



Research article

On two classes of two-parameter Hessenberg matrices

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Abstract: We investigate two classes of Hessenberg matrices defined by two parameters. For both classes, we explicitly compute the determinant, the inverse matrix, and the LU-factorization. For the matrix of the first class, we explicitly determine its spectrum, whereas the derivation of the eigenvalues for the second class is left as an open problem. Furthermore, we derive recurrence relations, various identities, and combinatorial sums satisfied by the determinants of matrices of both classes. Another contribution of this work is the formulation of the inverse matrix, the determinant, and the LU-factorization using the determinants of all leading principal Hessenberg submatrices. By specializing the parameters, we establish relationships between these Hessenberg determinants and certain integer or polynomial sequences as well as a factorization of the generalized Fibonacci numbers related to the first class.

Keywords: Hessenberg matrices; eigenvalue; recurrence; factorization; Zeilberger algorithm

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1. Introduction

A lower Hessenberg matrix $\mathbf{M}_n = [m_{ij}]$ of order n has the following form:

$$M_n = \begin{bmatrix} m_{1,1} & m_{1,2} & & & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & & \\ \vdots & & & \ddots & \\ m_{n-1,1} & m_{n-1,2} & \cdots & \cdots & m_{n-1,n} \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n-1} & m_{n,n} \end{bmatrix}.$$

Hessenberg matrices constitute a fundamental class of matrices with widespread applications in combinatorics, linear algebra, numerical analysis, and discrete mathematics. It is also worth noting

that a tridiagonal matrix is a special, symmetric-like case of a Hessenberg matrix, which underscores its profound significance in numerical linear algebra. According to [1], the n th order determinant of a Toeplitz–Hessenberg matrix is equivalent to a sum over integer partitions of n . Merca [1] utilized this relationship to derive various interesting combinatorial identities that incorporate both partitions and multinomial coefficients.

A general r -banded matrix $\mathbf{B}_{r,n}$ of order n is defined in the following way:

$$\mathbf{B}_{r,n} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 & \dots & a_1^r & 0 & \dots & 0 \\ a_1^{-2} & a_2^1 & a_2^2 & a_2^3 & \dots & a_2^r & \ddots & \vdots \\ a_1^{-3} & a_2^{-2} & a_3^1 & a_3^2 & a_3^3 & \dots & \ddots & 0 \\ \vdots & a_2^{-3} & a_3^{-2} & a_4^1 & a_4^2 & \ddots & \ddots & a_{n-r+1}^r \\ a_1^{-r} & \vdots & a_3^{-3} & a_4^{-2} & a_5^1 & \ddots & a_{n-3}^3 & \vdots \\ 0 & a_2^{-r} & \dots & \ddots & \ddots & \ddots & a_{n-2}^2 & a_{n-2}^3 \\ \vdots & \ddots & \ddots & \dots & a_{n-1}^{-3} & a_{n-2}^{-2} & a_{n-1}^1 & a_{n-1}^2 \\ 0 & \dots & 0 & a_{n-r+1}^{-r} & \dots & a_{n-2}^{-3} & a_{n-1}^{-2} & a_n^1 \end{bmatrix},$$

where a_j^i 's stand for arbitrary real numbers, $-r \leq i \leq r$, $1 \leq j \leq r \leq n$ (note that the i in the notation a_n^i does not denote the i th power of a_n). When $r = n$, the matrix $\mathbf{B}_{n,n}$ is reduced to an arbitrary square matrix. Kılıç and Stanica [2] established the LU-factorization and the inverse for a general r -banded matrix $\mathbf{B}_{r,n}$ of order n . Because their findings for $\mathbf{B}_{n,n}$ are applicable to arbitrary square matrices, they effectively proposed a novel inversion technique for nonsingular matrices based on *Hessenberg submatrices*. This aligns with a broader body of literature, where various Hessenberg-type matrices are analyzed to explicitly derive their eigenvalues and eigenvectors (see, e.g., [3, 4]).

On the other hand, matrices with explicitly determined eigenvalues, determinants, and inverses are highly sought after as test matrices to benchmark the precision and efficiency of modern computational algorithms. Although we hope the matrices introduced in this paper will serve as a valuable addition to the family of test matrices, our primary focus here is to investigate their algebraic properties.

In what represents the first instance where all eigenvalues of a specific class of Hessenberg matrices are explicitly determined, Chow [3] considered the lower Toeplitz–Hessenberg matrix $\mathbf{H}_q$ of order q of the form

$$\mathbf{H}_q = \begin{bmatrix} \alpha & 1 & & & 0 \\ \alpha^2 & \alpha & 1 & & \\ \alpha^3 & \alpha^2 & \alpha & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 1 \\ \alpha^q & \alpha^{q-1} & \dots & \alpha^2 & \alpha \end{bmatrix},$$

and it was stated that the matrix $\mathbf{H}_q$ has $m = \lfloor q/2 \rfloor$ eigenvalues 0, and the remaining $q - m$ eigenvalues are

$$4\alpha \cos^2 \frac{k\pi}{q+2}, \quad k = 1, 2, \dots, q - m.$$

In [5], the authors explored the same class of Hessenberg matrices, deriving explicit expressions for the pseudoinverse and Drazin inverse of each member. The corresponding eigenvectors for this

class were addressed in [4]. Furthermore, for a detailed analysis of the explicit relationships between eigenvalues, eigenmatrix entries, and the components of unreduced Hessenberg matrices, we refer to [6]. Let p and q be nonzero integers such that $D = p^2 - 4q \neq 0$. For $n \geq 2$, define the second-order linear recurrence $\{U_n\}$ by

$$U_n = pU_{n-1} - qU_{n-2}, \quad (1.1)$$

where $U_0 = 0$, and $U_1 = 1$. If $p = -q = 1$, then $U_n = F_n$ (the n th Fibonacci number).

In a related context, [7, 8] employed matrix methods to derive trigonometric factorizations for general second-order linear recurrences. The literature also reveals significant connections between various Hessenberg determinants and integer sequences. More recently, [9, 10] introduced various Hessenberg matrices whose determinants yield classical sequences, such as general Fibonacci and Lucas numbers, as well as q -Bernoulli and q -Cauchy numbers. For example, Komatsu and Kılıç [10] showed that

$$w_{kn+1} = \begin{vmatrix} w_{k+1} & -vw_k & 0 & \cdots & \cdots & 0 \\ w_k & w_{k+1} & -vw_k & & & \vdots \\ w_{k-1} & w_k & w_{k+1} & & & \vdots \\ \frac{w_{k-1}^2}{w_k} & w_{k-1} & & \ddots & & \vdots \\ \vdots & & & & w_{k+1} & -w_k & 0 \\ \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & & w_{k-1} & w_k & w_{k+1} & -vw_k \\ \frac{w_{k-1}^{n-2}}{w_k^{n-3}} & \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \cdots & \frac{w_{k-1}^2}{w_k} & w_{k-1} & w_k & w_{k+1} \end{vmatrix},$$

where $\{w_n\}$ is defined with for $n \geq 2$

$$w_n = uw_{n-1} + vw_{n-2}, \quad w_0 = 0, \quad w_1 = 1.$$

We investigate two classes of Hessenberg matrices defined by two parameters and derive their various algebraic properties. In Section 2, for the first class, we explicitly obtain its all eigenvalues and consequently its determinant. Furthermore, we derive a recurrences, some identities, and combinatorial sums satisfied by the determinant of this family. By specifying parameters, we derive a relationship between the determinant and the terms of general Fibonacci sequences. Notably, we derive the factorization of terms of the general Fibonacci sequences $\{U_{2n}\}$ and $\{U_{2n+1}\}$ as illustrative applications based on the product of the eigenvalues via Hessenberg matrices. In Section 3, finally, we provide the LU-factorization for this family. All aforementioned results, with the exception of the eigenvalues, are also established for the second-class Hessenberg matrices; the derivation of the eigenvalues for the second class is left as an open problem.

It is worth noting that lower Hessenberg matrices are inherently nonsymmetric ($\mathbf{M} \neq \mathbf{M}^T$), as all entries above the first superdiagonal are zero. Nevertheless, the matrices considered here exhibit a strong structural regularity through their constant diagonals and uniform off-diagonal patterns, which enables the explicit derivation of their algebraic properties.

