



Research article**On certain subclasses of bi-univalent functions associated with Jacobi polynomials and the Bell distribution with applications to image enhancement****Vanithakumari Balasubramaniam¹, Serkan Araci² and Selvaraj Palanisamy^{1,*}**

¹ Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham, Chennai 601103, Tamil Nadu, India; vanithagft@gmail.com (V.B.); selvamath89@gmail.com (S.P.)

² Department of Computer Engineering, Faculty of Engineering, Hasan Kalyoncu University, TR-27010, Gaziantep, Türkiye; serkan.araci@hku.edu.tr (S.A.)

* **Correspondence:** Email: selvamath89@gmail.com.

Abstract: In this research work, we introduced a novel subclass $\mathcal{B}_\lambda^{\theta, S}(\alpha, \beta; y, \zeta)$ of bi-univalent functions associated with the Bell distribution by using Jacobi polynomials. For this class, coefficient bounds $|a_2|$ and $|a_3|$, together with the corresponding Fekete–Szegő inequality were obtained. Based on these theoretical results, a simple and effective image enhancement algorithm (BJBIEA) model was proposed using a novel convolution strategy implemented through a 3×3 mask. The outcomes showed that the suggested method for general-purpose image enhancement is effective, dependable, and adaptable. The extension of this framework to detail enhancement, edge extraction, and image resolution quality will be the focus of future work.

Keywords: bi-univalent functions; Fekete–Szegő problem; Jacobi polynomials; Bell distribution; image enhancement; directional masks; convolution; Bell polynomials; Jacobi polynomials; grayscale image processing

Mathematics Subject Classification: 30C45

1. Introduction

A function is said to be analytic if it is differentiable at every point in its domain. Each normalized analytic function can be expressed as follows:

$$\tilde{f}(\zeta) = \zeta + \sum_{t=2}^{\infty} a_t \zeta^t, \quad (1.1)$$

where $\tilde{f}(\zeta)$ is an analytic function and its power series converges in the unit disk $\mathfrak{U} = \{\zeta : \zeta \in \mathbb{C}, |\zeta| < 1\}$. Let $\mathcal{A}$ be the set of all normalized analytic functions in $\mathfrak{U}$. A subclass $\mathcal{S}$ of $\mathcal{A}$ is defined by $\mathcal{S} = \{\tilde{f}(\zeta) \in \mathcal{A} : \tilde{f}(\zeta_1) = \tilde{f}(\zeta_2) \Rightarrow \zeta_1 = \zeta_2\}$. Let $\mathcal{S}$ be the set of all univalent functions. A function $\tilde{f}(\zeta)$ is called bi-univalent whenever its inverse function $\tilde{f}^{-1}(\omega)$ admits an analytic extension to the unit disk $|\omega| < 1$ in the ω -plane. In other words, Koebe One-quarter theorem [10] states that every univalent function defined on the unit disk maps itself onto a region that necessarily contains a disk of radius $1/4$. Thus, every such univalent function $\tilde{f} \in \mathcal{A}$ has an inverse $\tilde{f}^{-1}$ satisfying $\tilde{f}^{-1}(\tilde{f}(\zeta)) = \zeta$ and $\tilde{f}(\tilde{f}^{-1}(\omega)) = \omega$ for $|\omega| < r_0$, $r_0 \in (0, 1/4)$, where

$$\tilde{f}^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots$$

Let σ be the collection of all bi-univalent functions in $\mathfrak{U}$. Here, we give some examples of bi-univalent functions:

$$\tilde{f}(\zeta) = \frac{\zeta}{1-\zeta}, \quad g(\zeta) = -\log(1-\zeta), \quad h(\zeta) = \frac{1}{2} \log \frac{1+\zeta}{1-\zeta}$$

with their respective inverses

$$\tilde{f}^{-1}(\omega) = \frac{\omega}{1+\omega}, \quad g^{-1}(\omega) = e^{-\omega} - 1, \quad h^{-1}(\omega) = \frac{e^{2\omega} - 1}{e^{2\omega} + 1}.$$

This concept was first introduced by Lewin [12] in 1967, who also provided an estimate for the second coefficient of these functions, stating that $|a_2| < 1.51$. This estimation was later refined by Brannan and Clunie [3], who proposed $|a_2| \leq \sqrt{2}$. Over the years, the initial coefficients of bi-univalent functions have been extensively studied, contributing to a rich body of literature on the subject. However, much is still unknown about the bounds on the general coefficient $|a_n|$ for $n \geq 2$.

Table 1 Provides the chronological summary of known estimates for the second coefficient $|a_2|$ of bi-univalent functions in the class σ , together with the researchers who obtained them and the corresponding references. The last row lists works addressing the Maclaurin coefficients $|a_n|$ for $n \geq 3$.

Table 1. Summary of findings on the coefficient a_n for bi-univalent functions in σ .

Researchers	Findings	Estimates
Lewin (1967)	For all functions in σ	$ a_2 < 1.51$ [12]
Brannan and Clunie (1980)	For all functions in σ	$ a_2 \leq \sqrt{2}$ [3]
Netanyahu (1969)	Best estimate for $ a_2 $	$ a_2 = 4/3$ [14]
Various researchers	Estimates for the Maclaurin coefficient $ a_n $	See references [16, 17, 20].

Bell polynomials [5] are useful for making subclasses of analytic and bi-univalent functions because they are related to combinatorics and generating functions [6, 9, 22]. Jacobi polynomials form a classical form of polynomials defined on the interval $[-1, 1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, where α and $\beta > -1$. They have rich structural and recurrence properties and generalize several well-known polynomial families [18], such as the Legendre and Chebyshev polynomials. Jacobi polynomials offer a useful mathematical tool for simulating adaptive enhancement functions that boost image contrast and structural detail in the suggested Bell–Jacobi bi-univalent framework (BBJEA) model.

For non-negative $n, n + \alpha, n + \beta$, recurrence relations of Jacobi polynomials [13] are given by

$$\mathfrak{J}_n(y, \zeta) = 2^{\alpha+\beta} R^{-1} (1 - y + R)^{-\alpha} (1 + y + R)^{-\beta},$$

where $R(y, \zeta) = (1 - 2\zeta y + y^2)^{\frac{1}{2}}$, $\alpha > -1, \beta > -1, y \in [-1, 1]$, and $\zeta \in \mathfrak{U}$.

$$R(y, \zeta) = \sqrt{1 - 2\zeta y + y^2}.$$

Now, we have

$$R^{-1}(y, \zeta) = \frac{1}{R(y, \zeta)} = \frac{1}{\sqrt{1 - 2\zeta y + y^2}} = (1 - 2\zeta y + y^2)^{-1/2}$$

and

$$R - 1 = \sqrt{1 - 2\zeta y + y^2} - 1.$$

For a fixed y , the function $\mathfrak{J}_n(y, \zeta)$, which is analytic in $\mathfrak{U}$, is given by

$$\mathfrak{J}_n(y, \zeta) = \sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(y) \zeta^n, \quad (1.2)$$

where $P_n^{(\alpha, \beta)}(y)$ are Jacobi polynomials of degree n . The Jacobi polynomials $P_n^{(\alpha, \beta)}(y)$ satisfy a second-order linear homogeneous differential equation:

$$(1 - y^2)x'' + (\beta - \alpha - (\alpha + \beta + 2)y)x' + n(n + \alpha + \beta + 1)x = 0.$$

Jacobi polynomials are defined by the following recurrence equation:

$$P_n^{(\alpha, \beta)}(y) = (a_{n-1}\zeta - b_{n-1})P_{n-1}^{(\alpha, \beta)}(y) - c_{n-1}P_{n-2}^{(\alpha, \beta)}(y), n \geq 2,$$

where $a_n = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}$, $b_n = \frac{(2n+\alpha+\beta+1)(\beta^2-\alpha^2)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}$, and $c_n = \frac{(2n+\alpha+\beta+2)(n+\beta)(n+\alpha)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}$, with the initial values

$$P_0^{(\alpha, \beta)}(y) = 1, P_1^{(\alpha, \beta)}(y) = \frac{(\alpha - \beta) + (\alpha + \beta + 2)y}{2}$$

and $P_2^{(\alpha, \beta)}(y) = \frac{1}{8}((\alpha - \beta)^2 - (\alpha - \beta) - 4 + 2(\alpha - \beta)(\alpha + \beta + 3)y + (\alpha + \beta + 3)(\alpha + \beta + 4)y^2)$.

