



*Research article***Fractal attractors of iterated function systems in extended fuzzy $\mathcal{F}$ -metric spaces****Abhishikta Das¹, Dipti Barman¹, Mohammad Sajid^{2,*} and Tarapada Bag¹**¹ Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan-731235, West-Bengal, India² Department of Mechanical Engineering, College of Engineering, Qassim University, Buraydah, Saudi Arabia*** Correspondence:** Email: msajid@qu.edu.sa.

Abstract: Certain fractal and fuzzy dynamical systems involve auxiliary control functions that exhibit irregular features such as non-monotonicity or discontinuities, placing them outside the scope of classical fuzzy metric frameworks. To analyze these dynamics, we introduce the concept of an extended fuzzy $\mathcal{F}$ -metric space and establish basic and fixed-point results in this setting. As an application of the fixed-point results, we use the framework to analyze Hutchinson–Barnsley operators arising in iterated function systems, providing a convergence analysis, a numerical simulation, and visualizations of fractal attractors on the Cantor set. This perspective aligns fixed-point techniques more closely with the analytical demands of fractal and fuzzy dynamic modeling. The proposed framework may also be a flexible mathematical setting for studying nonlinear systems involving uncertainty, including potential applications in environmental modeling, energy systems, and resource management.

Keywords: extended fuzzy $\mathcal{F}$ -metric space; fuzzy Hutchinson–Barnsley operator; fuzzy contractive mapping; fuzzy fractal attractor; simulation; contraction inequality

Mathematics Subject Classification: 03E72, 28A80, 47H09, 47H10, 54A40, 54E50

1. Introduction

The study of fuzzy metric structures has evolved substantially since the frameworks of Kramosil–Michalek [1] and George–Veeramani [2], with numerous generalizations introduced to capture uncertainty and graded proximity in mathematical analysis, topology, and fixed-point theory. Among these developments, functional extensions of fuzzy metrics such as the fuzzy b -metric [3], the fuzzy quasi-metric [4], the fuzzy partial metric [5], have been introduced to enrich the underlying

distance structure. Further generalizations have also been developed to accommodate other forms of distance-like structures. The fuzzy G -metric [6] and fuzzy rectangular metric [7] modify the triangle inequality by involving additional intermediate points, whereas the distance structure of the fuzzy modular metric [8] is formulated based on the modular function. Another important direction concerns controlled fuzzy metric structures [9–11] where the classical restrictions on the underlying control functions can be relaxed while still retaining a well-behaved convergence structure. Several contributions have developed different forms of controlled fuzzy metrics [12, 13]. Their applications and further developments have also been investigated [14, 15]. In parallel, a substantial body of research work in this direction has focused on the development of fixed-point theorems for contractive mappings in fuzzy metric spaces. In particular, Banach-type [16], Kannan and Chatterjea-type [17], Suzuki-type [18] contractions are studied extensively. Further developments have considered other forms of contractive mappings and fixed-point conditions. Huang et al. [19] introduced a fuzzy F -contraction and established corresponding fixed-point results, while Plebaniak [20] introduced a generalized fuzzy metric and studied G - and GV -type Banach contractions together with corresponding fixed-point results. In another direction, Singh and Chauhan [21] studied the concept of compatibility for mappings in fuzzy metric spaces and related common fixed-point theorems under appropriate conditions. With these theoretical developments, fuzzy methodologies have found wide applications in nonlinear control systems, adaptive tracking, and secure control of cyber-physical systems [22–24]. Their ability to address uncertainty makes fuzzy frameworks relevant to environmental modeling, energy systems, resource allocation, and sustainable management of essential resources. In such nonlinear and iterative settings, fixed-point methods provide useful tools for establishing existence, uniqueness, and convergence, highlighting the potential relevance of fuzzy metric frameworks to sustainability-oriented problems. Moreover, recent studies have demonstrated the growing role of fuzzy-based frameworks in dynamical systems and control, including studies of fractional-order fuzzy neural networks, adaptive control of uncertain nonlinear systems, and stability analysis of complex networked models [25–27]. These developments further demonstrate the versatility and continuing relevance of fuzzy-based mathematical frameworks across diverse research areas. Such frameworks also provide useful analytical settings for nonlinear problems arising in differential equations, integral equations, and iterated function systems (IFS) under uncertainty.

Although fuzzy modular [8], fuzzy partial [5], and fuzzy quasi-metric [4] spaces provide effective frameworks for modeling different types of uncertainty, each is constructed by modifying the underlying metric structure in a specific direction, such as modular-type, partial-distance, or asymmetric behavior. In contrast, the concept of fuzzy $\mathcal{F}$ -metric spaces [28–30] further extends the classical fuzzy metric structure, allowing a more flexible approach to convergence and continuity through a functional inequality that generalizes the George–Veeramani (GV) model. In [28], Das et al. introduced the notion of fuzzy $\mathcal{F}$ -metric spaces by incorporating a pair (f, α) where $\alpha \in (0, 1]$ and $f \in \mathcal{F}$, the set of all functions $f: [0, 1] \rightarrow [0, 1]$ which satisfy the following conditions:

($\mathcal{F}1$) f is strictly increasing in $[0, 1]$;

($\mathcal{F}2$) For every sequence $\{t_n\}$ in $[0, 1]$, we have

$$\lim_{n \rightarrow \infty} t_n = 1 \iff \lim_{n \rightarrow \infty} f(t_n) = 1.$$

The condition ($\mathcal{F}1$) assumes that the functions in $\mathcal{F}$ must be strictly increasing, which imposes a monotonic rigidity that may not always be aligned with practical or theoretical needs. In the present

work, we introduce a broader formulation, termed the extended fuzzy $\mathcal{F}$ -metric space ($EF\mathcal{F}$ -MS), which generalizes the class of auxiliary functions $\mathcal{F}$ with another broader class $\mathcal{F}^*$. $\mathcal{F}^*$ replaces the strict monotonicity condition defining $\mathcal{F}$ with the weaker requirement of monotonicity, yielding the strictly larger class of auxiliary functions (see Remark 1.2).

To formalize this, we redefine the original class $\mathcal{F}$ as a more flexible family $\mathcal{F}^*$, consisting of all functions: $F: [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

$(\mathcal{F}^*1)$ $F(t)$ is a non-decreasing function of $t \in [0, 1]$;

$(\mathcal{F}^*2)$ $\lim_{t \rightarrow 1^-} F(t) = 1$.

Remark 1.1. From $(\mathcal{F}^*1)$ and $(\mathcal{F}^*2)$, it follows that $F(1) = 1$. Indeed, since F is non-decreasing and $\lim_{t \rightarrow 1^-} F(t) = 1$, we must have $F(1) \geq \lim_{t \rightarrow 1^-} F(t) = 1$. As the co-domain of F is $[0, 1]$, hence, $F(1) = 1$.

Remark 1.2. The class $\mathcal{F}$ is a proper subclass of $\mathcal{F}^*$, whereas the converse is generally false. Indeed, the function $F: [0, 1] \rightarrow [0, 1]$ defined by:

$$F(t) = \begin{cases} \frac{1}{2}, & 0 \leq t < \frac{3}{4}, \\ t, & \frac{3}{4} \leq t \leq 1, \end{cases}$$

for all $t \in [0, 1]$, belongs to $\mathcal{F}^*$ but not to $\mathcal{F}$.

This relaxation significantly enlarges the admissible class of functional controls while preserving essential convergence behavior and topological consistency of the underlying new fuzzy $\mathcal{F}$ -metric structure, thereby providing greater flexibility for modeling irregular fuzzy dynamical systems in which strict monotonicity of the auxiliary function may be unnecessarily restrictive. As a result, compared with existing fuzzy metric frameworks [2, 3, 6], this framework captures situations involving discontinuously graded proximity that the earlier model cannot accommodate.

Having established the relaxed conditions on the function class $\mathcal{F}^*$, we proceed to develop the corresponding structure of an $EF\mathcal{F}$ -MS. In the following sections, we present the formal definition, followed by several illustrative examples, topological characterizations, and key results that demonstrate the richness and utility of this framework. Unlike the classical framework where the fuzzy metric is required to be continuous with respect to the parameter $t \in (0, \infty)$, we omit this assumption. Beyond these foundational developments, the framework introduced here also accommodates a fixed-point theorem and its applications in settings where the underlying structure is inherently irregular, such as IFS-generated fractal sets [31, 32]. Fractal geometry extends classical Euclidean concepts by describing self-similar structures with intricate, non-integer dimensions [33, 34]. The Hutchinson–Barnsley operator [35–37] provides a powerful functional-analytic mechanism for constructing such attractors. Later, this framework was explored for studying attractors and their properties [38, 39]. These ideas have also been extended to hybrid and generalized metric frameworks [40–42]. Further studies have investigated generalized Hutchinson–Barnsley operators and related structures [43–45]. In the application section, we build upon the developments in IFS theory to show how the proposed framework yields new fixed-point results and convergence criteria for generalized Hutchinson–Barnsley operators. In addition, we present a numerical illustration based on constructing a fractal on the Cantor set using MATLAB simulations of the defined Hutchinson–Barnsley-type operator in this setting.

The main contributions of this paper can be summarized as follows. First, we introduce the notion of an $EF\mathcal{F}$ -MS by replacing the class $\mathcal{F}$ of auxiliary functions with the broader class $\mathcal{F}^*$. This strictly enlarges the admissible family of auxiliary functions and extends the previously introduced fuzzy $\mathcal{F}$ -metric spaces. Second, we investigate the induced topological structure and establish its fundamental properties, including Hausdorffness, convergence, completeness, and related results. Third, we develop Banach-type fixed point theorems in this generalized setting and demonstrate their applicability to generalized Hutchinson–Barnsley operators for IFS.

