



Research article

Stationarity, moment properties, and particle-EM estimation for periodic logGARCH-SV models

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Abstract: This paper introduces a periodic logarithmic generalized autoregressive conditional heteroskedasticity (logGARCH) stochastic volatility model designed to capture volatility dynamics that evolve across different phases of a recurrent cycle. By allowing all model coefficients to vary periodically, the proposed framework extends the standard logGARCH stochastic volatility specification and provides a flexible mechanism for modeling seasonal variations and time-dependent persistence patterns in latent volatility processes. The probabilistic properties of the model are investigated through a periodic stochastic recurrence representation. Sufficient conditions are established for the existence of a unique strictly periodically stationary and causal solution. In addition, higher-order moment properties are derived, and explicit expressions for periodic moments are obtained under suitable integrability conditions. For statistical inference, the model is reformulated within a periodic state-space framework in which the latent log-volatility process is treated as an unobserved state variable. To estimate both the periodically varying parameters and the latent volatility states, a particle-assisted expectation-maximization algorithm combining sequential Monte Carlo filtering and smoothing techniques is developed. The finite-sample performance of the proposed methodology is evaluated through extensive Monte Carlo experiments under both Gaussian and Student- t_{15} innovations.

Keywords: stochastic volatility model; periodic logGARCH model; stationarity; EM algorithm; higher order moments; particle filtering

Mathematics Subject Classification: 62M10, 62M05

1. Introduction

Over the past three decades, stochastic volatility models have received considerable attention in financial econometrics. This development has been driven largely by the need to describe the

time-varying nature of volatility in financial returns. Financial data commonly exhibit several well-known stylized features, including volatility clustering, asymmetric effects, long-memory dependence, and structural or cyclical variation. Traditional linear models do not always capture these features adequately. Taylor's contribution to stochastic volatility (SV) modeling was an important early step in the development of this field [1]. In 1982, he introduced a formulation in which financial returns are represented as the product of an innovation term and a latent, time-varying volatility component. This formulation provided a natural way to model volatility as an unobserved stochastic process and served as a basis for many subsequent developments in financial time-series modeling.

In contrast, generalized autoregressive conditional heteroskedasticity (GARCH) models are among the most widely used approaches for modeling time-varying volatility. Their popularity is largely due to their ability to capture volatility clustering in financial time series. Despite their empirical success, GARCH-type models describe volatility through a conditional variance determined by observed and past information. They therefore do not explicitly represent volatility as a latent stochastic process. SV models provide a different approach by treating volatility as an unobserved process that evolves over time. This formulation offers greater flexibility in describing nonlinear dynamics and other stylized features of financial markets, as discussed by Carnero et al. [2]. The SV framework has subsequently been extended to accommodate several important characteristics of financial data. These include leverage and asymmetric effects, as considered by Black [3] and Jacquier et al. [4] as well as long-memory dependence and continuous-time formulations studied by Ding et al. [5]. Multivariate SV models have also been developed to describe dependence and volatility transmission across financial markets (see, e.g., Harvey et al. [6]).

On the other hand, empirical studies have shown that many financial time series are influenced not only by stochastic fluctuations in volatility, but also by systematic seasonal and periodic patterns associated with market microstructure and trading-time effects. This observation motivated the development of periodic volatility models, most notably the periodic GARCH framework introduced by Bollerslev and Ghysels in 1996 [7]. These models incorporate periodically varying coefficients, allowing the volatility dynamics to evolve across different phases of the underlying cycle. As a result, they provide a more realistic representation of seasonal behavior in financial volatility, particularly in markets affected by recurring temporal effects. The development of periodic stochastic models has been accompanied by significant advances in nonlinear periodic time-series analysis and statistical inference. In particular, Ghezal investigated several classes of periodically varying nonlinear processes and developed estimation methodologies for models combining periodic and stochastic features. Among these contributions are quasi-maximum likelihood estimation procedures for periodic bilinear models and periodic autoregressive moving-average (PARMA) processes with periodic bilinear innovations, as well as periodic time-varying bilinear GARCH specifications. For these models, both asymptotic and finite-sample properties of the estimators were thoroughly analyzed [8, 9]. In a related line of research, minimum distance estimation methods were proposed for Markov-switching bilinear processes, providing additional inferential tools for nonlinear stochastic models characterized by regime-dependent dynamics [10]. Collectively, these studies have enriched the theoretical and statistical foundations of periodically varying nonlinear stochastic processes and have paved the way for the development of more sophisticated volatility models incorporating periodic structures. Subsequently, more sophisticated extensions were proposed by combining periodic structures with regime-switching mechanisms to capture transitions between distinct volatility states [11–13]. Further

developments of regime-switching specifications have also been considered under varying market conditions [14–16], while threshold mechanisms provide an alternative way to account for nonlinear changes in volatility dynamics [17–19]. Two closely related recent contributions by Ghezal and Alzeley [20, 21] further illustrated the development of stochastic-volatility models toward richer forms of dependence and temporal heterogeneity. In [20], a periodic threshold autoregressive stochastic volatility model (PTAR-SV) was introduced to combine periodic variation with threshold-dependent dynamics. The authors established strict periodic stationarity and higher-order moment conditions and considered parameter estimation using quasi-maximum likelihood based on prediction-error decomposition and Kalman filtering together with a Bayesian Griddy-Gibbs approach. An empirical application to euro-Algerian dinar spot rates further demonstrated the usefulness of the model in a financial setting. In [21], an asymmetric multivariate stochastic difference volatility model was proposed to account for cross-variable dependence and asymmetric volatility effects, with emphasis on its structural properties and estimation. These studies show how stochastic-volatility models can be extended to accommodate periodicity, threshold effects, asymmetry, and cross-variable dependence. However, the two frameworks do not incorporate the logarithmic feedback from past squared observations that is central to the logGARCH-SV setting considered here. The PTAR-SV model in [20] introduced periodicity through a threshold mechanism, whereas [21] focused on asymmetric dependence in a multivariate setting. The present study follows a different line by combining latent stochastic volatility, logarithmic feedback, and periodic parameter variation within a single framework. This formulation also requires a corresponding analysis of the causal representation and moment properties, together with a particle-assisted EM procedure for parameter estimation.

In parallel with these developments, logGARCH models have attracted growing attention in the recent econometric literature because of their ability to overcome several structural limitations associated with conventional GARCH specifications. In particular, the logarithmic formulation naturally guarantees the positivity of conditional variance while simultaneously simplifying the analytical structure of the model. Sucarrat et al. [22, 23] provided a useful theoretical and inferential basis for logGARCH models, showing that their logarithmic specification can be expressed in an autoregressive moving average (ARMA)-type form and studied within a quasi-maximum likelihood estimate (QMLE) framework. Their results clarified the structure of logGARCH models and made their statistical analysis more tractable. However, the models considered in these studies do not incorporate latent stochastic volatility or allow the model parameters to vary periodically over time. As a result, they do not directly address settings in which volatility is both driven by an unobserved stochastic component and subject to systematic changes across the phases of a recurring cycle. More recently, Ghezal [24] extended the logGARCH framework by allowing the asymmetric coefficients to vary periodically, showing that periodic effects can be incorporated into logarithmic volatility dynamics. This work provides a direct link between periodic volatility behavior and the logGARCH framework and is therefore closely related to the setting considered in this paper. However, the model remains within the conditional-variance framework, where volatility is determined by past observations and does not introduce a latent stochastic-volatility component. Thus, although periodicity is explicitly incorporated, the model does not address the joint presence of periodic dynamics and unobserved stochastic volatility that is central to our approach.

