



Research article

Copulas and complex analysis: Holomorphic extensions, rigidity and regularity

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Abstract: The rigidity of holomorphic functions leads us, after exploring the harmonic aspect of the copulas in our previous work, to the possibility of making them more flexible to better suit analytical treatment. This requires making concessions on the boundary conditions and exploring the price to be paid in terms of the loss of stochastic information. Two ways to address the complexification procedure are developed.

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1. Introduction

A well-known technique in mathematical analysis involves dealing with a fairly high degree of generalization. This is the case when one wishes to establish a result in real analysis and uses a complex variable. Classically, one either performs a regular transformation such as the Fourier or Laplace transformation, or one studies the possibility of analytic extension when the real variable is replaced by a complex variable, but one with a certain degree of regularity (see [1–3] and references therein).

A fruitful perspective in both probability theory and complex analysis is to interpret the fundamental objects as structure-preserving maps. This perspective is a natural continuation of Sani et al.'s results on general preserver problems [4, 5].

In complex analysis, holomorphic functions are the natural morphisms between complex domains. Let $\Omega \subset \mathbb{C}$ be an open set. A function $f : \Omega \rightarrow \mathbb{C}$ is said to be holomorphic if it is complex differentiable at all points of Ω . This condition is highly restrictive: Holomorphic functions are infinitely differentiable and analytic, and satisfy strong rigidity properties such as the identity principle and analytic continuation. Moreover, away from critical points, they are conformal, i.e., they preserve angles and the local complex structure.

Thus, holomorphic maps preserve the intrinsic complex structure of domains and may be regarded as the canonical morphisms in the category of complex manifolds.

Now we switch to probability theory, where copulas play an analogous structural role. We refer to [6–8] for a survey on this topic. For a recent contribution on perturbations that preserve the copula property, we refer to [9] and for a link with optimization theory, one may see [10].

Let us recall the elementary definition of a copula in the bivariate context.

Definition 1.1. A copula is a function

$$C : [0, 1]^2 \rightarrow [0, 1]$$

satisfying the following properties:

- *Uniform marginals:* For all $x, y \in [0, 1]$, $C(x, 1) = x$, $C(1, y) = y$;
- *Groundedness:* For all $x, y \in [0, 1]$, $C(x, 0) = C(0, y) = 0$;
- *2-increasing property:* For all rectangles $[x_1, x_2] \times [y_1, y_2] \subset [0, 1]^2$,

$$\Delta C := C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \geq 0.$$

These conditions ensure that C defines a probability measure on $[0, 1]^2$. The fundamental result underlying this theory is Sklar's theorem, which states that any bivariate distribution function F with marginals F_X and F_Y can be written as

$$F(x, y) = C(F_X(x), F_Y(y)),$$

for some copula C . Let us rigorously enunciate this important result which ensures that copulas describe all dependence structures.

Theorem 1.2 (Sklar). [11] *Let F be a bivariate distribution with marginals F_1, F_2 . Then a copula C exists such that*

$$F(x, y) = C(F_1(x), F_2(y)).$$

In this framework, the copula C precisely captures the dependence structure between the random variables, independently of their marginals. Therefore, copulas may be viewed as canonical objects acting on probability distributions while preserving marginal constraints.

This leads to the following conceptual parallel: While holomorphic functions are appealed to play the role of copulas, the complex structure may explain the dependence structure. However, conformal transformations and analytic invariance will be linked to marginal invariance.

Let us now clarify how copulas are linked to operator theory. As reliable sources, we refer to [12–14]. Copulas and Markov operators are deeply connected through the study of dependence structures and stochastic transitions. A copula is a function that couples marginal distributions into

a multivariate distribution according to Sklar's theorem, whereas a Markov operator describes the evolution of probability measures or random variables through transition mechanisms. The bridge between these two notions appears naturally when one interprets a copula as the transition structure of a Markov process on the unit interval.

More precisely, let C be a bivariate copula on $[0, 1]^2$. One may associate C with a Markov operator T_C acting on suitable spaces of measurable functions by

$$(T_C f)(x) = \int_0^1 f(y) \partial_2 \partial_1 C(x, y) dy,$$

whenever the partial derivative exists in a suitable sense. Such an operator preserves positivity and constants, and therefore satisfies the axioms of a Markov operator. Conversely, many Markov operators on spaces such as $L^1([0, 1])$ or $L^\infty([0, 1])$ determine copulas through their transition kernels.

This correspondence permits the translation of probabilistic properties of copulas into operator-theoretic properties. For example, independence corresponds to the product copula $\Pi(u, v) = uv$, whose associated operator reduces to the expectation operator. Similarly, the comonotonic copula $M(u, v) = \min(u, v)$ corresponds to a deterministic transition operator. Consequently, properties such as mixing, ergodicity, idempotence, and invariance may be investigated through spectral and functional-analytic methods.

A fundamental concept in this theory is the Markov product of copulas, introduced by Darsow et al. Given two copulas A and B , their Markov product is defined by

$$(A * B)(x, y) = \int_0^1 \partial_2 A(x, t) \partial_1 B(t, y) dt.$$

This operation corresponds exactly to the composition of the associated Markov operators. It therefore provides an algebraic framework for the description of dependence propagation in Markov chains.

The operator-theoretic viewpoint has proved fruitful in the study of stationary Markov processes, dependence measures, stochastic dynamics, and evolution copulas. It also creates natural connections with harmonic analysis, ergodic theory, and functional analysis.

In both settings, the relevant objects are not arbitrary functions, but maps constrained by strong structural conditions. In this sense, copulas may be interpreted as probabilistic analogs of holomorphic maps: They isolate and preserve an intrinsic structure while factoring out extraneous data. In the current paper, we will not develop the operator aspect but focus on the complexification procedure. This latter technique will be detailed below.

2. Preliminaries

A deeper analogy between the two theories arises from their respective reconstruction principles, which assert that a global object can be recovered from partial data together with a structural constraint.

In complex analysis, the Cauchy integral formula states that if f is holomorphic in a domain Ω and $\partial\Omega$ is a sufficiently regular boundary, then for all $z \in \Omega$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

This formula expresses a fundamental rigidity property: The values of a holomorphic function inside the domain are entirely determined by its values on the boundary. It follows that holomorphic functions are uniquely determined by the boundary data, and any local information propagates globally via analytic continuation.

In probability theory, Sklar's theorem provides an analogous reconstruction mechanism. Given a joint distribution function F with marginals F_X and F_Y , a copula C exists such that

$$F(x, y) = C(F_X(x), F_Y(y)).$$

Thus, the joint distribution is reconstructed from its marginal distributions F_X and F_Y , a copula C encoding the dependence structure.

