau = 0$, Eq (5.14) degenerates into

$$\frac{d\psi_2}{dt} = (\zeta + \xi + \mu)\psi_2 - \zeta\psi_3 - \xi\psi_4, \quad (5.19)$$

which coincides with the adjoint equation of the optimal control system without delay.

5.4. Characterization of the optimal control

The partial derivatives of the Hamiltonian for the control variables are

$$\frac{\partial \mathcal{H}}{\partial u_1} = -\nu_1 u_1 + C(\psi_3 - \psi_4), \quad (5.20)$$

$$\frac{\partial \mathcal{H}}{\partial u_2} = -\nu_2 u_2 + \delta S(\psi_3 - \psi_4). \quad (5.21)$$

Since the Hamiltonian is strictly concave in (u_1, u_2) , the unique maximum point without considering the control bounds is

$$\hat{u}_1 = \frac{C(\psi_3 - \psi_4)}{\nu_1}, \quad \hat{u}_2 = \frac{\delta S(\psi_3 - \psi_4)}{\nu_2}. \quad (5.22)$$

Projecting them onto the feasible control intervals gives the characterization of the optimal controls:

$$u_1^*(t) = \min \left\{ u_{1\max}, \max \left[0, \frac{C^*(t) [\psi_3(t) - \psi_4(t)]}{\nu_1} \right] \right\}, \quad (5.23)$$

$$u_2^*(t) = \min \left\{ u_{2\max}, \max \left[0, \frac{\delta S^*(t) [\psi_3(t) - \psi_4(t)]}{\nu_2} \right] \right\}. \quad (5.24)$$

Equation (5.23) raises u_1 when the gain from moving holders to spreaders is larger than its cost. Equation (5.24) raises u_2 when limiting $S \rightarrow C$ improves the active-state allocation. Larger ν_1 or ν_2 lowers the corresponding control intensity.

These projection formulas are necessary conditions supplied by the maximum principle. Strict concavity of the Hamiltonian in the controls alone does not establish global optimality or uniqueness for the nonlinear state problem.

5.5. Optimality system and numerical solution

The optimality system contains the controlled state equations, the adjoint equations with zero terminal values and advanced terms, and the two projection formulas. We solve it by a forward–backward sweep in Section 6. The steps are:

- (1) Specify initial controls $u_1^{(0)}(t)$ and $u_2^{(0)}(t)$ within the feasible control intervals.
- (2) According to the initial history functions and the current controls, solve the controlled time-delay system forward from $t = 0$ to $t = T$.
- (3) Using the obtained state trajectories and the terminal conditions, solve the adjoint system backward from $t = T$ to $t = 0$, and handle the advanced terms $\psi_3(t + \tau)$ and $\psi_4(t + \tau)$ piecewise by intervals.
- (4) Compute the updated candidate controls $\tilde{u}_i^{(n+1)}(t)$ from Eqs (5.23) and (5.24).
- (5) Adopt the relaxation iteration

$$u_i^{(n+1)}(t) = \rho_u \tilde{u}_i^{(n+1)}(t) + (1 - \rho_u) u_i^{(n)}(t), \quad 0 < \rho_u \leq 1, \quad (5.25)$$

until the relative errors of the state variables, the adjoint variables, and the control variables are simultaneously smaller than the preset tolerance.

The state DDE is advanced on a uniform grid by fourth-order Runge–Kutta integration with linear interpolation of delayed values. The adjoint system is integrated backward on the same grid, with the advanced values obtained by linear interpolation. The main runs use $\Delta t = 0.05$, relaxation factor $\rho_u = 0.5$, and stopping tolerance 2×10^{-6} for the maximum change in the control arrays and the scaled changes in the state and adjoint arrays. Positive-delay runs require $\Delta t \leq \tau$. After convergence, the state and adjoint equations are reintegrated using the accepted controls, and the projection residual is recorded separately. Grid refinement with $\Delta t = 0.10, 0.05$, and 0.025 is reported in Section 6.

In Section 6, we compare no control, fixed control, and optimal control. We examine the spreader and holder states, cumulative benefit, control cost, and the net objective.

6. Numerical simulation

For the numerical comparisons, we use illustrative rather than empirically calibrated parameters. Motivated by studies of intervention, game-transition settings, network environments, and behavioral-rate perturbations [19–21], we compare the two intervention channels, game-to-spreading coupling strengths, and behavioral-transition rates. Solver settings and convergence checks are stated in the corresponding subsections.

6.1. Strategy evolution of the ecological snowdrift game

The numerical simulations in this subsection use the baseline parameter set

$$b = 1, \quad c = 0.4, \quad m = 0, \quad Q = 0, \quad r_0 = 0.1, \quad \mu_G = 0.05, \quad (6.1)$$

unless otherwise stated. The game system is integrated from several feasible initial states satisfying $x_0 \geq 0$, $y_0 \geq 0$, and $x_0 + y_0 \leq 1 - Q$. The equilibrium values reported below are obtained without clipping the state variables, and numerical feasibility is checked throughout the integration.

The five initial states displayed in Figure 2 are $(0.1, 0.1)$, $(0.2, 0.3)$, $(0.4, 0.1)$, $(0.8, 0.1)$, and $(0.3, 0.5)$. The downstream coupling baseline uses $z_0 = (0.2, 0.1)$; under the baseline parameters, it selects the same attracting coexistence equilibrium.

Figure 2 shows the baseline dynamics from the time-domain and state-space perspectives. Although the selected initial states produce different transient paths, the cooperator and defector ecological densities converge to

$$E_G^* = (x_G^*, y_G^*) \approx (0.7033, 0.2344), \quad (6.2)$$

which lies strictly inside the feasible region because $x_G^* + y_G^* \approx 0.9378 < 1 - Q$. The phase portrait further shows that the representative trajectories are directed toward the same coexistence state. The Jacobian matrix at E_G^* has eigenvalues approximately -0.7095 and -0.0088 , supporting the local asymptotic stability of this equilibrium under the baseline parameters. These simulations provide numerical evidence of attraction from multiple feasible initial conditions; they are not used as a substitute for a global-stability proof.

Figure 3 shows how the payoff parameters reshape the stable cooperation–defection structure. For $m = Q = 0$, the payoff-balance condition at a positive coexistence equilibrium gives

$$\frac{y_G^*}{x_G^*} = \frac{c}{2(b - c)}. \quad (6.3)$$

For fixed c , a larger b reduces the relative share of defectors and moves the coexistence state toward cooperation. The equality $x_G^* = y_G^*$ occurs at $b = 3c/2$; for $c = 0.4$, this gives $b = 0.6$, which agrees with Figure 3(a). The total strategic density also increases with b over the investigated range, indicating that larger dissemination benefits support a denser strategy population under the environmental-capacity constraint.

When $b = 1$ is fixed and c increases, x_G^* decreases monotonically, whereas y_G^* exhibits a non-monotonic response. The equality $x_G^* = y_G^*$ occurs at $c = 2b/3 \approx 0.6667$. Beyond this point, defectors dominate the coexistence composition. However, y_G^* does not increase indefinitely: It reaches a maximum near $c \approx 0.913$ and then declines. This behavior results from two competing mechanisms. A larger cost increases the defector proportion in the strategic population, but it simultaneously reduces the total equilibrium density. At sufficiently high costs, the contraction of the total strategic population offsets the compositional shift toward defection.

The two-dimensional heatmaps in Figures 3(c)(d) extend the one-parameter results to the full valid b – c domain. High-benefit and low-cost combinations generate the largest cooperative ecological density. The line $c = 2b/3$ divides the valid region into a cooperation-dominant domain, where $x_G^* > y_G^*$, and a defection-dominant domain, where $y_G^* > x_G^*$. The baseline point $(1, 0.4)$ lies in the cooperation-dominant region, consistently yielding $x_G^* > y_G^*$.