Despite these substantial advances, an important gap still exists in the literature concerning the incorporation of periodic structures into modern logarithmic stochastic volatility frameworks,

particularly those based on the interaction between the observed process and latent volatility through the squared logarithm of past observations. A closely related contribution is the logGARCH stochastic volatility model proposed by Guerbyenne et al. [25] in 2024. Their model extends the classical autoregressive-stochastic volatility (AR-SV) framework by introducing a logarithmic feedback term involving past squared observations, thereby combining the latent-volatility structure of SV models with a logGARCH-type specification. This formulation provides a useful framework for describing the interaction between observed returns and unobserved volatility and represents an important development in the logGARCH-SV literature. However, the model assumes time-invariant coefficients and therefore does not allow the persistence or scale of volatility to change systematically across different phases of a recurring cycle. In this sense, the stochastic and logarithmic components are present, but the periodic dimension of volatility dynamics remains unexplored. Extending this framework to allow periodic parameter variation raises several additional questions, including the existence and characterization of periodic stationary solutions, the conditions for moment existence, and the development of a suitable estimation procedure for the resulting latent model. These issues provide the main motivation for the framework considered in this paper.

Although the literature has made important progress in modeling stochastic volatility, periodic heterogeneity, and logarithmic volatility dynamics, these features have generally been studied within different modeling frameworks. Periodic GARCH-type models allow the persistence and scale of volatility to vary across the phases of a cycle, but volatility remains represented through a conditional variance rather than as a latent stochastic process. SV models, in contrast, provide a direct representation of latent volatility and its random evolution, but most existing specifications use time-invariant parameters and therefore do not allow the volatility dynamics to change systematically from one phase of a cycle to another. LogGARCH models provide another useful perspective by describing volatility on the logarithmic scale and incorporating past observations through logarithmic feedback. However, the existing logGARCH-SV formulation is time-invariant and therefore does not account for systematic changes in volatility dynamics across recurring phases of a cycle. Similarly, periodic extensions of logGARCH models have mainly been developed within the conditional heteroskedasticity framework and do not explicitly introduce a latent stochastic-volatility component.

The estimation methodologies developed in these related studies likewise reflect the structure of each model, including QMLE, minimum-distance estimation, and likelihood-based approaches. Thus, an important gap remains in the literature: There is no unified framework that simultaneously allows for periodic parameter variation, latent stochastic volatility, and logarithmic feedback from past observations while also providing a probabilistic characterization and a practical estimation procedure for the resulting nonlinear latent model. This issue becomes particularly important when the persistence of volatility changes systematically across the phases of a recurring cycle. The main challenge is therefore to incorporate periodicity into a logGARCH-SV structure without losing a tractable stochastic representation. Such a model should allow the periodic dynamics to be characterized rigorously, provide conditions for the existence of a well-defined periodic stationary solution and its moments, and remain suitable for statistical inference. The model proposed in this paper addresses these requirements by integrating periodic coefficients, latent stochastic volatility, and logarithmic feedback into a common framework. Its periodic state-space representation is then used to develop a particle-assisted expectation-maximization (EM) procedure for parameter estimation.

Motivated by these considerations, the present paper introduces a periodic extension of the

logGARCH stochastic volatility framework, in which the model parameters evolve according to a regular temporal cycle. This extension is designed to capture systematic seasonal and cyclical variations in financial volatility while retaining the stochastic and logarithmic structure of the underlying model. The proposed framework combines the latent-volatility structure of SV models, the logarithmic feedback mechanism of logGARCH models, and the phase-dependent dynamics of periodic volatility models within a unified setting. This combination is particularly relevant for financial time series, in which volatility exhibits both stochastic persistence and systematic cyclical variation.

The remainder of this paper is organized as follows. Section 2 presents the complete mathematical formulation of the proposed model. Section 3 establishes several probabilistic properties of the model, with particular emphasis on strict and second order periodic stationarity as well as the existence of higher-order moments. In Section 4, the model parameters are estimated using the sequential Monte Carlo (SMC) method. Finally, Section 5 provides a numerical simulation study to assess the performance of the proposed estimation procedure and to examine its statistical efficiency under various scenarios.

2. A periodic logGARCH stochastic volatility model

In recent developments in stochastic volatility modeling within financial econometrics, increasing attention has been devoted to the construction of models capable of simultaneously capturing nonlinear volatility dynamics, intrinsic stochastic behavior, and the cyclical structures frequently observed in financial time series. In many empirical applications, particularly for high-frequency financial data, volatility exhibits not only persistent random fluctuations but also systematic seasonal patterns generated by trading seasonality, calendar effects, and recurrent market microstructure mechanisms. Although classical logGARCH-SV models provide a flexible framework for describing the interaction between returns and volatility through logarithmic volatility specifications, their time-invariant structure limits their ability to account for periodically evolving volatility dynamics.

The standard logGARCH stochastic volatility specification is defined by

$$\begin{cases} U_t = \rho_t \exp(V_t/2), \\ V_t = \alpha_1 + \alpha_2 \log U_{t-1}^2 + \alpha_3 V_{t-1} + \alpha_4 \delta_t, \end{cases} \quad (1)$$

where the latent process (V_t) follows AR-type dynamics driven by past observations and past volatility, incorporating a logarithmic transformation of squared returns, and (ρ_t) and (δ_t) are two mutually independent sequences of independent and identically distributed random variables with zero mean and unit variance. This structure introduces a stochastic innovation component into the volatility equation, thereby extending the classical GARCH framework, which relies on deterministic conditional variance dynamics. The persistence of volatility is captured through the memory structure embedded in the parameters governing the propagation of shocks over time.

However, empirical evidence suggests that financial volatility is not only characterized by stochastic persistence but also by systematic cyclical patterns associated with calendar effects such as days, weeks, or trading quarters. This motivates a generalization of the logGARCH-SV framework into a periodic setting, allowing parameters to vary across positions within a cycle.

To address this issue, we propose the following periodic logGARCH stochastic volatility model:

$$\begin{cases} U_{st+\tau} = \rho_{st+\tau} \exp(V_{st+\tau}/2), \\ V_{st+\tau} = \alpha_1(\tau) + \alpha_2(\tau) \log U_{st+\tau-1}^2 + \alpha_3(\tau) V_{st+\tau-1} + \alpha_4(\tau) \delta_{st+\tau}, \end{cases} \quad (2)$$

where $\tau \in \mathbb{S} = \{1, \dots, s\}$ and the coefficient functions $\alpha_i(\tau)$, $i = 1, 2, 3, 4$ are periodic in s , capturing the dependence of the volatility dynamics on the position within the cycle. Consequently, the latent process $(V_{st+\tau})$ represents the log-volatility of $(U_{st+\tau})$, allowing the model to adapt flexibly to periodic variations in financial volatility.

By reformulating the model in terms of the logarithm of the latent volatility process, a more compact autoregressive representation can be obtained:

$$V_{st+\tau} = \alpha_0(\tau) + (\alpha_2(\tau) + \alpha_3(\tau)) V_{st+\tau-1} + W_{st+\tau}, \quad \tau \in \mathbb{S}, \quad (3)$$

where

$$\alpha_0(\tau) = \alpha_1(\tau) + \alpha_2(\tau) E\{\log \rho_{\tau-1}^2\},$$

and $W_{st+\tau}$ collects the centered innovation component inherited from the return innovation and the stochastic-volatility innovation. More precisely,

$$W_{st+\tau} = \alpha_2(\tau)(\log \rho_{st+\tau-1}^2 - E\{\log \rho_{st+\tau-1}^2\}) + \alpha_4(\tau) \delta_{st+\tau}, \quad \tau \in \mathbb{S}.$$

Assumption on the innovation sequences

We assume that the innovation sequences (ρ_t) and (δ_t) are mutually independent and independent over time. Consequently, the collection of innovation variables entering $W_{st+\tau}$ at distinct time indices is independent across time. Because $W_{st+\tau}$ is a measurable function of the corresponding innovation variables, the sequence $(W_{st+\tau})$ is independent across distinct time indices. Moreover, because the coefficient functions are periodic, the resulting sequence is periodically distributed.