The article is organized as follows. In Section 2, we present essential preliminaries on fuzzy metric spaces, t -norms, and related generalizations. In Section 3, we provide the formal definitions of $EF\mathcal{F}$ -metric structures, accompanied by illustrative examples that demonstrate their properties and highlight the flexibility of this framework in comparison with classical fuzzy metric spaces. Section 4 is devoted to establishing Banach-type and related fixed-point results within this generalized framework. Section 5 explores applications of these results to fractal analysis, providing both theoretical insights and numerical illustrations. Finally, Section 6 presents the conclusions and outlines possible directions for future research.

For the convenience of reader, principal symbols used throughout the paper are summarized below (see Table 1).

Table 1. Symbols and notation.

$\mathcal{F}$	Class of admissible auxiliary functions satisfying $(\mathcal{F}1)$ – $(\mathcal{F}2)$
$\mathcal{F}^*$	Class of admissible auxiliary functions satisfying $(\mathcal{F}^*1)$ – $(\mathcal{F}^*2)$
$\mathcal{U}$	Underlying nonempty set
$\mathcal{M}_F$	$EF\mathcal{F}$ -metric on $\mathcal{U}$
$(\mathcal{U}, \mathcal{M}_F, F, \star)$	$EF\mathcal{F}$ -MS
$CB(\mathcal{U})$	Collection of nonempty, $EF\mathcal{F}$ -bounded, and $EF\mathcal{F}$ -closed subsets of $\mathcal{U}$
$\mathcal{H}_F$	Hausdorff $EF\mathcal{F}$ -distance on $CB(\mathcal{U})$
Ψ_F	$EF\mathcal{F}$ -Hutchinson Barnsley–type operator.

2. Preliminaries

In this section, we recall fundamental notions and properties of t -norms, GV fuzzy metric, fuzzy b -metric, fuzzy $\mathcal{F}$ -metric, and sequential topology, which we use throughout the article.

To begin, we recall the classical notion of a t -norm, which plays a central role in the structure of fuzzy metric spaces.

Definition 2.1. [46] A binary operation $\star: [0, 1]^2 \rightarrow [0, 1]$ is called a t -norm if it satisfies the following conditions:

- (i) $\star$ is commutative and associative;
- (ii) $\varrho \star 1 = \varrho$, $\forall \varrho \in [0, 1]$;
- (iii) $\varrho \star \gamma \leq \vartheta \star \delta$ whenever $\varrho \leq \vartheta$ and $\gamma \leq \delta$, $\forall \varrho, \vartheta, \gamma, \delta \in [0, 1]$.

If $\star$ is continuous on $[0, 1]^2$, then it is called a continuous t -norm.

Example 2.2. [46] Typical examples of t -norms include: (i) Standard intersection: $\varrho \star \vartheta = \min\{\varrho, \vartheta\}$; (ii) algebraic product: $\varrho \star \vartheta = \varrho \vartheta$; and (iii) bounded difference: $\varrho \star \vartheta = \max\{0, \varrho + \vartheta - 1\}$.

We now recall the notation of a fuzzy metric space in the sense of George–Veeramani [2].

Definition 2.3. [2] Let $\mathcal{U}$ be a nonempty set and let $\star$ be a continuous t -norm. The 3-tuple $(\mathcal{U}, \mathcal{M}, \star)$ is said to be a GV fuzzy metric space if the fuzzy set $\mathcal{M}$ on $\mathcal{U}^2 \times (0, \infty)$, for all $\varrho, \vartheta, z \in \mathcal{U}$, and $s, t > 0$ satisfies the following conditions:

- ($\mathcal{M}1$) $\mathcal{M}(\varrho, \vartheta, t) > 0$;
- ($\mathcal{M}2$) $\mathcal{M}(\varrho, \vartheta, t) = 1, \forall t > 0$ if and only if $\varrho = \vartheta$;
- ($\mathcal{M}3$) $\mathcal{M}(\varrho, \vartheta, t) = \mathcal{M}(\vartheta, \varrho, t)$;
- ($\mathcal{M}4$) $\mathcal{M}(\varrho, z, s + t) \geq \mathcal{M}(\varrho, \vartheta, t) \star \mathcal{M}(\vartheta, z, s)$;
- ($\mathcal{M}5$) $\mathcal{M}(\varrho, \vartheta, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

The next remark recalls some additional properties of GV-type fuzzy metrics that will be used later.

Remark 2.4. [2] The following statements hold in a fuzzy metric space $(\mathcal{U}, \mathcal{M}, \star)$:

- (i) If for $\varrho, \vartheta \in \mathcal{U}$, $\mathcal{M}(\varrho, \vartheta, t) > 1 - r$, for some $t > 0$ and $0 < r < 1$, we can find a l_0 , $0 < l_0 < t$ such that $\mathcal{M}(\varrho, \vartheta, l_0) > 1 - r$.
- (ii) For any $0 < r_2 < r_1 < 1$, we can find $0 < r_3 < 1$ such that $r_1 \star r_3 \geq r_2$ and for any $0 < r_4 < 1$, we can find $0 < r_5 < 1$ such that $r_5 \star r_5 \geq r_4$.

Lemma 2.5. [47] $\mathcal{M}(\varrho, \vartheta, \cdot)$ is a non-decreasing function on $(0, \infty)$, for all $\varrho, \vartheta \in \mathcal{U}$.

Next, we recall a fuzzy analogue of the classical b -metric space, introduced in [3].

Definition 2.6. [3] Let $\mathcal{U}$ be a nonempty set, $k \in [1, \infty)$, and $\star$ be a continuous t -norm. The quadruple $(\mathcal{U}, \mathcal{M}, \star, k)$ is said to be a fuzzy b -metric space if, for all $\varrho, \vartheta, z \in \mathcal{U}$ and $s, t > 0$, $\mathcal{M}$ satisfies the following properties:

- ($b\mathcal{M}1$) $\mathcal{M}(\varrho, \vartheta, 0) = 0$;
- ($b\mathcal{M}2$) $(\mathcal{M}(\varrho, \vartheta, t) = 1, \forall t > 0)$ if and only if $\varrho = \vartheta$;
- ($b\mathcal{M}2$) $\mathcal{M}(\varrho, \vartheta, t) = \mathcal{M}(\vartheta, \varrho, t)$;
- ($b\mathcal{M}4$) $\mathcal{M}(\varrho, z, k(t + s)) \geq \mathcal{M}(\varrho, \vartheta, t) \star \mathcal{M}(\vartheta, z, s)$;
- ($b\mathcal{M}5$) $\mathcal{M}(\varrho, \vartheta, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous and $\lim_{t \rightarrow \infty} \mathcal{M}(\varrho, \vartheta, t) = 1$.

We now recall the definition of a more general class of fuzzy metric spaces introduced in [28].

Definition 2.7. [28] Let $\mathcal{U}$ be a non-empty set, $\mathcal{M} : \mathcal{U}^2 \times (0, \infty) \rightarrow [0, 1]$ be a mapping and $\star$ be a continuous t -norm. If $(f, \alpha) \in \mathcal{F} \times (0, 1]$ exists such that M satisfies the following conditions:

- ($\mathcal{FM}1$) $\mathcal{M}(\varrho, \vartheta, t) > 0, \forall \varrho, \vartheta \in \mathcal{U}, \forall t > 0$;
- ($\mathcal{FM}2$) $(\mathcal{M}(\varrho, \vartheta, t) = 1, \forall t > 0)$ iff $\varrho = \vartheta$;
- ($\mathcal{FM}3$) $\mathcal{M}(\varrho, \vartheta, t) = \mathcal{M}(\vartheta, \varrho, t), \forall \varrho, \vartheta \in \mathcal{U}, \forall t > 0$

($\mathcal{FM}4$) for every $(\varrho, \vartheta) \in \mathcal{U}^2$ and $N \in \mathbb{N}$, $N \geq 2$ and for every $\{v_i\}_i^N \subseteq \mathcal{U}$ with $v_1 = \varrho$ and $v_N = \vartheta$,

$$\mathcal{M}(\varrho, \vartheta, t) < 1 \implies (f(\mathcal{M}(\varrho, \vartheta, t)))^\alpha \geq f(\mathcal{M}(v_1, v_2, t_1) \star \mathcal{M}(v_2, v_3, t_2) \star \cdots \star \mathcal{M}(v_{N-1}, v_N, t_{N-1})),$$

where $t = t_1 + t_2 + \dots + t_{N-1}$; $t_i > 0$ for $i = 1, 2, \dots, (N-1)$. Then, $\mathcal{M}$ is called a fuzzy $\mathcal{F}$ -metric on $\mathcal{U}$ and the 5-tuple $(\mathcal{U}, \mathcal{M}, f, \alpha, \star)$ is said to be a fuzzy $\mathcal{F}$ -metric space.

To complete this part, we recall an example of a fuzzy $\mathcal{F}$ -metric space, with the associated concept of the sequence convergence introduced in [28].