Figure 1 represents the image of the unit disk $\mathfrak{U}$ under the function $\mathfrak{J}_n(y, \zeta)$. For $\beta = 0, \alpha = 0$, and $y = 1$, we have the function $\mathfrak{f}_1(\zeta) = \frac{1}{\sqrt{\zeta^2 - 2\zeta + 1}}$ (see Figure 1(a)). For $\beta = 1, \alpha = 1$, and $y = 1$, we have the function $\mathfrak{f}_2(\zeta) = \frac{4}{(\zeta^2 - 2\zeta + 1)(2 + \sqrt{\zeta^2 - 2\zeta + 1})}$ (see Figure 1(b)).

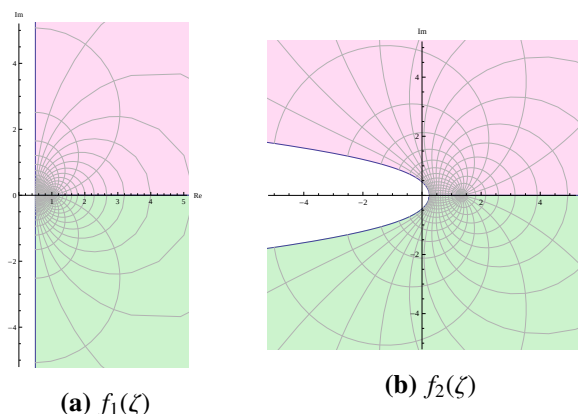


Figure 1. The image of $\mathfrak{U}$ under the function $\mathfrak{J}_n(y, \zeta)$.

A discrete probability distribution associated with Bell numbers, which list the number of partitions of a finite set, is known as the Bell distribution. This work builds flexible subclasses of bi-univalent functions for image-enhancement applications using the Bell distribution. Furthermore, its introduction by Castellares et al. [4] in 2018 represents an evolution, refining its characteristics beyond its origins in Bell numbers and enhancing its effectiveness in addressing intricate analytical challenges. In the Bell distribution, the probability function of a discrete random variable X can be defined as

$$P(X = m) = \frac{\lambda^m e^{(1-e^\lambda)} \mathfrak{B}_m}{m!}; \quad m = 0, 1, 2, 3, \dots, \quad (1.3)$$

where

$$\mathfrak{B}_m = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^m}{m!}, \quad (1.4)$$

where the Bell numbers are $m \geq 2$ and $\lambda \geq 0$. The Bell number $\mathfrak{B}_m$ in (1.4) is the m th moment of the Poisson distribution with a parameter of 1. The first few Bell numbers are $\mathfrak{B}_2 = 2$, $\mathfrak{B}_3 = 5$, $\mathfrak{B}_4 = 15$, and $\mathfrak{B}_5 = 52$. Now, we consider the power series with coefficients drawn from the Bell distribution [2, 11].

$$\mathfrak{B}(\lambda, \zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\lambda^{n-1} e^{e^{(-\lambda^2)+1}} \mathfrak{B}_n}{(n-1)!} \zeta^n, \quad \zeta \in \mathfrak{U}, \lambda > 0. \quad (1.5)$$

The linear operator $\mathbb{P}_\lambda : \mathcal{A} \rightarrow \mathcal{A}$ defined through convolution is given by

$$\mathbb{P}_\lambda \mathfrak{f}(\zeta) = \mathfrak{B}(\lambda, \zeta) * \mathfrak{f}(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\lambda^{n-1} e^{e^{(-\lambda^2)+1}} \mathfrak{B}_n}{(n-1)!} a_n \zeta^n, \quad \zeta \in \mathfrak{U}. \quad (1.6)$$

Definition 1. A function $\mathfrak{f} \in \sigma$ is said to belong to the class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$ whenever both $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ satisfy the following subordination conditions:

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda \mathfrak{f}(\zeta)}{\zeta} + \vartheta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))' + \varsigma \zeta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))'' < \mathfrak{J}_n(y, \zeta), \quad \zeta \in \mathfrak{U} \quad (1.7)$$

and

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda \mathfrak{g}(\omega)}{\omega} + \vartheta (\mathbb{P}_\lambda \mathfrak{g}(\omega))' + \varsigma \omega (\mathbb{P}_\lambda \mathfrak{g}(\omega))'' < \mathfrak{J}_n(y, \omega), \quad \omega \in \mathfrak{U}, \quad (1.8)$$

where $\zeta, \omega \in \mathfrak{U}$, $\vartheta, \varsigma \geq 0$, and $y \in [-1, 1]$.

Remark 1. By choosing specific values of the parameters ϑ , ς , and λ , the general class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$ reduces to several important subclasses, as follows:

(i) If $\mathfrak{f} \in \sigma$, defined by (1.1), belongs to the class $\mathcal{B}_\lambda^{\vartheta, 0}(\alpha, \beta; y, \zeta)$, then

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda \mathfrak{f}(\zeta)}{\zeta} + \vartheta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))' < \mathfrak{J}_n(y, \zeta)$$

and

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda \mathfrak{g}(\omega)}{\omega} + \vartheta (\mathbb{P}_\lambda \mathfrak{g}(\omega))' < \mathfrak{J}_n(y, \omega).$$

(ii) If $\mathfrak{f} \in \sigma$, defined by (1.1), belongs to the class $\mathcal{B}_\lambda^{1,0}(\alpha, \beta; y, \zeta)$, then

$$(\mathbb{P}_\lambda \mathfrak{f}(\zeta))' < \mathfrak{I}_n(y, \zeta)$$

and

$$(\mathbb{P}_\lambda \mathfrak{g}(\omega))' < \mathfrak{I}_n(y, \omega).$$

(iii) If $\mathfrak{f} \in \sigma$, defined by (1.1), belongs to the class $\mathcal{B}_\lambda^{0,0}(\alpha, \beta; y, \zeta)$, then

$$\frac{\mathbb{P}_\lambda \mathfrak{f}(\zeta)}{\zeta} < \mathfrak{I}_n(y, \zeta)$$

and

$$\frac{\mathbb{P}_\lambda \mathfrak{g}(\omega)}{\omega} < \mathfrak{I}_n(y, \omega).$$

(iv) If $\mathfrak{f} \in \sigma$, defined by (1.1), belongs to the class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$, then

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda \mathfrak{f}(\zeta)}{\zeta} + \vartheta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))' + \varsigma \zeta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))'' < \mathfrak{I}_n(y, \zeta)$$

and

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda \mathfrak{g}(\omega)}{\omega} + \vartheta (\mathbb{P}_\lambda \mathfrak{g}(\omega))' + \varsigma \omega (\mathbb{P}_\lambda \mathfrak{g}(\omega))'' < \mathfrak{I}_n(y, \omega),$$

where $\mathfrak{g} = \mathfrak{f}^{-1}$, $\zeta, \omega \in \mathfrak{U}$, $\vartheta, \varsigma \geq 0$, and $y \in [-1, 1]$.

Hence, by choosing suitable parameters, the general class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$ encompasses several interesting known and new subclasses.

Inspired by the increasing interest in applying geometric function theory to solve real-world problems, this work explores the application of coefficient estimates of bi-univalent functions in image enhancement. This approach provides a solid mathematical foundation for improving structural details, brightness, and image contrast.