2. First-class Hessenberg matrices

For a nonzero real number a and any real number b such that $a - 4b \neq 0$, we define a class of Hessenberg matrices $\mathbf{H}_n(a, b)$: For $n \geq 2$, define the $n \times n$ lower Hessenberg matrix $\mathbf{H}_n(a, b) = [h_{ij}]$,

denoted shortly by $\mathbf{H}_n$, with

$$h_{ij} = \begin{cases} a, & \text{if } i = j, \\ b, & \text{if } j = i + 1, \\ (-1)^{i+j-1}b, & \text{if } i > j, \\ 0, & \text{otherwise.} \end{cases}$$

For example, if $n = 5$, then

$$\mathbf{H}_5 = \begin{bmatrix} a-b & b & 0 & 0 & 0 \\ b & a-b & b & 0 & 0 \\ -b & b & a-b & b & 0 \\ b & -b & b & a-b & b \\ -b & b & -b & b & a-b \end{bmatrix}.$$

Let A_n be the $n \times n$ matrix defined as follows

$$\mathbf{A}_n = \begin{bmatrix} -a & & & & \\ -a & -a & & & \\ & \ddots & \ddots & & \\ & & & -a & -a \end{bmatrix}. \quad (2.1)$$

Then, we have the following result.

Theorem 2.1. *The matrix $\mathbf{H}_n(a, b)$ has $m = \lfloor n/2 \rfloor$ eigenvalues that are equal to a , and the remaining $n - m$ eigenvalues are*

$$a - 4b \cos^2 \frac{k\pi}{n+2}.$$

Proof. The eigenvalues of $H_n(a, b)$ are the roots of the characteristic equation

$$\det(H_n - \lambda I_n) = 0,$$

where I_n is the identity matrix of order n . If we multiply the matrix $H_n - \lambda I_n$ by the matrix A_n , then

$$\det(A_n \cdot (H_n - \lambda I_n)) = (-a)^n \det(H_n - \lambda I_n). \quad (2.2)$$

On the other hand, we observe that

$$A_n \cdot (H_n - \lambda I_n) = \begin{bmatrix} ab + a\lambda - a^2 & -ab & & & 0 \\ a\lambda - a^2 & a\lambda - a^2 & -ab & & \\ & a\lambda - a^2 & a\lambda - a^2 & \ddots & \\ & & \ddots & \ddots & -ab \\ 0 & & & a\lambda - a^2 & a\lambda - a^2 \end{bmatrix}.$$

Let T_n be the n th order determinant defined by

$$T_n := \begin{vmatrix} a\lambda - a^2 & -ab & & & 0 \\ a\lambda - a^2 & a\lambda - a^2 & -ab & & \\ & a\lambda - a^2 & a\lambda - a^2 & \ddots & \\ & & \ddots & \ddots & -ab \\ 0 & & & a\lambda - a^2 & a\lambda - a^2 \end{vmatrix}.$$

Expanding the left-hand side of (2.2) along the first column yields for $n > 1$

$$(ab + a\lambda - a^2)T_{n-1} + ab(a\lambda - a^2)T_{n-2} = 0. \quad (2.3)$$

Similarly, T_n satisfies the following three-term recurrence relation for $n > 1$:

$$T_n - (a\lambda - a^2)T_{n-1} - ab(a\lambda - a^2)T_{n-2} = 0, \quad (2.4)$$

with $T_0 = 1$ and $T_1 = a\lambda - a^2$.

By setting

$$\begin{aligned} w &= \left((a\lambda - a^2)^2 + 4ab(a\lambda - a^2) \right)^{\frac{1}{2}}, \\ \gamma &= \left((a\lambda - a^2) + w \right) / 2, \\ \sigma &= \left((a\lambda - a^2) - w \right) / 2, \end{aligned}$$

the solution to Eq (2.4) can be written as follows:

$$T_n = \frac{\gamma^{n+1} - \sigma^{n+1}}{w}. \quad (2.5)$$

Here, recall the following algebraic factorization of the difference of two powers:

$$x^{n+2} - y^{n+2} = \prod_{i=0}^{n+1} (x - z^i y), \quad (2.6)$$

where $z = e^{\frac{2\pi i}{n+2}}$ is a primitive $(n+2)$ -th root of unity.

Setting $x = \gamma$ and $y = \sigma$, we obtain

$$\gamma^{n+2} - \sigma^{n+2} = \prod_{i=0}^{n+1} (\gamma - z^i \sigma). \quad (2.7)$$

Because $a \neq 0$, we have $w \neq 0$. Dividing both sides by $w = \gamma - \sigma$ (noting that for $i = 0$, the term $\gamma - z^0 \sigma = \gamma - \sigma = w$), we separate the $i = 0$ term from the product:

$$\frac{\gamma^{n+2} - \sigma^{n+2}}{w} = \frac{(\gamma - \sigma) \prod_{i=1}^{n+1} (\gamma - z^i \sigma)}{w} = \prod_{i=1}^{n+1} (\gamma - z^i \sigma). \quad (2.8)$$

Thus we write

$$(ab + a\lambda - a^2)T_{n-1} + ab(a\lambda - a^2)T_{n-2} = \frac{\gamma^{n+2} - \sigma^{n+2}}{(a\lambda - a^2)w} = \frac{1}{(a\lambda - a^2)w} \prod_{i=1}^{n+1} (\gamma - z^i \sigma), \quad (2.9)$$

where $1, z, z^2, \dots, z^{n+1}$ are the roots of the equation $x^{n+2} - 1 = 0$. Using the relation

$$z^k + z^{n+2-k} = 2 \cos \frac{2k\pi}{n+2}, \quad k = 1, 2, \dots, p,$$

where $p = \lfloor (n+1)/2 \rfloor = n-m$, from (2.9), we obtain

$$\begin{aligned} & (ab + a\lambda - a^2)T_{n-1} + ab(a\lambda - a^2)T_{n-2} \\ &= \begin{cases} \frac{1}{(a^2 - a\lambda)} \prod_{k=1}^p \left((a^2 - a\lambda)^2 - 4ab(a^2 - a\lambda) \cos^2 \frac{k\pi}{n+2} \right) (a\lambda - a^2), & \text{if } n \text{ is even,} \\ \frac{-1}{(a^2 - a\lambda)} \prod_{k=1}^p \left((a^2 - a\lambda)^2 - 4ab(a^2 - a\lambda) \cos^2 \frac{k\pi}{n+2} \right), & \text{if } n \text{ is odd,} \end{cases} \\ &= \begin{cases} a^n (\lambda - a)^m \prod_{k=1}^p \left(\lambda - a + 4b \cos^2 \frac{k\pi}{n+2} \right), & \text{if } n \text{ is even,} \\ -a^n (\lambda - a)^m \prod_{k=1}^p \left(\lambda - a + 4b \cos^2 \frac{k\pi}{n+2} \right), & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

By (2.2), the claim is seen. $\square$

For convenience, we denote $\det \mathbf{H}_n(a, b)$ by $\tilde{H}_n(a, b)$ or shortly $\tilde{H}_n$. Because the eigenvalues of the matrix $\mathbf{H}_n$ are given by Theorem 2.1 and $n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$, we evaluate $\tilde{H}_n$ as the product of its eigenvalues using the following result.

Corollary 2.2. For $n > 1$,

$$\tilde{H}_n = a^{\lfloor n/2 \rfloor} \prod_{k=1}^{\lceil n/2 \rceil} \left[a - 4b \cos^2 \frac{k\pi}{n+2} \right].$$

Remark 2.3. (Condition number and numerical benchmarking) From the explicit spectrum in Theorem 2.1, the asymptotic scaling of the spectral condition number

$$\kappa_2(\mathbf{H}_n) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|}$$

as $n \rightarrow \infty$ depends crucially on the choice of parameters:

- (1) When $a > 4b > 0$, the minimum eigenvalue converges to $a - 4b > 0$, keeping the condition number bounded ($\kappa_2(\mathbf{H}_n) = O(1)$), making $\mathbf{H}_n(a, b)$ a well-conditioned test matrix.
- (2) At the critical boundary $a = 4b$, the smallest eigenvalue vanishes quadratically as

$$\lambda_{\min} = 4b \sin^2 \frac{\pi}{n+2} = O(n^{-2})$$

yielding an asymptotically ill-conditioned matrix with $\kappa_2(\mathbf{H}_n) = O(n^2)$. This property makes $\mathbf{H}_n(4b, b)$ an ideal benchmark candidate for testing the precision and sensitivity of algorithms dealing with ill-conditioned linear systems.