The analogy can be summarized as follows: The Cauchy formula reconstructs a holomorphic function from its boundary values, while Sklar's theorem reconstructs a joint distribution from its marginal laws.

In both cases, a global object is determined by lower-dimensional data together with a compatibility condition. This reveals a common structural paradigm: Global coherence arises from the interaction between partial information and a rigid underlying principle. In complex analysis, this principle is analyticity; in copula theory, it is the preservation of marginals together with the 2-increasing condition. In conclusion, both theories exhibit a remarkable interplay between rigidity, reconstruction, and structural invariance, suggesting a deeper unifying framework in which holomorphic functions and copulas play analogous roles in their respective domains.

3. A first attempt to regular complexification

3.1. Setting and notation

In this section, we consider a harmonic point of view consisting of extending copulas as real valued functions, whenever it is possible, to complex ones by classical quasi-holomorphic continuation and/or merely using Schwarz principle (see, among others, [15, 16]). To do so, we recall some facts on harmonic functions and clarify how they are linked to holomorphic functions.

We start with elementary definitions

Let $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ be an open connected set.

Definition 3.1. A function $u \in C^2(\Omega)$ is called harmonic if

$$\Delta u(x, y) := \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \quad \text{in } \Omega.$$

Definition 3.2. Let $u \in C^2(\Omega)$ be harmonic. A function $v \in C^2(\Omega)$ is called a harmonic conjugate of u if

$$f : z = x + iy \mapsto u(x, y) + iv(x, y)$$

is holomorphic in Ω .

It is well-known that this is equivalent to the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The main result linking harmonicity to holomorphy is given by the following theorem.

Theorem 3.3. [17] Let $u \in C^2(\Omega)$ be harmonic. Then for every $z_0 \in \Omega$, a neighbourhood O of z_0 and a function $v \in C^2(O)$ exist such that $u + iv$ is holomorphic in O .

With some additional geometrical properties of Ω , we have the more general result.

Theorem 3.4 (Global existence). If $\Omega \subset \mathbb{C}$ is simply connected and $u \in C^2(\Omega)$ is harmonic, then $v \in C^2(\Omega)$ exists, which is unique up to an additive constant, such that

$$f : z = x + iy \mapsto u(x, y) + iv(x, y)$$

is holomorphic in Ω .

It is easy, using the Cauchy–Riemann equations, to verify that for any holomorphic function $f = u + iv$ on Ω , the real and imaginary parts satisfy

$$\Delta(\operatorname{Re} f) = 0, \quad \Delta(\operatorname{Im} f) = 0.$$

Indeed, since f is holomorphic, it is, in particular, of class C^∞ (as recalled in the Introduction), so its mixed second-order partial derivatives commute by Schwarz’s theorem. Differentiating the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

once more, first with respect to x and then with respect to y , gives

$$u_{xx} = v_{yx}, \quad u_{yy} = -v_{xy}.$$

Adding these two identities and using $v_{xy} = v_{yx}$ yields

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

The same computation applied to the second Cauchy–Riemann equation, $v_x = -u_y$, differentiated with respect to x , and $v_y = u_x$, differentiated with respect to y , gives

$$v_{xx} = -u_{yx}, \quad v_{yy} = u_{xy},$$

and hence, again by $u_{xy} = u_{yx}$,

$$\Delta v = v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$$

This proves $\Delta(\operatorname{Re} f) = \Delta u = 0$ and $\Delta(\operatorname{Im} f) = \Delta v = 0$, which means that, in some sense, holomorphy implies harmonicity.

The converse fails in general, as shown by two classical examples: The map $z \mapsto \bar{z}$ (harmonic real and imaginary parts not satisfying Cauchy–Riemann conditions) and the multivalued argument function $u(x, y) = \log \sqrt{x^2 + y^2}$ (harmonic but without a global holomorphic conjugate).

For complex-valued harmonic functions, the natural definition of a harmonic function in literature is as follows.

Definition 3.5. Let $u, v : \Omega \rightarrow \mathbb{R}$ be real-valued functions on Ω , playing the role of the real and imaginary parts of a function

$$F = u + iv : \Omega \rightarrow \mathbb{C}.$$

The function F is called harmonic if both u and v are harmonic.

To state equivalence between harmonicity and holomorphy, the additional hypothesis given by Cauchy-Riemann equations are enough.

Proposition 3.6. *A complex-valued harmonic function is holomorphic if and only if its real and imaginary parts satisfy the Cauchy–Riemann equations.*

3.2. Harmonic copulas

We come back to copulas and especially harmonic ones. Harmonic static copulas concern static equilibrium states, whereas evolution copulas study dynamic relaxation toward equilibrium.

We survey harmonic copulas, emphasizing their analytic behavior and their interpretation as solutions to elliptic boundary value problems. We also recall key ideas from recent works on evolution copulas, notably those developed in [5, 18], where time-dependent copulas arise from evolution equations and semigroup methods.

Copulas provide a canonical way to encode dependence structures independently of marginal distributions. While classical families are typically constructed through parametric formulas, a more analytic approach consists in defining copulas via partial differential equations.

Among these, harmonic copulas form a particularly natural class: They are solutions of the Laplace equation on the unit square with boundary conditions compatible with copula constraints. This connects dependence modeling with potential theory and elliptic partial differential equations.

More recently, evolution copulas have been introduced to describe time-dependence structures, often governed by parabolic-type equations or semigroup dynamics. Let us make precise the definition of the harmonic copula.

Definition 3.7. A copula C is said to be harmonic if

$$\Delta C = \partial_{xx}C + \partial_{yy}C = 0 \quad \text{in } (0, 1)^2.$$

Thus, harmonic copulas are the solutions of a Dirichlet problem:

$$\begin{cases} \Delta C = 0 & \text{in } (0, 1)^2, \\ C = \varphi & \text{on } \partial([0, 1]^2), \end{cases}$$

where φ satisfies compatibility conditions induced by copula axioms.

As any harmonic function, the analytic behavior of a harmonic copula in the interior of the square is so regular, as clarified in the following proposition.

Remark 3.8. Every harmonic copula on $\Omega \subset I^2$ may be therefore seen as a real part of a holomorphic function on Ω .

Proposition 3.9. *Every harmonic copula is smooth in the interior:*

$$C \in C^\infty((0, 1)^2).$$

Proof. This follows from classical elliptic regularity theory for the Laplace equation. $\square$

Remark 3.10. This smoothness implies that harmonic copulas exclude singular dependence structures such as sharp tail concentration.

A central feature of harmonic maps is the mean value property:

$$C(a) = \frac{1}{|B_r|} \int_{B_r(a)} C(t) dt,$$

where B_r denotes the ball centered in a and $|B_r|$ its volume.