2. Coefficient estimates for the class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$

Theorem 1. If $\mathfrak{f} \in \sigma$, defined by (1.1), belongs to the class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$, then

$$|a_2| \leq \frac{|A| \sqrt{2|A|}}{\lambda e^{\frac{1-\lambda^2}{2}} \sqrt{|5(1+2\vartheta+6\varsigma)A^2 - 8(1+\vartheta+2\varsigma)^2 B e^{e(1-\lambda^2)}}|}} \quad (2.1)$$

and

$$|a_3| \leq \frac{2|A|}{5\lambda^2(1+2\vartheta+6\varsigma)e^{e(1-\lambda^2)}} + \frac{|A|^2}{4\lambda^2(1+\vartheta+2\varsigma)^2 e^{2e(1-\lambda^2)}}, \quad (2.2)$$

where

$$A = (\alpha - \beta) + (\alpha + \beta + 2)y$$

and

$$B = (\alpha - \beta)^2 - (\alpha - \beta) - 4 + 2(\alpha - \beta)(\alpha + \beta + 3)y + (\alpha + \beta + 3)(\alpha + \beta + 4)y^2.$$

Further, $\zeta, \omega \in \mathfrak{U}$, $\vartheta, \varsigma \geq 0$, $y \in [-1, 1]$, and $\mathfrak{g} = f^{-1}$.

Proof. Let $\mathfrak{f} \in \mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$, and then there exist analytic functions $\psi(\zeta)$ and $\tau(\omega) : \mathfrak{U} \rightarrow \mathfrak{U}$, given by (1.7) and (1.8), such that

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda \mathfrak{f}(\zeta)}{\zeta} + \vartheta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))' + \varsigma \zeta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))'' = \mathfrak{J}_n(y, \psi(\zeta)) \quad (2.3)$$

and

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda g(\omega)}{\omega} + \vartheta (\mathbb{P}_\lambda g(\omega))' + \varsigma \omega (\mathbb{P}_\lambda g(\omega))'' = \mathfrak{J}_n(y, \tau(\omega)). \quad (2.4)$$

Define the functions $\psi(\zeta)$ and $\tau(\omega)$ as

$$\psi(\zeta) = c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3 + \cdots \quad (\zeta \in \mathfrak{U})$$

and

$$\tau(\omega) = d_1 \omega + d_2 \omega^2 + d_3 \omega^3 + \cdots \quad (\omega \in \mathfrak{U}),$$

which are analytic in $\mathfrak{U}$, with $\psi(0) = 0$, $\tau(0) = 0$, $|\psi(\zeta)| < 1$ and $|\tau(\omega)| < 1$.

Since

$$|\psi(\zeta)| = |c_1 \zeta + c_2 \zeta^2 + c_3 \zeta^3 + \cdots| < 1 \quad (\zeta \in \mathfrak{U}) \quad (2.5)$$

and

$$|\tau(\omega)| = |d_1 \omega + d_2 \omega^2 + d_3 \omega^3 + \cdots| < 1 \quad (\omega \in \mathfrak{U}), \quad (2.6)$$

then

$$|c_i| \leq 1, \quad |d_i| \leq 1 \quad (i = 1, 2, 3, \dots). \quad (2.7)$$

Since

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda \mathfrak{f}(\zeta)}{\zeta} + \vartheta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))' + \varsigma \zeta (\mathbb{P}_\lambda \mathfrak{f}(\zeta))'' = 1 + [P_1^{(\alpha, \beta)}(y) c_1] \zeta + [\mathbb{P}_1^{(\alpha, \beta)}(y) c_2 + P_2^{(\alpha, \beta)}(y) c_1^2] \zeta^2 + \cdots \quad (2.8)$$

and

$$(1 - \vartheta) \frac{\mathbb{P}_\lambda g(\omega)}{\omega} + \vartheta (\mathbb{P}_\lambda g(\omega))' + \varsigma \omega (\mathbb{P}_\lambda g(\omega))'' = 1 + [\mathbb{P}_1^{(\alpha, \beta)}(y) d_1] \omega + [\mathbb{P}_1^{(\alpha, \beta)}(y) d_2 + P_2^{(\alpha, \beta)}(y) d_1^2] \omega^2 + \cdots, \quad (2.9)$$

we get the following equations:

$$2\lambda(1 + \vartheta + 2\varsigma) e^{(1-\lambda^2)} a_2 = \mathbb{P}_1^{(\alpha, \beta)}(y) c_1, \quad (2.10)$$

$$\frac{5}{2} \lambda^2 (1 + 2\vartheta + 6\varsigma) e^{(1-\lambda^2)} a_3 = \mathbb{P}_1^{(\alpha, \beta)}(y) c_2 + \mathbb{P}_2^{(\alpha, \beta)}(y) c_1^2, \quad (2.11)$$

$$-2\lambda(1 + \vartheta + 2\varsigma) e^{(1-\lambda^2)} a_2 = \mathbb{P}_1^{(\alpha, \beta)}(y) d_1, \quad (2.12)$$

$$\frac{5}{2} \lambda^2 (1 + 2\vartheta + 6\varsigma) e^{(1-\lambda^2)} (2a_2^2 - a_3) = \mathbb{P}_1^{(\alpha, \beta)}(y) d_2 + \mathbb{P}_2^{(\alpha, \beta)}(y) d_1^2. \quad (2.13)$$

Adding (2.10) and (2.12), we get the following equation:

$$c_1 = -d_1. \quad (2.14)$$

Further, squaring and adding (2.10) and (2.12), we have

$$8\lambda^2(1 + \vartheta + 2\varsigma)^2 e^{2e^{(1-\lambda^2)}} a_2^2 = [\mathbb{P}_1^{(\alpha, \beta)}(y)]^2 (c_1^2 + d_1^2). \quad (2.15)$$

Then, the addition of (2.11) and (2.13) gives

$$5\lambda^2(1 + 2\vartheta + 6\varsigma) e^{e^{(1-\lambda^2)}} a_2^2 = \mathbb{P}_1^{(\alpha, \beta)}(y)(c_2 + d_2) + \mathbb{P}_2^{(\alpha, \beta)}(y)(c_1^2 + d_1^2). \quad (2.16)$$

From the above equations, we obtain

$$\left[\frac{5\lambda^2(1 + 2\vartheta + 6\varsigma) e^{e^{(1-\lambda^2)}} [\mathbb{P}_1^{(\alpha, \beta)}(y)]^2 - 8\lambda^2(1 + \vartheta + 2\varsigma)^2 e^{2e^{(1-\lambda^2)}} \mathbb{P}_2^{(\alpha, \beta)}(y)}{[\mathbb{P}_1^{(\alpha, \beta)}(y)]^2} \right] a_2^2 = \mathbb{P}_1^{(\alpha, \beta)}(y)(c_2 + d_2). \quad (2.17)$$

A small computation leads to

$$|a_2| \leq \frac{|A| \sqrt{2|A|}}{\lambda e^{\frac{1}{2}e^{(1-\lambda^2)}} \sqrt{|5(1 + 2\vartheta + 6\varsigma)A^2 - 8(1 + \vartheta + 2\varsigma)^2 B e^{e^{(1-\lambda^2)}}|}}. \quad (2.18)$$

To obtain the bound for $|a_3|$, we subtract (2.13) from (2.11), and using (2.14), we have

$$a_3 - a_2^2 = \frac{\mathbb{P}_1^{(\alpha, \beta)}(y)(c_2 - d_2)}{5\lambda^2(1 + 2\vartheta + 6\varsigma) e^{e^{(1-\lambda^2)}}}. \quad (2.19)$$

Then, with the aid of (2.15), we get

$$a_3 = \frac{\mathbb{P}_1^{(\alpha, \beta)}(y)(c_2 - d_2)}{5\lambda^2(1 + 2\vartheta + 6\varsigma) e^{e^{(1-\lambda^2)}}} + \frac{[\mathbb{P}_1^{(\alpha, \beta)}(y)]^2 (c_1^2 + d_1^2)}{8\lambda^2(1 + \vartheta + 2\varsigma)^2 e^{2e^{(1-\lambda^2)}}}. \quad (2.20)$$