Under the above assumption, the sequence $(W_{st+\tau})$ forms a collection of independent periodically distributed random variables with zero mean and periodic variance given by

$$\sigma_W^2(\tau) = \alpha_2^2(\tau) \text{Var}(\log \rho_{st+\tau-1}^2) + \alpha_4^2(\tau), \quad \tau \in \mathbb{S}.$$

This construction shows that the proposed model retains the basic structure of logGARCH-SV, with the crucial addition of introducing a periodic dimension in all coefficients, allowing for the capture of seasonal variations in both the level and continuity of volatility. The proposed model represents a natural extension of the classical model obtained by replacing constant coefficients with periodically varying coefficients depending on τ . As a result, the conventional notion of time-invariant stability is replaced by a periodic stability structure that evolves across different phases of the cycle. This generalization makes it possible to reformulate the latent volatility equation as a periodic autoregressive process of order one (see Eq (3)), thereby providing a more flexible framework for describing cyclical persistence and time-varying volatility dynamics.

The formulation developed in this section provides the starting point for the theoretical and inferential analysis that follows. The periodic logGARCH-SV specification is first reduced to a periodic autoregressive representation, which makes the stability and causal structure of the latent

process analytically tractable. This representation is then used in Section 3 to establish periodic stationarity and moment properties, and Section 4 exploits the corresponding state-space formulation for particle-assisted maximum-likelihood estimation.

Figure 1 summarizes the main analytical and estimation steps developed in this paper. The framework begins with the proposed periodic logGARCH-SV model and its stochastic recurrence representation, followed by the derivation of a causal periodically stationary solution. The model is then expressed in periodic state-space form, with the observed process and the latent log-volatility process serving as the measurement and state components, respectively. Particle filtering and smoothing are used within the particle-assisted EM framework to approximate the conditional quantities needed for parameter estimation. The final step is the Monte Carlo simulation study, which evaluates the numerical performance of the proposed estimation procedure. Thus, Figure 1 provides a roadmap of the paper, linking the model formulation, theoretical results, estimation method, and numerical assessment before these elements are examined in detail in the following sections.

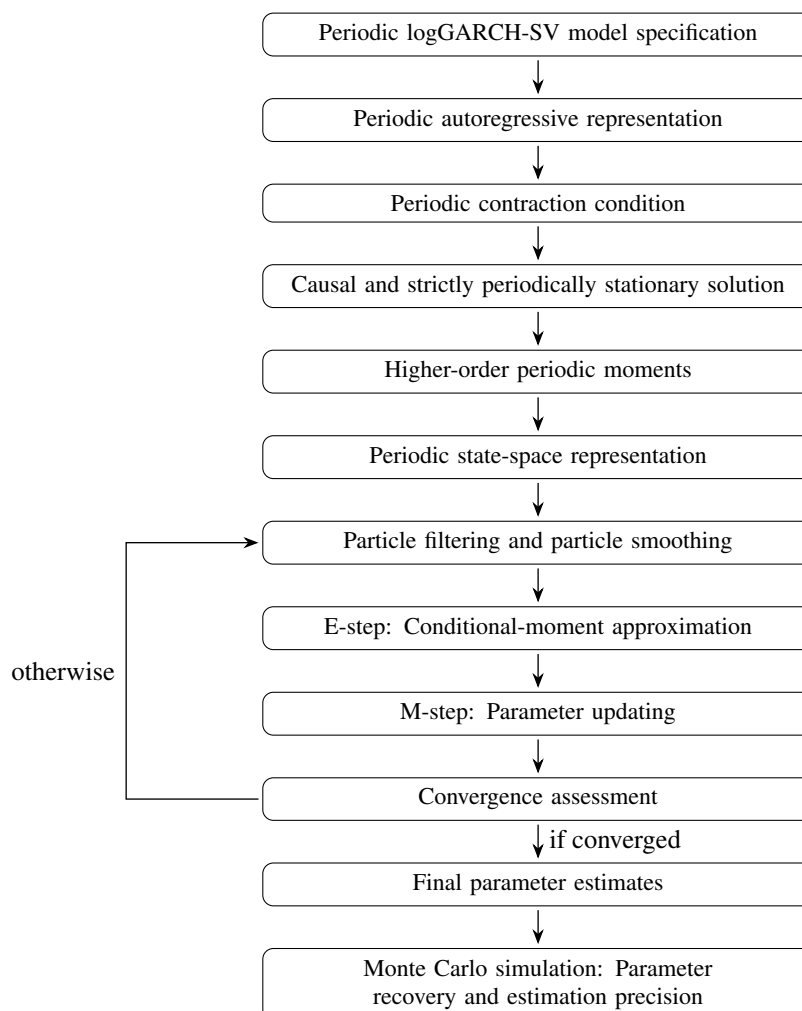


Figure 1. Overall analytical and estimation framework of the proposed periodic logGARCH-SV model.

3. Probabilistic structure and periodic stationarity of the logGARCH-SV model

Incorporating periodic feedback effects into the logGARCH stochastic volatility framework leads to a considerably richer probabilistic structure compared with standard periodic stochastic volatility specifications. More precisely, the interaction between the innovation sequence and the latent log-volatility process induces a nonlinear dependence structure in which current volatility behavior depends recursively on past observations through logarithmic feedback mechanisms. As a consequence, the proposed formulation exhibits enhanced flexibility in capturing persistent and periodically varying volatility dynamics frequently encountered in financial time series.

Despite this additional structural complexity, the proposed periodic logGARCH-SV model preserves an analytically tractable representation. More specifically, the logarithmic formulation of the volatility dynamics enables the original nonlinear system to be reformulated as a periodic first-order autoregressive process driven by random innovations. This transformation plays a central role in the theoretical analysis of the model, as the existence of a causal strictly periodically stationary solution to the original process becomes intrinsically linked to the existence of a periodically stationary solution for the associated latent autoregressive equation.

The probabilistic analysis in this section establishes the theoretical foundation required for the subsequent estimation procedure. Starting from the periodic autoregressive representation obtained in Section 2, we derive sufficient conditions for the existence of a strictly periodically stationary and causal solution and characterize its higher-order moments. These results ensure that the latent volatility process is well-defined under the proposed periodic dynamics and provide the probabilistic conditions needed for the state-space formulation and statistical inference developed in Section 4.

An important advantage of this representation is that it allows the probabilistic behavior of the model to be analyzed within the framework of periodic stochastic recurrence equations. In particular, the periodically varying coefficients governing the propagation of shocks determine whether the influence of past innovations decays sufficiently fast over time. Consequently, the stationarity properties of the process are no longer described by a single persistence parameter, as in classical stationary models, but rather by a global periodic persistence mechanism evolving across the different phases of the cycle.

The following theorem establishes sufficient conditions for the existence and uniqueness of a strictly periodically stationary solution that is ergodic in the periodic sense for the proposed periodic logGARCH-SV model. In addition, it provides an explicit nonanticipative representation of the solution and derives further conditions which ensure periodic second-order stationarity. These results form the theoretical foundation of the proposed framework and are crucial for guaranteeing the statistical validity of subsequent inference and estimation procedures.

Theorem 3.1. *Consider the periodic logGARCH-SV process defined in Eq (2). Under a periodic contraction condition induced by the model coefficients, the system admits a well-defined stochastic recursion whose solution can be represented as a strictly periodically stationary and causal (non-anticipative) process of the form*

$$U_{st+\tau} = \rho_{st+\tau} \exp\left(\frac{1}{2} \sum_{k=0}^{\infty} \left(\prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau-l) \right\} \right) (\alpha_0(\tau-k) + W_{st+\tau-k}) \right). \quad (4)$$

The convergence of this representation holds almost surely provided that the following periodic

Lyapunov-type condition is satisfied:

$$\prod_{\tau=1}^s \left| \sum_{i=2}^3 \alpha_i(\tau) \right| < 1. \quad (5)$$

Moreover, under the additional exponential integrability condition,

$$\prod_{k=0}^{\infty} E \left\{ \exp \left(\left(\prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau - l) \right\} \right) (\alpha_0(\tau - k) + W_{st+\tau-k}) \right) \right\} < \infty, \quad \text{for } \tau \in \mathbb{S}, \quad (6)$$

which ensures controlled accumulation of periodic innovations, the solution further admits periodic second-order stationarity.