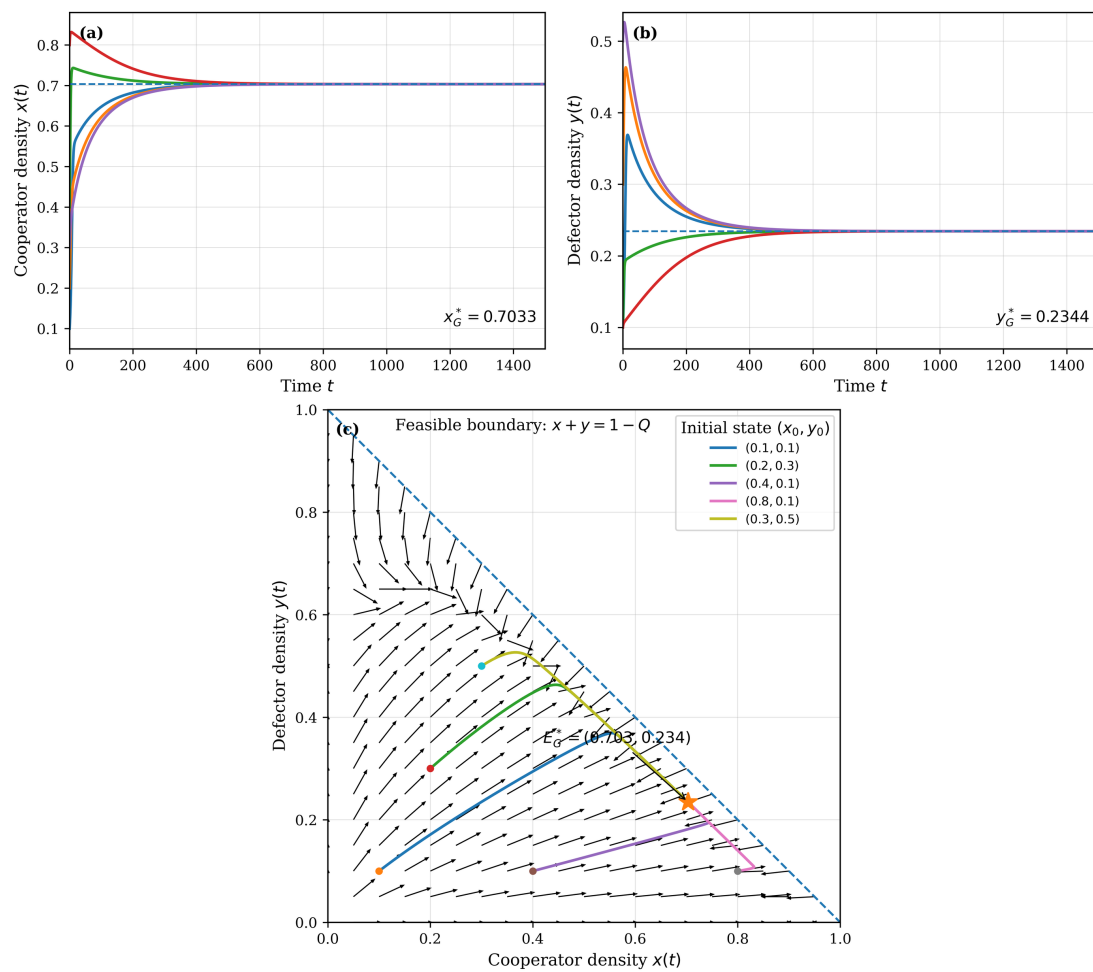


Figure 2. Baseline evolutionary dynamics of the eco-evolutionary snowdrift game. (a) Time evolution of the cooperator ecological density $x(t)$ under representative feasible initial states; (b) time evolution of the defector ecological density $y(t)$; and (c) phase portrait and normalized vector field in the feasible region. The trajectories converge numerically to the coexistence equilibrium $E_G^* = (0.7033, 0.2344)$.

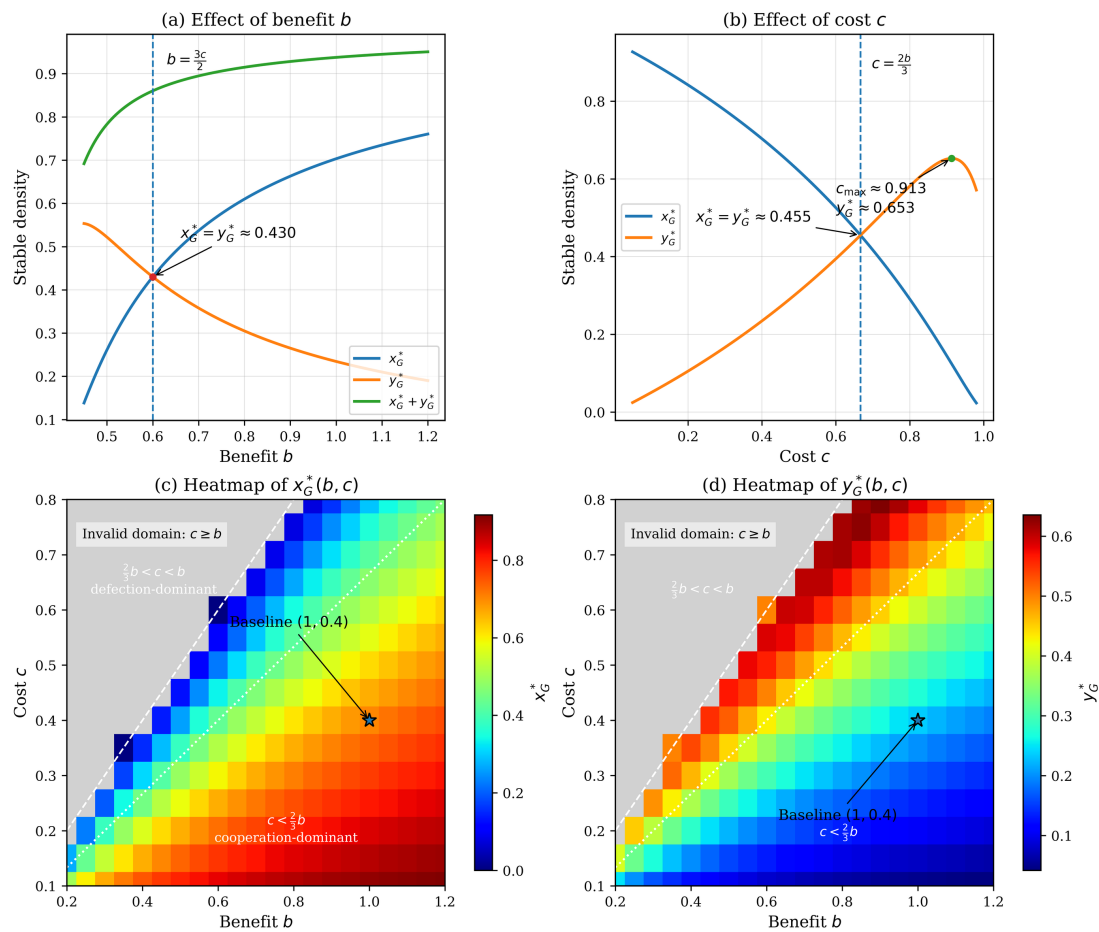


Figure 3. Effects of the benefit b and cost c on the stable ecological densities. (a) Dependence of x_G^* , y_G^* , and $x_G^* + y_G^*$ on b for $c = 0.4$; (b) dependence of x_G^* and y_G^* on c for $b = 1$; (c) heatmap of $x_G^*(b, c)$; and (d) heatmap of $y_G^*(b, c)$. The gray area denotes the invalid domain $c \geq b$, the dotted line $c = 2b/3$ separates cooperation-dominant and defection-dominant coexistence states, and the star marks the baseline point $(b, c) = (1, 0.4)$.

Figure 4(a) reveals a structural transition induced by the same-strategy encounter tendency m . As m increases, the cooperative ecological density rises and the defector density decreases. At

$$m_c \approx 0.1906, \quad (6.4)$$

the coexistence branch reaches $y_G^* = 0$ and connects to the pure-cooperation boundary branch. Here, “pure cooperation” means that the defector ecological density vanishes while x_G^* remains positive; this does not imply $x_G^* = 1$. Since this analysis reveals the equilibrium transition numerically and does not verify all non-degeneracy conditions for a specific codimension-one bifurcation, the result is described conservatively as a strategy or equilibrium transition.