After taking the modulus for $|a_3|$ and using (2.15), we get

$$|a_3| \leq \frac{2|A|}{5\lambda^2(1 + 2\vartheta + 6\varsigma) e^{e^{(1-\lambda^2)}}} + \frac{|A|^2}{4\lambda^2(1 + \vartheta + 2\varsigma)^2 e^{2e^{(1-\lambda^2)}}}. \quad (2.21)$$

Corollary 1.1. Let $\mathfrak{f} \in \sigma$, defined by (1.1), belong to the class $\mathcal{B}_\lambda^{\theta, \varsigma}(0, 0; y, \zeta)$. Then

$$|a_2| \leq \frac{|2y| \sqrt{4|y|}}{\lambda e^{\frac{1}{2}e^{(1-\lambda^2)}} \sqrt{|20(1 + 2\vartheta + 6\varsigma)y^2 - 8(1 + \vartheta + 2\varsigma)^2(12y^2 - 4)e^{e^{(1-\lambda^2)}}|}}$$

and

$$|a_3| \leq \frac{4|y|}{5\lambda^2(1 + 2\vartheta + 6\varsigma) e^{e^{(1-\lambda^2)}}} + \frac{|2y|^2}{4\lambda^2(1 + \vartheta + 2\varsigma)^2 e^{2e^{(1-\lambda^2)}}}.$$

Corollary 1.2. Let $\mathfrak{f} \in \sigma$, defined by (1.1), belong to the class $\mathcal{B}_\lambda^{\theta, 0}(\alpha, \beta; y, \zeta)$. Then

$$|a_2| \leq \frac{|A| \sqrt{2|A|}}{\lambda e^{\frac{1}{2}e^{(1-\lambda^2)}} \sqrt{|5(1 + 2\vartheta)A^2 - 8(1 + \vartheta)^2 B e^{e^{(1-\lambda^2)}}|}}$$

and

$$|a_3| \leq \frac{2|A|}{5\lambda^2(1 + 2\vartheta) e^{e^{(1-\lambda^2)}}} + \frac{|A|^2}{4\lambda^2(1 + \vartheta)^2 e^{2e^{(1-\lambda^2)}}}.$$

Corollary 1.3. Let $\mathfrak{f} \in \sigma$, defined by (1.1), belong to the class $\mathcal{B}_\lambda^{1,0}(\alpha, \beta; y, \zeta)$. Then

$$|a_2| \leq \frac{|A| \sqrt{2|A|}}{\lambda e^{\frac{1}{2}e^{(1-\lambda^2)}} \sqrt{|15A^2 - 32Be^{e^{(1-\lambda^2)}}|}}$$

and

$$|a_3| \leq \frac{2|A|}{15\lambda^2 e^{e^{(1-\lambda^2)}}} + \frac{|A|^2}{16\lambda^2 e^{2e^{(1-\lambda^2)}}}.$$

Corollary 1.4. Let $\mathfrak{f} \in \sigma$, defined by (1.1), belong to the class $\mathcal{B}_\lambda^{0,0}(\alpha, \beta; y, \zeta)$. Then

$$|a_2| \leq \frac{|A| \sqrt{2|A|}}{\lambda e^{\frac{1}{2}e^{(1-\lambda^2)}} \sqrt{|5A^2 - 8Be^{e^{(1-\lambda^2)}}|}}$$

and

$$|a_3| \leq \frac{2|A|}{5\lambda^2 e^{e^{(1-\lambda^2)}}} + \frac{|A|^2}{4\lambda^2 e^{2e^{(1-\lambda^2)}}}.$$

2.1. Fekete–Szegő problem for the functions $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$

In this section, we derive a sharp bound for the functional modulus

$$|a_3 - \eta a_2^2|, \quad 0 \leq \eta \leq 1,$$

where η is a real parameter. This problem is known as the classical Fekete–Szegő problem [7, 19]. We also obtain bounds for this linear functional for functions belonging to the class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$.

Theorem 2. Let $\mathfrak{f} \in \sigma$, given by (1.1), belong to the class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(\alpha, \beta; y, \zeta)$. Then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|A|}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e^{(1-\lambda^2)}}}, & \text{if } 0 \leq |\eta - 1| \leq \phi, \\ \frac{2|A|^3|1 - \eta|}{\lambda^2 e^{e^{(1-\lambda^2)}} |5(1 + 2\vartheta + 6\varsigma)A^2 - 8(1 + \vartheta + 2\varsigma)^2 B e^{e^{(1-\lambda^2)}}|}, & \text{if } |\eta - 1| \geq \phi, \end{cases}$$

where

$$\phi = \frac{|5(1 + 2\vartheta + 6\varsigma)A^2 - 8(1 + \vartheta + 2\varsigma)^2 B e^{e^{(1-\lambda^2)}}|}{5(1 + 2\vartheta + 6\varsigma)|A|^2},$$

with

$$A = (\alpha - \beta) + (\alpha + \beta + 2)y$$

and

$$B = (\alpha - \beta)^2 - (\alpha - \beta) - 4 + 2(\alpha - \beta)(\alpha + \beta + 3)y + (\alpha + \beta + 3)(\alpha + \beta + 4)y^2.$$

Proof. From (2.19), we have

$$a_3 - \eta a_2^2 = (1 - \eta)a_2^2 + \frac{\mathbb{P}_1^{(\alpha, \beta)}(y)(c_2 - d_2)}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e(1-\lambda^2)}}. \quad (2.22)$$

Using (2.17) in (2.22), we get

$$a_3 - \eta a_2^2 = \frac{(1 - \eta)(c_2 + d_2)[\mathbb{P}_1^{(\alpha, \beta)}(y)]^3}{\lambda^2 e^{e(1-\lambda^2)} \left(5(1 + 2\vartheta + 6\varsigma)[\mathbb{P}_1^{(\alpha, \beta)}(y)]^2 - 8(1 + \vartheta + 2\varsigma)^2 e^{e(1-\lambda^2)} \mathbb{P}_2^{(\alpha, \beta)}(y) \right)} + \frac{\mathbb{P}_1^{(\alpha, \beta)}(y)(c_2 - d_2)}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e(1-\lambda^2)}}. \quad (2.23)$$

By using (2.23), we have

$$a_3 - \eta a_2^2 = \mathbb{P}_1^{(\alpha, \beta)}(y) \left[h(\eta) + \frac{1}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e(1-\lambda^2)}} \right] c_2 + \mathbb{P}_1^{(\alpha, \beta)}(y) \left[h(\eta) - \frac{1}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e(1-\lambda^2)}} \right] d_2,$$

where

$$h(\eta) = \frac{(1 - \eta)[\mathbb{P}_1^{(\alpha, \beta)}(y)]^2}{\lambda^2 e^{e(1-\lambda^2)} \left[5(1 + 2\vartheta + 6\varsigma)[\mathbb{P}_1^{(\alpha, \beta)}(y)]^2 - 8(1 + \vartheta + 2\varsigma)^2 e^{e(1-\lambda^2)} \mathbb{P}_2^{(\alpha, \beta)}(y) \right]}.$$

Taking the modulus, we have

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|A|}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e(1-\lambda^2)}}, & \text{if } 0 \leq |h(\eta)| \leq \frac{1}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e(1-\lambda^2)}}, \\ 2|A||h(\eta)|, & \text{if } |h(\eta)| \geq \frac{1}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e(1-\lambda^2)}}. \end{cases}$$