Now, we give two linear recurrence relations satisfied by $\tilde{H}_n$ with the following result.

Theorem 2.4. *i) For $n > 0$, $\{\tilde{H}_n\}$ satisfies the following second-order linear recurrence:*

$$\tilde{H}_n = a\tilde{H}_{n-1} - ab\tilde{H}_{n-2},$$

where $\tilde{H}_0 = 1$ and $\tilde{H}_{-1} = 1/a$.

ii) For $n > 1$, $\{\tilde{H}_n\}$ satisfies the following third-order linear recurrence:

$$\tilde{H}_n = (a - b)\tilde{H}_{n-1} - ab^2\tilde{H}_{n-3}, \quad (2.10)$$

where $\tilde{H}_1 = a - b$, $\tilde{H}_0 = 1$ and $\tilde{H}_{-1} = 1/a$.

Proof. i) By the definition of matrix H_n , if we subtract the $(n - 1)$ th row from the n th row, then we obtain

$$\tilde{H}_n = \left| \begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & & b \\ \hline 0 & \cdots & 0 & a & a \end{array} \right|,$$

which, by the definition of H_n and expanding along the last column first and the last row later, gives us

$$\tilde{H}_n = a\tilde{H}_{n-1} - ab\tilde{H}_{n-2},$$

as claimed.

ii) By the definition of matrix $\mathbf{H}_n$, if we expand $\tilde{H}_n$ along the last column, we get

$$\tilde{H}_n = (a - b)\tilde{H}_{n-1} - b \left| \begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & & b \\ \hline b & -b & b & \cdots & -b \\ & & & & b \end{array} \right|,$$

which, by subtracting the $(n - 2)$ th row from the $(n - 1)$ th row, gives us

$$\tilde{H}_n = (a - b)\tilde{H}_{n-1} - b \left| \begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & & b \\ \hline 0 & 0 & 0 & \cdots & a \\ & & & & 0 \end{array} \right|.$$

Then, by expanding according to the last column, we get

$$\tilde{H}_n = (a - b)\tilde{H}_{n-1} - b^2 \left| \begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & & b \\ \hline 0 & 0 & 0 & \cdots & 0 \\ & & & & a \end{array} \right|,$$

which, by expanding along the last row, gives us

$$\tilde{H}_n = (a - b)\tilde{H}_{n-1} - ab^2\tilde{H}_{n-3},$$

as claimed. $\square$

By the characteristic equation of the matrix $\mathbf{H}_n$ and its roots, we state the Binet formula for $\tilde{H}_n$ without proof as follows.

Corollary 2.5. For $n > 0$,

$$\tilde{H}_n = \frac{1}{2} \left[(a - 2b) \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) + (\alpha^n + \beta^n) \right],$$

where $\alpha = (a + \sqrt{a(a - 4b)})/2$, and $\beta = (a - \sqrt{a(a - 4b)})/2$.

The determinant $\tilde{H}_n$ can be expressed as a combinatorial sum by the following result.

Corollary 2.6. For $n > 0$,

$$\tilde{H}_n = \sum_{m=0}^{\lfloor (n+1)/2 \rfloor} (-1)^m \binom{n+1-m}{m} a^{n-m} b^m.$$

This combinatorial sum related to $\tilde{H}_n$ can be proven by showing that the combinatorial sum satisfies the same recurrence given in Theorem 2.4 (i).

We also present some identities including terms of $\{\tilde{H}_n\}$ in the following result. We use some of them for future results.

Theorem 2.7. i) For $1 \leq k \leq n$,

$$\tilde{H}_n = \tilde{H}_k \tilde{H}_{n-k} - \tilde{H}_{n-k-2} \tilde{H}_{k-2} a^2 b^2; \quad (2.11)$$

ii) For $n > 0$,

$$\sum_{k=0}^n b^k \tilde{H}_{n-k} = a \tilde{H}_{n-1}; \quad (2.12)$$

iii) For $n \geq m + 1$ and $m \geq 0$,

$$\tilde{H}_n = a^{m+1} \tilde{H}_{n-m-1} - ab \sum_{k=0}^m a^k \tilde{H}_{n-k-2}, \quad (2.13)$$

where $\tilde{H}_0 = 1$ and $\tilde{H}_{-1} = a^{-1}$.

Proof. We prove the identity (iii) as a showcase by induction. For $n = 1$ and $m = 0$, consider

$$a^{0+1} \tilde{H}_0 - ab \sum_{k=0}^0 a^k \tilde{H}_{-k-1} = a \tilde{H}_0 - ab \tilde{H}_{-1} = \tilde{H}_1$$

as claimed. Now, assume the claim holds for $n \leq r$ such that $r > 1$. We aim to show that it holds for $n = r + 1$; that is,

$$\tilde{H}_{r+1} = a^{m+1} \tilde{H}_{r-m} - ab \sum_{k=0}^m a^k \tilde{H}_{r-k-1}.$$

By the induction hypothesis, assume the claim is true for $n = r$ and $n = r - 1$; that is,

$$\tilde{H}_{r-1} = a^{m+1} \tilde{H}_{r-m-2} - ab \sum_{k=0}^m a^k \tilde{H}_{r-k-3},$$

$$\tilde{H}_r = a^{m+1} \tilde{H}_{r-m-1} - ab \sum_{k=0}^m a^k \tilde{H}_{r-k-2}.$$

Now, consider the linear combination of the two identities above:

$$\begin{aligned} a\tilde{H}_r - ab\tilde{H}_{r-1} &= a^{m+2} \tilde{H}_{r-m-1} - a^2b \sum_{k=0}^m a^k \tilde{H}_{r-k-2} - a^{m+2}b\tilde{H}_{r-m-2} + a^2b^2 \sum_{k=0}^m a^k \tilde{H}_{r-k-3} \\ &= a^{m+2} (\tilde{H}_{r-m-1} - b\tilde{H}_{r-m-2}) - a^2b \sum_{k=0}^m a^k (\tilde{H}_{r-k-2} - b\tilde{H}_{r-k-3}), \end{aligned}$$

which, by the recurrence relation of $\{\tilde{H}_n\}$, equals

$$\begin{aligned} a^{m+2} \left(\frac{1}{a} \tilde{H}_{r-m} \right) - a^2b \sum_{k=0}^m a^k \left(\frac{1}{a} \tilde{H}_{r-k-1} \right) &= a^{m+1} \tilde{H}_{r-m} - ab \sum_{k=0}^m a^k \tilde{H}_{r-k-1} \\ &= \tilde{H}_{r+1} \end{aligned}$$

as claimed.

The claims (i) and (ii) can be proven using the Binet formula of $\{\tilde{H}_n\}$ or induction. $\square$

Now, we explicitly give the inverse matrix H_n^{-1} by the following result.

Theorem 2.8. For $n > 1$, the entries of the inverse matrix $\mathbf{H}_n^{-1} = [H_{i,j}^{-1}]$ are explicitly given by

$$H_{i,j}^{-1} = \frac{(-1)^{i+j}}{\tilde{H}_n} \begin{cases} b^{j-i} \tilde{H}_{n-j} \tilde{H}_{i-1}, & j \geq i, \\ ba^{i-j+1} \tilde{H}_{j-2} \tilde{H}_{n-i-1}, & i > j, \end{cases}$$

where $\tilde{H}_0 = 1$, and $\tilde{H}_{-1} = a^{-1}$.