Proof. The proof relies on the recursive structure of the latent log-volatility equation associated with the periodic logGARCH-SV process. From the periodic autoregressive representation given in Eq (3), the latent dynamics can be formulated as a periodic stochastic recurrence equation driven by the innovation sequence $(W_{st+\tau})$. By iterating the recursion, one obtains a stochastic series representation involving successive products of the periodically varying coefficients:

$$\left(\sum_{i=2}^3 \alpha_i(\tau) \right).$$

The convergence of this series is therefore determined by the asymptotic behavior of these products along the periodic cycle. Under Condition (5), the corresponding periodic contraction property ensures that the influence of past innovations decays exponentially as the lag tends to infinity. Consequently, the infinite stochastic expansion appearing in Eq (4) converges almost surely, thereby establishing the existence of a causal solution. Moreover, because the innovation sequences are periodically distributed and independent over time, the resulting process inherits the periodic distributional structure of the driving innovations. It follows that the obtained solution is strictly periodically stationary and ergodic in the periodic sense. Finally, Condition (6) provides the exponential integrability required to control the second-order behavior of the stochastic exponential representation. In particular, it guarantees the finiteness of the second moments, thereby establishing periodic second-order stationarity. $\square$

Remark 3.1. The exponential integrability condition in Theorem 3.1 can be verified directly from the distributions of the innovation sequences. A convenient sufficient condition is that for every $\tau \in \mathbb{S}$, the random variables $\log \rho_{st+\tau}^2$ and $\delta_{st+\tau}$ possess finite moment-generating functions in a neighborhood of the origin. More precisely, assume that there exists $\varepsilon > 0$ such that

$$E \{ |\rho_t|^{2q} \} < \infty \quad \text{and} \quad E \{ e^{q\delta_t} \} < \infty, \quad |q| \leq \varepsilon.$$

Because

$$E \{ e^{q \log \rho_t^2} \} = E \{ |\rho_t|^{2q} \},$$

this condition guarantees the required exponential integrability of the innovation term W_t for all sufficiently small arguments. Combined with the periodic contraction condition in Eq (5), it provides a readily verifiable sufficient condition for the convergence of the exponential moments appearing in Eq (6).

Remark 3.2. The above requirement is mild for the innovation distributions considered in the simulation study. If ρ_t is Gaussian, then $E\{|\rho_t|^r\} < \infty$ for every $r > 0$, where a Gaussian δ_t has a finite moment-generating function for every real argument. Hence, the required exponential integrability condition is automatically satisfied for the Gaussian specification whenever the periodic contraction condition holds. For Student- t , innovations, the condition is slightly more restrictive but remains readily interpretable: $E\{|\rho_t|^r\} < \infty$ whenever $r < \nu$. Thus, for the Student- t_{15} specification used in the simulations, all moments of ρ_t up to order 15 exist, which is more than sufficient for the moment orders considered in this paper. This shows that the exponential integrability assumption is not unduly restrictive for the innovation distributions commonly employed in financial volatility modeling.

Remark 3.3. An important aspect of Condition (5) is that it preserves the same fundamental periodic stability structure encountered in periodic logGARCH models without stochastic volatility components. In particular, the periodic contraction criterion

$$\prod_{\tau=1}^s \left| \sum_{i=2}^3 \alpha_i(\tau) \right| < 1$$

depends exclusively on the persistence mechanism governing the propagation of past log-volatility effects throughout the cycle. Consequently, the introduction of the additional stochastic volatility innovation term $W_{st+\tau}$ does not modify the essential stability regime underlying the periodic logGARCH dynamics. More precisely, if the stochastic volatility innovation component is removed from the latent equation, the proposed periodic logGARCH-SV specification reduces to a standard periodic logGARCH-type process. In this reduced framework, Condition (5) coincides with the classical periodic stationarity condition ensuring the existence of a unique causal periodically stationary solution. Therefore, the stochastic volatility extension preserves the core probabilistic stability structure of periodic logGARCH processes while enriching the model through an additional stochastic perturbation acting on the latent volatility dynamics. This observation reveals an important structural property of the proposed framework: The stochastic volatility component increases the flexibility of the volatility dynamics without altering the fundamental periodic contraction mechanism responsible for stationarity and ergodicity.

Example 3.1. Consider the periodic extension with period $s = 2$. Let the periodic persistence coefficients be defined as

$$\phi_1 = \alpha_2(1) + \alpha_3(1) := a + b$$

and

$$\phi_2 = \alpha_2(2) + \alpha_3(2) := (2a) + (b + 1).$$

The periodic stationarity condition derived from Eq (5) can then be written as: $|\phi_1\phi_2| < 1$. This condition shows that stationarity depends on the joint interaction between the two phases of the period rather than on a single constant coefficient. In contrast, for the standard specification, the stationarity condition ensuring the existence of a well-defined stationary solution reduces to $|\phi_1| < 1$. The following Figure 2 presents the stationarity region for both the periodic and nonperiodic models.

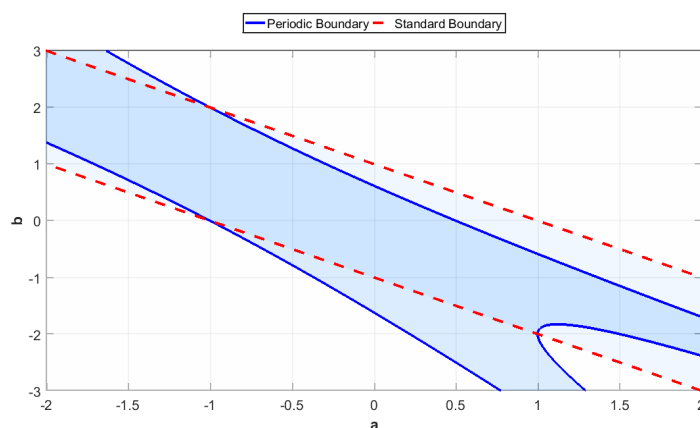


Figure 2. Stationarity regions of the periodic and standard logGARCH-SV models.

Figure 2 illustrates the stationarity region associated with the proposed periodic logGARCH-SV model and compares it with the corresponding region obtained under the standard nonperiodic specification. The dashed red lines represent the classical stationarity boundary determined by the local condition ($|\phi_1| < 1$), whereas the solid blue curves correspond to the periodic stability boundary induced by the periodic contraction condition ($|\phi_1\phi_2| < 1$). The figure reveals that periodic stationarity is governed by the cumulative interaction between the different phases of the cycle rather than by the stability properties of each phase considered separately. As a consequence, the periodic model admits a substantially larger admissible parameter region than does its nonperiodic counterpart. In particular, it includes parameter configurations for which one regime may violate the classical local stability condition while the overall periodic dynamics remain stable over a complete cycle. This geometric representation highlights a fundamental feature of periodically varying systems: Stability is determined by the aggregate effect of the periodic coefficients across the entire cycle rather than by the behavior of individual regimes alone. Therefore, the figure provides a clear visual illustration of the additional flexibility introduced by the periodic structure and emphasizes the distinction between local stability and periodic stability in stochastic volatility dynamics.

The analysis of higher-order moments in stochastic volatility models plays a central role in understanding the tail behavior and long-term stability of financial time series. In standard logGARCH-SV frameworks, the existence of finite moments is typically established through recursive moment conditions that control the accumulation of shocks over time. However, in a cyclical environment, this approach becomes more involved, as the dynamics are no longer driven by a single homogeneous iterative mechanism, but rather by a collection of phase-dependent dynamics interacting over a full seasonal cycle. This introduces an additional layer of complexity, as stability is no longer determined by a one-step contraction property but by the cumulative behavior of the system across periodic regimes. Within this framework, we establish a theorem on the existence of higher-order moments based on a Lyapunov-type periodic contraction mechanism.