Corollary 2.1. Let $\mathfrak{f} \in \sigma$, defined by (1.1), belong to the class $\mathcal{B}_\lambda^{\vartheta, \varsigma}(0, 0; y, \zeta)$. Then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{4|y|}{5\lambda^2(1 + 2\vartheta + 6\varsigma)e^{e(1-\lambda^2)}}, & \text{if } 0 \leq |\eta - 1| \leq \phi_0, \\ \frac{16|y|^3|1 - \eta|}{\lambda^2 e^{e(1-\lambda^2)} \left| 20(1 + 2\vartheta + 6\varsigma)y^2 - 8(1 + \vartheta + 2\varsigma)^2(12y^2 - 4)e^{e(1-\lambda^2)} \right|}, & \text{if } |\eta - 1| \geq \phi_0, \end{cases}$$

where

$$\phi_0 = \frac{\left| 20(1 + 2\vartheta + 6\varsigma)y^2 - 8(1 + \vartheta + 2\varsigma)^2(12y^2 - 4)e^{e(1-\lambda^2)} \right|}{20(1 + 2\vartheta + 6\varsigma)y^2}.$$

Corollary 2.2. Let $\mathfrak{f} \in \sigma$, defined by (1.1), belong to the class $\mathcal{B}_\lambda^{\vartheta, 0}(\alpha, \beta; y, \zeta)$. Then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|A|}{5\lambda^2(1 + 2\vartheta)e^{e(1-\lambda^2)}}, & \text{if } 0 \leq |\eta - 1| \leq \phi_1, \\ \frac{2|A|^3|1 - \eta|}{\lambda^2 e^{e(1-\lambda^2)} \left| 5(1 + 2\vartheta)A^2 - 8(1 + \vartheta)^2 B e^{e(1-\lambda^2)} \right|}, & \text{if } |\eta - 1| \geq \phi_1, \end{cases}$$

where

$$\phi_1 = \frac{\left| 5(1 + 2\vartheta)A^2 - 8(1 + \vartheta)^2 B e^{e(1-\lambda^2)} \right|}{5(1 + 2\vartheta)|A|^2}.$$

Corollary 2.3. Let $f \in \sigma$, defined by (1.1), belong to the class $\mathcal{B}_\lambda^{1,0}(\alpha, \beta; y, \zeta)$. Then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|A|}{15\lambda^2 e^{e(1-\lambda^2)}}, & \text{if } 0 \leq |\eta - 1| \leq \phi_2, \\ \frac{2|A|^3|1-\eta|}{\lambda^2 e^{e(1-\lambda^2)} |15A^2 - 32Be^{e(1-\lambda^2)}}}, & \text{if } |\eta - 1| \geq \phi_2, \end{cases}$$

where

$$\phi_2 = \frac{|15A^2 - 32Be^{e(1-\lambda^2)}|}{15|A|^2}.$$

Corollary 2.4. Let $f \in \sigma$, defined by (1.1), belong to the class $\mathcal{B}_\lambda^{0,0}(\alpha, \beta; y, \zeta)$. Then

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|A|}{5\lambda^2 e^{e(1-\lambda^2)}}, & \text{if } 0 \leq |\eta - 1| \leq \phi_3, \\ \frac{2|A|^3|1-\eta|}{\lambda^2 e^{e(1-\lambda^2)} |5A^2 - 8Be^{e(1-\lambda^2)}}}, & \text{if } |\eta - 1| \geq \phi_3, \end{cases}$$

where

$$\phi_3 = \frac{|5A^2 - 8Be^{e(1-\lambda^2)}|}{5|A|^2}.$$

The articles [1, 15] demonstrate the process of image enhancement using the Sakaguchi-type function in the spatial domain, implemented on MATLABTM software. However, this approach is limited to a specific domain. To overcome this limitation, we have improved upon this idea by using a subclass of bi-univalent functions in the spatial domain, implemented with the Python OpenCV module. This approach allows us to work within a broader domain, providing various ideas based on different aspects of image enhancement.

3. Applications of the Bell–Jacobi bi-univalent image enhancement algorithm (BJBIEA) model in image enhancement

In this research, we present the Bell–Jacobi bi-univalent image-enhancement algorithm (BJBIEA) model based on Jacobi polynomials involving the Bell distribution, which can be used in digital image processing and signal processing. The proposed algorithm (BJBIEA) model achieves remarkable results. This section presents a mathematical study based on the coefficient derived for the given class $\mathcal{B}_\lambda^{\theta, \varsigma}(\alpha, \beta; y, \zeta)$ using the coefficient a_n . The mask window coefficients a_1, a_2, a_3 are represented in four directional 3×3 matrices corresponding to the orientations $0^\circ, 45^\circ, 90^\circ$, and 135° . For example, the horizontal mask is given by

$$M_{0^\circ} = \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_{45^\circ} = \begin{bmatrix} a_3 & 0 & 0 \\ a_2 & 0 & 0 \\ a_1 & 0 & 0 \end{bmatrix}, \quad M_{90^\circ} = \begin{bmatrix} 0 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & a_3 & 0 \end{bmatrix}, \quad M_{135^\circ} = \begin{bmatrix} a_3 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1 \end{bmatrix},$$

$$a_1 = 1, \quad a_2 = \frac{|P_1^{(\alpha,\beta)}(y)| \sqrt{2|P_1^{(\alpha,\beta)}(y)|}}{e \sqrt{|15[P_1^{(\alpha,\beta)}(y)]^2 - 32[P_2^{(\alpha,\beta)}(y)]e|}}, \quad a_3 = \frac{2|P_1^{(\alpha,\beta)}(y)|}{5e} + \frac{|P_1^{(\alpha,\beta)}(y)|^2}{4e},$$

while the remaining masks are obtained by appropriate rotation of these coefficients. The use of multiple orientations prevents directional bias and allows structural features to be enhanced more uniformly.

To ensure that the convolution process remains stable, each mask is normalized by the sum of the absolute values of its entries,

$$\tilde{M} = \frac{M}{\sum_{i,j} |M_{ij}|}.$$

This normalization step guarantees that the overall magnitude of the response remains controlled and that no artificial amplification of image energy occurs. The boundedness of the higher-order terms limits the curvature and oscillatory behavior of the mapping. From a geometric viewpoint, this means that the function cannot exhibit excessive local variation. In digital image analysis, local variation in intensity is captured through discrete convolution operators. A convolution mask may be viewed as a finite-dimensional analogue of a differential operator, responding to changes in pixel intensity across neighboring locations. If the mask weights are chosen in accordance with analytically bounded coefficients, then the resulting discrete operator inherits a degree of regularity. For this reason, the coefficients a_1, a_2 , and a_3 derived in Section 3 are employed as the fundamental weights in the construction of the enhancement masks. In this way, the analytic properties of the subclass influence the computational procedure in a direct and transparent manner.

Let $I(x, y)$ denote a grayscale image defined on a finite grid. Each normalized mask is applied to the image via discrete convolution,

$$I_\theta(x, y) = (\tilde{M}_\theta * I)(x, y), \quad \theta \in \{0^\circ, 45^\circ, 90^\circ, 135^\circ\}.$$

Since each directional mask emphasizes variations along a specific orientation, relying on a single response could introduce anisotropic artifacts. To mitigate this, we define the aggregated response

$$I_{\text{avg}}(x, y) = \frac{1}{4} \sum_{\theta} I_\theta(x, y).$$

This averaging procedure produces a more balanced enhancement, reinforcing structural features while reducing orientation-specific bias.

Although the averaged response highlights structural details, direct substitution may alter the overall intensity distribution more than desired. To regulate the enhancement strength, we introduce a convex blending model,

$$I_{\text{final}}(x, y) = \mu I_{\text{avg}}(x, y) + (1 - \mu)I(x, y), \quad \mu \in (0, 1).$$

In the present implementation, the parameter is fixed as $\mu = 0.7$, which provides a moderate enhancement effect while preserving the essential characteristics of the original image.

Because the final output is a convex combination of the original and enhanced responses, its deviation from the original remains controlled. The intensities are then restricted to the interval $[0, 255]$ to ensure compatibility with digital image representation.