Proof. Let, denote $H_n H_n^{-1}$ by $P_n = [P_{i,j}]$. Then, we show that $P_n = I_n$, the identity matrix of order n . We will consider three subcases:

(i) $P_{i,i} = 1$;

(ii) $P_{i,j} = 0$ for $i > j$;

(iii) $P_{i,j} = 0$ for $j > i$.

i) For $2 \leq i \leq n-1$, consider

$$\begin{aligned} P_{i,i} &= \sum_{k=1}^{i-1} H_{i,k} H_{k,i}^{-1} + H_{i,i} H_{i,i}^{-1} + H_{i,i+1} H_{i+1,i}^{-1} \\ &= \sum_{k=1}^{i-1} (-1)^{i+k-1} b \frac{(-b)^{i-k} \tilde{H}_{n-i} \tilde{H}_{k-1}}{\tilde{H}_n} + (a-b) \frac{\tilde{H}_{n-i} \tilde{H}_{i-1}}{\tilde{H}_n} + b \left(-\frac{a^2 b \tilde{H}_{i-2} \tilde{H}_{n-i-2}}{\tilde{H}_n} \right) \\ &= -\frac{\tilde{H}_{n-i}}{\tilde{H}_n} \sum_{k=1}^{i-1} b^{i-k+1} \tilde{H}_{k-1} + (a-b) \frac{\tilde{H}_{n-i} \tilde{H}_{i-1}}{\tilde{H}_n} - \frac{a^2 b^2 \tilde{H}_{i-2} \tilde{H}_{n-i-2}}{\tilde{H}_n} \end{aligned}$$

$$\begin{aligned}
&= \frac{\tilde{H}_{n-i}}{\tilde{H}_n} \left[- \sum_{k=0}^{i-2} b^{i-k} \tilde{H}_k + (a-b) \tilde{H}_{i-1} \right] - a^2 b^2 \frac{\tilde{H}_{i-2} \tilde{H}_{n-i-2}}{\tilde{H}_n} \\
&= \frac{\tilde{H}_{n-i}}{\tilde{H}_n} \left[a \tilde{H}_{i-1} - \sum_{k=0}^{i-1} b^{i-k} \tilde{H}_k \right] - a^2 b \frac{\tilde{H}_{i-2} \tilde{H}_{n-i-2}}{\tilde{H}_n} \\
&= \frac{\tilde{H}_{n-i}}{\tilde{H}_n} \left[a \tilde{H}_{i-1} - \sum_{k=0}^i b^{i-k} \tilde{H}_k + \tilde{H}_i \right] - a^2 b \frac{\tilde{H}_{i-2} \tilde{H}_{n-i-2}}{\tilde{H}_n},
\end{aligned}$$

which, by (2.12) and (2.11) given in Theorem 2.7, equals

$$\begin{aligned}
\frac{\tilde{H}_{n-i}}{\tilde{H}_n} \left[a \tilde{H}_{i-1} - \sum_{k=0}^i b^{i-k} \tilde{H}_k + \tilde{H}_i \right] - a^2 b \frac{\tilde{H}_{i-2} \tilde{H}_{n-i-2}}{\tilde{H}_n} &= \frac{\tilde{H}_{n-i}}{\tilde{H}_n} [a \tilde{H}_{i-1} - a \tilde{H}_{i-1} + \tilde{H}_i] - a^2 b^2 \frac{\tilde{H}_{i-2} \tilde{H}_{n-i-2}}{\tilde{H}_n} \\
&= \frac{\tilde{H}_{n-i} \tilde{H}_i - a^2 b^2 \tilde{H}_{i-2} \tilde{H}_{n-i-2}}{\tilde{H}_n} = \frac{1}{\tilde{H}_n} \tilde{H}_n = 1,
\end{aligned}$$

as claimed.

In particular, using (2.10) given in Theorem 2.4, consider the entry $P_{1,1}$ as

$$\begin{aligned}
P_{1,1} &= H_{1,1} H_{1,1}^{-1} + H_{1,2} H_{2,1}^{-1} \\
&= (a-b) \frac{1}{\tilde{H}_n} \tilde{H}_{n-1} \tilde{H}_0 - b \frac{1}{\tilde{H}_n} a^2 b \tilde{H}_{-1} \tilde{H}_{n-3} \\
&= \frac{1}{\tilde{H}_n} [(a-b) \tilde{H}_{n-1} - a^2 b^2 \tilde{H}_{n-3}] \\
&= \frac{1}{\tilde{H}_n} \tilde{H}_n \\
&= 1,
\end{aligned}$$

as claimed.

ii) For $i > j$, consider

$$P_{i,j} = \sum_{k=1}^n H_{n,k} H_{k,j}^{-1} = \sum_{k=1}^j H_{n,k} H_{k,j}^{-1} + \sum_{k=j+1}^n H_{n,k} H_{k,j}^{-1},$$

which, after some algebraic rearrangement, yields

$$\begin{aligned}
&b \left[(-1)^{i+j} a^{i-j+2} \tilde{H}_{n-1-i} \tilde{H}_{j-2} + (-1)^{i+j-1} \tilde{H}_{n-j} \sum_{k=0}^{j-1} b^k \tilde{H}_{j-1-k} + (-1)^{i+j-1} a^2 b \tilde{H}_{j-2} \sum_{k=0}^{i-j} a^k \tilde{H}_{n-j-2-k} \right] \\
&= b \left[(-1)^{i+j} a^{i-j+2} \tilde{H}_{n-1-i} \tilde{H}_{j-2} + (-1)^{i+j-1} \tilde{H}_{n-j} a \tilde{H}_{j-2} + (-1)^{i+j-1} a^2 b \tilde{H}_{j-2} \sum_{k=0}^{i-j} a^k \tilde{H}_{n-j-2-k} \right] \\
&= (-1)^{i+j} b \tilde{H}_{j-2} a \left[a^{i-j+1} \tilde{H}_{n-1-i} - ab \sum_{k=0}^{i-j} a^k \tilde{H}_{n-j-2-k} - \tilde{H}_{n-j} \right],
\end{aligned}$$

which, by substituting $m = i - j$ and $n = n - j$ into Eq (2.13), yields

$$P_{i,j} = (-1)^{i+j} b \tilde{H}_{j-2} \cdot 0 = 0,$$

as required.

iii) For $j > i$, by Eq (2.12), consider

$$\begin{aligned} \sum_{k=1}^n H_{i,k} H_{k,j}^{-1} &= \sum_{k=1}^{i+1} H_{i,k} H_{k,j}^{-1} = \sum_{k=1}^{i-1} H_{i,k} H_{k,j}^{-1} + H_{i,i} H_{i,j}^{-1} + H_{i,i+1} H_{i+1,j}^{-1} \\ &= \sum_{k=1}^{i-1} (-1)^{i+k-1} b \cdot \frac{(-1)^{k+j}}{\tilde{H}_n} b^{j-k} \tilde{H}_{n-j} \tilde{H}_{k-1} \\ &\quad + (a-b) \cdot \frac{(-1)^{i+j}}{\tilde{H}_n} b^{j-i} \tilde{H}_{n-j} \tilde{H}_{i-1} + b \cdot \frac{(-1)^{i+1+j}}{\tilde{H}_n} b^{j-i-1} \tilde{H}_{n-j} \tilde{H}_i \\ &= \frac{(-1)^{i+j-1}}{\tilde{H}_n} \left[\sum_{k=1}^{i-1} b^{j-k+1} \tilde{H}_{n-j} \tilde{H}_{k-1} - (a-b) b^{j-i} \tilde{H}_{n-j} \tilde{H}_{i-1} + b^{j-i} \tilde{H}_{n-j} \tilde{H}_i \right] \\ &= \frac{(-1)^{i+j-1} \tilde{H}_{n-j}}{\tilde{H}_n} \left[\sum_{k=0}^{i-2} b^{j-k} \tilde{H}_k - a b^{j-i} \tilde{H}_{i-1} + b^{j-i+1} \tilde{H}_{i-1} + b^{j-i} \tilde{H}_i \right] \\ &= \frac{(-1)^{i+j-1} \tilde{H}_{n-j}}{\tilde{H}_n} \left[b^{j-i} \sum_{k=0}^i b^{i-k} \tilde{H}_k - a b^{j-i} \tilde{H}_{i-1} \right] \\ &= \frac{(-1)^{i+j-1} \tilde{H}_{n-j} b^{j-i}}{\tilde{H}_n} \left[\sum_{k=0}^i b^{i-k} \tilde{H}_k - a \tilde{H}_{i-1} \right] \\ &= \frac{(-1)^{i+j-1} \tilde{H}_{n-j} b^{j-i}}{\tilde{H}_n} \left[a \tilde{H}_{i-1} - a \tilde{H}_{i-1} \right] \\ &= 0, \end{aligned}$$

as claimed.