Theorem 3.2. Let $(U_t)_{t \in \mathbb{Z}}$ denote the causal strictly periodically stationary solution of the periodic logGARCH-SV model (2). Assume that the innovation sequence (ρ_t) satisfies $E\{\rho_t^{2m}\} < \infty$ for the considered order $m \geq 1$. Suppose further that, for every $\tau \in \mathbb{S}$,

$$\prod_{k=0}^{\infty} E \left\{ \exp \left(m \left(\prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau - l) \right\} \right) (\alpha_0(\tau - k) + W_{st+\tau-k}) \right) \right\} < \infty.$$

A sufficient distribution-based condition for the latter requirement is that for all $\tau \in \mathbb{S}$ and $k \geq 0$,

$$E \left\{ |\rho_t|^{2m} \prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau - l) \right\} \right\} < \infty,$$

and

$$E \left\{ \exp \left(m \left(\prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau - l) \right\} \right) |\alpha_4(\tau - k)| |\delta_t| \right) \right\} < \infty.$$

Then, the process admits a finite periodic moment of order $2m$, that is, $E\{U_t^{2m}\} < \infty$. Moreover, for every $\tau \in \mathbb{S}$, the corresponding periodic moment is given by the convergent expansion

$$E \{ U_{st+\tau}^{2m} \} = E \{ \rho_{st+\tau}^{2m} \} \prod_{k=0}^{\infty} E \left\{ \exp \left(m \left(\prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau - l) \right\} \right) (\alpha_0(\tau - k) + W_{st+\tau-k}) \right) \right\}.$$

Proof. Starting from the causal representation established in Theorem 3.1, we have for every $\tau \in \mathbb{S}$,

$$U_{st+\tau} = \rho_{st+\tau} \exp \left(\frac{1}{2} \sum_{k=0}^{\infty} \left(\prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau - l) \right\} \right) (\alpha_0(\tau - k) + W_{st+\tau-k}) \right).$$

Raising both sides to the power $2m$ yields

$$U_{st+\tau}^{2m} = \rho_{st+\tau}^{2m} \left\{ \exp \left(m \sum_{k=0}^{\infty} \left(\prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau - l) \right\} \right) (\alpha_0(\tau - k) + W_{st+\tau-k}) \right) \right\}.$$

Taking expectations and exploiting the independence between the innovation sequence (ρ_t) and the volatility innovations (W_t) , we obtain

$$E \{ U_{st+\tau}^{2m} \} = E \{ \rho_{st+\tau}^{2m} \} E \left\{ \exp \left(m \sum_{k=0}^{\infty} \left(\prod_{l=0}^{k-1} \left\{ \sum_{i=2}^3 \alpha_i(\tau - l) \right\} \right) (\alpha_0(\tau - k) + W_{st+\tau-k}) \right) \right\}.$$

Because $E\{\rho_t^{2m}\} < \infty$, it remains to establish the finiteness of the exponential term. By the assumed periodic Lyapunov-type condition, the exponential moment associated with the stationary representation remains finite. Therefore, $E\{U_{st+\tau}^{2m}\} < \infty$, which establishes the existence of a periodic moment of order $2m$. Moreover, because the random variables $W_{st+\tau-k}$ are independent across distinct lags, the exponential factors appearing in the decomposition of the stochastic exponential are also independent. Therefore, using the multiplicative property of the exponential function together with

independence, the expectation can be factorized into the corresponding infinite product. Furthermore, combining the above decomposition with the causal representation of the latent process yields

$$E\{U_{st+\tau}^{2m}\} = E\{\rho_{st+\tau}^{2m}\} \prod_{k=0}^{\infty} E\left\{\exp\left(m\left(\prod_{l=0}^{k-1}\left\{\sum_{i=2}^3 \alpha_i(\tau-l)\right\}\right)(\alpha_0(\tau-k) + W_{st+\tau-k})\right)\right\},$$

which is finite under the stated condition. Hence, the periodic logGARCH-SV process possesses finite higher-order moments of order $2m$. $\square$

Remark 3.4. An immediate consequence of Theorem 3.2 for the particular case $m = 1$ is the existence of finite second-order moments of the periodic logGARCH-SV process. In this case, the periodic variance corresponding to phase $\tau \in \mathbb{S}$ can be written as

$$\text{Var}\{U_{st+\tau}\} = \prod_{k=0}^{\infty} E\left\{\exp\left(\left(\prod_{l=0}^{k-1}\left\{\sum_{i=2}^3 \alpha_i(\tau-l)\right\}\right)(\alpha_0(\tau-k) + W_{st+\tau-k})\right)\right\}.$$

The resulting representation separates the unconditional periodic variance into two distinct components: the deterministic contribution arising from the periodic intercepts and the stochastic contribution generated by the volatility innovations. Because both the intercepts and persistence coefficients vary across the cycle, the deterministic component is also phase-dependent. Thus, the unconditional variance at a given phase reflects not only the accumulated effect of the periodic intercepts but also the propagation of past volatility shocks through the corresponding sequence of persistence coefficients. This structure differs from the usual constant-coefficient logGARCH-type setting, where the variance dynamics are governed by fixed parameters. In the proposed model, the variance at each phase depends on the periodic sequence of coefficients, so the relative contributions of the deterministic and stochastic components can change from one phase to another. The model therefore accommodates recurrent changes in both the level and persistence of volatility while maintaining finite second-order moments under the stated integrability conditions.

Remark 3.5. Consider Theorem 3.2 with $m = 2$. Under the corresponding periodic Lyapunov-type condition, the fourth-order periodic moments of the process (U_t) exist and remain finite. In particular, for every $\tau \in \mathbb{S}$, the fourth moment can be expressed as

$$E\{U_{st+\tau}^4\} = E\{\rho_{st+\tau}^4\} \prod_{k=0}^{\infty} E\left\{\exp\left(2\left(\prod_{l=0}^{k-1}\left\{\sum_{i=2}^3 \alpha_i(\tau-l)\right\}\right)(\alpha_0(\tau-k) + W_{st+\tau-k})\right)\right\}.$$

Consequently, the periodic kurtosis coefficient associated with the process (U_t) is given by

$$\kappa_U(\tau) = \frac{E\{U_{st+\tau}^4\}}{(E\{U_{st+\tau}^2\})^2}.$$

Using the moment representation established above, we obtain

$$\kappa_U(\tau) = \kappa_\rho(\tau) \prod_{k=0}^{\infty} \kappa_k(\tau),$$

where

$$\kappa_{\rho}(\tau) = \frac{E\{\rho_{st+\tau}^4\}}{(E\{\rho_{st+\tau}^2\})^2}$$

denotes the kurtosis coefficient of the innovation process, and

$$\kappa_k(\tau) = \frac{E\left\{\exp\left(2\left(\prod_{l=0}^{k-1}\left\{\sum_{i=2}^3\alpha_i(\tau-l)\right\}\right)\left(\alpha_0(\tau-k)+W_{st+\tau-k}\right)\right)\right\}}{E\left\{\exp\left(\left(\prod_{l=0}^{k-1}\left\{\sum_{i=2}^3\alpha_i(\tau-l)\right\}\right)\left(\alpha_0(\tau-k)+W_{st+\tau-k}\right)\right)\right\}^2}$$

represents the contribution of the k th periodic volatility component to the overall kurtosis. Because the distribution of $W_{st+\tau-k}$ is nondegenerate, Jensen's inequality implies that $\kappa_k(\tau) > 1$ for every $k \geq 0$. Therefore, $\kappa_U(\tau) > \kappa_{\rho}(\tau)$. This result demonstrates that the periodic logGARCH-SV mechanism amplifies the tail behavior of the underlying innovation process. More specifically, the interaction between periodic persistence and stochastic volatility induces additional kurtosis, producing tails that are heavier than those generated by the innovations alone. Consequently, even when the innovation sequence exhibits only moderate kurtosis, the resulting process (U_t) may display substantially greater kurtosis and stronger tail risk. This feature makes the proposed model particularly well-suited for modeling financial time series characterized by extreme observations, volatility clustering, and heavy-tailed behavior.