Algorithm 1 Bell–Jacobi bi-univalent image-enhancement algorithm (BJBIEA)

Input: Grayscale input image I ; coefficients a_1, a_2, a_3 ; blending parameter μ

Output: Enhanced image I_{final}

1. Construct four directional 3×3 masks using the coefficients a_1, a_2, a_3
 2. Normalize each mask to ensure a stable response
 3. Apply each directional mask to the image through convolution
 4. Compute the average of the four directional responses
 5. Blend the averaged response with the original image using parameter μ
 6. Adjust intensity values to remain within the valid range
 7. **return** I_{final}
-

Remark. The Bell–Jacobi bi-univalent image-enhancement framework transforms analytic coefficient constraints into a stable discrete spatial operator. The theoretical boundedness of the subclass guarantees regulated enhancement while maintaining critical image structures. So, the method combines mathematical rigour with stability and consistency in the real world.

4. Comparative analysis

This section outlines the image dataset, implementation protocol, and evaluation criteria that were used to test the proposed Bell–Jacobi framework. We chose three grayscale images: A natural scene (Dog), a periodic geometric texture (Argyle), and a medical radiographic image (X-ray). These images show different types of structures. The natural image shows that the texture and edges remain stable, the Argyle image confirms that the direction remains consistent and the sharp boundary stays intact, and the X-ray image explains how the enhancement works in low-contrast medical settings. We only looked at Figure 2, which show changes in structural intensity by processing all the images in grayscale. Using convolution masks based on the theoretical coefficients $|a_1|$, $|a_2|$, and $|a_3|$ averaging, the proposed Bell–Jacobi bi-univalent image-enhancement algorithm (BJBIEA) was put into action in the spatial domain to make the output better. The final enhanced output was obtained using a convex blending strategy. Four directional 3×3 masks (0° , 45° , 90° , 135°) were used and adjusted to make sure that the intensity amplification stayed within certain limits. The enhanced output was finally made by averaging the directional responses.

$$I_{\text{final}} = \mu I_{\text{avg}} + (1 - \mu)I,$$

where the blending parameter was set to $\mu = 0.7$ for all tests. Using a fixed parameter makes sure that results can be repeated and prevents adaptive tuning, which could affect the results of a comparison. To keep the output image compatible with standard digital image representation, the intensity range was limited to the acceptable range of $[0, 255]$. It is important to stress that all of the images used the same coefficients, masks, and blending settings. Because of this, the enhancement process is deterministic and can be repeated exactly. Figure 3 shows the performance of the proposed enhancement framework was evaluated using standard quantitative image-quality measures, including the peak signal-to-noise ratio (PSNR), structural similarity index measure (SSIM) [21], mean squared error (MSE), root mean squared error (RMSE), mean absolute error (MAE), and the pearson correlation coefficient (PCC).

These metrics were computed according to their conventional mathematical definitions. The PSNR and MSE assess overall intensity deviation between the original and enhanced images, the SSIM evaluates structural similarity, the PCC measures global linear correlation, and the RMSE and MAE quantify average error magnitude. The combined use of these measures provides a comprehensive assessment of both intensity fidelity and structural preservation.

Tables 2 and 3 show that the suggested model combines directional filtering with linear blending to guarantee balanced contrast enhancement, in contrast to traditional methods that result in over-brightness or noise amplification. Strong fidelity to the original image is demonstrated by the Dog image's high PSNR value of 40.0789 dB. The framework exhibits better overall performance when compared to current methods like [1, 8]. Strong pixel-level correlation and outstanding structural preservation are confirmed by mesh representation Figure 4 SSIM values greater than 0.97 and PCC values greater than 0.996. Furthermore, robust stability and little reconstruction error are indicated by low MSE, RMSE, and MAE values.

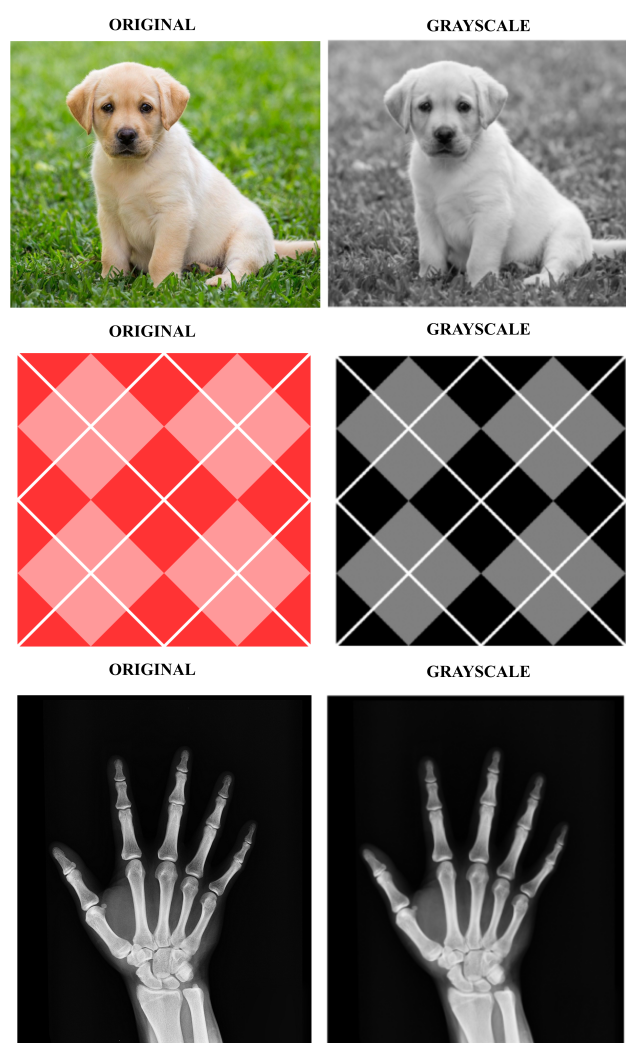
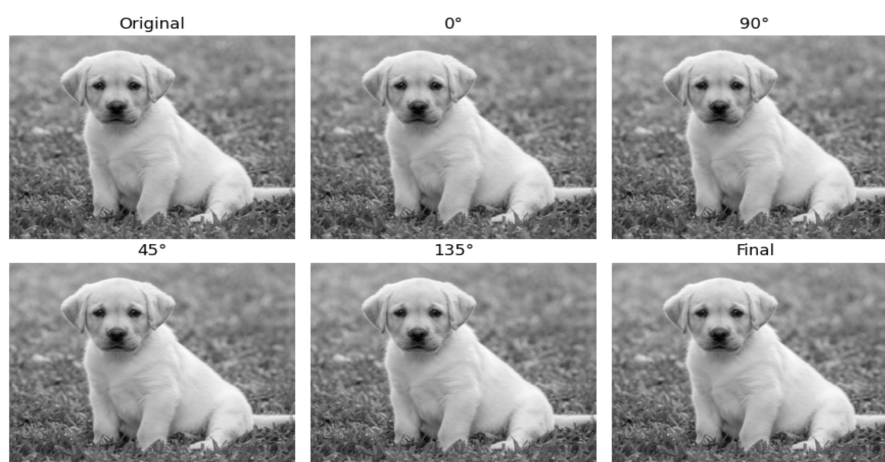
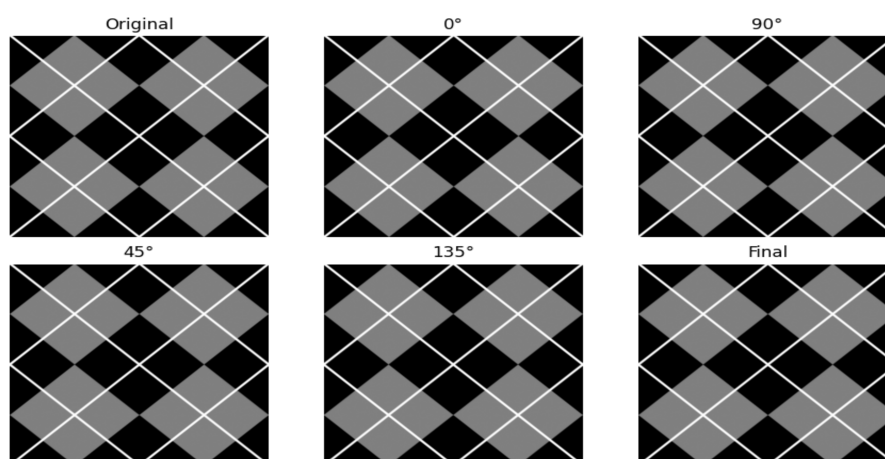