Harmonic copulas model moderate, smooth dependence, and exhibit a smoothing effect in the following sense: Local irregularities in boundary data are averaged out in the interior. We aim to study the behavior on the boundaries since the behavior of a harmonic copula is entirely determined by its boundary values.

Unfortunately, we immediately face a rigidity obstruction. Every copula shares the same boundary data $C(x, 0) = C(0, y) = 0$, $C(x, 1) = x$, $C(1, y) = y$, requiring C to be harmonic on the whole open square $(0, 1)^2$ therefore forces C to solve a Dirichlet problem whose boundary data does not depend on the choice of copula. Since the independence copula $\Pi(x, y) = xy$ is itself harmonic and already satisfies this boundary data, the uniqueness of the Dirichlet problem on $(0, 1)^2$ forces $C = \Pi$. Thus Π is the only copula that can be harmonic on the whole square. The main goal of this section is to present a method for circumventing this rigidity by relaxing harmonicity to hold only on a subset $D_\varepsilon \subsetneq (0, 1)^2$ touching part, but not all, of $\partial(I^2)$, rather than on $(0, 1)^2$ itself, as illustrated in Figure 1 below.

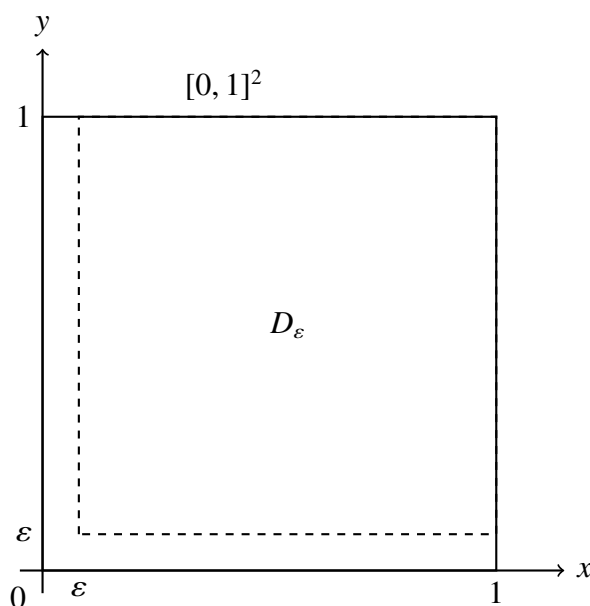


Figure 1. The window $D_\varepsilon = [\varepsilon, 1]^2$ inside the unit square $[0, 1]^2$.

In fact, one may construct a large family of copulas (different from Π) that are harmonic on D_ε , so seen as real parts of holomorphic functions on D_ε , but not harmonic on the whole square (see Example 3.12).

Proposition 3.11. *A harmonic function C , which satisfies the boundary condition in Definition (1.1), defines a copula if and only if*

$$\frac{\partial^2 C}{\partial x \partial y} \geq 0$$

in the sense of distributions.

Example 3.12. Let define

$$H(x, y) = \sin(\pi(1 - x)) \sinh(\pi(1 - y)), \quad (x, y) \in D_\varepsilon = [\varepsilon, 1]^2.$$

The function H is harmonic on D_ε , and vanishes on the two sides $\{x = 1\}$ and $\{y = 1\}$. Indeed, a direct computation gives

$$\frac{\partial^2 H}{\partial x^2} = -\pi^2 H,$$

while

$$\frac{\partial^2 H}{\partial y^2} = \pi^2 H.$$

Therefore

$$\Delta H = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0.$$

Hence, H is harmonic. Moreover, $H(1, y) = H(x, 1) = 0$.

Let $\alpha \in \mathbb{R}$ and define

$$C_\varepsilon(x, y) = xy + \alpha H(x, y).$$

Since both Π and H are harmonic, the function C_ε is harmonic on D_ε .

Let

$$\chi \in C^\infty([0, 1]),$$

such that:

- $0 \leq \chi \leq 1$.
- $\chi(t) = 1$ for $t \in [\varepsilon, 1]$.
- $\chi(t) = 0$ for $t \in [0, \frac{\varepsilon}{2}]$.

Define

$$\eta(x, y) := \chi(x)\chi(y).$$

Now we are able to give a more general construction process. Let us define the function $\widetilde{C} : I^2 \rightarrow \mathbb{R}$ by

$$\widetilde{C}(x, y) = xy + \eta(x, y)(C_\varepsilon(x, y) - xy).$$

Equivalently,

$$\widetilde{C}(x, y) = xy + \alpha \eta(x, y) H(x, y).$$

Explicitly,

$$\widetilde{C}(x, y) = xy + \alpha \chi(x)\chi(y) \sin(\pi(1 - x)) \sinh(\pi(1 - y))$$

for every $(x, y) \in I^2$.

The function $\widetilde{C}$ satisfies:

- (1) $\widetilde{C} = C_\varepsilon$ on D_ε .
- (2) $\widetilde{C}(x, 0) = \widetilde{C}(0, y) = 0$.
- (3) $\widetilde{C}(x, 1) = x$, $\widetilde{C}(1, y) = y$.
- (4) $\widetilde{C} \in C^\infty(I^2)$.

(5) For sufficiently small $|\alpha|$, $\tilde{C}$ is 2-increasing on I^2 . Indeed, we compute

$$\frac{\partial^2 \tilde{C}}{\partial x \partial y} = 1 + \alpha \frac{\partial^2 (\eta H)}{\partial x \partial y}.$$

Since ηH is smooth on the compact square I^2 , $K > 0$ exists such that

$$\left| \frac{\partial^2 (\eta H)}{\partial x \partial y} \right| \leq K.$$

Therefore, if

$$|\alpha| < \frac{1}{K},$$

then

$$\frac{\partial^2 \tilde{C}}{\partial x \partial y} > 0$$

on I^2 . Thus $\tilde{C}$ is 2-increasing.

Therefore, many infinite copulas exist which are harmonic on D_ε but not harmonic on I^2 . Moreover, for very small values of ε , the copula takes small values in the strips $I^2 \setminus D_\varepsilon$.

We now construct a copula which is harmonic on an interior square K_ε , while its harmonic component admits no harmonic continuation across any nonempty open arc of ∂K_ε . Thus, every regular boundary point of K_ε is an obstruction point for harmonic continuation, as depicted in Figure 2 below.