The case $i = j = n$ can be followed by the case above. Thus, we omit it here. This concludes the proof for all cases. $\square$

For example, if $n = 5$, then we have that

$$\mathbf{H}_5^{-1} = \frac{1}{\tilde{H}_5} \begin{bmatrix} \tilde{H}_4 \tilde{H}_0 & -b \tilde{H}_3 \tilde{H}_0 & b^2 \tilde{H}_2 \tilde{H}_0 & -b^3 \tilde{H}_1 \tilde{H}_0 & b^4 \tilde{H}_0 \tilde{H}_0 \\ -a^2 b \tilde{H}_{-1} \tilde{H}_2 & \tilde{H}_3 \tilde{H}_1 & -b \tilde{H}_2 \tilde{H}_1 & b^2 \tilde{H}_1 \tilde{H}_1 & -b^3 \tilde{H}_0 \tilde{H}_1 \\ a^3 b \tilde{H}_{-1} \tilde{H}_1 & -a^2 b \tilde{H}_0 \tilde{H}_1 & \tilde{H}_2 \tilde{H}_2 & -b \tilde{H}_1 \tilde{H}_2 & b^2 \tilde{H}_0 \tilde{H}_2 \\ -a^4 b \tilde{H}_{-1} \tilde{H}_0 & a^3 b \tilde{H}_0 \tilde{H}_0 & -a^2 b \tilde{H}_1 \tilde{H}_0 & \tilde{H}_1 \tilde{H}_3 & -b \tilde{H}_0 \tilde{H}_3 \\ a^5 b \tilde{H}_{-1} \tilde{H}_{-1} & -a^4 b \tilde{H}_0 \tilde{H}_{-1} & a^3 b \tilde{H}_4 \tilde{H}_{-1} & -a^2 b \tilde{H}_2 \tilde{H}_{-1} & \tilde{H}_0 \tilde{H}_4 \end{bmatrix}.$$

Remark 2.9. (Practical implementation and overflow prevention) The exact entries of H_n^{-1} in Theorem 2.8 are expressed in terms of products of subdeterminants $\tilde{H}_k$. Because $\tilde{H}_k$ grows exponentially with k (governed by the dominant root $\alpha = (a + \sqrt{a(a-4b)})/2$ of the characteristic polynomial), evaluating $\tilde{H}_k$ directly for large n in standard floating-point arithmetic may lead to

numerical overflow. To prevent overflow in numerical benchmarking implementations, one should work with scaled quantities $\hat{H}_k = \tilde{H}_k/\alpha^k$ or compute the entrywise logarithms

$$\ln |H_{i,j}^{-1}| = \ln |\tilde{H}_{n-j}| + \ln |\tilde{H}_{i-1}| - \ln |\tilde{H}_n| + (j-i) \ln |b|.$$

This logarithmic or normalized representation allows exact and overflow-free generation of $\mathbf{H}_n^{-1}$ even for very large matrix dimensions n .

As a consequence of Theorem 2.1, we give a relationship between the determinant of the matrix $\mathbf{H}_n$ and the terms of the sequence $\{U_n\}$. To this end, we define the $n \times n$ lower Hessenberg matrix $\mathbf{G}_n(a, b) = [g_{ij}]$, or briefly $\mathbf{G}_n$, with $g_{ii} = a - b$ for $1 \leq i \leq n - 1$, $g_{i,i+1} = b$ for $1 \leq i \leq n - 1$, $g_{ij} = (-1)^{i+j-1} b$ for $1 \leq j < i < n$, and $g_{n,j} = (-1)^{n+j} b$ for $1 \leq j \leq n$. By definition, we state that $g_{nn} = b$.

For even n , it has the form

$$\begin{aligned} \mathbf{G}_n(a, b) &= \begin{bmatrix} a-b & b & & & & & & 0 \\ b & a-b & b & & & & & \\ -b & b & a-b & b & & & & \\ b & -b & b & a-b & b & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ b & -b & b & \cdots & b & a-b & b & \\ -b & b & -b & \cdots & -b & b & a-b & b \\ -b & b & -b & & -b & b & -b & b \end{bmatrix} \\ &= \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline -b & b & \cdots & -b & b \end{array} \right]. \end{aligned}$$

Denote $\det G_n(a, b)$ by $\tilde{G}_n(a, b)$, or shortly, $\tilde{G}_n$. Then, we have the following theorem.

Theorem 2.10. For $n \geq 2$,

$$\tilde{H}_n(p^2, q) = p^{n-1} U_{n+2},$$

where U_n denotes the n th general Fibonacci number, and p and q are the coefficients in the definition of $\{U_n\}$.

Proof. (Induction on n) If $n = 2$, then

$$\tilde{H}_2(p^2, q) = p^4 - 2p^2q = pU_4,$$

as claimed. Suppose that $n \geq 3$ is an even integer, and the claim holds for n . We prove that the claim holds for $n + 1$. Expanding the $\tilde{H}_{n+1}(p^2, q)$ according to the last column, by the definitions of $\tilde{H}_n$ and $\tilde{G}_n$ we write

$$\tilde{H}_{n+1}(p^2, q) = (p^2 - q) \tilde{H}_n(p^2, q) - q \tilde{G}_n(p^2, q).$$

For an even integer n , subtracting the $(n - 1)$ th row from the n -th row of $G_n(a, b)$ yields:

$$\tilde{G}_n(a, b) = \begin{vmatrix} a-b & b & 0 & \dots & 0 & 0 & 0 \\ -b & a-b & b & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ * & * & \dots & a-b & b & 0 & 0 \\ * & * & \dots & -b & a-b & b & 0 \\ 0 & 0 & \dots & 0 & 0 & -a & 0 \end{vmatrix}_{n \times n}.$$

Expanding this determinant first along the n th row and subsequently along the $(n - 1)$ th column of the resulting minor directly gives

$$\tilde{G}_n(a, b) = (-1)^{2n-1}(-a) \cdot b \cdot \tilde{H}_{n-2}(a, b) = ab\tilde{H}_{n-2}(a, b).$$

Thus, by our assumption, we write

$$\begin{aligned} \tilde{H}_{n+1}(p^2, q) &= (p^2 - q)\tilde{H}_n(p^2, q) - p^2q^2\tilde{H}_{n-2}(p^2, q) \\ &= (p^2 - q)p^{n-1}U_{n+2} - p^2q^2p^{n-3}U_n. \end{aligned}$$

From the recursion of $\{U_n\}$, we rewrite it as

$$\begin{aligned} \tilde{H}_{n+1}(p^2, q) &= (p^{n+1} - p^{n-1}q)U_{n+2} - p^{n-1}q^2U_n \\ &= (p^{n+1} - p^{n-1}q)(pU_{n+1} - qU_n) - p^{n-1}q^2U_n \\ &= p^{n+1}(pU_{n+1} - qU_n) - p^nqU_{n+1} = p^nU_{n+3}. \end{aligned}$$

The odd case can be treated in a similar manner. The proof is therefore complete. $\square$

Special sequence cases of the determinantal sequence $\{\tilde{H}_n(a, b)\}$ can be seen Tables 1 and 2.

Table 1. Special sequence cases of the determinantal sequence $\{\tilde{H}_n(a, b)\}$.

a	b	$\tilde{H}_n(a, b)$	Sequence type
p^2	q	$p^{n-1}U_{n+2}$	Generalized Fibonacci sequence
1	-1	F_{n+2}	Fibonacci sequence
2	-1	P_{n+2}	Pell sequence
1	-2	J_{n+2}	Jacobsthal sequence

Table 2. Special polynomial cases of the determinantal sequence $\{\tilde{H}_n(a, b)\}$.

a	b	$\tilde{H}_n(a, b)$	Polynomial type
$2x$	$1/2$	$\tilde{H}_{2n-1}(a, b) \rightarrow \frac{1}{2}x^{2n-2}U_{2n}(x)$	Second-kind Chebyshev
$2x$	$1/2$	$\tilde{H}_{2n}(a, b) \rightarrow x^{2n} \sum_{k=0}^n (-1)^k U_{2(n-k)}(x)$	Alternating sums of second-kind Chebyshev

As applications of the above results, we shall present two trigonometric factorizations of terms of the sequence $\{U_n\}$ by the following result.