Example 2.8. [28] Let $\mathcal{U} = \mathbb{R}$ and $\mathcal{M}: \mathcal{U}^2 \times (0, \infty) \rightarrow [0, 1]$ be a function defined by

$$\mathcal{M}(\varrho, \vartheta, t) = \left(\frac{t}{t+1} \right)^{|\varrho - \vartheta|^2},$$

for all $\varrho, \vartheta \in \mathcal{U}$ and $t \in (0, \infty)$. Then $\mathcal{M}$ is a fuzzy $\mathcal{F}$ -metric on $\mathcal{U}$ with respect to $f(\zeta) = \zeta^2$, for all $\zeta \in [0, 1]$, $\alpha = \frac{1}{2}$, and the ‘product’ t -norm.

Definition 2.9. [28] Let $\{\varrho_n\}$ be a sequence in $(\mathcal{U}, \mathcal{M}, f, \alpha, \star)$. Then, $\{\varrho_n\}$ is called

- (i) An $\mathcal{F}$ -convergent sequence if some $\varrho \in \mathcal{U}$ exists such that $\lim_{n \rightarrow \infty} \mathcal{M}(\varrho_n, \varrho, t) = 1$, $\forall t > 0$;
- (ii) An $\mathcal{F}$ -Cauchy sequence if $\lim_{m, n \rightarrow \infty} \mathcal{M}(\varrho_m, \varrho_n, t) = 1$, $\forall t > 0$.

Definition 2.10. [28] A fuzzy $\mathcal{F}$ -metric space $(\mathcal{U}, \mathcal{M}, f, \alpha, \star)$ is called $\mathcal{F}$ -complete if every $\mathcal{F}$ -Cauchy sequence converges to some point in $\mathcal{U}$.

Proposition 2.11. [28] In a fuzzy $\mathcal{F}$ -metric space $(\mathcal{U}, \mathcal{M}, f, \alpha, \star)$, every finite subset of $\mathcal{U}$ is $\mathcal{F}$ -bounded.

Finally, for later use, we recall the notions of sequentially open and closed sets.

Definition 2.12. [48, 49] Let $\mathcal{U}$ be a topological space. A subset B of $\mathcal{U}$ is called

- (i) Sequentially open, whenever a sequence $\{\varrho_n\} \subseteq \mathcal{U}$ converging to some $\varrho \in B$, $N \in \mathbb{N}$ exists such that $\varrho_n \in B$, for all $n \geq N$;
- (ii) Sequentially closed if every convergent sequence in B has its limit in B .

3. Fundamentals of the $EF\mathcal{F}$ -metric structure

This section establishes the foundational framework required for our subsequent results. We begin by defining the $EF\mathcal{F}$ -metric with illustrative examples that clarify its structure and behavior. We then examine the topology induced by an $EF\mathcal{F}$ -metric and describe the associated notions of convergence for sequences.

3.1. Basics of the $EF\mathcal{F}$ -metric

With the newly defined class $\mathcal{F}^*$, we are now in a position to extend the fuzzy $\mathcal{F}$ -metric framework.

Definition 3.1. Let $\mathcal{U}$ be a nonempty set, $\star$ be a continuous t -norm, and $\mathcal{M}_F: \mathcal{U}^2 \times (0, \infty) \rightarrow [0, 1]$ be a fuzzy set which satisfies the following properties:

($EF\mathcal{F}$ - $\mathcal{M}_F1$) $\mathcal{M}_F(\varrho_1, \varrho_2, t) > 0$, $\forall \varrho_1, \varrho_2 \in \mathcal{U}$, $\forall t > 0$;

($EF\mathcal{F}$ - $\mathcal{M}_F2$) $(\mathcal{M}_F(\varrho_1, \varrho_2, t) = 1, \forall t > 0) \iff \varrho_1 = \varrho_2$;

($EF\mathcal{F}$ - $\mathcal{M}_F3$) $\mathcal{M}_F(\varrho_1, \varrho_2, t) = \mathcal{M}_F(\varrho_2, \varrho_1, t)$, $\forall \varrho_1, \varrho_2 \in \mathcal{U}$, $\forall t > 0$;

($EF\mathcal{F}$ - $\mathcal{M}_F4$) There is an $F \in \mathcal{F}^*$ such that

$$\mathcal{M}_F(\varrho_1, \varrho_2, s+t) \geq F(\mathcal{M}_F(\varrho_1, \varrho_3, s) \star \mathcal{M}_F(\varrho_3, \varrho_2, t)), \quad \forall \varrho_1, \varrho_2, \varrho_3 \in \mathcal{U}, \forall s, t > 0.$$

This function $\mathcal{M}_F$ is called an $EF\mathcal{F}$ -metric on $\mathcal{U}$ with respect to the considered function F and $(\mathcal{U}, \mathcal{M}_F, F, \star)$ is called an $EF\mathcal{F}$ -MS.

Remark 3.2. It is evident that by selecting the function $F(t) = t$, for all $t \in [0, 1]$, every GV-type fuzzy metric space can be regarded as a particular case of the $EF\mathcal{F}$ -MS whose respective $EF\mathcal{F}$ -metric $\mathcal{M}_F(\varrho_1, \varrho_2, \cdot)$ is continuous on $(0, \infty)$. However, in general, not every $EF\mathcal{F}$ -MS satisfies the axioms of a GV fuzzy metric space. A counterexample is provided later.

Remark 3.3. For $N = 2$, $f(\zeta) = \zeta^2$ in $\mathcal{F}$ and $\alpha = \frac{1}{2}$, every fuzzy $\mathcal{F}$ -metric space is an extended fuzzy $\mathcal{F}$ -metric space. However, an $EF\mathcal{F}$ -MS is not a fuzzy $\mathcal{F}$ -metric space in general. An explicit example is given later.

Example 3.4. Let $(\mathcal{U}, \|\cdot\|)$ be a normed linear space. Define the functions $\mathcal{M}_F: \mathcal{U}^2 \times (0, \infty) \rightarrow [0, 1]$ by

$$\mathcal{M}_F(\varrho_1, \varrho_2, t) = \exp\left(-\frac{\|\varrho_1 - \varrho_2\|^3}{t}\right), \quad \forall \varrho_1, \varrho_2 \in \mathcal{U}, \forall t > 0$$

and $F: [0, 1] \rightarrow [0, 1]$ by $F(t) = t^4$, for all $t \in [0, 1]$. Then clearly $F \in \mathcal{F}^*$. The conditions ($EF\mathcal{F}$ - $\mathcal{M}_F1$)–($EF\mathcal{F}$ - $\mathcal{M}_F3$) follow immediately from the definition of $\mathcal{M}_F$. Therefore, we only verify the nontrivial condition ($EF\mathcal{F}$ - $\mathcal{M}_F4$). Let $\varrho_1, \varrho_2, \varrho_3 \in \mathcal{U}$ and $s, t > 0$. Then, using the inequality

$$(a+b)^3 \leq 4(a^3 + b^3), \quad \forall a, b \in [0, \infty)$$

and properties of the exponential function, we have

$$\frac{\|\varrho_1 - \varrho_2\|^3}{(s+t)} \leq 4 \frac{\|\varrho_1 - \varrho_3\|^3 + \|\varrho_2 - \varrho_3\|^3}{(s+t)} \leq 4 \frac{\|\varrho_1 - \varrho_3\|^3}{s} + \frac{\|\varrho_2 - \varrho_3\|^3}{t}$$

or

$$\exp\left(-\frac{\|\varrho_1 - \varrho_2\|^3}{(s+t)}\right) \geq \exp\left(-4 \frac{\|\varrho_1 - \varrho_3\|^3}{s}\right) \cdot \exp\left(-4 \frac{\|\varrho_2 - \varrho_3\|^3}{t}\right)$$

or

$$\mathcal{M}_F(\varrho_1, \varrho_2, s+t) \geq (\mathcal{M}_F(\varrho_1, \varrho_3, s) \cdot \mathcal{M}_F(\varrho_2, \varrho_3, t))^4 = F(\mathcal{M}_F(\varrho_1, \varrho_3, s) \star \mathcal{M}_F(\varrho_2, \varrho_3, t)).$$

Therefore, $\mathcal{M}_F$ is an $EF\mathcal{F}$ -metric on $\mathcal{U}$ with respect to the considered function F under the ‘product’ t -norm.

Remark 3.5. Though the function $\mathcal{M}_F$ of Example 3.4 satisfies conditions $(\mathcal{M}1)$ – $(\mathcal{M}3)$ and $(\mathcal{M}5)$ of Definition 2.3, ensuring the compatibility of having a GV fuzzy metric structure, it is not a GV fuzzy metric. To demonstrate it, we consider the special case where $\mathcal{U} = \mathbb{R}$ and choose the points $q_1 = 0, q_2 = 2, q_3 = 1$ with $s = t = 1$. We then compute

$$\mathcal{M}_F(q_1, q_2, s + t) = \exp(-4) < \exp(-2) = \mathcal{M}_F(q_1, q_3, s) \star \mathcal{M}_F(q_3, q_2, t).$$

This inequality violates the inequality $(\mathcal{M}4)$ required in the definition of GV fuzzy metric spaces. Hence, $\mathcal{M}_F$ cannot be regarded as a GV fuzzy metric.