4. Periodic state-space representation and parameter estimation

Although the previous sections established the probabilistic foundations of the proposed periodic logGARCH-SV model, practical implementation requires the estimation of the unknown parameter functions governing both the periodic feedback mechanism and the stochastic volatility component. Owing to the latent nature of the volatility process, direct likelihood-based inference is generally infeasible. Indeed, the observed process provides only partial information about the underlying state dynamics, thereby giving rise to an incomplete-data estimation problem. Compared with classical stationary logGARCH-SV specifications, the proposed framework introduces an additional layer of complexity through periodically varying coefficients. As a consequence, both the conditional dynamics and the second-order properties evolve across the different phases of the cycle. Therefore, statistical inference requires estimation procedures capable of jointly accommodating latent stochastic volatility and periodically varying parameters, a feature absent from standard stationary formulations. The causal representation established in the previous section provides the theoretical basis for the estimation methodology developed herein and guarantees the well-definedness of the latent volatility process. Building upon this representation, the model can be reformulated as a periodically varying state-space system in which the latent log-volatility process serves as an unobserved state variable, and the observed process forms the measurement equation. The theoretical results established in Section 3 provide the foundation for statistical inference. Because the resulting state-space model is nonlinear, and its coefficients vary periodically, the conditional quantities required for likelihood-based inference are not available in closed form. We therefore employ particle filtering and smoothing to approximate the relevant conditional moments within an expectation-maximization (EM) framework. The resulting

particle-assisted EM procedure provides estimates of the periodic parameters, whose finite-sample performance is assessed through the Monte Carlo experiments in Section 5. Within this framework, likelihood-based estimation is implemented through a particle-assisted EM procedure specifically adapted to the nonlinear state-space structure induced by the periodic coefficients. The EM framework treats the latent volatility trajectory as missing data, and particle filtering and particle smoothing provide numerical approximations to the conditional distributions and moments required in the E-step. The periodicity of the coefficients is retained explicitly throughout the filtering, smoothing, and parameter-update steps so that the resulting procedure accommodates both the nonlinear latent dynamics and the phase-dependent structure of the model. The EM algorithm treats the latent volatility path as missing data, and particle filtering and smoothing techniques provide numerical approximations of the conditional distributions required during the iterative optimization procedure. This combination yields a flexible and computationally efficient framework for handling the nonlinear and latent structure of the periodic logGARCH-SV model. The objective of this section is twofold. First, we derive a periodic state-space representation associated with the proposed model. Second, we develop a particle-assisted EM estimation procedure that enables the joint estimation of the periodically varying parameters and the latent stochastic volatility process within a unified framework.

Starting from the observation equation of the proposed periodic logGARCH-SV model,

$$U_t = \rho_t \exp(V_t/2);$$

taking logarithms of the squared observations yields

$$u_t = \log U_t^2 = \log \rho_t^2 + V_t.$$

To make the periodic structure of the observation equation explicit, for each phase $\tau \in \mathbb{S}$, define

$$\beta_\tau := E \left\{ \log \rho_{st+\tau}^2 \right\}$$

and

$$\lambda_{st+\tau} := \log \rho_{st+\tau}^2 - \beta_\tau.$$

Hence, $E\{\lambda_{st+\tau}\} = 0$, and the observation equation can be written as

$$u_{st+\tau} = V_{st+\tau} + \lambda_{st+\tau} + \beta_\tau, \quad \tau \in \mathbb{S}. \quad (7)$$

Thus, β_τ represents the phase-specific deterministic component of the observation equation, whereas $\lambda_{st+\tau}$ is the corresponding centered measurement innovation. The latent log-volatility dynamics are governed by the periodic state equation

$$V_{st+\tau} = \alpha_1(\tau) + \alpha_2(\tau)u_{st+\tau-1} + \alpha_3(\tau)V_{st+\tau-1} + \alpha_4(\tau)\delta_{st+\tau}. \quad (8)$$

Equations (7) and (8) define a periodically varying state-space representation in which the observed process (u_t) forms the observation equation, and the latent process (V_t) constitutes the hidden state variable. Because the volatility process (V_t) is not directly observable, the likelihood function cannot be evaluated analytically. To address this issue, we adopt an EM framework in which the latent volatility

trajectory is treated as missing data. Let $\mathbf{u} = (u_1, \dots, u_{sn})$ denote the observed log-squared process and $\mathbf{V} = (V_0, \dots, V_{sn})$ the corresponding latent state sequence. For the parameter vector

$$\boldsymbol{\theta} := \{\alpha_1(\tau), \alpha_2(\tau), \alpha_3(\tau), \alpha_4(\tau), \tau \in \mathbb{S}\},$$

the complete-data Gaussian quasi-likelihood is constructed from the periodic state-space representation (7) and (8). Consequently, for a realization of the periodic stationary model, we construct a conditional Gaussian quasi-likelihood criterion for the complete data. This choice is motivated by the conditional structure of the state-space representation and is used as a computationally tractable approximation rather than as an exact likelihood under arbitrary innovation distributions. In particular, the Gaussian criterion provides a convenient quadratic approximation for the latent state transition, and the observation contribution follows from the log-squared transformation of the measurement equation. The resulting criterion is therefore interpreted as a Gaussian quasi-likelihood and does not require the underlying innovations to be Gaussian:

$$\begin{aligned} -2l_{\boldsymbol{\theta}}(\mathbf{u}, \mathbf{V}) = & (1 + 2sn) \log(2\pi) + (V_0 - E\{V_0\})^2 (\text{Var}(V_0))^{-1} + \log(\text{Var}(V_0)) \\ & + \sum_{t=0}^{n-1} \sum_{\tau=0}^{s-1} \left((V_{st+\tau} - \alpha_1(\tau) - \alpha_2(\tau)u_{st+\tau-1} - \alpha_3(\tau)V_{st+\tau-1})^2 \alpha_4^{-2}(\tau) \right) \\ & + n \sum_{\tau=0}^{s-1} \log \alpha_4^2(\tau) - \sum_{t=0}^{n-1} \sum_{\tau=0}^{s-1} (u_{st+\tau} - V_{st+\tau}) + \sum_{t=0}^{n-1} \sum_{\tau=0}^{s-1} \exp(u_{st+\tau} - V_{st+\tau}). \end{aligned}$$

Accordingly, the quantity $l_{\boldsymbol{\theta}}(\mathbf{u}, \mathbf{V})$ should be understood as a Gaussian quasi-log-likelihood criterion used within the particle-assisted EM procedure rather than as an exact likelihood under unrestricted innovation distributions. At iteration r of the EM algorithm, the E-step consists of evaluating the conditional expectation of the complete-data log-likelihood given the observed sample and the current parameter estimate $\boldsymbol{\theta}(r-1)$. Because the smoothing distribution of the latent state process is unavailable in closed form, sequential Monte Carlo techniques are employed to approximate the required conditional moments. The corresponding auxiliary function is defined by

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}(r-1)) := E\{-2l_{\boldsymbol{\theta}}(\mathbf{u}, \mathbf{V}) \mid \mathbf{u}, \mathbf{V}, \boldsymbol{\theta}(r-1)\}.$$