Theorem 2.11. For $k > 1$,

$$U_{2k+1} = \prod_{j=1}^k \left[p^2 - 4q \cos^2 \left(\frac{\pi j}{2k+1} \right) \right]$$

and

$$U_{2k} = p \prod_{j=1}^{k-1} \left[p^2 - 4q \cos^2 \left(\frac{\pi j}{2k} \right) \right].$$

Proof. We consider two cases. First, suppose that n is odd such that $n = 2k + 1$. By Theorem 2.1, we have that

$$\tilde{H}_{2k+1}(p^2, q) = p^{2k} \prod_{j=1}^{k+1} \left[p^2 - 4q \cos^2 \left(\frac{\pi j}{2k+3} \right) \right]. \quad (2.14)$$

By Theorem 2.10, we also know that

$$\tilde{H}_{2k+1}(p^2, q) = p^{2k} U_{2k+3}. \quad (2.15)$$

Considering Eqs (2.14) and (2.15), we obtain

$$U_{2k+3} = \prod_{j=1}^{k+1} \left[p^2 - 4q \cos^2 \left(\frac{\pi j}{2k+3} \right) \right].$$

Now, we consider the case n is even such that $n = 2k$. By Theorems 2.1 and 2.10, we write

$$\tilde{H}_{2k}(p^2, q) = p^{2k-1} U_{2k+2} = p^{2k} \prod_{j=1}^k \left[p^2 - 4q \cos^2 \left(\frac{\pi j}{2k+2} \right) \right].$$

Thus, we obtain

$$U_{2k+2} = p \prod_{j=1}^k \left[p^2 - 4q \cos^2 \left(\frac{\pi j}{2k+2} \right) \right].$$

The proof is then complete. $\square$

It should be noted that the factorizations of terms of the sequence $\{U_n\}$ given in Theorem 2.11 are already known via alternative methods, but here, we prove by using a Hessenberg matrix property for the first time.

Now, we provide the LU-factorization of the matrix $\mathbf{H}_n(a, b)$. We first assume that the LU-factorization exists if and only if all leading principal minors $\tilde{H}_k$ are nonzero for all k , which means that all eigenvalues of $\mathbf{H}_n$ must be nonzero; that is, $a \neq 4b \cos^2((k\pi)/(n+2))$ for $1 \leq k \leq n$. Now, we define two triangular matrices under these assumptions. First, we define the $n \times n$ lower triangular matrix $\mathbf{L} = (L_{i,j})$ with

$$L_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ (-1)^{i+j+1} ab \frac{\tilde{H}_{j-2}}{\tilde{H}_j}, & \text{if } i > j, \\ 0, & \text{otherwise,} \end{cases}$$

and second define the $n \times n$ upper triangular matrix $\mathbf{U} = (U_{i,j})$ with

$$U_{i,j} = \begin{cases} \frac{\tilde{H}_j}{\tilde{H}_{j-1}}, & \text{if } i = j, \\ b, & \text{if } j = i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

For example, for $n = 5$, the matrices $\mathbf{L}$ and $\mathbf{U}$ have the forms

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ ab\frac{\tilde{H}_{-1}}{\tilde{H}_1} & 1 & 0 & 0 & 0 \\ -ab\frac{\tilde{H}_{-1}}{\tilde{H}_1} & ab\frac{\tilde{H}_0}{\tilde{H}_2} & 1 & 0 & 0 \\ ab\frac{\tilde{H}_{-1}}{\tilde{H}_1} & -ab\frac{\tilde{H}_0}{\tilde{H}_2} & ab\frac{\tilde{H}_1}{\tilde{H}_3} & 1 & 0 \\ -ab\frac{\tilde{H}_{-1}}{\tilde{H}_1} & ab\frac{\tilde{H}_0}{\tilde{H}_2} & -ab\frac{\tilde{H}_1}{\tilde{H}_3} & ab\frac{\tilde{H}_2}{\tilde{H}_4} & 1 \end{bmatrix}$$

and

$$\mathbf{U} = \begin{bmatrix} \frac{\tilde{H}_1}{\tilde{H}_0} & b & 0 & 0 & 0 \\ 0 & \frac{\tilde{H}_2}{\tilde{H}_1} & b & 0 & 0 \\ 0 & 0 & \frac{\tilde{H}_3}{\tilde{H}_2} & b & 0 \\ 0 & 0 & 0 & \frac{\tilde{H}_4}{\tilde{H}_3} & b \\ 0 & 0 & 0 & 0 & \frac{\tilde{H}_5}{\tilde{H}_4} \end{bmatrix}.$$

Assuming that all leading principal minors of the matrix $\mathbf{H}_n$ are nonzero, the following result holds.

Theorem 2.12. For $n > 0$, the LU-factorization of matrix $\mathbf{H}_n$ is

$$\mathbf{H}_n = \mathbf{L} \cdot \mathbf{U},$$

where the matrices $\mathbf{L}$ and $\mathbf{U}$ are defined as before.

Proof. If $i = j$, then, using the recurrence relation, consider

$$\begin{aligned} (\mathbf{LU})_{i,i} &= \sum_{k=1}^n L_{i,k} U_{k,i} = L_{i,i-1} U_{i-1,i} + L_{i,i} U_{i,i} \\ &= ab \frac{\tilde{H}_{j-3}}{\tilde{H}_{j-1}} \cdot b + 1 \cdot \frac{\tilde{H}_j}{\tilde{H}_{j-1}} = \frac{ab^2 \tilde{H}_{j-3} + \tilde{H}_j}{\tilde{H}_{j-1}}, \end{aligned}$$

which by the recurrence (2.10) given in Theorem 2.4, equals

$$(\mathbf{LU})_{i,i} = \frac{(a-b)\tilde{H}_{j-1}}{\tilde{H}_{j-1}} = a-b,$$

as desired. If $j = i + 1$, then consider

$$(\mathbf{LU})_{i,i+1} = \sum_{k=1}^n L_{i,k} U_{k,i+1} = L_{i,i} U_{i,i+1} = 1 \cdot b = b,$$

as required. Finally, for the case $i > j$, consider

$$\begin{aligned}
 (\mathbf{LU})_{i,j} &= \sum_{k=1}^n L_{i,k} U_{k,j} = L_{i,j-1} U_{j,j-1} + L_{i,j} U_{j,j} \\
 &= (-1)^{i+j} ab \frac{\tilde{H}_{j-3}}{\tilde{H}_{j-1}} \cdot b + (-1)^{i+j+1} ab \frac{\tilde{H}_{j-2}}{\tilde{H}_j} \cdot \frac{\tilde{H}_j}{\tilde{H}_{j-1}} \\
 &= \left[-\tilde{H}_{j-3}b + \tilde{H}_{j-2} \right] (-1)^{i+j+1} \frac{ab}{\tilde{H}_{j-1}}, \\
 &= \left[a\tilde{H}_{j-2} - ab\tilde{H}_{j-3} \right] (-1)^{i+j+1} \frac{b}{\tilde{H}_{j-1}},
 \end{aligned}$$

which, by the recurrence relation of $\{\tilde{H}_n\}$, gives us

$$(\mathbf{LU})_{i,j} = \tilde{H}_{j-1} (-1)^{i+j+1} \frac{b}{\tilde{H}_{j-1}} = b (-1)^{i+j+1},$$

as desired. Thus, the proof is complete. $\square$

As a consequence, we have the following result, as the matrix is a unit lower triangular matrix:

$$\begin{aligned}
 \det \mathbf{H}_n &= \det (\mathbf{LU}) = \det \mathbf{L} \cdot \det \mathbf{U} = \det \mathbf{U} \\
 &= \prod_{k=1}^n U_{k,k} = \frac{\tilde{H}_1}{\tilde{H}_0} \frac{\tilde{H}_2}{\tilde{H}_1} \cdots \frac{\tilde{H}_n}{\tilde{H}_{n-1}} = \tilde{H}_n.
 \end{aligned}$$

Remark 2.13. (Numerical stability consideration) Although Theorem 2.12 theoretically requires $a \neq 4b \cos^2((k\pi)/(n+2))$ for all $1 \leq k \leq n$ to ensure nonzero leading principal minors, in practical floating-point implementations, if (a, b) is chosen such that a is extremely close to any of these critical values $4b \cos^2((k\pi)/(n+2))$, the leading principal submatrices become ill-conditioned ($\tilde{H}_k \approx 0$). In such cases, standard LU-factorization without pivoting may experience numerical instability due to division by near-zero pivot elements ($\tilde{H}_k/\tilde{H}_{k-1}$). Therefore, when utilizing $\mathbf{H}_n(a, b)$ in numerical testing environments, parameters (a, b) should be selected sufficiently far from these singular spectral boundaries to guarantee numerical robustness.