Moreover, it is not a fuzzy $\mathcal{F}$ -metric space either. To justify this, we take $q_1 = 0, q_2 = 1$ and choose $t = 1$. Let $\{v_i\}_{i=0}^N \subset \mathcal{U}$ be defined by: $v_i = \frac{i}{N}$, $i = 0, 1, \dots, N$. Then $v_0 = q_1, v_N = q_2$ and for every $i = 0, 1, \dots, (N-1)$, we have

$$\mathcal{M}_F(v_i, v_{i+1}, t_i) = \exp\left(-\frac{\|v_i - v_{i+1}\|^3}{t_i}\right) = \exp\left(\frac{-1}{t_i N^3}\right).$$

Suppose $(f, \alpha) \in \mathcal{F} \times (0, 1]$ exists such that $(\mathcal{F} \mathcal{M}4)$ holds. Taking $t_i = \frac{1}{N}$, $i = 1, 2, \dots, N$, we have $\sum_{i=1}^N t_i = 1 = t$. Then,

$$(f(\mathcal{M}_F(q_1, q_2, t)))^\alpha \geq f(\mathcal{M}_F(v_0, v_1, t_1) \star \mathcal{M}_F(v_1, v_2, t_2) \star \dots \star \mathcal{M}_F(v_{N-1}, v_N, t_N))$$

implies

$$f\left(\exp(-1)\right)^\alpha \geq f\left(\exp\left(\frac{-1}{t_1 N^3}\right) \cdot \dots \cdot \exp\left(\frac{-1}{t_N N^3}\right)\right) = f\left(\exp\left(\frac{-1}{N^2}\right)\right).$$

Letting $N \rightarrow \infty$, we get

$$f(\exp(-1))^\alpha \geq f(1) = 1,$$

which is a contradiction to the properties of f .

Remark 3.6. In general, every EF $\mathcal{F}$ -MS may not be a fuzzy b-metric [3]. This highlights the broader generality and flexibility of our new framework.

Example 3.7. We define a function

$$\mathcal{M}_F(q, y, t) = \begin{cases} 1/3, & 0 < t < D_1(q, y), \\ 1/6, & D_1(q, y) \leq t \leq D_2(q, y), \\ 1/10, & D_2(q, y) < t < \infty, \\ 1, & q = y, \forall t > 0, \end{cases}$$

where D_1 , and D_2 are two metrics on $\mathbb{R}^2$ defined by

$$D_1(q, y) = \max\{|q_i - y_i| : i = 1, 2\} \quad \text{and} \quad D_2(q, y) = \sum_{i=1}^2 |q_i - y_i|,$$

for all $q = (q_1, q_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then clearly $\mathcal{M}_F$ satisfies $(EF \mathcal{F}\text{-}\mathcal{M}_F1)$ – $(EF \mathcal{F}\text{-}\mathcal{M}_F3)$. We prove that $\mathcal{M}_F$ satisfies $(EF \mathcal{F}\text{-}\mathcal{M}_F4)$. We define $F: [0, 1] \rightarrow [0, 1]$ by $F(\zeta) = \zeta^2$, for all $\zeta \in [0, 1]$. Then $F \in \mathcal{F}^*$ and we take ‘product’ as the t -norm ‘ $\star$ ’.

Another inequality to remember is that

$$D_1(\varrho, y) \leq D_2(\varrho, y), \forall \varrho, y \in \mathbb{R}^2.$$

We take $\varrho, y, \varpi \in \mathbb{R}^2$ and $s, t > 0$. The possible cases in the verification of $(EF\mathcal{F}-\mathcal{M}_F4)$ are summarized in Table 2.

Table 2. Case analysis for the verification of $(EF\mathcal{F}-\mathcal{M}_F4)$.

Condition on s, t	$(\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(\varpi, y, t))$	Estimate
$s > D_2(\varrho, \varpi), t > D_2(\varpi, y)$	$= (1/10, 1/10)$	$s + t > D_2(\varrho, y)$
$s > D_2(\varrho, \varpi), D_1(\varpi, y) \leq t \leq D_2(\varpi, y)$	$= (1/10, 1/6)$	$s + t > D_1(\varrho, y)$
$D_1(\varrho, \varpi) \leq s \leq D_2(\varrho, \varpi), D_1(\varpi, y) \leq t \leq D_2(\varpi, y)$	$= (1/6, 1/6)$	$s + t > D_1(\varrho, y)$
At least one of s, t satisfies $0 < \cdot \leq D_1$	$\geq (1/10, 1/3)$	$s + t > 0$

Without loss of generality, we may assume that $\rho \neq y$. If $\rho = y$, then $\mathcal{M}_F(\rho, y, s + t) = 1$ and the condition is obvious. We now discuss the three general patterns.

Pattern I: Both parameters are outside the threshold D_2 . Suppose that $s > D_2(\varrho, \varpi), t > D_2(\varpi, y)$. Then, $\mathcal{M}_F(\varrho, \varpi, s) = \mathcal{M}_F(\varpi, y, t) = 1/10$ and hence $F(\mathcal{M}_F(\varrho, \varpi, s) \star \mathcal{M}_F(\varpi, y, t)) = 1/100^2$.

Moreover, $s + t > D_2(\varrho, \varpi) + D_2(\varpi, y) \geq D_2(\varrho, y)$, and hence $\mathcal{M}_F(\varrho, y, s + t) = 1/10 > F(\mathcal{M}_F(\varrho, \varpi, s) \star \mathcal{M}_F(\varpi, y, t))$.

Pattern II: At least one parameter lies in the threshold region $D_1 \leq \cdot \leq D_2$. In this case, we have the following cases:

(a) When $s > D_2(\rho, \bar{\omega}), D_1(\bar{\omega}, y) \leq t \leq D_2(\bar{\omega}, y)$, then $\mathcal{M}_F(\rho, \bar{\omega}, s) = \frac{1}{10}$ and $\mathcal{M}_F(\bar{\omega}, y, s) = \frac{1}{6}$. So $\mathcal{M}_F(\rho, \bar{\omega}, s) \star \mathcal{M}_F(\bar{\omega}, y, t) = \frac{1}{60}$, which implies $F(\mathcal{M}_F(\rho, \bar{\omega}, s) \star \mathcal{M}_F(\bar{\omega}, y, t)) = \frac{1}{60^2}$.

Moreover, $s + t > D_2(\rho, \bar{\omega}) + D_1(\bar{\omega}, y) \geq D_1(\rho, \bar{\omega}) + D_1(\bar{\omega}, y) \geq D_1(\rho, y)$.

(b) When $D_1(\rho, \bar{\omega}) \leq s \leq D_2(\rho, \bar{\omega}), D_1(\bar{\omega}, y) \leq t \leq D_2(\bar{\omega}, y)$, then $\mathcal{M}_F(\rho, \bar{\omega}, s) = \frac{1}{6} = \mathcal{M}_F(\bar{\omega}, y, t)$. So $\mathcal{M}_F(\rho, \bar{\omega}, s) \star \mathcal{M}_F(\bar{\omega}, y, t) = \frac{1}{36}$, which implies $F(\mathcal{M}_F(\rho, \bar{\omega}, s) \star \mathcal{M}_F(\bar{\omega}, y, t)) = \frac{1}{36^2}$.

Moreover, $s + t \geq D_1(\rho, \bar{\omega}) + D_1(\bar{\omega}, y) \geq D_1(\rho, y)$. Therefore, $\mathcal{M}_F(\rho, y, s + t) \in \{\frac{1}{6}, \frac{1}{10}\}$ and hence $\mathcal{M}_F(\varrho, y, s + t) > F(\mathcal{M}_F(\varrho, \varpi, s) \star \mathcal{M}_F(\varpi, y, t))$.

Pattern III: At least one parameter lies inside the threshold region D_1 . Here, we discuss the following three sub-cases:

(a) If $s > D_2(\rho, \bar{\omega}), 0 < t < D_1(\bar{\omega}, y)$, then $\mathcal{M}_F(\rho, \bar{\omega}, s) = \frac{1}{10}$ and $\mathcal{M}_F(\bar{\omega}, y, t) = \frac{1}{3}$.

(b) If $D_1(\rho, \bar{\omega}) \leq s \leq D_2(\rho, \bar{\omega}), 0 < t < D_1(\bar{\omega}, y)$, then $\mathcal{M}_F(\rho, \bar{\omega}, s) = \frac{1}{6}$ and $\mathcal{M}_F(\bar{\omega}, y, t) = \frac{1}{3}$.

(c) If $0 < s < D_1(\rho, \bar{\omega}), 0 < t < D_1(\bar{\omega}, y)$, then $\mathcal{M}_F(\rho, \bar{\omega}, s) = \frac{1}{3} = \mathcal{M}_F(\bar{\omega}, y, t)$.

Therefore, $F(\mathcal{M}_F(\rho, \bar{\omega}, s) \star \mathcal{M}_F(\bar{\omega}, y, t)) \in \{\frac{1}{30^2}, \frac{1}{18^2}, \frac{1}{9^2}\}$ and since $s + t > 0$, $\mathcal{M}_F(\rho, y, s + t) \in \{\frac{1}{3}, \frac{1}{6}, \frac{1}{10}\}$. Hence, $\mathcal{M}_F(\varrho, y, s + t) > F(\mathcal{M}_F(\varrho, \varpi, s) \star \mathcal{M}_F(\varpi, y, t))$.

Therefore, in all cases, $\mathcal{M}_F$ satisfies $(EF\mathcal{F}-\mathcal{M}_F4) : F(\mathcal{M}_F(\varrho, \varpi, s) \star \mathcal{M}_F(\varpi, y, t)) \leq \mathcal{M}_F(\varrho, y, s + t)$. Thus $\mathcal{M}_F$ is an $EF\mathcal{F}$ -MS on $\mathbb{R}^2$ with respect to the defined $F \in \mathcal{F}^*$ and the ‘product’ t -norm.