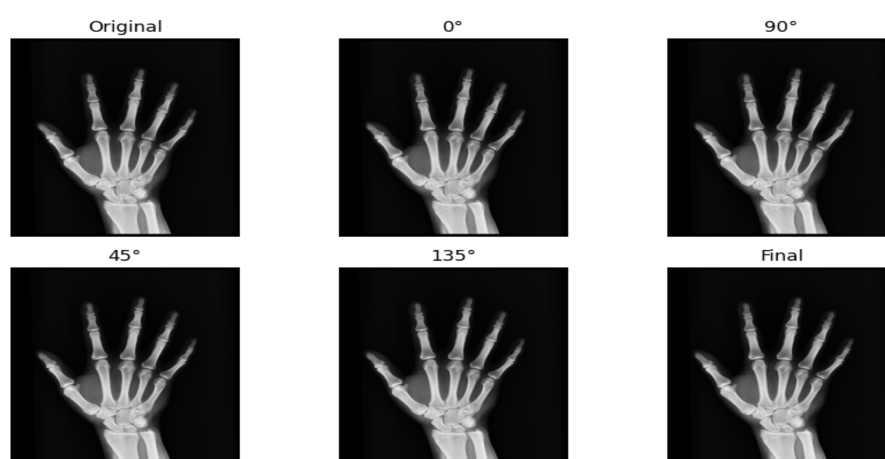
Figure 2. Original grayscale input images: Dog, Argyle, and X-ray.



(a) Dog image



(b) Argyle image



(c) X-ray image

Figure 3. Enhancement of the images at different angles.

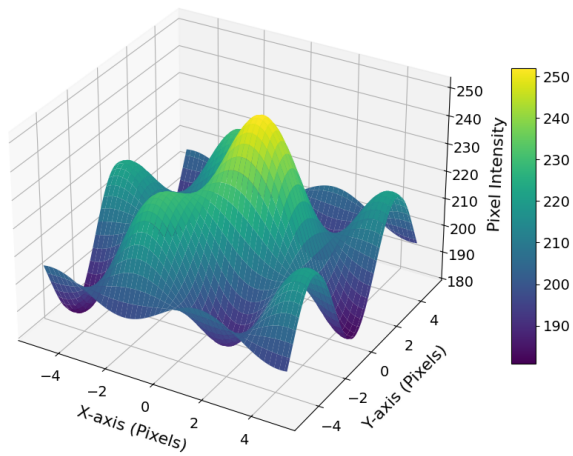
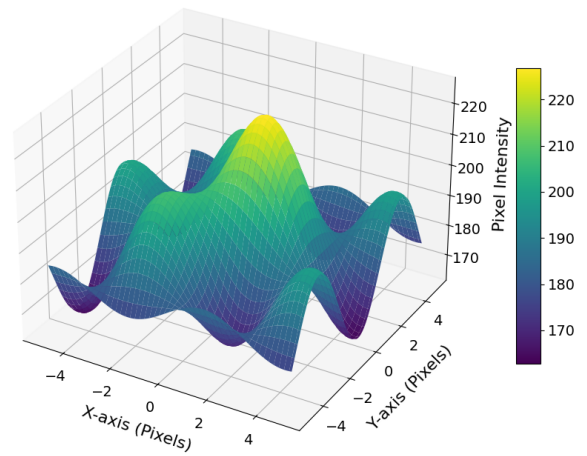
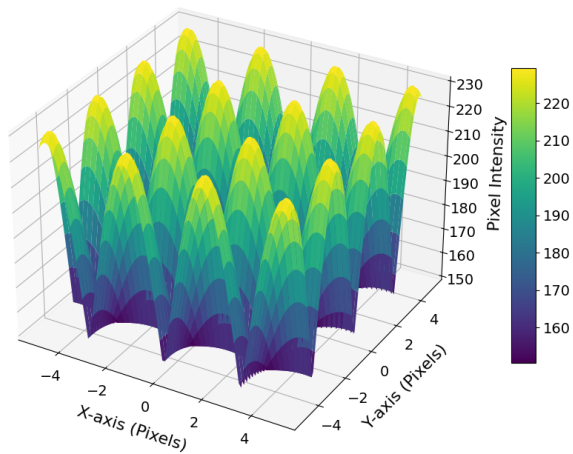
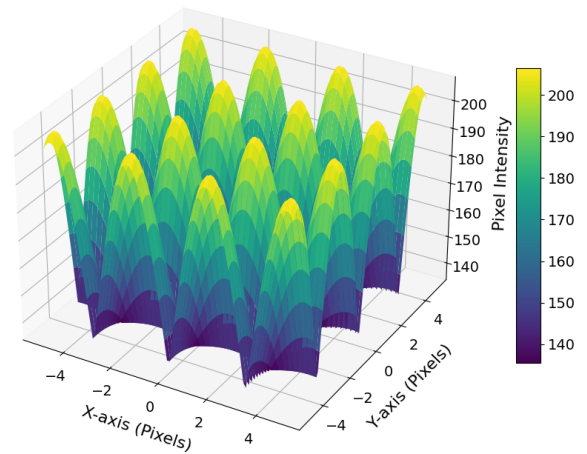
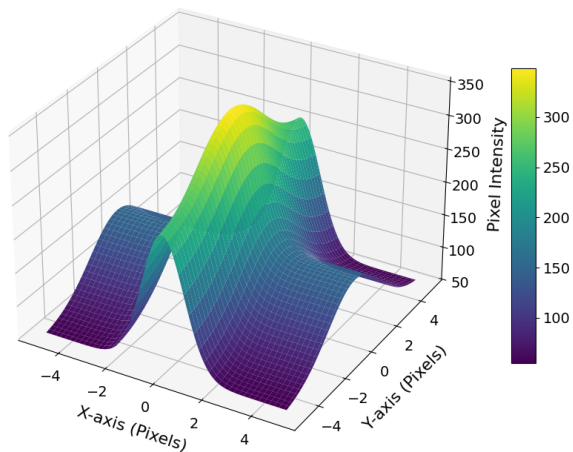
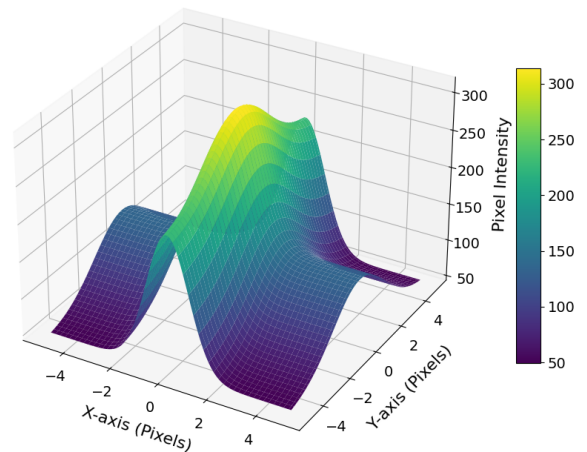
Mesh Representation of DOG - Original**Mesh Representation of DOG - Grayscale****Mesh Representation of ARGYLE - Original****Mesh Representation of ARGYLE - Grayscale****Mesh Representation of XRAY - Original****Mesh Representation of XRAY - Grayscale****Figure 4.** Original grayscale input images: Dog, Argyle, and X-ray.

Table 2. Quantitative performance of the proposed BJBIEA method.