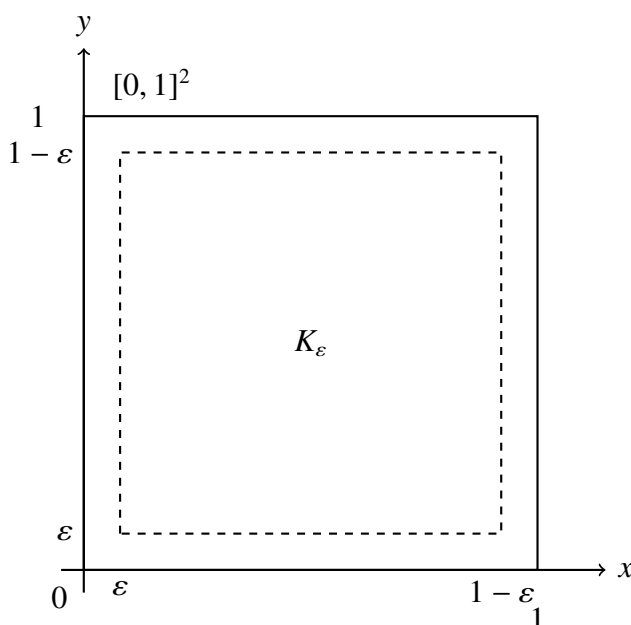


Figure 2. The window $K_\varepsilon = [\varepsilon, 1 - \varepsilon]^2$ inside the unit square $[0, 1]^2$.

Example 3.13. Consider

$$0 < \varepsilon < \frac{1}{2}, \quad \Omega_\varepsilon := (\varepsilon, 1 - \varepsilon)^2 \quad \text{and} \quad K_\varepsilon := \overline{\Omega_\varepsilon} = [\varepsilon, 1 - \varepsilon]^2.$$

We regard Ω_ε as a planar domain in $\mathbb{C}$ and write $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$.

Let $\chi_0 \in C^\infty([0, 1])$ satisfy

$$0 \leq \chi_0(t) \leq 1 \text{ for all } t \in [0, 1], \quad \chi_0(t) = \begin{cases} 1, & t \in [\varepsilon, 1 - \varepsilon], \\ 0, & t \in \left[0, \frac{\varepsilon}{2}\right] \cup \left[1 - \frac{\varepsilon}{2}, 1\right]. \end{cases}$$

Set

$$\eta_0(x, y) := \chi_0(x)\chi_0(y).$$

By the Riemann mapping theorem, a biholomorphism exists

$$\Phi : \Omega_\varepsilon \longrightarrow \mathbb{D}.$$

Since Ω_ε is a Jordan domain, Carathéodory's theorem implies that Φ extends continuously and homeomorphically to K_ε .

We choose the Riemann map so that one of the four corners, denoted by z_1 , satisfies

$$\Phi(z_1) = -1.$$

This is possible by composing any Riemann map with a suitable rotation of $\mathbb{D}$.

Extension across the regular boundary arcs. Consider the Cayley transform

$$\psi(\zeta) := i \frac{1 - \zeta}{1 + \zeta}, \quad \zeta \in \mathbb{D}.$$

It maps $\mathbb{D}$ biholomorphically onto the upper half-plane and maps $\partial\mathbb{D} \setminus \{-1\}$ onto $\mathbb{R}$.

Since z_1 is the only boundary point mapped to -1 , the function

$$\psi \circ \Phi$$

is well defined and holomorphic on Ω_ε , continuous up to every open edge of ∂K_ε , and has real boundary values there. By the geometric Schwarz reflection principle, $\psi \circ \Phi$ therefore admits a holomorphic continuation across every regular boundary point

$$z_0 \in \partial K_\varepsilon \setminus \{\text{four corners}\}.$$

Since ψ^{-1} is a Möbius transformation, the same conclusion holds for Φ .

Moreover, at every such regular boundary point z_0 , the continuation of Φ is locally biholomorphic. In particular, we have

$$\Phi'(z_0) \neq 0.$$

Behavior at the corners. We next describe the local behavior of Φ at the four corners.

Let z_0 be a corner of K_ε . The interior angle at z_0 is $\pi/2$. Introduce the local change of variables

$$w = (z - z_0)^2.$$

This maps the corner sector biholomorphically onto a half-disk, while the two sides meeting at z_0 are mapped onto the two halves of the real diameter.

If $z_0 \neq z_1$, then $\Phi(z_0) \neq -1$, and the preceding Cayley transform can be applied after this change of variables. If $z_0 = z_1$, so that $\Phi(z_0) = -1$, we instead use

$$\psi_2(\zeta) := i \frac{1 + \zeta}{1 - \zeta},$$

whose pole is at 1. In both cases, the corresponding transformed map has real boundary values on the real diameter of the half-disk. The Schwarz reflection principle (see [19]) therefore yields a holomorphic continuation of the transformed map to a neighborhood of $w = 0$.

Consequently, a function $\tilde{\Phi}$, holomorphic in a neighborhood of $w = 0$, exists such that

$$\Phi(z) = \tilde{\Phi}((z - z_0)^2)$$

for z in the original corner sector. In particular,

$$\Phi'(z_0) = 0.$$

Thus Φ admits a holomorphic local continuation at every corner, although this continuation is not locally biholomorphic there.

The lacunary holomorphic function. Consider the lacunary power series

$$g(\zeta) := \sum_{k=0}^{\infty} e^{-\sqrt{2^k}} \zeta^{2^k}, \quad |\zeta| < 1.$$

Define

$$H_{\text{lac}}(x, y) := \operatorname{Re}(g(\Phi(x + iy))), \quad (x, y) \in \Omega_\varepsilon.$$

As shown in the proof below,

$$g \in C^\infty(\overline{\mathbb{D}}).$$

Hence, by Whitney's extension theorem, g admits a C^∞ extension G to a neighborhood of $\overline{\mathbb{D}}$. The local holomorphic continuations of Φ across the regular boundary points and at the corners then imply that

$$G \circ \Phi$$

is smooth in a neighborhood of every point of K_ε . Since $G = g$ on $\overline{\mathbb{D}}$, this gives

$$H_{\text{lac}} \in C^\infty(K_\varepsilon).$$

A second application of Whitney's extension theorem therefore provides a function

$$\widehat{H}_{\text{lac}} \in C^\infty(I^2),$$

such that

$$\widehat{H}_{\text{lac}} = H_{\text{lac}} \quad \text{on } K_\varepsilon.$$

For $\alpha \in \mathbb{R}$, define

$$\widetilde{C}_{\text{lac}}(x, y) := xy + \alpha \eta_0(x, y) \widehat{H}_{\text{lac}}(x, y), \quad (x, y) \in I^2.$$

Then $\alpha_0 > 0$ exists such that, for every

$$0 < |\alpha| < \alpha_0,$$

the function $\widetilde{C}_{\text{lac}}$ is a copula on I^2 , is harmonic on Ω_ε , and its harmonic component admits no harmonic continuation across any nonempty open arc of ∂K_ε .

We explain the construction in several steps.