Next, we show that $\mathcal{M}_F$ is not a fuzzy b -metric space. To this end, choose $\varrho_0 = (0, 0)$, $\varpi_0 = (2, 2)$, $y_0 = (2, 0)$, and $s_0 = t_0 = 1.5$. Then $D_1(\varrho_0, \varpi_0) = 2$, $D_2(\varrho_0, \varpi_0) = 4$ implies $\mathcal{M}_F(\varrho_0, \varpi_0, s_0) = \frac{1}{3}$ and

$D_1(y_0, \varpi_0) = 2, D_2(y_0, \varpi_0) = 2$ implies $\mathcal{M}_F(y_0, \varpi_0, t_0) = \frac{1}{3}$. Hence, we have

$$\mathcal{M}_F(\varrho_0, \varpi_0, s_0) \star \mathcal{M}_F(y_0, \varpi_0, t_0) = \frac{1}{9}.$$

Since $D_1(\varrho_0, y_0) = 2, D_2(\varrho_0, y_0) = 2$, then for every $k \geq 1, 3k \geq 3 > 2$ implies

$$\mathcal{M}_F(\varrho_0, y_0, k(s_0 + t_0)) = \frac{1}{10} < \frac{1}{9} = \mathcal{M}_F(\varrho_0, \varpi_0, s_0) \star \mathcal{M}_F(y_0, \varpi_0, t_0).$$

Therefore, the inequality (b $\mathcal{M}$ 4) fails; consequently, $\mathcal{M}_F$ is not a fuzzy b -metric on $\mathbb{R}^2$.

Example 3.8. Let $(\mathcal{U}, d)$ be a metric space, and we define a mapping by:

$$\mathcal{M}_F(\varrho, y, t) = \begin{cases} t/2(t + d(\varrho, y)), & 0 < t < d(\varrho, y), \\ 1/2, & d(\varrho, y) \leq t \leq 2d(\varrho, y), \\ t/t + d(\varrho, y), & 2d(\varrho, y) < t < \infty, \\ 1, & \varrho = y, \end{cases}$$

for all $\varrho, y \in \mathcal{U}$ and $t > 0$.

Then following the properties of metric d and the definition of $\mathcal{M}_F$, (EF $\mathcal{F}$ - $\mathcal{M}_F$ 1)-(EF $\mathcal{F}$ - $\mathcal{M}_F$ 3) follows directly. We have to prove only that $\mathcal{M}_F$ satisfies (EF $\mathcal{F}$ - $\mathcal{M}_F$ 4). For, we define $F \in \mathcal{F}^*$ by:

$$F(t) = \begin{cases} \frac{t}{3}, & 0 \leq t \leq \frac{1}{4}, \\ \frac{1}{12}, & \frac{1}{4} \leq t \leq \frac{1}{3}, \\ \frac{5}{2}t - \frac{3}{4}, & \frac{1}{3} \leq t \leq \frac{1}{2}, \\ t, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

To check (EF $\mathcal{F}$ - $\mathcal{M}_F$ 4), take $\varrho, y, \varpi \in \mathcal{U}$, $s, t > 0$, and ‘min’ as t -norm. The possible cases in the verification of (EF $\mathcal{F}$ - $\mathcal{M}_F$ 4) are summarized in Table 3.

Table 3. Case analysis for the verification of (EF $\mathcal{F}$ - $\mathcal{M}_F$ 4).

Condition on s, t	$\min(\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(\varpi, y, t))$	Estimate
$s > 2d(\varrho, \varpi), t > 2d(y, \varpi)$	$\geq 1/2$	$s + t > 2d(\varrho, y)$
$s > 2d(\varrho, \varpi), d(\varpi, y) < t \leq 2d(\varpi, y)$	$= 1/2$	$s + t > d(\varrho, y)$
$d(\varrho, \varpi) < s \leq d(\varrho, \varpi), d(\varpi, y) < t \leq 2d(\varpi, y)$	$= 1/2$	$s + t > d(\varrho, y)$
At least one of s, t satisfies $0 < \cdot \leq d$	$< 1/2$	$s + t > 0$

Without loss of generality, we may assume that $\rho \neq y$. If $\rho = y$, then $\mathcal{M}_F(\rho, y, s + t) = 1$, and the condition is obvious. We now discuss the three general patterns.

Pattern I. Both parameters are outside the threshold $2d$. When $s > 2d(\varrho, \varpi), t > 2d(\varpi, y)$ then

$$\mathcal{M}_F(\varrho, \varpi, s) = \frac{s}{s + d(\varrho, \varpi)} (\geq \frac{1}{2}) \quad \text{and} \quad \mathcal{M}_F(y, \varpi, t) = \frac{t}{t + d(y, \varpi)} (\geq \frac{1}{2}).$$

Again, we have $s + t > 2d(\varrho, \varpi) + 2d(\varpi, y) \geq 2d(\varrho, y)$. Then

$$\mathcal{M}_F(\varrho, y, s + t) = \frac{s + t}{(s + t) + d(\varrho, y)} \geq \min \left\{ \frac{s}{s + d(\varrho, \varpi)}, \frac{t}{t + d(y, \varpi)} \right\} = F(\mathcal{M}_F(\varrho, \varpi, s) \star \mathcal{M}_F(y, \varpi, t))$$

follows from the proof of [2, Example 2.9].

Pattern II. At least one of the parameters lies within the threshold $d \leq \cdot \leq 2d$. Here, we discuss two main subcases.

(A) When $s > 2d(\varrho, \varpi)$, $d(y, \varpi) \leq t \leq 2d(y, \varpi)$, then

$$\mathcal{M}_F(\varrho, \varpi, t) = \frac{t}{t + d(\varrho, \varpi)} (\geq \frac{1}{2}) \quad \text{and} \quad \mathcal{M}_F(y, \varpi, t) = \frac{1}{2}.$$

Hence, $\mathcal{M}_F(\varrho, \varpi, s) > \mathcal{M}_F(y, \varpi, t)$. In this case, $s + t > 2d(\varrho, \varpi) + d(y, \varpi) > d(\varrho, \varpi) + d(y, \varpi) \geq d(\varrho, y)$.

(B) When $d(\varrho, \varpi) \leq s \leq 2d(\varrho, \varpi)$ and $d(y, \varpi) \leq t \leq 2d(y, \varpi)$, then $\mathcal{M}_F(\varrho, \varpi, s) = \frac{1}{2} = \mathcal{M}_F(y, \varpi, t)$. Here, $s + t \geq d(\varrho, \varpi) + d(\varpi, y) \geq d(\varrho, y)$.

In both cases, $\min \left\{ \mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t) \right\} = \frac{1}{2}$ implies $F(\min \left\{ \mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t) \right\}) = \frac{1}{2}$. For $s + t \geq d(\varrho, y)$ the following cases arise.

(a) If $s + t > 2d(\varrho, y)$, then $\mathcal{M}_F(\varrho, y, s + t) = \frac{s+t}{(s+t)+d(\varrho, y)} > \frac{1}{2} = \min \left\{ \mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t) \right\}$.

(b) If $d(\varrho, y) \leq s + t \leq 2d(\varrho, y)$, then $\mathcal{M}_F(\varrho, \varpi, s + t) = \frac{1}{2} = \min \left\{ \mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t) \right\}$. Hence $\mathcal{M}_F$ satisfies: $\mathcal{M}_F(\varrho, y, s + t) \geq F(\min \{ \mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(\varpi, y, t) \})$.

Pattern III: At least one parameter lies inside the threshold region d . Here, we discuss three subcases.

(A) $s > 2d(\varrho, \varpi)$, $0 < t < d(\varpi, y)$. Then $\mathcal{M}_F(y, \varpi, t) = \frac{t}{2(t+d(y, \varpi))} (< \frac{1}{4})$.

(B) $d(\varrho, \varpi) \leq s \leq 2d(\varrho, \varpi)$, $0 < t < d(y, \varpi)$. Then $\mathcal{M}_F(\varrho, \varpi, s) = \frac{1}{2}$ & $\mathcal{M}_F(y, \varpi, t) = \frac{t}{2(t+d(y, \varpi))} (< \frac{1}{4})$.

Therefore, in both the cases, $\min \left\{ \mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t) \right\} = \mathcal{M}_F(y, \varpi, t)$ and hence

$$F \left(\min \left\{ \mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t) \right\} \right) = \frac{\mathcal{M}_F(y, \varpi, t)}{3} \leq \frac{1}{12}.$$

Here, $s + t > 2d(\varrho, \varpi) > 0$, which yields the following cases:

(a) If $0 < s + t < d(\varrho, y)$, then $\mathcal{M}_F(\varrho, y, s + t) = \frac{s+t}{2[(s+t)+d(\varrho, y)]}$. Again, $\min \left\{ \mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t) \right\} = \mathcal{M}_F(y, \varpi, t)$ implies $\frac{t}{2(t+d(y, \varpi))} \leq \frac{s}{s+d(\varrho, \varpi)}$, that is

$$td(\varrho, \varpi) \leq st + 2sd(\varpi, y). \quad (3.1)$$

We now have

$$\begin{aligned} \mathcal{M}_F(\varrho, y, s + t) - \frac{1}{3} \mathcal{M}_F(y, \varpi, t) &= \frac{s + t}{2((s + t) + d(\varrho, y))} - \frac{t}{3 \cdot 2(t + d(y, \varpi))} \\ &\geq \frac{2t(s + t) + 3(s + t)d(y, \varpi) - t\{d(\varrho, \varpi) + d(\varpi, y)\}}{6\{s + t + d(\varrho, y)\}(t + d(y, \varpi))} \end{aligned}$$

$$\begin{aligned} &\geq \frac{2t^2 + 2st + 2td(y, \varpi) + 3sd(y, \varpi) - 2\{st + sd(y, \varpi)\}}{6\{s + t + d(\varrho, y)\}(t + d(y, \varpi))} \\ &> 0, \end{aligned}$$

which holds using the triangle inequality axiom of d and (3.1).