Metric	X-ray.jpg	Argyle.png	Dog.jpg
PSNR (dB)	38.9874	36.9788	40.0789
SSIM	0.9718	0.9905	0.9811
PCC	0.9989	0.9964	0.9984
MSE	8.210038	13.037556	6.385436
RMSE	2.8653	3.6108	2.5269
MAE	1.237993	0.614015	1.469841

Table 3. Comparison of the PSNR and SSIM values for different enhancement methods.

Dataset	Methods	PSNR	SSIM
Dog	Aarthy et al. [1]	23.7637	0.9236
	Hankel [8]	27.37	0.9368
	Proposed	40.0789	0.9811
Argyle	Aarthy et al. [1]	23.4281	0.9660
	Hankel [8]	35.68	0.9744
	Proposed	36.9788	0.9905
X-Ray	Aarthy et al. [1]	29.1316	0.9244
	Hankel [8]	33.86	0.9328
	Proposed	38.9874	0.9718

5. Conclusions

In this research work, we introduce a novel subclass $\mathcal{B}_\lambda^{\theta, \varsigma}(\alpha, \beta; y, \zeta)$ of bi-univalent functions associated with the Bell distribution by using Jacobi polynomials. Based on the obtained coefficient restrictions, we proposed a new computational framework, namely, the Bell–Jacobi bi-univalent image enhancement algorithm (BJBIEA), formulated through a structured 3×3 spatial-domain mask model. The proposed approach improves brightness, contrast, and visual clarity while preserving the structural consistency of the image. Experimental results demonstrate the stability and effectiveness of the method when compared with existing coefficient-based enhancement techniques. This research connects the geometric function theory and real-world image-processing techniques and opens the path for further research on image enhancement, sharpening, and edge detection. Future work will involve extending the proposed framework toward detail enhancement, edge extraction, and image resolution quality.

Author contributions

Vanithakumari Balasubramaniam: Conceptualization, investigation, validation, methodology, data interpretation and visualization, writing—original draft, resource management for the project;

Selvaraj Palanisamy: Conceptualization, investigation, validation, methodology, data interpretation and visualization, writing—original draft, experimentation, supervision; Seran Araci: Validation, methodology, manuscript review and editing. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

The authors gratefully acknowledge Sri Mata Amritanandamayi Devi (Amma), Chancellor, Amrita Vishwa Vidyapeetham, for her inspiration and for providing financial support for the article processing charges (APC) of this publication.

Conflict of interests

The authors declare that they possess no financial interests or personal connections that could have affected the work described in this paper.

Data availability

The data sets of Dog, Argyle, and X-ray are available in the Kaggle repository: <https://www.kaggle.com/c/ultrasound-nerve-segmentation/data/?select=sample>.

References

1. B. Aarthy, B. S. Keerthi, Enhancement of various images using coefficients obtained from a class of Sakaguchi type functions, *Sci. Rep.*, **13** (2023), 18722. <https://doi.org/10.1038/s41598-023-45938-y>
2. E. T. Bell, Exponential numbers, *Am. Math. Mon.*, **41** (1934), 411–419. <https://doi.org/10.1080/00029890.1934.11987615>
3. D. A. Brannan, J. Clunie, W. E. Kirwan, Coefficient estimates for a class of star-like functions, *Can. J. Math.*, **22** (1970), 476–485. <https://doi.org/10.4153/CJM-1970-055-8>
4. F. Castellares, S. L. P. Ferrari, A. J. Lemonte, On the Bell distribution and its associated regression model for count data, *Appl. Math. Model.*, **56** (2018), 172–185. <https://doi.org/10.1016/j.apm.2017.12.014>
5. L. Chen, T. Kim, D. S. Kim, H. Lee, S. H. Lee, Probabilistic degenerate central Bell polynomials, *Math. Comp. Model. Dyn.*, **30** (2024), 523–542. <https://doi.org/10.1080/13873954.2024.2358899>
6. U. Duran, Central Bell-based type 2 Bernoulli polynomials of order β , *Fund. J. Math. Appl.*, **8** (2025), 55–64.
7. M. Fekete, G. Szegő, Eine bemerkung über ungerade schlichte funktionen, *J. Lond. Math. Soc.*, **1–8** (1933), 85–89. <https://doi.org/10.1112/jlms/s1-8.2.85>
8. B. Kanwal, A. Iman, S. Kanwal, A. K. Alkhalifa, Estimation of Hankel inequalities of symmetric starlike functions in crescent-shaped domains and their application in image processing, *Sci. Rep.*, **15** (2025), 27402. <https://doi.org/10.1038/s41598-025-12935-2>

9. D. S. Kim, S. Bu, H. Lee, M. Khalil, T. Kim, Spivey's type recurrence relation for Lah–Bell polynomials, *Math. Comp. Model. Dyn.*, **31** (2025), 2547873. <https://doi.org/10.1080/13873954.2025.2547873>
10. P. Koebe, Über die konforme Abbildung des Einheitskreises, *J. Reine Angew. Math.*, **147** (1916), 205–237.
11. V. Kumar, N. E. Cho, V. Ravichandran, H. M. Srivastava, Sharp coefficient bounds for starlike functions associated with the Bell numbers, *Math. Slovaca*, **69** (2019), 1053–1064. <https://doi.org/10.1515/ms-2017-0289>
12. M. Lewin, On a coefficient problem for bi-univalent functions, *P. Am. Math. Soc.*, **18** (1967), 63–68. <https://doi.org/10.2307/2035225>
13. M. Markova, V. Guldan, *Jacobi polynomials and some related functions*, In: Mathematical Methods in Engineering, Dordrecht: Springer, 2014, 219–227. https://doi.org/10.1007/978-94-007-7183-3_20
14. E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Ration. Mech. An.*, **32** (1969), 100–112. <https://doi.org/10.1007/BF00247676>
15. E. K. Nithiyanandham, B. S. Keerthi, Image edge detection enhancement using coefficients of Sakaguchi type functions mapped onto petal-shaped domain, *Heliyon*, **10** (2024), e31430. <https://doi.org/10.1016/j.heliyon.2024.e31430>
16. G. Saravanan, S. Baskaran, B. Vanithakumari, L. Alnaji, T. G. Shaba, I. Al-Shbeil, et al., Bernoulli polynomials for a new subclass of Te-univalent functions, *Heliyon*, **10** (2024), e33953. <https://doi.org/10.1016/j.heliyon.2024.e33953>
17. G. Saravanan, S. Baskaran, B. Vanithakumari, S. Bulut, Sakaguchi type functions defined by balancing polynomials, *Math. Bohem.*, **150** (2025), 71–83. <http://doi.org/10.21136/MB.2024.0173-23>
18. G. Saravanan, B. Sudharsanan, B. Vanithakumari, S. Altınkaya, Bi-univalent characteristics for a particular class of Bazilevič functions generated by convolution using balancing polynomials, *Jordan J. Math. Stat.*, **18** (2025), 381–392. Available from: <https://jjms.yu.edu.jo/index.php/jjms/article/view/1417>.
19. H. M. Srivastava, N. Raza, E. S. A. AbuJarad, G. Srivastava, M. H. AbuJarad, Fekete-Szegő inequality for classes of (p,q)-Starlike and (p,q)-convex functions, *RACSAM Rev. R. Acad. A*, **113** (2019), 3563–3584. <https://doi.org/10.1007/s13398-019-00713-5>
20. B. Vanithakumari, G. Saravanan, B. Sudharsanan, S. Yalçın, Horadam polynomials for a new subclass of Sakaguchi-type bi-univalent functions defined by (p, q) -derivative operator, *Commun. Korean Math. S.*, **39** (2024), 461–470.
21. Z. Wang, A. C. Bovik, Structural similarity index measure, *IEEE Signal Proc. Let.*, **9** (2002), 81–84.
22. P. Xue, Y. Ma, T. Kim, D. S. Kim, W. Zhang, Probabilistic degenerate poly-Bell polynomials associated with random variables, *Math. Comp. Model. Dyn.*, **31** (2025), 2497367. <https://doi.org/10.1080/13873954.2025.2497367>