3. Second-class Hessenberg matrices

We define the second-class Hessenberg matrices and then evaluate their determinants explicitly. For $n > 1$, we define the lower Hessenberg matrix $\mathbf{W}_n(a, b) = [W_{ij}]$, briefly denoted by $\mathbf{W}_n$, with $W_{1,1} = a + b$, $W_{i,i} = a - b$ for $2 \leq i \leq n$, $W_{i,i+1} = b$ for $1 \leq i \leq n - 1$, and $W_{ij} = (-1)^{i+j-1} b$ for $i > j$. Clearly, if n is odd, then the matrix $\mathbf{W}_n$ takes the form

$$\mathbf{W}_n(a, b) = \begin{bmatrix} a+b & b & & & & 0 \\ b & a-b & b & & & \\ -b & b & a-b & b & & \\ b & -b & \ddots & a-b & \ddots & \\ \vdots & \ddots & \ddots & b & \ddots & b \\ -b & \dots & b & -b & b & a-b \end{bmatrix}.$$

For convenience, we denote $\det \mathbf{W}_n(a, b)$ by $\tilde{W}_n(a, b)$, or shortly, $\tilde{W}_n$.

The modification of the top-left entry $W_{1,1} = a + b$ (in contrast to $h_{1,1} = a - b$ in H_n) is primarily motivated by generating companion sequences of Lucas type. Although the matrix H_n naturally yields terms of generalized Fibonacci sequences U_n , this boundary adjustment modifies the initial conditions of the determinantal recurrence while preserving its order, thereby capturing Lucas-type sequences such as the Pell–Lucas numbers (see Table 3).

Table 3. Special cases of the determinantal sequence $\{\tilde{W}_n(a, b)\}$.

a	b	$\tilde{W}_n(a, b)$	Sequence type
1	−1	$-F_{n-1}$	Fibonacci sequence
−1	1	$(-1)^{n+1} F_{n-1}$	Alternating Fibonacci sequence
2	−1/2	$Q_n/2$	Pell–Lucas sequence

We derive a recurrence relation and a combinatorial sum for the determinant $\tilde{W}_n$ by the following result.

Theorem 3.1. For $n > 2$,

i) $\{\tilde{W}_n\}_{n>2}$ satisfies the following second–order linear recurrence:

$$\tilde{W}_n = a\tilde{W}_{n-1} - ab\tilde{W}_{n-2}$$

with $\tilde{W}_1 = a + b$, $\tilde{W}_2 = a^2 - 2b^2$.

ii)

$$\tilde{W}_n = \sum_{m=0}^{n+1} (-1)^m \frac{n+2-4m}{n+2-m} \binom{n+2-m}{m} a^{n-m} b^m. \quad (3.1)$$

Proof. i) By the definition of matrix W_n , if we subtract the $(n-1)$ th row from the n th row, then we obtain

$$\tilde{W}_n = \left| \begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & & b \\ \hline 0 & \cdots & 0 & a & a \end{array} \right|,$$

which, by the definition of W_n and expanding according to the last column first and the last row later, gives us

$$\tilde{W}_n = a\tilde{W}_{n-1} - ab\tilde{W}_{n-2}$$

as claimed.

ii) Denote the sum given in (3.1) by SUM_n , that is,

$$\text{SUM}_n := \sum_{m=0}^{n+1} (-1)^m \frac{n+2-4m}{n+2-m} \binom{n+2-m}{m} a^{n-m} b^m.$$

It will suffice to show that $\{\text{SUM}_n\}$ has the same recurrence relation and the same two initial conditions as $\{\tilde{W}_n\}$. Applying Zeilberger's algorithm via the **Mathematica** package to the sum SUM_n , the input command and its corresponding output recurrence relation are obtained as follows:

$$\text{Zb}\left[\frac{(-1)^m a^{n-m} b^m (n+2-4m) \text{Binomial}[n+2-m, m]}{n+2-m}, \{m, 0, n+1\}, n\right]$$

and

$$ab \text{SUM}[n] - a \text{SUM}[n+1] + \text{SUM}[n+2] = 0.$$

Thus, the Zeilberger algorithm gives the recursion

$$\text{SUM}_n = a \text{SUM}_{n-1} - ab \text{SUM}_{n-2}.$$

For further details on the execution and theoretical background of Zeilberger's algorithm, we refer to [11]. Because the first two initials of $\{\text{SUM}_n\}$ and $\{\tilde{W}_n\}$ are the same, we have the conclusion. $\square$

Now, we shall explicitly formulate the inverse matrix $\mathbf{W}_n^{-1}$ by the following result without proof to keep the manuscript in readable form. One can follow the proof of Theorem 2.8 and could prove it in a similar fashion. For later use, we also define $\tilde{W}_0 = (a+2b)/a$ and $\tilde{W}_{-1} = 1/a$.

Theorem 3.2. For $n > 0$, the inverse matrix $\mathbf{W}_n^{-1} = [W_{i,j}^{-1}]$ is given by

$$W_{i,j}^{-1} = \frac{(-1)^{i-j}}{\tilde{W}_n} \begin{cases} b^{j-i} \tilde{H}_{n-j}, & \text{if } i = 1, \\ b^{j-i} \tilde{W}_{i-1} \tilde{H}_{n-j}, & \text{if } j \geq i \geq 2, \\ a^{i-j+1} b \tilde{W}_{j-2} \tilde{H}_{n-i-1}, & \text{if } i > j, \end{cases}$$

where $\tilde{W}_n$ and $\tilde{H}_n$ stand for the determinants of the matrices $\mathbf{W}_n$ and $\mathbf{H}_n$, respectively.

For $n = 5$, we have that

$$W_{i,j}^{-1} = \frac{(-1)^{i-j}}{\tilde{W}_5} \begin{bmatrix} \tilde{H}_4 & b\tilde{H}_3 & b^2\tilde{H}_2 & b^3\tilde{H}_1 & b^4\tilde{H}_0 \\ a^2b\tilde{W}_{-1}\tilde{H}_2 & \tilde{W}_1\tilde{H}_3 & b\tilde{W}_1\tilde{H}_2 & b^2\tilde{W}_1\tilde{H}_1 & b^3\tilde{W}_1\tilde{H}_0 \\ a^3b\tilde{W}_{-1}\tilde{H}_1 & a^2b\tilde{W}_0\tilde{H}_1 & \tilde{W}_2\tilde{H}_2 & b\tilde{W}_2\tilde{H}_1 & b^2\tilde{W}_2\tilde{H}_0 \\ a^4b\tilde{W}_{-1}\tilde{H}_0 & a^3b\tilde{W}_0\tilde{H}_0 & a^2b\tilde{W}_1\tilde{H}_0 & \tilde{W}_3\tilde{H}_1 & b\tilde{W}_3\tilde{H}_0 \\ a^5b\tilde{W}_{-1}\tilde{H}_{-1} & a^4b\tilde{W}_0\tilde{H}_{-1} & a^3b\tilde{W}_1\tilde{H}_{-1} & a^2b\tilde{W}_2\tilde{H}_{-1} & \tilde{W}_4\tilde{H}_0 \end{bmatrix}.$$

Now we give the LU-factorization of the matrix $\mathbf{W}_n$. Define $n \times n$ unit lower triangular matrix $\mathcal{L} = (\mathcal{L}_{i,j})$ with

$$\mathcal{L}_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ (-1)^{i+j+1} ab \frac{\tilde{W}_{j-2}}{\tilde{W}_j}, & \text{if } i > j, \\ 0, & \text{otherwise,} \end{cases}$$

and define $n \times n$ upper triangular matrix $\mathcal{U} = (U_{i,j})$ with

$$\mathcal{U}_{i,j} = \begin{cases} \tilde{W}_1, & \text{if } i = j = 1, \\ \frac{\tilde{W}_j}{\tilde{W}_{j-1}}, & \text{if } i = j \geq 2, \\ b, & \text{if } j = i + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Assuming that all leading principal minors of the matrix $\mathbf{W}_n$ are nonzero, the following result holds. Its proof could be similarly followed by proof of Theorem 2.12.