(b) If $d(\varrho, y) \leq s + t \leq 2d(\varrho, y)$, then $\mathcal{M}_F(\varrho, y, s + t) = \frac{1}{2} > F(\min\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t)\})$.

(c) If $2d(\varrho, y) < s + t < \infty$, then $\mathcal{M}_F(\varrho, y, s + t) = \frac{s+t}{(s+t)+d(\varrho, y)} > \frac{2}{3} > F(\min\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t)\})$.

(C) $0 < s < d(\varrho, \varpi)$, $0 < t < d(\varpi, y)$.

Then $\mathcal{M}_F(\varrho, \varpi, s) = \frac{s}{2(s+d(\varrho, \varpi))} (< \frac{1}{4})$, $\mathcal{M}_F(y, \varpi, t) = \frac{t}{2(t+d(\varpi, y))} (< \frac{1}{4})$ and hence

$$F\left(\min\left\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t)\right\}\right) < \frac{1}{3} \min\left\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(y, \varpi, t)\right\} < \frac{1}{12}.$$

Here, $s + t > 0$. Then the following subcases arise.

(a) If $0 < s + t < d(\varrho, y)$, then $\mathcal{M}_F(\varrho, y, s + t) = \frac{s+t}{2\{(s+t)+d(\varrho, y)\}}$.

Now, we assume that $\min\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(\varpi, y, t)\} = \mathcal{M}_F(\varrho, \varpi, s)$ then

$$sd(y, \varpi) \leq td(\varrho, \varpi). \quad (3.2)$$

Hence

$$\begin{aligned} \mathcal{M}_F(\varrho, y, s + t) - \frac{1}{3} \mathcal{M}_F(\varrho, \varpi, s) &= \frac{(s + t)}{2((s + t) + d(\varrho, y))} - \frac{s}{3(s + d(\varrho, \varpi))} \\ &\geq \frac{2s(s + t) + 3sd(\varrho, \varpi) + 3td(\varrho, \varpi) - s\{d(\varrho, \varpi) + d(y, \varpi)\}}{2((s + t) + d(\varrho, y)) \cdot 3(s + d(\varrho, \varpi))} \\ &\geq \frac{s(s + t) + sd(\varrho, \varpi) + 3sd(y, \varpi) - 2sd(y, \varpi)}{2((s + t) + d(\varrho, y)) \cdot 3(s + d(\varrho, \varpi))} \\ &> 0, \end{aligned}$$

which holds using the triangle inequality of d and (3.2).

If $\min\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(\varpi, y, t)\} = \mathcal{M}_F(\varpi, y, t)$, then by similar arguments, $\mathcal{M}_F(\varrho, y, s + t) > \frac{\mathcal{M}_F(\varpi, y, t)}{3}$.

(b) If $d(\varrho, y) \leq s + t \leq 2d(\varrho, y)$, then $\mathcal{M}_F(\varrho, y, s + t) = \frac{1}{2} > F(\min\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(\varpi, y, t)\})$.

(c) If $2d(\varrho, y) < s + t < \infty$, then $\mathcal{M}_F(\varrho, y, s + t) = \frac{s+t}{(s+t)+d(\varrho, y)} > \frac{2}{3} > F(\min\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(\varpi, y, t)\})$.

Therefore, in this case, $\mathcal{M}_F(\varrho, y, s + t) > F(\min\{\mathcal{M}_F(\varrho, \varpi, s), \mathcal{M}_F(\varpi, y, t)\})$.

Hence, $\mathcal{M}_F$ satisfies (EF $\mathcal{F}$ - $\mathcal{M}_F$ 4): $\mathcal{M}_F(\varrho, y, s + t) \geq F(\mathcal{M}_F(\varrho, \varpi, s) \star \mathcal{M}_F(\varpi, y, t))$ with respect to the defined F and 'min' t -norm. Henceforth, $(\mathcal{U}, \mathcal{M}_F, F, \min)$ is an EF $\mathcal{F}$ -MS.

Remark 3.9. The examples above illustrate that the function $\mathcal{M}_F$ defining the EF $\mathcal{F}$ -metric need not satisfy the continuity of the function $t \rightarrow \mathcal{M}_F(\varrho, y, t)$ nor the non-decreasing property with respect to the parameter $t \in (0, \infty)$. Consequently, it does not satisfy the continuity requirements imposed in GV fuzzy metric spaces or fuzzy b -metric spaces. Therefore, this example remains a valid EF $\mathcal{F}$ -MS but falls outside the existing frameworks in which continuity and monotonicity with respect to the parameter t are imposed as part of the defining axioms.

We next introduce the following properties as *optional additional conditions*, which may be imposed when required in specific analytical or application-oriented settings

$$\begin{aligned} (EF\mathcal{F}\text{-}\mathcal{M}_F5) \quad & \mathcal{M}_F(\varrho, y, \cdot) \text{ is non-decreasing on } (0, \infty), \forall \varrho, y \in \mathcal{U}, \\ (EF\mathcal{F}\text{-}\mathcal{M}_F6) \quad & \mathcal{M}_F(\varrho, y, \cdot) \text{ is continuous on } (0, \infty), \forall \varrho, y \in \mathcal{U}. \end{aligned}$$

Structure overview. For the reader's convenience, we briefly summarize the main objects introduced in this section. Starting from the notion of an $EF\mathcal{F}$ -MS (Definition 3.1), we introduce the associated $EF\mathcal{F}$ -open balls and then define the notions of $EF\mathcal{F}$ -open and $EF\mathcal{F}$ -closed sets, and it is subsequently proved that every $EF\mathcal{F}$ -MS is Hausdorff. Furthermore, we establish the notions of $EF\mathcal{F}$ -convergent and $EF\mathcal{F}$ -Cauchy sequences. These concepts provide the fundamental framework used throughout the remainder of the paper.

3.2. Topology induced by the $EF\mathcal{F}$ -metric

In this subsection, we introduce the basic topological concepts that arise from the fuzzy metric $\mathcal{M}_F$. Using ball-based neighborhoods, we construct a topology $\tau_{\mathcal{M}_F}$ on $\mathcal{U}$ and show that every $EF\mathcal{F}$ -MS becomes a Hausdorff space under this topology. These structural results form the groundwork needed for analyzing convergence and continuity in subsequent parts.

Definition 3.10. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS. The set $B_{\mathcal{M}_F}(\varrho, r, t)$ where $\varrho \in \mathcal{U}$, $t \in (0, \infty)$, and $0 < r < 1$, defined by $B_{\mathcal{M}_F}(\varrho, r, t) = \{y \in \mathcal{U} : \mathcal{M}_F(\varrho, y, t) > 1 - r\}$ is called an $EF\mathcal{F}$ -open ball with center ϱ and radius r .

The following proposition mirrors the standard method used to define a topology in GV fuzzy metric spaces. There, the topology is constructed by taking sets that contain an ‘open ball’ around each of their points. Similarly, in the context of $EF\mathcal{F}$ -metric spaces, we propose the following result.

Proposition 3.11. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS satisfying $(EF\mathcal{F}\text{-}\mathcal{M}_F5)$. Then $\tau_{\mathcal{M}_F} = \{A \subseteq \mathcal{U} : \text{for each } \varrho \in A, r \in (0, 1) \text{ and } t > 0 \text{ exist such that } B_{\mathcal{M}_F}(\varrho, r, t) \subseteq A\}$ is a topology on $\mathcal{U}$.

Proof. Trivially $\emptyset, \mathcal{U} \in \tau_{\mathcal{M}_F}$. Further, let $A, B \in \tau_{\mathcal{M}_F}$ and $\varrho \in A \cap B$. Then $\varrho \in A$ and $\varrho \in B$. Hence $t_1, t_2 > 0$ and $r_1, r_2 \in (0, 1)$ exist such that $B_{\mathcal{M}_F}(\varrho, r_1, t_1) \subset A$ and $B_{\mathcal{M}_F}(\varrho, r_2, t_2) \subset B$.

Let $r = \min\{r_1, r_2\}$, $t = \min\{t_1, t_2\}$ and take $y \in B_{\mathcal{M}_F}(\varrho, r, t)$. Then $\mathcal{M}_F(\varrho, y, t) > 1 - r$.

Since $t_i \geq t$ and $r_i \geq r$, for each $i = 1, 2$, applying $(EF\mathcal{F}\text{-}\mathcal{M}_F5)$, we get

$$\mathcal{M}_F(\varrho, y, t_i) \geq \mathcal{M}_F(\varrho, y, t) > 1 - r \geq 1 - r_i.$$

This implies $y \in B_{\mathcal{M}_F}(\varrho, r_i, t_i)$, $i = 1, 2$. Consequently, $B_{\mathcal{M}_F}(\varrho, r, t) \subset B_{\mathcal{M}_F}(\varrho, r_1, t_1) \cap B_{\mathcal{M}_F}(\varrho, r_2, t_2) \subset A \cap B$ and hence $A \cap B \in \tau_{\mathcal{M}_F}$.