• **Step 1: The natural boundary of g .**

Set

$$n_k := 2^k, \quad a_k := e^{-\sqrt{2^k}}.$$

Then

$$\frac{n_{k+1}}{n_k} = 2 > 1,$$

so the exponents satisfy the Hadamard gap condition. Moreover,

$$|a_k|^{1/n_k} = \exp\left(-\frac{\sqrt{2^k}}{2^k}\right) = \exp(-2^{-k/2}) \longrightarrow 1.$$

Therefore, the radius of convergence of g is exactly 1. The Hadamard gap theorem consequently implies that $\partial\mathbb{D}$ is a natural boundary for g . In particular, g admits no holomorphic continuation across any point of $\partial\mathbb{D}$.

• **Step 2: Smoothness of g up to $\partial\mathbb{D}$.**

Let $m \geq 0$ be an integer. Termwise differentiation gives

$$g^{(m)}(\zeta) = \sum_{k: 2^k \geq m} a_k (2^k)(2^k - 1) \cdots (2^k - m + 1) \zeta^{2^k - m}.$$

For $|\zeta| \leq 1$,

$$|g^{(m)}(\zeta)| \leq \sum_{k=0}^{\infty} e^{-2^{k/2}} 2^{km}.$$

For every fixed m , the series on the right converges, since $e^{-2^{k/2}}$ decays faster than any exponential in k , whereas 2^{km} grows exponentially in k . Hence, the series defining $g^{(m)}$ converges uniformly on $\overline{\mathbb{D}}$. It follows that $g \in C^\infty(\overline{\mathbb{D}})$.

Notice that this does not contradict Step 1: Smoothness up to the boundary does not imply holomorphic continuation across the boundary.

• **Step 3: Regularity and harmonicity of H_{lac} .**

Since g and Φ are holomorphic on $\mathbb{D}$ and Ω_ε , respectively, the function $g \circ \Phi$ is holomorphic on Ω_ε . Hence

$$H_{\text{lac}} = \operatorname{Re}(g \circ \Phi)$$

is harmonic on Ω_ε .

By Step 2, g is C^∞ on $\overline{\mathbb{D}}$. Let G be a C^∞ extension of g to a neighborhood of $\overline{\mathbb{D}}$. The local holomorphic continuation of Φ across each regular boundary point and its local representation

$$\Phi(z) = \tilde{\Phi}((z - z_0)^2)$$

at each corner imply that $G \circ \Phi$ is smooth up to every point of K_ε . Therefore

$$H_{\text{lac}} \in C^\infty(K_\varepsilon).$$

By Whitney's extension theorem, $\widehat{H}_{\text{lac}} \in C^\infty(I^2)$ exists such that

$$\widehat{H}_{\text{lac}} = H_{\text{lac}} \quad \text{on } K_\varepsilon.$$

• **Step 4: The copula property.**

Since χ_0 vanishes in neighborhoods of 0 and 1, the function η_0 vanishes in a neighborhood of ∂I^2 . Consequently,

$$\tilde{C}_{\text{lac}}(x, 0) = \tilde{C}_{\text{lac}}(0, y) = 0,$$

and

$$\tilde{C}_{\text{lac}}(x, 1) = x, \quad \tilde{C}_{\text{lac}}(1, y) = y.$$

Thus the required boundary conditions are satisfied.

Set

$$M := \left\| \partial_{xy}^2 (\eta_0 \widehat{H}_{\text{lac}}) \right\|_{L^\infty(I^2)} < \infty$$

and define

$$\alpha_0 := \frac{1}{1 + M}.$$

For $0 < |\alpha| < \alpha_0$,

$$\begin{aligned} \partial_{xy}^2 \tilde{C}_{\text{lac}} &= 1 + \alpha \partial_{xy}^2 (\eta_0 \widehat{H}_{\text{lac}}) \\ &\geq 1 - |\alpha| M > 0. \end{aligned}$$

Hence $\tilde{C}_{\text{lac}}$ is 2-increasing on I^2 . Together with the boundary conditions above, this proves that $\tilde{C}_{\text{lac}}$ is a copula on I^2 .

• **Step 5: Harmonicity on Ω_ε .**

Since

$$\chi_0 \equiv 1 \quad \text{on} \quad [\varepsilon, 1 - \varepsilon],$$

we have

$$\eta_0 \equiv 1 \quad \text{on} \quad \Omega_\varepsilon.$$

Therefore

$$\tilde{C}_{\text{lac}} = xy + \alpha H_{\text{lac}} \quad \text{on} \quad \Omega_\varepsilon.$$

Since

$$\Delta(xy) = 0$$

and H_{lac} is harmonic on Ω_ε , it follows that

$$\Delta \widetilde{C}_{\text{lac}} = 0 \quad \text{on} \quad \Omega_\varepsilon.$$

• **Step 6: Failure of harmonic continuation across the regular boundary.**

Let

$$z_0 \in \partial K_\varepsilon \setminus \{\text{four corners}\}.$$

Suppose, for contradiction, that $\widetilde{C}_{\text{lac}}$ admits a harmonic continuation to a neighborhood U of z_0 . Since $\alpha \neq 0$ and xy is harmonic, the function

$$H := \frac{1}{\alpha}(\widetilde{C}_{\text{lac}} - xy)$$

is a harmonic continuation of H_{lac} to U .

Shrinking U if necessary, we may assume that U is simply connected and that $U \cap \Omega_\varepsilon$ is connected. Hence H admits a harmonic conjugate on U . Thus a holomorphic function

$$F : U \longrightarrow \mathbb{C}$$

exists such that

$$\operatorname{Re} F = H.$$

On $U \cap \Omega_\varepsilon$,

$$\operatorname{Re} F = H_{\text{lac}} = \operatorname{Re}(g \circ \Phi).$$

Therefore

$$F - g \circ \Phi$$

is holomorphic on $U \cap \Omega_\varepsilon$ and has identically zero real part. Since $U \cap \Omega_\varepsilon$ is connected, the function is constant and purely imaginary. Hence, for some $c \in \mathbb{R}$,

$$F - g \circ \Phi = ic$$

on $U \cap \Omega_\varepsilon$, and therefore

$$g \circ \Phi = F - ic.$$

Thus $g \circ \Phi$ admits a holomorphic continuation across z_0 .

Since z_0 is a regular boundary point, the continuation of Φ is locally biholomorphic there and

$$\Phi'(z_0) \neq 0.$$

Let

$$\zeta_0 := \Phi(z_0) \in \partial \mathbb{D}.$$

After shrinking the neighborhood if necessary, Φ has a holomorphic local inverse near ζ_0 . Composing the holomorphic continuation of $g \circ \Phi$ with this local inverse yields a holomorphic continuation of g across ζ_0 .