Substituting the complete log-likelihood into the above expression yields the expected complete-data criterion

$$\begin{aligned} Q(\boldsymbol{\theta}, \boldsymbol{\theta}(r-1)) = & (1 + 2sn) \log(2\pi) + ((V_{0,n} - E\{V_0\})^2 p_{0,n}) (\text{Var}(V_0))^{-1} \\ & + n \sum_{\tau=0}^{s-1} \log \alpha_4^2(\tau) + \log(\text{Var}(V_0)) \\ & + \sum_{t=0}^{n-1} \sum_{\tau=0}^{s-1} \left[(p_{st+v,n} + \alpha_3^2(\tau) p_{st+v,n} - 2\alpha_3(\tau) p_{st+v,n}^{(st+v-1)}) \right. \\ & \quad \left. + (V_{st+\tau,n} - \alpha_1(\tau) - \alpha_2(\tau)u_{st+\tau-1} - \alpha_3(\tau)V_{st+\tau-1,n})^2 \right] / \alpha_4^{-2}(\tau) \\ & - \sum_{t=0}^{n-1} \sum_{\tau=0}^{s-1} (u_{st+\tau} - V_{st+\tau,n}) + \sum_{t=0}^{n-1} \sum_{\tau=0}^{s-1} E\{\exp(u_{st+\tau} - V_{st+\tau}) \mid \mathbf{u}, \mathbf{V}, \boldsymbol{\theta}(r-1)\}, \end{aligned}$$

which depends on the smoothed moments

$$V_{t,n} = E \{V_t | u, \mathbf{V}, \boldsymbol{\theta}(r-1)\},$$

$$p_{t,n} = E \{(V_{t,n} - V_t)^2 | u, \mathbf{V}, \boldsymbol{\theta}(r-1)\},$$

and

$$p_{t,n}^{(t-1)} = E \{(V_{t,n} - V_t)(V_{t-1,n} - V_{t-1}) | u, \mathbf{V}, \boldsymbol{\theta}(r-1)\}.$$

These quantities summarize the conditional distribution of the latent volatility process and are obtained numerically through particle filtering and particle smoothing procedures.

In the M-step, the updated parameter vector $\boldsymbol{\theta}(r)$ is obtained by maximizing $Q(\boldsymbol{\theta}, \boldsymbol{\theta}(r-1))$ with respect to $\boldsymbol{\theta}$. Owing to the nonlinear structure of the periodic logGARCH-SV model and the presence of latent states, closed-form maximizers are generally unavailable. Therefore, particle-based approximations of the conditional moments are employed to compute the parameter updates iteratively until convergence. The particle filtering and smoothing procedures used here follow the general sequential Monte Carlo framework described in Ghezal and Alzeley [20], and the state propagation, particle weighting, smoothing quantities, and conditional expectations are adapted to the periodic logGARCH-SV state-space representation developed in the present paper. In particular, the particle filter provides an approximation to the filtering distribution of the latent log-volatility, whereas the particle smoother is used to approximate the conditional moments and exponential terms required in the E-step.

5. Simulation study

To evaluate the finite-sample performance of the proposed particle-assisted EM estimation procedure, a Monte Carlo simulation study was conducted for the periodic logGARCH-SV model developed in the previous sections. The primary objective was to assess the ability of the estimation method to accurately recover the periodically varying parameters and to investigate how its performance evolves as the sample size increases. To this end, artificial time series were generated from the periodic logGARCH-SV process under two different parameter configurations which satisfy the theoretical conditions that ensure the existence of a strictly periodically stationary solution.

The simulation study therefore constitutes the final computational stage of the proposed framework: Observations are generated from the periodic model under parameter configurations which satisfy the theoretical conditions established in Section 3, and the particle-assisted EM procedure developed in Section 4 is subsequently applied to the simulated samples. The resulting estimates are compared with the true parameter values to assess finite-sample parameter recovery and estimation precision.

The periodicity length was set to $s = 2$, corresponding to the two phases of the periodic coefficient functions. Two total sample sizes, $sn = 2000$ and $sn = 3500$, were considered, corresponding respectively to $n = 1000$ and $n = 1750$ complete cycles. This choice allows the effect of increasing the number of observations, while keeping the periodic structure fixed, to be examined directly. For each parameter configuration, the particle-assisted EM algorithm was implemented using $M = 195$ particles throughout the simulation experiments. The innovation structure was specified explicitly in all simulation experiments. The sequences (ρ_t) and (δ_t) were generated independently across time and independently of each other. Two innovation distributions were considered. In the Gaussian case, both

innovation sequences were generated from standard normal distributions. In the heavy-tailed case, both sequences were generated from standardized Student- t_{15} distributions, with the scaling chosen so that each innovation has zero mean and unit variance. Thus, the two simulation settings differ only in the tail behavior of the innovations, and preserving the moment normalization imposed in the model specification. The estimation procedure was repeated independently 1000 times. In each replication, the particle-assisted EM algorithm was iterated until numerical convergence. Convergence was monitored using the relative change in the parameter vector between two consecutive EM iterations and the corresponding relative change in the EM auxiliary function. Specifically, the algorithm was stopped when both the maximum relative change in the parameter estimates and the relative change in the auxiliary function were below the prescribed tolerance $\varepsilon = 10^{-6}$, or when a maximum of 500 EM iterations was reached. The same convergence criterion was used for all simulation designs to ensure a consistent assessment of the estimation procedure. Because the proposed model is a nonlinear state-space system with periodically varying coefficients, the conditional moments required in the E-step are not available in closed form. They were therefore approximated using particle filtering and particle smoothing, and these particle-based approximations were used to construct the EM auxiliary function. The resulting procedure can consequently be viewed as a particle-assisted EM algorithm specifically tailored to the nonlinear periodic state-space structure of the proposed model. For each replication, parameter estimates were obtained and subsequently summarized by their empirical means and standard deviations across the Monte Carlo runs. The results are presented in Tables 1 and 2, which provide the true parameter values (TV) employed in the simulation design along with the corresponding parameter estimates obtained under Gaussian and standardized Student- t_{15} innovations for both the standard and periodic logGARCH-SV specifications. Standard errors are reported in parentheses. Overall, the numerical results indicate that the proposed estimation procedure successfully recovers the underlying periodic parameters and that the accuracy of the estimates improves as the sample size increases. These findings demonstrate that the proposed particle-assisted EM procedure is capable of accurately recovering the underlying periodic parameters and performs satisfactorily in finite samples.

Table 1. Parameter estimation results for Configuration I: Comparison between standard and periodic logGARCH-SV models under Gaussian and standardized Student- t_{15} innovations for $sn = 2000$.

Parameters	Tv	Periodic	logGARCH-SV	Standard	logGARCH-SV
		Gaussian	Student- t_{15}	Gaussian	Student- t_{15}
$\alpha_1(1)$	1.500	1.485(0.071)	1.476(0.087)	1.384(0.128)	1.379(0.135)
$\alpha_1(2)$	1.500	1.521(0.083)	1.532(0.101)	–	–
$\alpha_2(1)$	0.500	0.492(0.059)	0.481(0.094)	0.438(0.087)	0.433(0.099)
$\alpha_2(2)$	–0.025	–0.032(0.035)	–0.039(0.048)	–	–
$\alpha_3(1)$	0.750	0.740(0.057)	0.736(0.073)	0.665(0.098)	0.661(0.105)
$\alpha_3(2)$	0.250	0.241(0.064)	0.232(0.081)	–	–
$\alpha_4(1)$	1.000	1.013(0.036)	1.022(0.053)	1.082(0.106)	1.088(0.122)
$\alpha_4(2)$	0.550	0.564(0.049)	0.569(0.058)	–	–

Table 2. Parameter estimation results for Configuration II: Comparison between standard and periodic logGARCH-SV models under Gaussian and standardized Student- t_{15} innovations for $sn = 3500$.