Theorem 3.3. For $n > 0$,

$$\mathbf{W}_n = \mathcal{L} \cdot \mathcal{U},$$

where the matrices $\mathcal{L}$ and $\mathcal{U}$ are defined as before.

For example, for $n = 5$, the matrices $\mathcal{L}$ and $\mathcal{U}$ have the forms

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ ab \frac{\tilde{W}_{-1}}{\tilde{W}_1} & 1 & 0 & 0 & 0 \\ -ab \frac{\tilde{W}_{-1}}{\tilde{W}_1} & ab \frac{\tilde{W}_0}{\tilde{W}_2} & 1 & 0 & 0 \\ ab \frac{\tilde{W}_{-1}}{\tilde{W}_1} & -ab \frac{\tilde{W}_0}{\tilde{W}_2} & ab \frac{\tilde{W}_1}{\tilde{W}_3} & 1 & 0 \\ -ab \frac{\tilde{W}_{-1}}{\tilde{W}_1} & ab \frac{\tilde{W}_0}{\tilde{W}_2} & -ab \frac{\tilde{W}_1}{\tilde{W}_3} & ab \frac{\tilde{W}_2}{\tilde{W}_4} & 1 \end{bmatrix}$$

and

$$\mathcal{U} = \begin{bmatrix} \tilde{W}_1 & b & 0 & 0 & 0 \\ 0 & \frac{\tilde{W}_2}{\tilde{W}_1} & b & 0 & 0 \\ 0 & 0 & \frac{\tilde{W}_3}{\tilde{W}_2} & b & 0 \\ 0 & 0 & 0 & \frac{\tilde{W}_4}{\tilde{W}_3} & b \\ 0 & 0 & 0 & 0 & \frac{\tilde{W}_5}{\tilde{W}_4} \end{bmatrix}.$$

As a consequence, we have the following result because $\mathcal{L}$ is a unit lower triangular matrix:

$$\begin{aligned} \det \mathbf{W}_n &= \det (\mathcal{L}\mathcal{U}) = \det \mathcal{L} \cdot \det \mathcal{U} = \det \mathcal{U} \\ &= \prod_{k=1}^n \mathcal{U}_{k,k} = \tilde{W}_1 \frac{\tilde{W}_2}{\tilde{W}_1} \cdots \frac{\tilde{W}_n}{\tilde{W}_{n-1}} \\ &= \tilde{W}_n \end{aligned}$$

as wanted.

4. Two open problems

We leave two open problems. The first open problem concerns the eigenvectors of the matrix $\mathbf{H}_n(a, b)$.

Before the second open problem, we want to add a brief discussion about the eigenvalues of the matrix $\mathbf{W}_n$. By following the proof given for the matrix $\mathbf{H}_n$, if we multiply $\mathbf{W}_n - \lambda \mathbf{I}_n$ by the matrix $\mathbf{A}_n$ and denote its determinant by $K_n(\lambda)$, or briefly K_n , then we get

$$\begin{aligned} K_n &:= \det (\mathbf{A}_n \cdot (\mathbf{W}_n - \lambda \mathbf{I}_n)) \\ &= \begin{vmatrix} -a(a+b-\lambda) & -ab & & & \\ -a(a+b-\lambda)-ab & a(\lambda-a) & -ab & & \\ & a(\lambda-a) & a(\lambda-a) & \ddots & \\ & & \ddots & \ddots & -ab \\ & & & a(\lambda-a) & a(\lambda-a) \end{vmatrix}. \end{aligned}$$

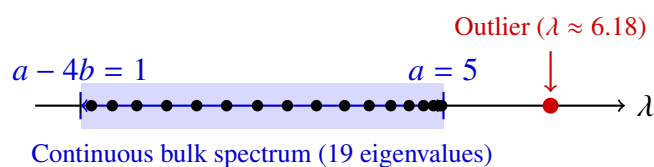


Figure 1. Numerical eigenvalue spectrum distribution of $W_{20}(5, 1)$, illustrating the continuous bulk interval $[1, 5]$ and the isolated outlier eigenvalue resulting from the boundary perturbation $W_{1,1} = a + b$.

Then, we see that $\{K_n\}_{n \geq 2}$ satisfies the three-term recurrence:

$$K_n - a(\lambda - a)K_{n-1} + aba(\lambda - a)K_{n-2} = 0 \quad (4.1)$$

with initials

$$K_1 = -a(a + b - \lambda) \text{ and } K_2 = a^2(-2a\lambda + \lambda^2 + a^2 - 2b^2).$$

We note that whereas the recurrence (4.1) for K_n is a linear second-order recurrence with constant coefficients, obtaining an explicit closed-form expression for the roots of $K_n(\lambda) = 0$ remains challenging. Unlike the first class H_n , the boundary entries in W_n yield perturbed initial conditions (K_1 and K_2). This perturbation breaks the trigonometric symmetry, preventing the characteristic polynomial from reducing to classical orthogonal polynomial structures (such as Chebyshev polynomials) or standard Toeplitz–Hessenberg determinant evaluations. Consequently, deriving explicit formulas for the eigenvalues of $\mathbf{W}_n$ remains an interesting open problem.

Remark 4.1. (Empirical and asymptotic behavior of $\mathbf{W}_n(a, b)$ eigenvalues) Although an exact closed-form expression for the roots of $K_n(\lambda) = 0$ remains an open problem due to the perturbed boundary terms K_1 and K_2 , numerical experiments and asymptotic analysis reveal key properties of the spectrum:

- (1) **Bulk distribution:** As $n \rightarrow \infty$, the bulk of the eigenvalues of $\mathbf{W}_n(a, b)$ share the same continuous support $[a - 4b, a]$ (for $a \geq 4b > 0$) as $\mathbf{H}_n(a, b)$, distributed according to the limiting spectral density characteristic of second-order linear recurrences.
- (2) **Outlier eigenvalues:** The boundary perturbation $W_{1,1} = a + b$ introduces at most two isolated outlier eigenvalues outside the main spectral interval $[a - 4b, a]$. Numerical simulations show that as n increases, these outliers rapidly stabilize to fixed limit values outside the continuous band, and the remaining $n - 2$ eigenvalues fill the interval $[a - 4b, a]$ densely.

This spectral phenomenon is illustrated in Figure 1 for $\mathbf{W}_{20}(5, 1)$, where 19 eigenvalues densely occupy the bulk band $[1, 5]$, and a single outlier eigenvalue emerges at $\lambda \approx 6.18$ due to $W_{1,1} = 6$.

5. Conclusions

In this paper, we investigated two classes of two-parameter lower Hessenberg matrices, explicitly deriving their determinants, inverses, and LU-factorizations, as well as the full spectrum for the first class. The main limitation lies in the perturbed boundary condition of the second class, which prevents a straightforward closed-form eigenvalue derivation. Future research may focus on addressing the two open problems regarding the remaining eigenvalues and eigenvectors, as well as exploring these structures as numerical test matrices.

Author contributions

Abdullah Alazemi: formal analysis, writing-original draft, editing; Emrah Kılıç: methodology, formal analysis, writing-review, editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare no conflicts of interest.

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