This contradicts Step 1, which states that every point of $\partial \mathbb{D}$ is a natural boundary point for g .

Consequently, H_{lac} admits no local harmonic continuation across any regular boundary point of K_ε . In particular, it admits no harmonic continuation across any nonempty open arc of ∂K_ε .

Remark 3.14. The example exhibits a sharp distinction between smooth boundary regularity and harmonic extendibility: $H_{\text{lac}} \in C^\infty(K_\varepsilon)$, so it can not be continued harmonically across any regular boundary arc. The obstruction is not a lack of smoothness but the natural boundary of the underlying lacunary function g .

3.3. Approximation by complexifiable copulas on the expanding domains

The examples above show that infinitely many copulas exist, distinct from the product copula Π , that are harmonic on substantial subdomains of the unit square. However, such harmonicity needs not extend to a holomorphic complexification beyond the natural domain of harmonicity. This suggests a weaker question: Can an arbitrary copula be approximated by copulas that are holomorphically complexifiable on domains progressively covering $(0, 1)^2$?

Let $(\mathcal{U}_\varepsilon)_{\varepsilon>0}$ be a family of domains in $\mathbb{C}^2$, and set

$$D_\varepsilon := \mathcal{U}_\varepsilon \cap (0, 1)^2.$$

Assume that

$$0 < \varepsilon' < \varepsilon \implies \mathcal{U}_\varepsilon \subset \mathcal{U}_{\varepsilon'}, \quad \bigcup_{\varepsilon>0} D_\varepsilon = (0, 1)^2.$$

Let $(C_\varepsilon)_{\varepsilon>0}$ be a family of copulas such that, for every $\varepsilon > 0$, a holomorphic function

$$F_\varepsilon : \mathcal{U}_\varepsilon \longrightarrow \mathbb{C}$$

exists satisfying

$$C_\varepsilon = \operatorname{Re} F_\varepsilon \quad \text{on } D_\varepsilon.$$

Proposition 3.15 (Harmonicity of the limiting copula). *If*

$$C_\varepsilon \longrightarrow C$$

locally uniformly on $(0, 1)^2$ as $\varepsilon \rightarrow 0^+$, for some copula C , then C is harmonic on $(0, 1)^2$. Consequently, by the rigidity result established in Remark 3.14, $C = \Pi$.

Remark 3.16. The preceding proposition shows that

$$C_\varepsilon \longrightarrow C \text{ locally uniformly on } (0, 1)^2 \implies C = \Pi.$$

Thus, approximability by holomorphically complexifiable copulas on expanding domains is already a characterization of the product copula Π . In particular, this conclusion does not depend on any specific evolution equation governing the family $(C_\varepsilon)_{\varepsilon>0}$.

4. A second level of complexification

In this section, the main tool used is Hartogs' extension theorem. We will use this latter result to explain why such multi-valued extensions cannot occur. Our technique here is based on the extension of real functions to holomorphic functions in $\mathbb{C}^2$.

As above, let $\Omega \subset \mathbb{R}^2$ be an open connected set and let $f : \Omega \rightarrow \mathbb{R}$ be a given function. We investigate whether f can be extended to a function that preserves regularity, say harmonicity

$$F : U \subset \mathbb{C}^2 \rightarrow \mathbb{C}$$

such that

$$F(x, y) = f(x, y), \quad \forall (x, y) \in \Omega.$$

A common algebraic extension is so trivial

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^2$ be open and let $f : \Omega \rightarrow \mathbb{R}$ be an arbitrary function. Set*

$$U := \{(z_1, z_2) \in \mathbb{C}^2 : (\operatorname{Re} z_1, \operatorname{Re} z_2) \in \Omega\}.$$

Then U is an open subset of $\mathbb{C}^2$ containing Ω (embedded in $\mathbb{C}^2$ via $(x_1, x_2) \mapsto (x_1, x_2)$), and the function

$$F : U \rightarrow \mathbb{R}, \quad F(z_1, z_2) := f(\operatorname{Re} z_1, \operatorname{Re} z_2)$$

satisfies $F(x_1, x_2) = f(x_1, x_2)$ for all $(x_1, x_2) \in \Omega$, i.e., F is an extension of f to U .

But such a trivial algebraic extension is, in general, not holomorphic since it does not satisfy the Cauchy–Riemann equations even if the function f is smooth enough (belongs to the C^∞ class.)

A positive answer to this elementary problem is stated for real-analytic functions that we redefine here in order to make the paper self-contained (see [20] or [21]).

Definition 4.2. A function $f : \Omega \rightarrow \mathbb{R}$ is called *real analytic* if for every point $(x_0, y_0) \in \Omega$, a neighborhood exists where

$$f(x, y) = \sum_{m, n \geq 0} a_{m, n} (x - x_0)^m (y - y_0)^n,$$

with a convergent power series.

Theorem 4.3 (Holomorphic extension). *Let f be real analytic on $\Omega \subset \mathbb{R}^2$. Then for every point $(x_0, y_0) \in \Omega$, a neighbourhood $U \subset \mathbb{C}^2$ and a holomorphic function*

$$F : U \rightarrow \mathbb{C}$$

exist such that

$$F(x, y) = f(x, y)$$

for all $(x, y) \in \Omega \cap U$.

Construction. If

$$f(x, y) = \sum_{m, n \geq 0} a_{m, n} x^m y^n,$$

define

$$F(z_1, z_2) := \sum_{m, n \geq 0} a_{m, n} z_1^m z_2^n.$$

Then F is holomorphic in a neighborhood of $\mathbb{R}^2$ and coincides with f on real points. $\square$

Example 4.4 (Positive and negative cases). There are many examples; we only mention the best known, namely the exponential function and polynomial functions. For a negative statement, the function $f(x, y) = |x|$ is not real-analytic at $x = 0$, hence, it does not admit a holomorphic extension.

To summarize all purposes on regular continuation, we note that harmonicity of a function f is not sufficient to guarantee the existence of a holomorphic extension in $\mathbb{C}^2$. Additional compatibility conditions are required. Precisely, real analyticity is the key condition ensuring local holomorphic extension.

We now discuss the problem of extending a real harmonic function

$$f(x_1, x_2)$$

to a function of several complex variables

$$F(z_1, z_2),$$

while preserving harmonic or analytic properties. Special attention is devoted to the framework of harmonic copulas, where harmonicity interacts with probabilistic constraints. We present several natural constructions based on holomorphic, pluriharmonic, and analytic extensions.

Harmonic functions occupy a central position in analysis, partial differential equations, probability theory, and mathematical physics. In recent developments concerning harmonic copulas and evolution copulas, one is naturally led to investigate whether a real harmonic function defined on a subset of $\mathbb{R}^2$ admits a meaningful extension to several complex variables.