Parameters	Tv	Periodic	logGARCH-SV	Standard	logGARCH-SV
		Gaussian	Student- t_{15}	Gaussian	Student- t_{15}
$\alpha_1(1)$	2.000	1.990(0.043)	1.987(0.050)	1.942(0.079)	1.934(0.092)
$\alpha_1(2)$	1.000	1.008(0.052)	1.009(0.056)	–	–
$\alpha_2(1)$	–0.500	–0.495(0.026)	–0.491(0.032)	–0.447(0.058)	–0.436(0.075)
$\alpha_2(2)$	–0.025	–0.028(0.018)	–0.033(0.026)	–	–
$\alpha_3(1)$	0.750	0.746(0.029)	0.738(0.044)	0.693(0.062)	0.674(0.079)
$\alpha_3(2)$	–0.250	–0.244(0.031)	–0.241(0.048)	–	–
$\alpha_4(1)$	0.300	0.310(0.027)	0.316(0.035)	0.373(0.079)	0.376(0.083)
$\alpha_4(2)$	0.550	0.556(0.022)	0.562(0.040)	–	–

Tables 1 and 2 report the Monte Carlo estimation results obtained using the proposed particle-assisted EM procedure under two distinct parameter configurations and two innovation distributions, namely Gaussian and standardized Student- t_{15} . The reported values correspond to the empirical means of the parameter estimates computed over 1000 independent replications, and the numbers in parentheses represent the associated empirical standard deviations. Overall, the results demonstrate that the proposed estimation methodology successfully recovers the true parameter values of the periodic logGARCH-SV model. For both parameter configurations, the estimated periodic coefficients remain close to their corresponding true values, indicating satisfactory finite-sample performance of the particle-assisted EM algorithm. In particular, the estimates associated with the periodic specification exhibit only minor deviations from the true parameters, even under the heavier-tailed standardized Student- t_{15} innovations. A comparison between Tables 1 and 2 further reveals the beneficial effect of increasing the sample size. When the number of observations increases from ($sn = 2000$) to ($sn = 3500$), the empirical standard deviations decrease substantially for nearly all parameters. This reduction in estimation variability reflects improved estimation precision and is consistent with the expected enhancement in estimation accuracy as the sample size grows. The results also highlight the importance of incorporating periodicity into the volatility dynamics. Because the standard logGARCH-SV model imposes common coefficients across all phases, its parameter estimates are not directly one-to-one comparable with the phase-specific estimates of the periodic specification. The comparison in Tables 1 and 2 should therefore be interpreted as a comparison between a correctly specified periodic model and a restricted nonperiodic model that ignores phase-specific heterogeneity. Under this interpretation, the results show that the periodic specification recovers the phase-specific parameters with small deviations from their true values, whereas the restricted standard specification generally exhibits larger deviations in its common coefficients from the underlying periodic structure. This distinction is particularly relevant for the feedback and persistence coefficients, represented by $\alpha_2(\tau)$ and $\alpha_3(\tau)$, whose true values differ across phases in the periodic data-generating process. A single common coefficient in the standard specification cannot reproduce this phase-specific heterogeneity. Consequently, its estimates should be viewed as restricted or pooled representations of the underlying

periodic dynamics rather than as direct estimates of any particular phase-specific coefficient. The larger deviations observed for the standard specification therefore reflect model misspecification arising from the omission of periodic heterogeneity rather than merely inferior numerical estimation. An additional observation concerns the periodic stability condition established in Theorem 3.1. For Configuration I (Table 1), the regime-specific persistence measures are

$$\left| \sum_{i=2}^3 \alpha_i(1) \right| = |0.500 + 0.750| = 1.250$$

and

$$\left| \sum_{i=2}^3 \alpha_i(2) \right| = |-0.025 + 0.250| = 0.225.$$

Hence, the first regime is locally explosive because its persistence coefficient exceeds unity, whereas the second regime remains stable. Nevertheless, the periodic stability condition is satisfied because

$$\prod_{\tau=1}^s \left| \sum_{i=2}^3 \alpha_i(\tau) \right| = 0.28125 < 1.$$

This illustrates one of the fundamental features of periodically varying systems: Periodic stationarity may still hold even when one individual regime exhibits explosive behavior, provided that the cumulative effect over a complete cycle remains contractive. In contrast, for Configuration II (Table 2), both regimes are individually stable because

$$\left| \sum_{i=2}^3 \alpha_i(1) \right| = |-0.500 + 0.750| = 0.250,$$

and

$$\left| \sum_{i=2}^3 \alpha_i(2) \right| = |-0.025 - 0.250| = 0.275.$$

Consequently,

$$\prod_{\tau=1}^s \left| \sum_{i=2}^3 \alpha_i(\tau) \right| = 0.06875 < 1,$$

confirming that the periodic contraction condition is satisfied both at the regime level and over the complete cycle. The successful recovery of the parameters in both configurations therefore provides numerical support for the theoretical stability results established in Section 3 and demonstrates that the proposed estimation procedure remains effective under different periodic persistence structures. Moreover, the comparison between Gaussian and standardized Student- t_{15} innovations suggests that the proposed methodology remains robust in the presence of heavier-tailed disturbances. The standardized Student- t_{15} specification generally produces larger empirical standard deviations than does the Gaussian specification, reflecting the increased sampling variability associated with heavier-tailed innovations. This pattern is consistently observed across the periodic parameter estimates in both simulation configurations. Taken together, these results provide strong numerical

evidence that the proposed periodic logGARCH-SV framework, coupled with the particle-assisted EM estimation procedure, accurately captures periodically varying volatility dynamics and yields substantial improvements relative to the corresponding nonperiodic specification.

The results in Tables 1 and 2 should therefore be interpreted as evidence of finite-sample performance rather than as a formal ranking of competing estimation methods. A statistical comparison based on procedures such as the Wilcoxon signed-rank or Friedman tests would require a broader experimental design in which several alternative estimators are evaluated under the same simulation scenarios and according to common performance measures. Such a comparison is beyond the specific parameter-recovery objective of the present study. Nevertheless, we recognize the value of formal convergence diagnostics and multimethod statistical comparisons in the evaluation of computational estimation procedures, as highlighted in recent methodological studies (Fan et al., [26]; Li et al., [27, 28]; Peng et al., [29]). A broader comparison incorporating alternative estimators, nonparametric tests, and additional numerical stability diagnostics would therefore be a useful direction for future work.

6. Conclusions

This paper proposed a new periodic logGARCH-SV framework designed to capture both stochastic persistence and recurring cyclical variations frequently observed in financial volatility series. By allowing all structural coefficients to vary periodically over the cycle, the proposed model substantially extends the standard logGARCH-SV specification and provides a flexible mechanism for describing seasonal heterogeneity in volatility dynamics. From a theoretical perspective, the model was reformulated as a periodic stochastic recurrence equation, which enabled the derivation of a causal representation for the latent volatility process. Sufficient conditions which ensure the existence of a unique strictly periodically stationary and causal solution were established through a periodic contraction condition. In addition, higher-order moment properties were investigated, leading to explicit expressions for periodic moments under suitable integrability assumptions. Together, these results provide a rigorous probabilistic characterization of the proposed periodic volatility framework. To address the challenging inference problem arising from latent volatility and periodically varying coefficients, a periodic state-space representation was developed. Within this framework, estimation was carried out using a particle-assisted EM procedure that combines particle filtering and particle smoothing techniques. This methodology makes it possible to approximate the otherwise intractable conditional distributions required for likelihood-based inference while maintaining computational feasibility.

The Monte Carlo experiments confirmed the effectiveness of the proposed estimation procedure. Across different parameter configurations and innovation distributions, the periodic logGARCH-SV model accurately recovered the underlying parameters and consistently outperformed the corresponding nonperiodic specification. The simulation results further showed that estimation precision improves with increasing sample size and remains satisfactory under heavier-tailed Student- t_{15} innovations. An additional finding concerns the periodic stability mechanism. The numerical experiments demonstrated that the proposed framework remains effective both in settings involving locally explosive periodic regimes and in settings where all regimes are individually stable, provided that the overall periodic contraction condition is satisfied. This observation highlights a

distinctive feature of periodically varying systems, namely that periodic stationarity is determined by the cumulative effect over a complete cycle rather than by the stability properties of individual regimes alone.

Overall, the proposed periodic logGARCH-SV model provides a theoretically rigorous and tractable framework for studying volatility processes with recurrent changes in volatility level and persistence. The present study focuses on the theoretical properties and estimation of the model, whereas empirical applications constitute a natural direction for further research. In particular, applying the model to high-frequency and intraday financial data could provide an opportunity to assess its empirical relevance and forecasting performance in settings with pronounced calendar or seasonal patterns. Other directions include multivariate extensions and regime-switching periodic specifications, which would allow for richer forms of dependence and temporal heterogeneity.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

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