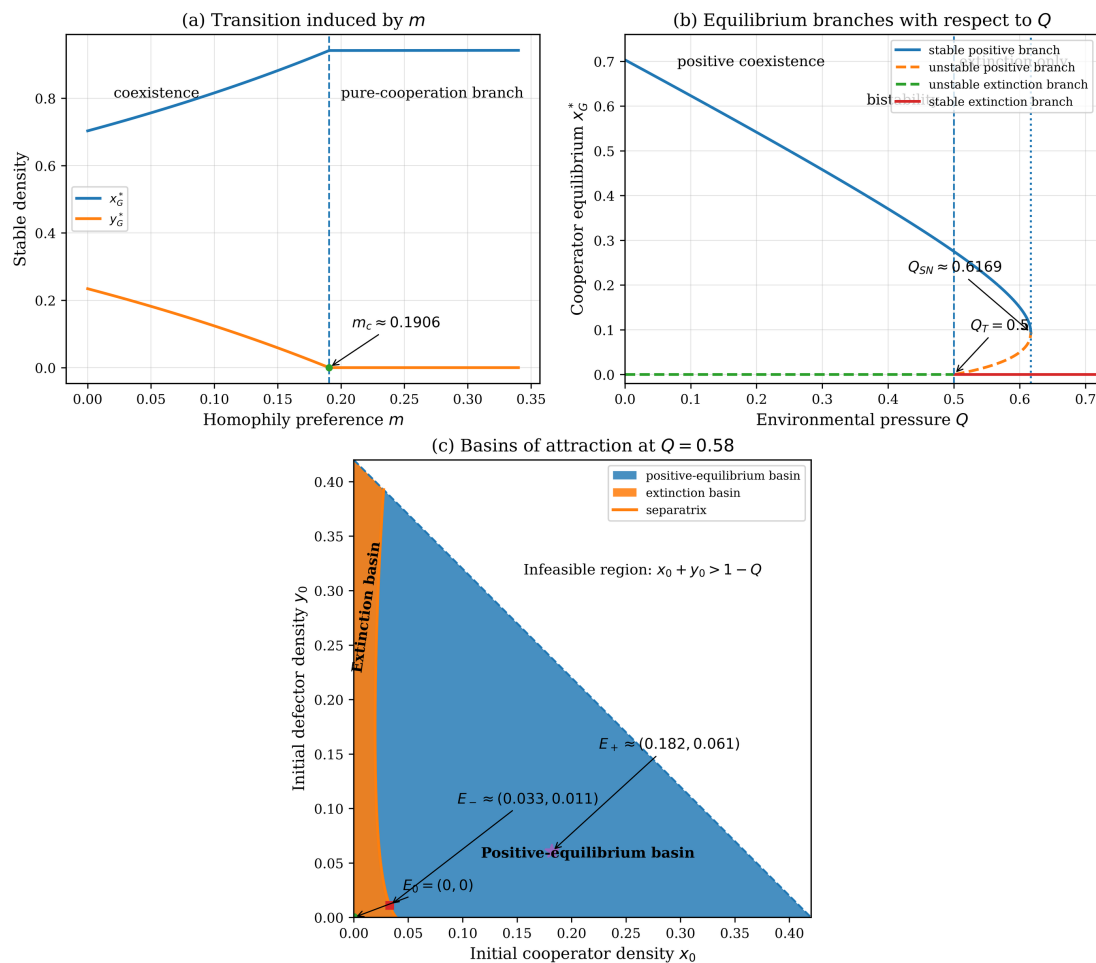


Figure 4. Strategy transition and environmental-pressure-induced bistability in the eco-evolutionary snowdrift game. (a) Transition from a cooperation–defection coexistence equilibrium to a pure-cooperation boundary equilibrium as m crosses $m_c \approx 0.1906$; (b) positive-equilibrium and extinction branches for Q , where $Q_T = 0.5$ is the extinction-stability threshold and $Q_{SN} \approx 0.6169$ is the saddle-node threshold; and (c) basins of attraction at $Q = 0.58$, with the smooth separatrix obtained from the stable manifold of the saddle equilibrium E_G^- .

The external environmental pressure Q produces a qualitatively different mechanism. Under the baseline payoff parameters, the coexistence relation is $y_G^* = x_G^*/3$, and the positive equilibria satisfy

$$\frac{4}{3}(x_G^*)^2 + \left(Q - \frac{13}{15}\right)x_G^* + \left(\frac{Q}{10} - \frac{1}{20}\right) = 0. \quad (6.5)$$

The extinction equilibrium changes stability at

$$Q_T = 1 - \frac{\mu_G}{r_0} = 0.5, \quad (6.6)$$

whereas the two positive branches collide and disappear at the saddle-node threshold

$$Q_{SN} = \frac{17}{15} - \frac{2\sqrt{15}}{15} \approx 0.6169. \quad (6.7)$$

For $Q < Q_T$, the extinction equilibrium is unstable, and the stable positive coexistence state is the relevant long-term outcome. For $Q_T < Q < Q_{SN}$, the stable extinction equilibrium and the stable positive equilibrium coexist, producing bistability. For $Q > Q_{SN}$, the positive equilibria no longer exist, and extinction is the only stable outcome.

To visualize the initial-state dependence in the bistable interval, Figure 4(c) sets $Q = 0.58$. The system then has the stable extinction equilibrium $E_G^0 = (0, 0)$, an unstable saddle equilibrium

$$E_G^- \approx (0.0330, 0.0110), \quad (6.8)$$

and a stable positive equilibrium

$$E_G^+ \approx (0.1820, 0.0607). \quad (6.9)$$

The stable manifold of E_G^- forms the separatrix between the two basins. Initial conditions on one side converge to extinction, whereas those on the other side recover to the positive coexistence equilibrium. Thus, in the bistable regime, the external pressure alone does not determine the final state; the initial cooperative and defector ecological densities are also decisive.

The coexistence roots, their Jacobian classification, and the numerical stable-manifold procedure are given in Section 3.3. Accordingly, the equilibrium supplied to the spreading layer is conditional on the game parameters, the initial state, and the corresponding basin of attraction.

Figures 2–4 show three parts of the game dynamics: baseline coexistence, smooth changes in equilibrium composition, and structural changes with initial-condition dependence. The stable densities then enter the spreading model through $\lambda_S = \omega_D y_G^*$ and $\lambda_C = \omega_C x_G^*$. They are not dissemination-compartment sizes. Their effect is carried by the two exit channels and depends on x_G^* and y_G^* .

6.2. Threshold verification and equilibrium stability

Unless otherwise stated, the spreading-layer simulations use

$$B = 1, \quad \mu = 0.2, \quad \zeta = 0.3, \quad \xi = 0.2, \quad \delta = \sigma = \varepsilon = \theta = 0.1, \quad (6.10)$$

with $\omega_D = \omega_C = 0.5$ and the game-layer baseline mappings

$$\lambda_S = \omega_D y_G^* \approx 0.1172, \quad \lambda_C = \omega_C x_G^* \approx 0.3517, \quad (6.11)$$

and $\tau = 0$. The representative initial state is $X(0) = (4, 0.6, 0.8, 0.4, 0.2)^T$. Figure 5 summarizes convergence on opposite sides of the threshold: $\kappa = 0.2$ gives $R_0(0) \approx 0.9259 < 1$, whereas $\kappa = 0.5$ gives $R_0(0) \approx 2.3148 > 1$.