More precisely, let

$$f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$$

be a harmonic function satisfying

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} = 0.$$

The purpose of this note is to discuss possible extensions

$$F : U \subset \mathbb{C}^2 \rightarrow \mathbb{C}$$

such that

$$F(x_1, x_2) = f(x_1, x_2)$$

on the real slice, while preserving harmonicity or related analytic structures.

5. Holomorphic extension by power series

Suppose that f is real analytic on an open set $\Omega \subset \mathbb{R}^2$. Then, locally,

$$f(x_1, x_2) = \sum_{m,n \geq 0} a_{m,n} x_1^m x_2^n.$$

A natural complexification is obtained by replacing the real variables with complex variables:

$$F(z_1, z_2) = \sum_{m,n \geq 0} a_{m,n} z_1^m z_2^n.$$

The function F is holomorphic in (z_1, z_2) on a suitable neighborhood of $\mathbb{C}^2$.

Proposition 5.1. *If f is real analytic and harmonic on Ω , then its holomorphic extension F satisfies*

$$\frac{\partial^2 F}{\partial z_1^2} + \frac{\partial^2 F}{\partial z_2^2} = 0.$$

Proof. The Laplace equation satisfied by f is an analytic identity. By the uniqueness of the analytic continuation, the same identity remains valid after complexification. $\square$

This representation provides a powerful bridge between harmonic analysis and complex analysis. The bridge is edified naturally with the classical expression of the Laplacian operator in terms of complex coordinates. Indeed, for every

$$z = x + iy, \quad \bar{z} = x - iy,$$

the differential operators become $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$.

A direct computation yields the factorization

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

Consequently, a function u is harmonic if and only if

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

This factorization explains why harmonicity is naturally connected with holomorphy. In fact, when f is holomorphic, it does not depend on the conjugate $\bar{z}$. So its Laplacian is automatically null.

6. Multivalued functions and the removable singularity phenomenon in several variables

In the following, explain why the classical notion of multivalued holomorphic functions arising from isolated singularities in one complex variable essentially disappears in several complex variables, as a consequence of Hartogs' extension theorem.

In one complex variable, multivalued holomorphic functions naturally appear through analytic continuation around isolated singularities. Typical examples are $\log z$, $\sqrt{z}$, z^α . The mechanism behind this phenomenon is the nontrivial topology of the punctured plane $\mathbb{C} \setminus \{0\}$. Indeed, continuation around the origin changes the branch of the function, that is the particular single-valued continuous determination of f obtained by restricting it to a simply connected subdomain of $\mathbb{C} \setminus \{0\}$ (typically the plane cut along a ray from the origin, called a branch cut). In contrast, several complex variables exhibit a radically different behavior. Hartogs' extension theorem is a pioneering result for this type of functions.

Theorem 6.1 (Hartogs). *Let $\Omega \subset \mathbb{C}^n$, ($n \geq 2$), be a domain and let $K \subset \Omega$ be compact such that $\Omega \setminus K$ is connected.*

Then every holomorphic function on $\Omega \setminus K$ extends uniquely to a holomorphic function on Ω .

An immediate consequence is the following.

Remark 6.2. In $\mathbb{C}^n$, $n \geq 2$, isolated singularities of holomorphic functions are removable.

Indeed, taking $K = \{a\}$, every holomorphic function on $\Omega \setminus \{a\}$ extends holomorphically through the point a .

After a deep understanding of the second complexification method, Hartogs' theorem stopped any way to pursue the analysis, since every copula obtained by this method is smooth enough and has no chance to be a candidate for residual calculus. The reason is structural: Section 3's escape route relied on a one-variable lacunary power series with a natural boundary, a genuine non-removable singularity, but once one moves to $\mathbb{C}^2$, Hartogs' theorem forces every isolated singularity to be removable, ruling out monodromy, branching, or multivaluedness. That method is therefore a one-variable-only phenomenon that simply does not extrapolate to the several-variable framework: The holomorphic regularity imposed by the second complexification structurally forecloses the very mechanism that made Section 3's construction work.

7. Perspective: Towards p -harmonic and fractional copulas

The phenomenon above lies at the heart of modern complex geometry, analytic continuation, and pluri-potential theory. In our future work, we will show how complexification plays a central role in integral calculus and in the determination of some classical statistical parameters like Kendall's τ and Spearman ρ . Another field to explore with complexification will be the $p(x)$ -Laplacian.

The rigidity theorem that Π is the only bivariate copula harmonic on the whole of $[0, 1]^2$ invites a natural nonlinear extension: Replace the linear evolution $\partial_t C = \Delta C$ by the p -Laplacian flow

$$\partial_t C(t, x, y) = \Delta_p C(t, x, y) := \operatorname{div}(|\nabla C(t, x, y)|^{p-2} \nabla C(t, x, y)), \quad p > 1. \quad (7.1)$$

The case $p = 2$ is precisely the evolution studied in [5], where the semigroup structure of the heat equation was exploited to write $C(t, \cdot, \cdot)$ explicitly via the heat kernel and to track which dependence parameters survive the flow. For $p \neq 2$, the operator Δ_p is no longer linear, no heat kernel representation is available, and the associated semigroup, if it exists at all, is at best nonlinear (of contraction semigroup type on L^2), so the explicit techniques of [5] would need to be replaced by variational and comparison-principle arguments. It is natural to ask whether Π remains the unique p -harmonic copula on $[0, 1]^2$ for every $p > 1$, and, in particular, to examine the degenerate limits $p \rightarrow 1$ (total variation type flow) and $p \rightarrow \infty$ (the infinity Laplacian absolutely minimizing Lipschitz extensions), which could reveal whether our rigidity phenomenon is a special feature of the linear case $p = 2$ or persists across the whole family.

A complementary direction is to replace the local operator Δ (or Δ_p) by a nonlocal or fractional-order one, $\partial_t^\alpha C = \Delta C$ or $\partial_t C = (-\Delta)^s C$. The fractional-calculus machinery developed with El-Mennaoui et al. in [22] where the Riemann Liouville semigroup of fractional integration is shown to admit a boundary group of purely imaginary order on the little Hölder space $\operatorname{lip}_0^\alpha([0, 1], X)$ is directly relevant here, since copulas are Hölder continuous functions on $[0, 1]^2$ and this framework can be transported to give rigorous meaning to a fractional in time copula evolution $\partial_t^\alpha C$. Finally, when singular or point-mass initial boundary data are needed for such a fractional evolution problem, the fractional order Dirac delta distribution recently constructed by Muddassar et al. [23], obtained by applying a generalized fractional derivative to the Heaviside function, provides a natural candidate for

prescribing $C(0, \cdot, \cdot) = \delta_{\text{GFD}}$ and testing whether rigidity persists under genuinely singular, non classical initial dependence structures.