Next, suppose $A_i \in \tau_{\mathcal{M}_F}$ for each $i \in I$ and $\varrho \in \bigcup_{i \in I} A_i$. Thus $i_0 \in I$ exists such that $\varrho \in A_{i_0}$. So $t > 0$ and $r \in (0, 1)$ exist such that $B_{\mathcal{M}_F}(\varrho, r, t) \subset A_{i_0} \subset \bigcup_{i \in I} A_i$. Thus $\bigcup_{i \in I} A_i \in \tau_{\mathcal{M}_F}$. Hence, $\tau_{\mathcal{M}_F}$ is a topology. $\square$

Definition 3.12. In an $EF\mathcal{F}$ -MS $(\mathcal{U}, \mathcal{M}_F, F, \star)$, a subset A in $\mathcal{U}$ is said to be a $EF\mathcal{F}$ -open set if $A \in \tau_F$ and a $EF\mathcal{F}$ -closed set if $\mathcal{U} \setminus A \in \tau_F$.

Proposition 3.13. Every $EF\mathcal{F}$ -MS $(\mathcal{U}, \mathcal{M}_F, F, \star)$ is a Hausdorff space.

Proof. Let $\varrho, y \in \mathcal{U}$ with $\varrho \neq y$. Then $l_0 > 0$ exists such that

$$0 < \mathcal{M}_F(\varrho, y, l_0) < 1. \quad (3.3)$$

Let $\gamma_n = 1 - \frac{\mathcal{M}_F(\varrho, y, l_0)}{n}$, $n \in \mathbb{N}$. Then $\gamma_n \rightarrow 1$ as $n \rightarrow \infty$.

Now consider the $EF\mathcal{F}$ -open balls $B_{\mathcal{M}_F}(\varrho, 1 - \gamma_n, \frac{l_0}{2})$ and $B_{\mathcal{M}_F}(y, 1 - \gamma_n, \frac{l_0}{2})$, $n \in \mathbb{N}$. We claim, for some $r \in \mathbb{N}$ that

$$B_{\mathcal{M}_F}(\varrho, 1 - \gamma_r, \frac{l_0}{2}) \cap B_{\mathcal{M}_F}(y, 1 - \gamma_r, \frac{l_0}{2}) = \emptyset.$$

Suppose to the contrary that

$$B_{\mathcal{M}_F}(\varrho, 1 - \gamma_n, \frac{l_0}{2}) \cap B_{\mathcal{M}_F}(y, 1 - \gamma_n, \frac{l_0}{2}) \neq \emptyset,$$

for each $n \in \mathbb{N}$. We then can find a sequence $\{z_n\}$ in $\mathcal{U}$ such that

$$z_n \in B_{\mathcal{M}_F}(\varrho, 1 - \gamma_n, \frac{l_0}{2}) \cap B_{\mathcal{M}_F}(y, 1 - \gamma_n, \frac{l_0}{2}),$$

for all $n \in \mathbb{N}$. Therefore, we have

$$\mathcal{M}_F(\varrho, z_n, \frac{l_0}{2}) > \gamma_n \text{ and } \mathcal{M}_F(y, z_n, \frac{l_0}{2}) > \gamma_n. \quad (3.4)$$

Hence, using $(EF\mathcal{F}-\mathcal{M}_F4)$, we get

$$\mathcal{M}_F(\varrho, y, l_0) \geq F\left(\mathcal{M}_F(\varrho, z_n, \frac{l_0}{2}) \star \mathcal{M}_F(z_n, y, \frac{l_0}{2})\right) \geq F(\gamma_n \star \gamma_n),$$

using $(\mathcal{F}^*1)$ and (3.4). Taking the limit as $n \rightarrow \infty$, we have

$$\mathcal{M}_F(\varrho, y, l_0) \geq \lim_{n \rightarrow \infty} F(\gamma_n \star \gamma_n) = 1,$$

where the last equality follows from the continuity of $\star$, which yields $\lim_{n \rightarrow \infty} (\gamma_n \star \gamma_n) = 1$ together with the condition $(\mathcal{F}^*2)$.

Consequently, $\mathcal{M}(\varrho, y, l_0) = 1$, which is a contradiction to (3.3). Hence, some $r \in \mathbb{N}$ exists such that

$$B_{\mathcal{M}_F}(\varrho, 1 - \gamma_r, \frac{l_0}{2}) \cap B_{\mathcal{M}_F}(y, 1 - \gamma_r, \frac{l_0}{2}) = \emptyset.$$

Therefore, the $EF\mathcal{F}$ -open balls separate distinct points, and the proof is complete. $\square$

3.3. Convergence, Cauchy sequences, and completeness in $EF\mathcal{F}$ -metric spaces

This subsection introduces convergence and Cauchy sequences in $EF\mathcal{F}$ -metric spaces. The discussion highlights how the notion of convergence extends naturally from classical fuzzy metric spaces, emphasizing the essential features that distinguish this generalized setting.

We define convergence and Cauchy sequences in an $EF\mathcal{F}$ -MS in the same manner as in the classical fuzzy $\mathcal{F}$ -metric framework [28], with the function $f \in \mathcal{F}$ replaced by $F \in \mathcal{F}^*$. Explicitly, a sequence $\{\varrho_n\} \subset \mathcal{U}$ is said to have the following characteristics:

(i) It will converge to $\varrho \in X$ if

$$\lim_{n \rightarrow \infty} \mathcal{M}_F(\varrho_n, \varrho, t) = 1, \forall t > 0$$

and called an $EF\mathcal{F}$ -convergent sequence.

(ii) It will be an $EF\mathcal{F}$ -Cauchy sequence if

$$\lim_{m, n \rightarrow \infty} \mathcal{M}_F(\varrho_m, \varrho_n, t) = 1, \forall t > 0.$$

Equivalently, the convergence condition above can be expressed as follows: For every $r \in (0, 1)$ and $t > 0$, $N \in \mathbb{N}$ exists such that $\mathcal{M}_F(\varrho_n, \varrho, t) > 1 - r$ whenever $n \geq N$. Similarly, the Cauchy condition is equivalent to the requirement that, for every $r \in (0, 1)$ and $t > 0$, $N \in \mathbb{N}$ exists such that $\mathcal{M}_F(\varrho_n, \varrho_m, t) > 1 - r$ whenever $n, m \geq N$. The proofs of these equivalent formulations are standard in fuzzy metric spaces (see [2]).

Proposition 3.14. *Let $\{\varrho_n\}$ be a sequence in an $EF\mathcal{F}$ -MS $(\mathcal{U}, \mathcal{M}_F, F, \star)$. Then the following hold:*

- (i) *The limit of $\{\varrho_n\}$ is unique, if it exists.*
- (ii) *If $\{\varrho_n\}$ converges, it is an $EF\mathcal{F}$ -Cauchy sequence.*

Proof.

(i) Let $\varrho_1, \varrho_2 \in \mathcal{U}$ be such that $\{\varrho_n\}$ converges to both ϱ_1 and ϱ_2 . Then

$$\lim_{n \rightarrow \infty} \mathcal{M}_F(\varrho_n, \varrho_1, t) = 1 = \lim_{n \rightarrow \infty} \mathcal{M}_F(\varrho_n, \varrho_2, t), \quad \forall t > 0. \quad (3.5)$$

Suppose $x \neq y$. Then $l_0 > 0$ exists such that

$$\mathcal{M}_F(\varrho_1, \varrho_2, l_0) < 1. \quad (3.6)$$

Now using $(EF\mathcal{F}-\mathcal{M}_F4)$, we can write

$$\mathcal{M}_F(\varrho_1, \varrho_2, l_0) \geq F\left(\mathcal{M}_F\left(\varrho_1, \varrho_n, \frac{l_0}{2}\right) \star \mathcal{M}_F\left(\varrho_n, \varrho_2, \frac{l_0}{2}\right)\right), \quad \forall n \in \mathbb{N},$$

which yields

$$\mathcal{M}_F(\varrho_1, \varrho_2, l_0) \geq \lim_{n \rightarrow \infty} F\left(\mathcal{M}_F\left(\varrho_1, \varrho_n, \frac{l_0}{2}\right) \star \mathcal{M}_F\left(\varrho_n, \varrho_2, \frac{l_0}{2}\right)\right) = 1,$$

where the last equality follows from the continuity of $\star$, which yields

$$\lim_{n \rightarrow \infty} \left(\mathcal{M}_F\left(\varrho_1, \varrho_n, \frac{l_0}{2}\right) \star \mathcal{M}_F\left(\varrho_n, \varrho_2, \frac{l_0}{2}\right) \right) = 1$$

together with the condition $(\mathcal{F}^*2)$ and (3.5). This contradicts our assumption (3.6).