Figure 5(a) shows that when $R_0(0) < 1$, the active information states $H(t)$, $S(t)$, and $C(t)$, together with the exit state $R(t)$, decay to zero, whereas the information-ignorant state tends to $I_0 = B/\mu = 5$. The short initial decrease of $I(t)$ reflects transient contacts with the initially present spreaders; after the active spreading chain becomes unsustainable, the contact term vanishes and $I(t)$ returns to the recruitment–exit balance. Figure 5(b) further shows that the system-level error decreases across several orders of magnitude to the numerical tolerance level. Because $R_0(0) \approx 0.9259$ is close to the threshold, the convergence is relatively slow.

For $\kappa = 0.5$, the model gives

$$E^*(0) \approx (2.1600, 0.8114, 0.5259, 0.2859, 1.2168). \quad (6.12)$$

Figure 5(c) shows damped transients followed by convergence to the positive equilibrium. The Jacobian eigenvalues have negative real parts, and the fourth-order Routh–Hurwitz conditions hold. The simulation therefore agrees with the local stability result for E^* at $\tau = 0$. The small oscillations in Figure 5(d) arise from damped complex modes and changes in the dominant component of the infinity norm; they do not indicate persistent oscillatory behavior.

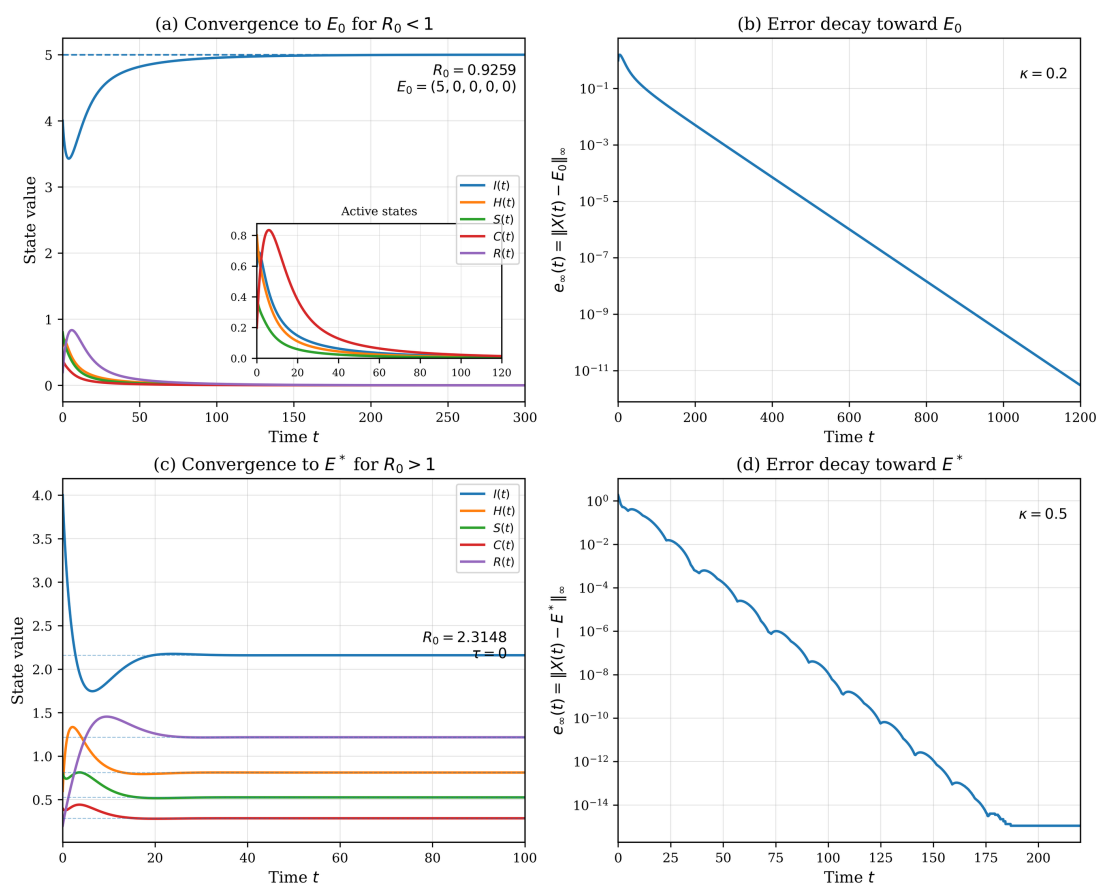


Figure 5. Threshold-dependent convergence of the information-dissemination system at $\tau = 0$. (a) State trajectories for $R_0(0) \approx 0.9259 < 1$, converging to the information-free equilibrium $E_0 = (5, 0, 0, 0, 0)$; (b) decay of the infinity-norm error $e_\infty(t) = \|X(t) - E_0\|_\infty$; (c) state trajectories for $R_0(0) \approx 2.3148 > 1$, converging to the information-spreading equilibrium E^* ; and (d) decay of $e_\infty(t) = \|X(t) - E^*\|_\infty$.

The figure is a numerical check, not a substitute for the analytical stability proofs. In particular, $R_0(0) > 1$ first guarantees the existence of a positive information-spreading equilibrium at zero delay; stability also requires the characteristic-polynomial conditions.

6.3. Sensitivity analysis of the basic reproduction number

The composite contact parameter satisfies $\kappa = \alpha\langle k \rangle$, where α is the effective contact efficiency along one network connection and $\langle k \rangle$ is the network average degree. To separate the contact and behavioral-conversion mechanisms, this subsection fixes $\langle k \rangle = 5$ and examines the joint effect of α and the beneficiary-to-spreader conversion rate ζ . At zero delay, the basic reproduction number is

$$R_0(0; \alpha, \zeta) = \frac{\alpha\langle k \rangle B[m_3\zeta + \sigma\xi]}{\mu(\zeta + \xi + \mu)(m_2m_3 - \delta\sigma)}. \quad (6.13)$$

The scan ranges are $0.02 \leq \alpha \leq 0.12$ and $0.05 \leq \zeta \leq 0.50$.

Figure 6(a) shows a smooth monotone increase of $R_0(0)$ in both parameter directions. For fixed ζ , $R_0(0)$ is linear in α , because increasing the contact efficiency proportionally strengthens the incidence term $\alpha\langle k \rangle IS$. For fixed α , increasing ζ raises the fraction of beneficiaries who enter the spreading state, thereby strengthening the regenerative spreading chain; the incremental effect gradually weakens over the investigated range because ζ also appears in the total outflow rate from H .

The contour $R_0(0) = 1$ in Figure 6(b) separates extinction from positive persistence. The baseline point is well inside the $R_0(0) > 1$ region. Moreover, contact efficiency and beneficiary-to-spreader conversion can move the system across the zero-delay threshold. Their joint increase makes sustained spreading possible over a wider parameter range. The next two figures show what these changes do to the state variables.

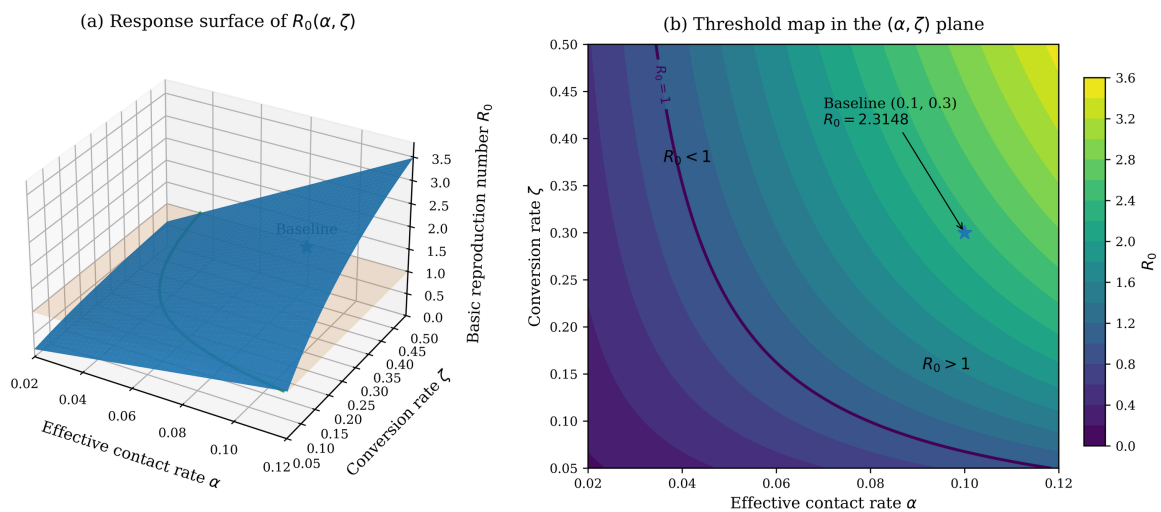


Figure 6. Joint sensitivity of the basic reproduction number to the effective contact efficiency α and the beneficiary-to-spreader conversion rate ζ for $\langle k \rangle = 5$ at zero delay. (a) Smooth three-dimensional response surface $R_0(0; \alpha, \zeta)$ together with the threshold plane $R_0(0) = 1$; and (b) two-dimensional threshold map in the (α, ζ) plane. The star denotes the baseline point $(\alpha, \zeta) = (0.1, 0.3)$, where $R_0(0) \approx 2.3148$.