8. Conclusions

In this paper, we investigated the possibility of enriching the theory of copulas by embedding it within the framework of complex analysis, motivated by a structural analogy between holomorphic functions and copulas as rigidity-constrained, structure-preserving maps. We first showed that this analogy is sharply constrained: Every copula shares the same boundary data on the unit square, so the only copula that can be harmonic on the whole open square $(0, 1)^2$ is the independence copula Π , by uniqueness for the associated Dirichlet problem (Section 3). We further showed that any copula obtained as a locally uniform limit of copulas that are holomorphically complexifiable on domains expanding to $(0, 1)^2$ must coincide with Π , so this rigidity persists in the limit (Proposition 3.15).

We then examined a second route to complexification, based on extending real-valued functions on $\mathbb{R}^2$ directly to $\mathbb{C}^2$. While real-analytic harmonic functions admit such an extension compatible with the Laplace equation (Section 5), Hartogs' extension theorem shows that every isolated singularity of a holomorphic function of two or more complex variables is removable (Section 6). Consequently, the one-variable mechanism that produced a genuine non-removable singularity in Section 3 cannot survive this second complexification: The increased holomorphic regularity in $\mathbb{C}^2$ structurally forecloses the very obstruction that made the first construction work. This dichotomy delineates a sharp boundary between what one-variable and several-variable complexification techniques can achieve for copulas.

Taken together, these results show that the rigidity of holomorphic functions and the rigidity of copulas are not merely analogous in spirit but interact in a precise, quantifiable way once complexification is attempted. They also open several directions for future work, discussed in Section 7, including nonlinear evolutions of copulas governed by the p -Laplacian and fractional-order extensions of the Dirichlet/heat-semigroup approach to copula dynamics.

Author contributions

Rachid Jaafar: Conceptualization, methodology, formal analysis, writing – original draft; Ahmed Hfa: Methodology, formal analysis, writing – review and editing; Mohamed Elmaazouz: Validation, writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. H. Eltayeb, S. Mesloub, A note on multidimensional Sumudu-Generalized Laplace decomposition method and singular pseudo-hyperbolic equations, *J. Funct. Spaces*, **2026** (2026), 4236537. <https://doi.org/10.1155/jofs/4236537>
2. M. Jleli, M. A. Ragusa, B. Samet, Nonlinear Liouville-type theorems for generalized Baouendi-Grushin operator on Riemannian manifolds, *Adv. Differ. Equ.*, **28** (2023), 143–168.
3. L. B. Wu, Q. Ma, X. H. Ding, Structure-preserving numerical methods for the two dimensional nonlinear fractional wave equation, *Filomat*, **38** (2024), 4187–4207.
4. L. Karbil, A. Sani, I. Daoudi, Copulas and preserver problems, *Int. J. Math. Trends Technol.*, **66** (2020), 105–116. <https://doi.org/10.14445/22315373/IJMTT-V66I11P509>
5. O. Elamrani, M. El Maazouz, A. Sani, An analytic treatment of evolution copulas, *Asia Pac. J. Math.*, **11** (2024), 48. <https://doi.org/10.28924/APJM/11-48>
6. M. Fréchet, Sur les tableaux de corrélation dont les marges sont données, *Revue Inst. Int. de Stat.*, **28** (1960), 10–32. <https://doi.org/10.2307/1401846>
7. A. Sklar, Fonctions de répartition à n dimensions et leurs marges, *Publ. Inst. Stat. Univ. Paris*, **8** (1959), 229–231.
8. R. B. Nelsen, *An introduction to Copulas*, New York: Springer, 2006. <https://doi.org/10.1007/0-387-28678-0>
9. O. Elamrani, A. Sani, A procedure of perturbation leading to a new class of asymmetric copulas, *Mathematics*, **14** (2026), 1093. <https://doi.org/10.3390/math14071093>
10. R. Jaafar, A. Hfa, A. Sani, Mathematical analysis and computational approximation of extremal points of the subset of Copulas, *Stats*, **9** (2026), 66. <https://doi.org/10.3390/stats9030066>
11. F. Durante, J. Fernández-Sánchez, C. Sempi, Sklar's theorem obtained via regularization techniques, *Nonlinear Anal.*, **75** (2012), 769–774. <https://doi.org/10.1016/j.na.2011.09.006>
12. W. F. Darsow, B. Nguyen, E. T. Olsen, Copulas and Markov processes, *Illinois J. Math.*, **36** (1992), 600–642. <https://doi.org/10.1215/ijm/1255987328>
13. P. Jaworski, F. Durante, W. K. Härdle, T. Rychlik, Copula theory and its applications, In: *Lecture notes in statistics*, Heidelberg: Springer, **198** (2010). <https://doi.org/10.1007/978-3-642-12465-5>
14. P. Mikusiński, M. D. Taylor, Markov operators and n -copulas, *Ann. Polon. Math.*, **96** (2009), 75–95.
15. L. Hörmander, *An introduction to complex analysis in several variables*, Amsterdam: North-Holland Pub. Co., 1973.
16. R. M. Range, *Holomorphic functions and integral representations in several complex variables*, New York: Springer, 1986. <https://doi.org/10.1007/978-1-4757-1918-5>
17. J. B. Conway, Functions of one complex variable I, In: *Graduate texts in mathematics*, New York: Springer, **11** (1978). <https://doi.org/10.1007/978-1-4612-6313-5>
18. Y. Yoshizawa, N. Ishimura, Evolution of multivariate copulas in continuous and discrete processes, *Intell. Syst. Account. Financ. Manag.*, **25** (2018), 44–59. <https://doi.org/10.1002/isaf.1420>

19. S. G. Krantz, *Complex analysis: The geometric viewpoint*, Mathematical Association of America, 2004.
20. E. M. Stein, R. Shakarchi, *Complex analysis*, Princeton: Princeton University Press, 2003.
21. W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$* , Heidelberg: Springer, 2008.
<https://doi.org/10.1007/978-3-540-68276-9>
22. O. El-Mennaoui, V. Keyantuo, A. Sani, Fractional integration of imaginary order in vector-valued Hölder spaces, *Arch. Math.*, **120** (2023), 493–506. <https://doi.org/10.1007/s00013-023-01841-6>
23. M. Muddassar, A. Jhangeer, N. Siddiqui, M. S. Mehmood, L. Khan, T. Jabeen, On the construction and analysis of a fractional-order Dirac Delta distribution with application, *Axioms*, **14** (2025), 728.
<https://doi.org/10.3390/axioms14100728>



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