6.4. Dynamic responses to key spreading parameters

6.4.1. Effect of the network average degree

To highlight the complex-network interpretation, the single-edge contact efficiency is fixed at $\alpha = 0.1$, while the average degree is varied. Since $\kappa = \alpha\langle k \rangle$, the zero-delay threshold $R_0(0) = 1$

corresponds to

$$\langle k \rangle_c \approx 2.1600. \quad (6.14)$$

Representative values $\langle k \rangle = 1.5, 2.0, 2.2, 3.0$, and 5.0 are selected to cover states below, near, and above this threshold.

For $\langle k \rangle < \langle k \rangle_c$, the spreader density vanishes. The trajectory with $\langle k \rangle = 2.0$ decays more slowly than that with $\langle k \rangle = 1.5$, reflecting the slower recovery expected near the threshold. When the average degree slightly exceeds the threshold, as for $\langle k \rangle = 2.2$, the system converges to a small but positive spreader equilibrium. Further increases in network connectivity enlarge the long-term spreader density. In the positive-equilibrium region,

$$S^*(\langle k \rangle) = \mu K - \frac{\mu}{\alpha \langle k \rangle}, \quad (6.15)$$

where $K = R_0(0)/\kappa$ is constant for the other baseline parameters. Thus, S^* rises with $\langle k \rangle$, but the increase becomes smaller. The branch diagram in Figure 7(b) visualizes the change from eventual spreading extinction to positive persistence at the network-connectivity threshold.

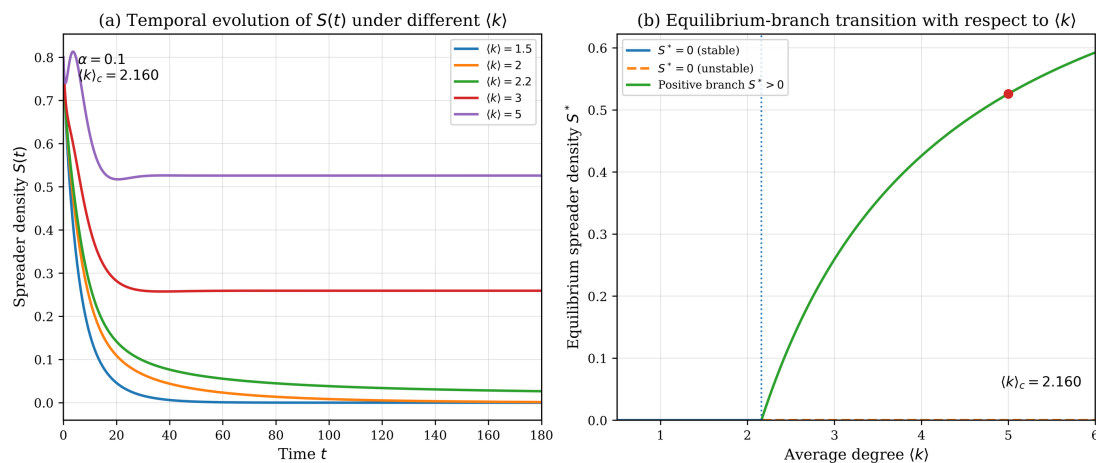


Figure 7. Effects of network connectivity on information dissemination. (a) Temporal evolution of the spreader density $S(t)$ for representative average degrees; and (b) equilibrium branches of S^* for $\langle k \rangle$. The dotted vertical line marks $\langle k \rangle_c \approx 2.1600$, and the point on the positive branch marks the baseline value $\langle k \rangle = 5$.

6.4.2. Effect of the beneficiary-to-spreader conversion rate

Parameter ζ directly regulates the transition $H \rightarrow S$. To examine how this behavioral conversion redistributes the active information states, we fix $\kappa = 0.5$ and set $\zeta = 0.05, 0.10, 0.20, 0.30$, and 0.50 . The threshold value obtained from $R_0(0; \zeta_c) = 1$ is

$$\zeta_c \approx 0.0677. \quad (6.16)$$

When $\zeta = 0.05 < \zeta_c$, the active states may show a short transient peak, but all decay to zero because the spreading chain cannot regenerate. Above the threshold, S^* increases monotonically with ζ , since a larger fraction of beneficiaries is directly converted into spreaders. The holder equilibrium C^* also increases but gradually saturates: The indirect inflow δS becomes stronger, whereas the direct inflow ξH is limited by the changing beneficiary level and the holder exit channels.

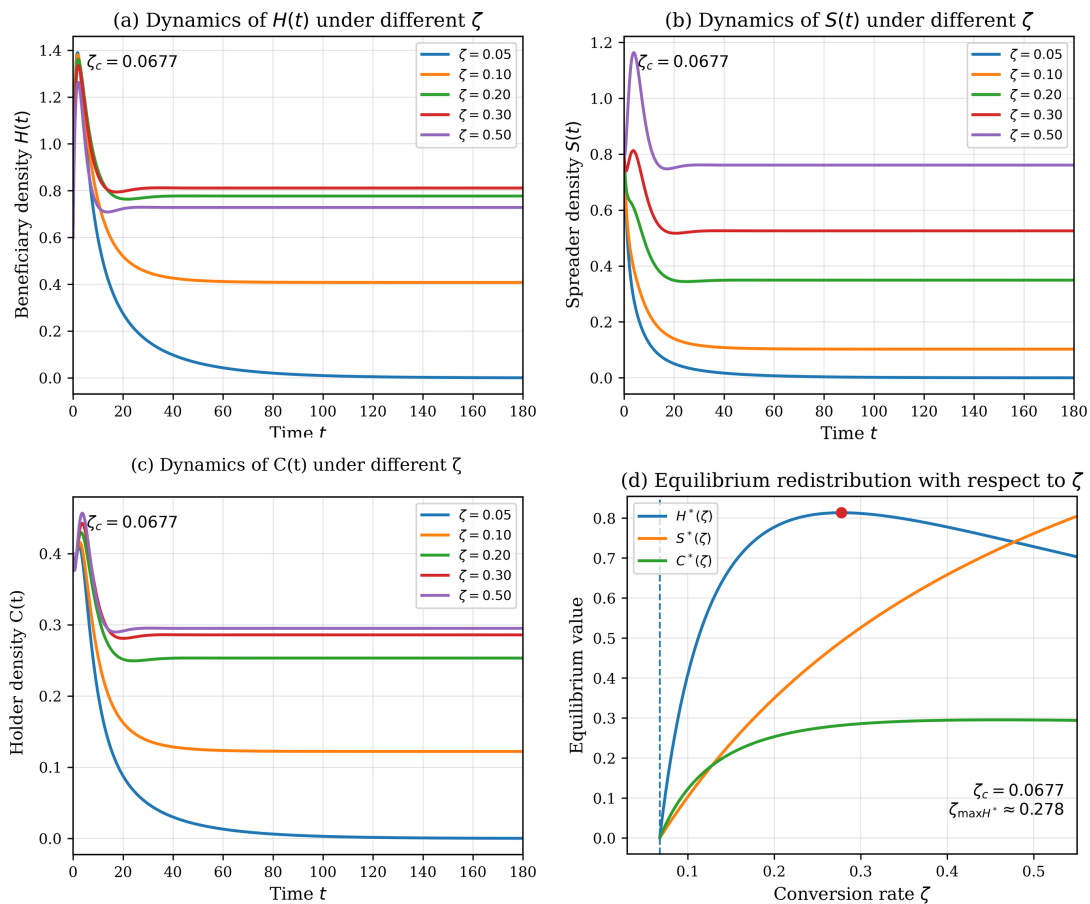


Figure 8. Dynamic and equilibrium redistribution induced by the beneficiary-to-spreader conversion rate ζ at zero delay. (a)–(c) Temporal evolution of the beneficiary, spreader, and holder densities under representative ζ values; and (d) equilibrium curves $H^*(\zeta)$, $S^*(\zeta)$, and $C^*(\zeta)$. The vertical line marks $\zeta_c \approx 0.0677$, and the point marks the maximum of H^* near $\zeta \approx 0.278$.

The response of H^* is not monotone. When ζ rises from a low level, stronger spreading creates more beneficiaries through new contacts, so H^* first increases. At larger ζ , beneficiaries leave H more quickly through $H \rightarrow S$, and this outflow becomes stronger. The maximum occurs near $\zeta \approx 0.278$. A larger conversion rate therefore does not raise every active state. It shifts more of the active population toward spreading. Figures 6–8 show the threshold, network, and state-allocation effects separately.

6.5. Delay-dependent persistence threshold and delayed dynamics

6.5.1. Baseline parameters and persistence critical delay

The positive-delay simulations use the same baseline game equilibrium and spreading parameters as the zero-delay experiments. In particular,

$$(x_G^*, y_G^*) = (0.70331865, 0.23443955), \quad (\lambda_S, \lambda_C) = (0.11721978, 0.35165933),$$

and $B = 1$, $\mu = 0.2$, $\zeta = 0.3$, $\xi = 0.2$, $\delta = \sigma = \varepsilon = \theta = 0.1$, and $\kappa = 0.5$. The complete focal-state history is

$$X(\vartheta) = (4, 0.6, 0.8, 0.4, 0.2)^T, \quad \vartheta \in [-\tau, 0]. \quad (6.17)$$

The pending cohorts are determined by the survival-consistent completion construction in Section 2. Across delay values, Eq (6.17) fixes the five focal-state histories; the finite-horizon comparisons are therefore conditional on this history design rather than on a fixed augmented initial population.

For these parameters,

$$R_0(0) = 2.31478403, \quad \tau_p = \frac{\ln R_0(0)}{\mu} = 4.19658197. \quad (6.18)$$

6.5.2. Dynamics below and above the persistence threshold

At $\tau = 0.8\tau_p \approx 3.3573$, $R_0(\tau) \approx 1.1828$ and the solution converges to

$$E^*(\tau) \approx (4.22735, 0.22076, 0.07311, 0.03974, 0.16915).$$

The terminal infinity-norm error at $t = 300$ is 1.61×10^{-4} . At $\tau = 1.2\tau_p \approx 5.0359$, $R_0(\tau) \approx 0.8455$; the active states converge to zero and $I(t)$ converges to $B/\mu = 5$. The terminal infinity-norm error at $t = 450$ is 1.12×10^{-4} . Figure 9 shows that both simulations remain nonnegative without numerical projection.

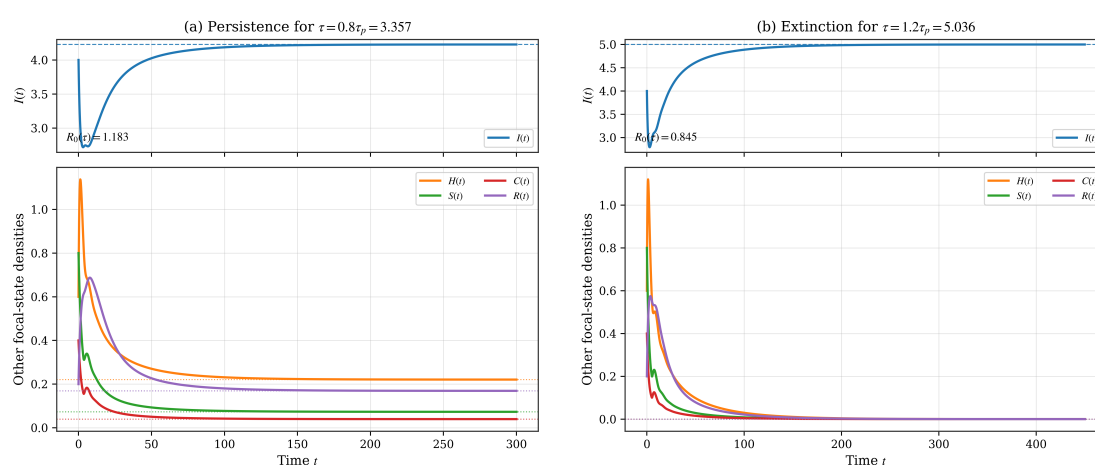


Figure 9. Delay-dependent transition across the persistence threshold. The ignorant state is plotted separately to resolve the smaller focal states; dotted or dashed lines mark theoretical equilibrium values. Below τ_p , the solution converges to the positive equilibrium; above τ_p , the active dissemination states converge to extinction.

6.5.3. Conversion-rate effect on the persistence threshold

Figure 10 summarizes the conversion-rate effect. In panel (a), $R_0(\tau) = e^{-\mu\tau}R_0(0)$ is plotted for three representative rates. The crossings with $R_0(\tau) = 1$ show directly that $d\tau_p/d\zeta > 0$. Panel (b) gives the corresponding positive spreader-equilibrium branches, each of which decreases continuously to zero as $\tau \uparrow \tau_p(\zeta)$.

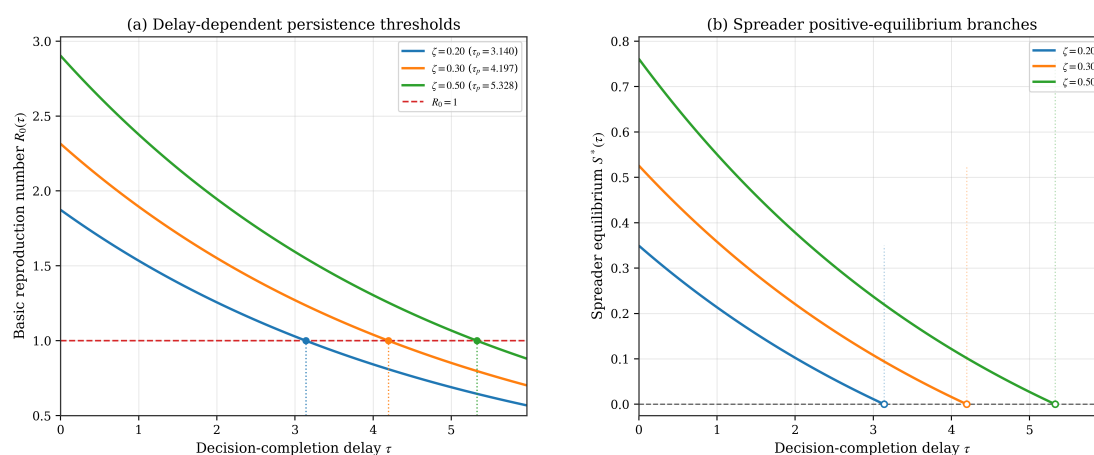


Figure 10. Effect of the beneficiary-to-spreader conversion rate on the delay-dependent persistence threshold and the positive spreader equilibrium. (a) Analytical curves of $R_0(\tau)$ for representative values of ζ ; the circles mark $R_0(\tau_p) = 1$. (b) Corresponding positive spreader-equilibrium branches, which approach zero as $\tau \uparrow \tau_p$. No oscillatory envelope is plotted because the corrected model has no nonzero imaginary-axis crossing while the positive equilibrium exists.

The positive-equilibrium characteristic equation has no nonzero imaginary-axis root while the equilibrium exists. Thus, the delay comparison is organized around the persistence threshold τ_p rather than an oscillatory stability switch.

6.6. Numerical results of optimal control

We use the spreading-layer baseline value $\zeta = 0.3$ unless stated otherwise. The initial state is $X(0) = (4, 0.6, 0.8, 0.4, 0.2)^T$, the control horizon is $T = 100$, and $\varrho = \nu_1 = \nu_2 = 1$. The controls satisfy $0 \leq u_1 \leq u_{1\max} = 1 \text{ time}^{-1}$ and $0 \leq u_2 \leq u_{2\max} = 1$. Here, u_1 promotes $C \rightarrow S$, while u_2 suppresses $S \rightarrow C$. For comparison, moderate fixed control uses $(u_1/u_{1\max}, u_2/u_{2\max}) = (0.5, 0.5)$, and strong fixed control uses $(1, 1)$.

6.6.1. Joint optimal control and state responses

For $\tau = 0$, the state and adjoint equations are integrated by the fourth-order Runge–Kutta method, and the controls are updated through a projected forward–backward sweep. The algorithm converges after 15 iterations, with a final maximum control change of approximately 9.17×10^{-6} .

Because u_1 is a rate and u_2 is a dimensionless suppression proportion, the control profiles are interpreted relative to their own upper bounds. With the numerical bounds equal to one, this normalization does not alter the plotted numerical curves.

Figure 11 summarizes the joint optimal-control results: Panel (a) shows that u_1^* is larger than u_2^* over most of the control period. Both controls fall to zero near T because $\psi_i(T) = 0$. Their mean fractions of the corresponding upper bounds are 0.4045 and 0.1438. Strong fixed control gives the largest spreader population, but its cumulative cost is 100, and the net objective is only 8.24. The corresponding values under no control, moderate fixed control, and joint optimal control are about 54.93, 64.97, and 72.14. The optimal strategy therefore does not seek the largest $S(t)$. It balances spreading benefit and control

cost.

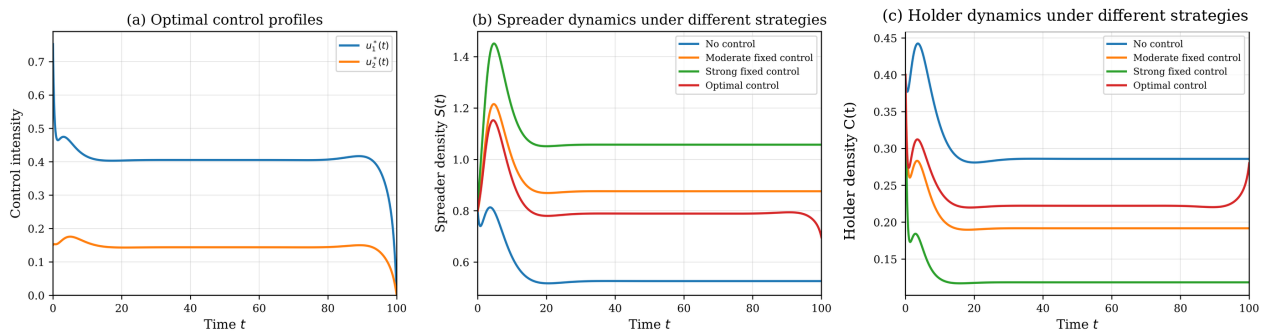


Figure 11. Joint optimal-control profiles and state responses at $\tau = 0$. (a) Optimal controls $u_1^*(t)$ and $u_2^*(t)$; (b) spreader dynamics under no control, moderate fixed control, strong fixed control, and joint optimal control; and (c) the information-holder dynamics.

6.6.2. Contributions of the two control channels

To identify the relative contribution of the two controls, the u_1 -only, u_2 -only, and joint optimal-control problems are solved separately rather than being obtained by deleting one control from the joint trajectory.

The terminal objective values in Figure 12 are approximately $J_{\text{no}} = 54.93$, $J_{u_1} = 71.13$, $J_{u_2} = 56.30$, and $J_{\text{joint}} = 72.14$. Hence,

$$J_{\text{joint}} > J_{u_1} > J_{u_2} > J_{\text{no}}. \quad (6.19)$$

The u_1 -only trajectory is already close to the joint-control trajectory. Using u_2 alone gives only a small improvement over no control. Thus, u_1 is the main spreading channel, while u_2 adds a smaller gain by adjusting the balance between spreaders and holders.

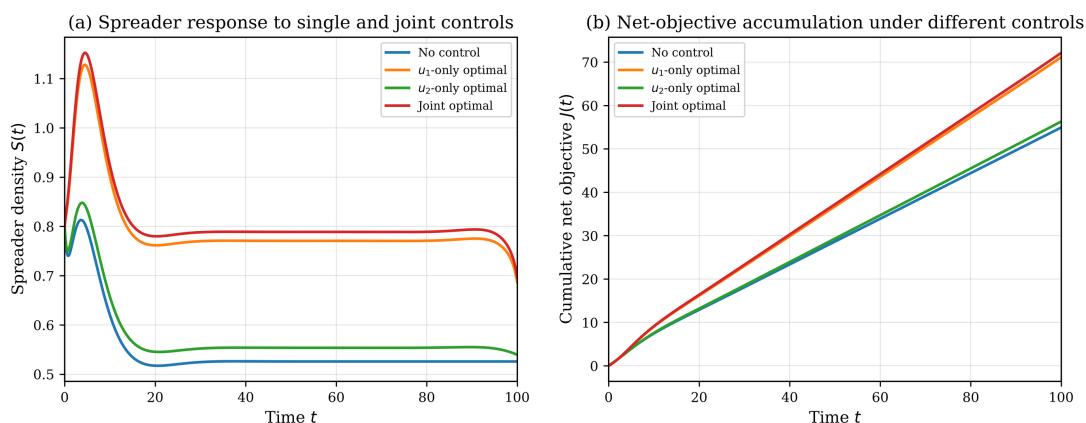


Figure 12. Comparison of single-channel and joint optimal controls. (a) Spreader dynamics under no control, u_1 -only control, u_2 -only control, and joint control; and (b) accumulation of the net objective $J(t)$ under the four strategies.

6.6.3. Effect of the behavioral decision-completion delay on control performance

For positive delay, the state system uses the survival-weighted completion inflows, and the adjoint system retains the corresponding advanced terms. The horizon is $T = 100$, the complete focal-state history is Eq (6.17), and the costs and bounds are the same as in the zero-delay experiment.

The computed delay-dependent results are summarized in Table 4.

Table 4. Delay-dependent computed control results.

τ	$R_0(\tau)$	J_0	J^*	$J^* - J_0$	$(\overline{u_1/u_{1\max}}, \overline{u_2/u_{2\max}})$
0	2.3148	54.9309	72.1360	17.2052	(0.4046, 0.1439)
1	1.8952	38.4122	50.8446	12.4323	(0.3550, 0.1186)
2	1.5516	25.4666	34.2898	8.8232	(0.3077, 0.0970)
3	1.2704	15.8511	21.7273	5.8762	(0.2587, 0.0772)
4	1.0401	9.8008	13.0109	3.2101	(0.1951, 0.0548)
5	0.8516	6.7065	8.2075	1.5010	(0.1201, 0.0318)

Figure 13 summarizes delay-dependent performance: Panel (a) shows that the finite-horizon no-control and optimal-control objectives decrease over the tested delays. Nevertheless, $J^*(\tau) > J_0(\tau)$ at every tested value. At $\tau = 5 > \tau_p$, the uncontrolled system has no positive equilibrium; the reported control gain is therefore a transient benefit over $[0, T]$ and is not interpreted as restoration of an uncontrolled positive equilibrium. Panel (b) shows that the mean fraction of the u_1 bound remains above that of u_2 under the specified bounds.

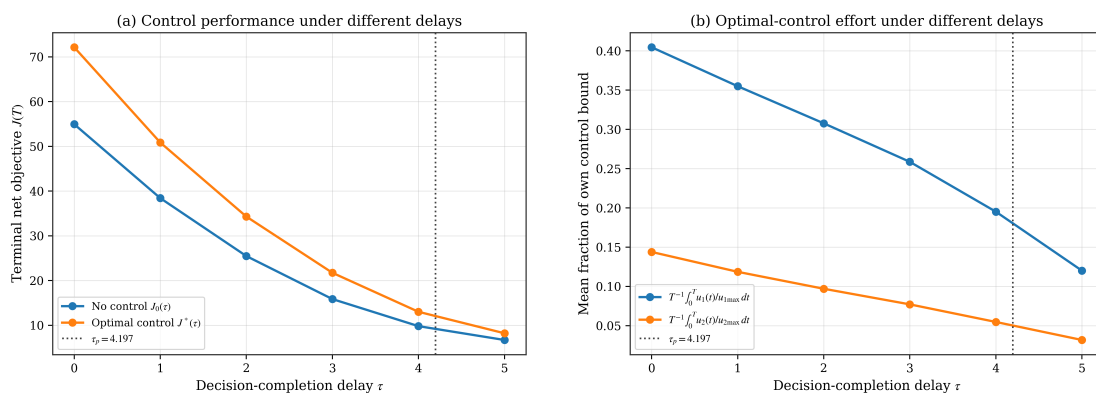


Figure 13. Effect of the decision-completion delay under the specified focal histories, cost weights, and control bounds.

6.6.4. Sensitivity of the two control channels

The u_1 -only, u_2 -only, and joint-control problems are solved independently in six scenarios. The coupling-strength alternatives use $\omega_C = \omega_D = 0.3$ and 0.7 ; the behavioral alternatives use $(\delta, \sigma) = (0.15, 0.05)$ and $(0.05, 0.15)$. All alternatives other than the explicitly identified $\tau = 2$ case use zero delay. The cost weights and control bounds are held at their baseline values.

The resulting objective values and gain ratios are reported in Table 5.

Table 5. Control-channel sensitivity at the common step $\Delta t = 0.05$.

Scenario	J_0	J_{u_1}	J_{u_2}	J_{joint}	$\Delta J_{u_1} / \Delta J_{u_2}$
Baseline, $\tau = 0$	54.9309	71.1251	56.3040	72.1360	11.79
Baseline, $\tau = 2$	25.4666	33.8155	26.0509	34.2898	14.29
Weak game-to-spreading coupling	66.9500	89.9601	68.9794	91.2705	11.34
Strong game-to-spreading coupling	45.6662	57.2206	46.6299	57.9933	11.99
High holding tendency	43.0164	62.4210	45.3716	64.3168	8.24
Easy reactivation	67.5916	80.4251	68.0414	80.7354	28.54

Here, $\Delta J_{u_i} = J_{u_i} - J_0$. The gain ratios range from 8.24 to 28.54. This ranking is conditional on the chosen cost weights and intervention bounds. In the baseline case, u_1 can add up to 1 time^{-1} to the holder-to-spreader rate, whereas u_2 can remove at most $\delta = 0.1 \text{ time}^{-1}$ from the reverse rate. Thus, the experiments compare the specified available interventions and do not establish a universal channel ranking under equal physical capacities or alternative cost weights.

The time-grid check at $\tau = 0, 3$, and 5 uses $\Delta t = 0.10, 0.05$, and 0.025 . The largest observed objective difference between the coarsest and finest grids is 8.55×10^{-4} ; the largest differences in the normalized mean controls are 2.56×10^{-6} for u_1 and 1.03×10^{-6} for u_2 . Three zero-delay initial control guesses yield objectives differing by less than 9.0×10^{-10} . These checks support numerical consistency but do not prove global optimality.

7. Conclusions

We coupled an ecological snowdrift game with a survival-consistent delayed *IHSCR* spreading model. The game gives stable cooperation and defection densities, and these densities set the exit rates of spreaders and holders. We then studied equilibrium selection, the delay-dependent threshold $R_0(\tau)$, equilibrium stability, and optimal control. The main findings are summarized below.

- (1) The game shows smooth equilibrium changes and structural transitions. A larger b moves coexistence toward cooperation. A larger c lowers x_G^* , while y_G^* changes non-monotonically. At $m_c \approx 0.1906$, the system moves from coexistence to a pure-cooperation boundary state. Environmental pressure gives two thresholds, $Q_T = 0.5$ and $Q_{SN} \approx 0.6169$. Between them, extinction and positive coexistence are stable, so the initial strategy densities matter. These game states affect spreading through λ_S and λ_C ; they are not compartment sizes. Thus, in a multistable region, the equilibrium passed to the spreading layer must be selected by the game parameters, initial state, and basin of attraction.
- (2) The decision delay is represented by a survival-consistent completion process. For every continuous nonnegative history, the state system is positively invariant, and the augmented population including the pending cohorts satisfies $\dot{\tilde{N}} = B - \mu \tilde{N}$. The positive-delay threshold is $R_0(\tau) = e^{-\mu\tau} R_0(0)$. The information-free equilibrium is globally asymptotically stable when $R_0(\tau) \leq 1$, while a unique positive equilibrium exists when $R_0(\tau) > 1$.
- (3) If $R_0(0) > 1$, the persistence critical delay is $\tau_p = \mu^{-1} \ln R_0(0)$. While the positive equilibrium exists, its characteristic equation has no nonzero purely imaginary root; hence, the corrected model does not exhibit the previously reported delay-induced Hopf stability switch. Under the

baseline parameters, $R_0(0) = 2.3148$ and $\tau_p = 4.1966$, and the numerical trajectories on the two sides of τ_p converge to the positive and information-free equilibria, respectively.

- (4) The contact efficiency α and the conversion rate ζ raise $R_0(0)$. With $\alpha = 0.1$, the critical average degree is $\langle k \rangle_c \approx 2.1600$. Crossing it changes the spreader state from extinction to positive persistence. A larger ζ raises S^* , while C^* approaches a plateau. The beneficiary state H^* first rises and then falls, with a maximum near $\zeta \approx 0.278$. Network connectivity and behavioral conversion therefore change the threshold and the long-term distribution of active states. A larger ζ also increases the persistence critical delay τ_p .
- (5) Joint control balances spreading benefit and intervention cost. At $\tau = 0$, the net objective is about 72.14, compared with 54.93 under no control and 64.97 under moderate fixed control. Strong fixed control gives more spreading benefit, but its high cost reduces the net objective to about 8.24. The single-channel tests show that u_1 is the main control and u_2 is auxiliary. For each tested delay in $[0, 5]$, the optimal objective remains above the corresponding no-control value over the finite horizon, although both values decrease as the delay grows under the specified focal histories. Across six coupling-strength, delay, and behavioral-transition scenarios, the gain from u_1 alone is 8.24–28.54 times the gain from u_2 alone. This ranking is conditional on the stated cost weights, control bounds, and histories.

The model has several limits. The game affects the spreading system in one direction, so feedback from spreading to strategy choice is not included. The spreading model also uses a mean-field network and cannot describe community structure, multilayer links, or strong individual differences. Most parameters come from theoretical assumptions and numerical experiments rather than platform data. In future work, researchers can add two-way feedback, heterogeneous networks, random effects, distributed decision times, and real data for calibration.

Author contributions

Xinwen Zhang: Conceptualization, Methodology, Formal analysis, Software, Visualization, Writing—original draft.

Rongyu Zhang: Validation, Investigation, Visualization, Writing—review and editing.

Xuechao Zhang: Conceptualization, Supervision, Project administration, Writing—review and editing.

All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

During revision, ChatGPT (OpenAI) assisted with English-language editing. The authors reviewed and verified all outputs and take full responsibility for the accuracy, originality, and integrity of the article.

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Conflict of interest

The authors declare no conflicts of interest.

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