(ii) Suppose that $\{\varrho_n\}$ converges to some $\varrho \in \mathcal{U}$. Then

$$\lim_{n \rightarrow \infty} \mathcal{M}_F(\varrho_n, \varrho, t) = 1, \forall t > 0. \quad (3.7)$$

Now using $(EF\mathcal{F}\text{-}\mathcal{M}_F4)$, we can write

$$\mathcal{M}_F(\varrho_n, \varrho_m, s) \geq F\left(\mathcal{M}_F\left(\varrho_n, \varrho, \frac{s}{2}\right) \star \mathcal{M}_F\left(\varrho, \varrho_m, \frac{s}{2}\right)\right), \quad \forall m, n \in \mathbb{N}, s > 0$$

or

$$\lim_{m, n \rightarrow \infty} \mathcal{M}_F(\varrho_n, \varrho_m, s) \geq \lim_{m, n \rightarrow \infty} F\left(\mathcal{M}_F\left(\varrho_n, \varrho, \frac{s}{2}\right) \star \mathcal{M}_F\left(\varrho, \varrho_m, \frac{s}{2}\right)\right) = 1, \quad \forall s > 0,$$

where the last equality follows from the continuity of $\star$, which yields

$$\lim_{n \rightarrow \infty} \left(\mathcal{M}_F\left(\varrho_n, \varrho, \frac{s}{2}\right) \star \mathcal{M}_F\left(\varrho, \varrho_m, \frac{s}{2}\right)\right) = 1$$

together with $(\mathcal{F}^*2)$ and (3.7). This proves that $\{\varrho_n\}$ is an $EF\mathcal{F}$ -Cauchy sequence in $\mathcal{U}$. $\square$

Definition 3.15. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -metric space. We say that $\mathcal{M}_F$ is sequentially continuous with respect to the first and second variables if, for every $\varrho, y \in \mathcal{U}$, every sequences $\{\varrho_n\}, \{y_n\} \subset \mathcal{U}$ satisfying $\varrho_n \rightarrow \varrho, y_n \rightarrow y$ in the $EF\mathcal{F}$ -metric sense, and every $t > 0$, $\lim_{n \rightarrow \infty} \mathcal{M}_F(\varrho_n, y_n, t) = \mathcal{M}_F(\varrho, y, t)$.

Remark 3.16. The sequential continuity of $\mathcal{M}_F$ with respect to its first and second variables does not follow, in general, from the defining axioms of an $EF\mathcal{F}$ -metric space. To justify, we consider Example 3.8 with $\mathcal{U} = \mathbb{R}$ and d as the usual metric on $\mathbb{R}$. Let $\varrho = 0, y = 1, \varrho_n = -\frac{1}{n}$, and $n \in \mathbb{N}$. Then $d(\varrho_n, \varrho) = \frac{1}{n}$, for every $n \in \mathbb{N}$.

We claim that $\varrho_n \rightarrow \varrho$ in the $EF\mathcal{F}$ -sense. Indeed, let $t > 0$ be arbitrary. Then $N \in \mathbb{N}$ exists such that $\frac{2}{N} < t$. Hence, for all $m \geq N$, we have

$$\frac{2}{m} \leq \frac{2}{N} < t \implies 2d(\varrho_m, \varrho) < t.$$

Consequently, by the definition of $\mathcal{M}_F$

$$\mathcal{M}_F(\varrho_m, \varrho, t) = \frac{t}{t + d(\varrho_m, \varrho)} = \frac{t}{t + \frac{1}{m}}, \quad \forall m \geq N,$$

and hence

$$\lim_{m \rightarrow \infty} \mathcal{M}_F(\varrho_m, \varrho, t) = 1.$$

Since $t > 0$ is arbitrary, it follows that $\varrho_n \rightarrow \varrho$ in the $EF\mathcal{F}$ -sense.

However, if we fix $t = 1$, then since $d(\varrho, y) = 1$, $\mathcal{M}_F(\varrho, y, 1) = \frac{1}{2}$, and $d(\varrho_n, y) = \left|1 + \frac{1}{n}\right| > 1$ implies

$$\mathcal{M}_F(\varrho_n, y, 1) = \frac{1}{2(1 + d(\varrho_n, y))} = \frac{1}{2\left(2 + \frac{1}{n}\right)} = \frac{1}{4 + \frac{2}{n}},$$

for every $n \in \mathbb{N}$.

Thus

$$\lim_{n \rightarrow \infty} \mathcal{M}_F(\varrho_n, y, 1) = \frac{1}{4} \neq \frac{1}{2} = \mathcal{M}_F(\varrho, y, 1).$$

Hence, although $\varrho_n \rightarrow \varrho$ in the $EF\mathcal{F}$ -sense, we have $\mathcal{M}_F(\varrho_n, y, 1) \not\rightarrow \mathcal{M}_F(\varrho, y, 1)$.

Therefore, $\mathcal{M}_F$ need not be sequentially continuous in its first variable. By the symmetry of $\mathcal{M}_F$, the same example also shows that sequential continuity in the second variable does not follow automatically.

This example illustrates that sequential continuity may be imposed separately as an additional hypothesis in specific application-oriented settings.

Definition 3.17. An $EF\mathcal{F}$ -MS $(\mathcal{U}, \mathcal{M}_F, F, \star)$ is called an $EF\mathcal{F}$ -complete $EF\mathcal{F}$ -MS if and only if every $EF\mathcal{F}$ -Cauchy sequence converges to some element in $\mathcal{U}$.

We now introduce boundedness, which we will use in the subsequent study in this framework.

Definition 3.18. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS. A subset G of $\mathcal{U}$ is called $EF\mathcal{F}$ -bounded if $t > 0$ and $0 < r < 1$ exist such that $\mathcal{M}_F(\varrho, y, t) > 1 - r$, for all $\varrho, y \in G$.

Remark 3.19. Following the argument used in the proof of [28, Proposition 2.11], every finite subset of $\mathcal{U}$ is $EF\mathcal{F}$ -bounded.

4. Fixed point theorems in $EF\mathcal{F}$ -MS

Motivated by the rich and evolving fuzzy metric fixed-point theory, the present work focuses on Banach-type contractions in $EF\mathcal{F}$ -MS; that is, we establish fuzzy analogues of classical Banach contraction principles adapted to the structure induced by the $EF\mathcal{F}$ -metric $\mathcal{M}_F$, aiming to derive analogous fixed-point results in a broader and more flexible framework. In particular, the study investigates how the relaxed structure of $EF\mathcal{F}$ -MS allows for the convergence of iterative schemes, paving the way for potential applications in fractal constructions, numerical simulations, and nonlinear analysis.

Definition 4.1. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS and $T: \mathcal{U} \rightarrow \mathcal{U}$ be a mapping. Then T is called an $EF\mathcal{F}$ -Banach type contraction mapping if $a \in (0, 1)$ exists such that

$$\left(\frac{1}{\mathcal{M}_F(T\mu, T\nu, t)} - 1 \right) \leq a \left(\frac{1}{\mathcal{M}_F(\mu, \nu, t)} - 1 \right) \quad (4.1)$$

holds for all $\mu, \nu \in \mathcal{U}$ and $t > 0$.

The continuity of $\mathcal{M}_F(\varrho, y, t)$ with respect to $t > 0$ guarantees continuity for each fixed pair (ϱ, y) . However, in the subsequent convergence arguments, the first and second coordinates vary with the sequence indices. Therefore, continuity in t does not, in general, provide the uniform control required to interchange limits and to compare values of $\mathcal{M}_F$ at nearby positive parameters. To overcome this difficulty, we impose the following uniform continuity condition on the family of functions generated by arbitrary sequences in $\mathcal{U}$.

Definition 4.2. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS. We say that $\mathcal{M}_F$ has the equicontinuity property on $(0, \infty)$ if, for every $t' > 0$ and every sequence $\{\varrho_k\}, \{y_k\} \subseteq \mathcal{U}$, the family $\{t \rightarrow \mathcal{M}_F(\varrho_k, y_k, t) : k \in \mathbb{N}\}$ is equicontinuous at t' . Equivalently, for every $t' > 0$ and every $\epsilon > 0$, $\delta > 0$ exists such that

$$|t - t'| < \delta \implies |\mathcal{M}_F(\varrho_k, y_k, t) - \mathcal{M}_F(\varrho_k, y_k, t')| < \epsilon, \quad \text{for all } k \in \mathbb{N}.$$

We are now in a position to establish the following Banach-type fixed point theorem in the $EF\mathcal{F}$ -metric setting.

Theorem 4.3. *Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -complete $EF\mathcal{F}$ -MS such that $(EF\mathcal{F}-\mathcal{M}_F5)$ hold. Further assume that, F is continuous on $[0, 1]$, $F(u) > 0$, for all $u \in (0, 1)$, and $\mathcal{M}_F$ satisfies the following.*

(i) *Equicontinuity at every $t' > 0$;*

(ii) *For every $0 < t < t'$ and every $0 < r < 1$, $\phi_{t,t'} \in (0, 1)$, independent of ϱ, y exists such that*

$$\mathcal{M}_F(\varrho, y, t') > 1 - r \implies \mathcal{M}_F(\varrho, y, t) > 1 - \phi_{t,t'}(r),$$

for all $\varrho, y \in \mathcal{U}$. If $T : \mathcal{U} \rightarrow \mathcal{U}$ is an $EF\mathcal{F}$ -Banach type contraction mapping, then T has a unique fixed point in $\mathcal{U}$.

The additional assumptions in (i) and (ii) are introduced to address two distinct technical issues that arise in the proof. Condition (i) provides uniform control of the dependence of $\mathcal{M}_F(\varrho_k, y_k, \cdot)$ on t when the points x_k and y_k vary with k . Condition (ii) is required because, under the definition of $EF\mathcal{F}$ -boundedness, the uniform estimate is initially available only at a single time $t' > 0$. The monotonicity in $(EF\mathcal{F}-\mathcal{M}_F5)$ transfers this estimate to $t \geq t'$, but does not provide a corresponding lower estimate for $0 < t < t'$.

In the GV fuzzy metric structure, for the classical metric d -induced fuzzy metric

$$\mathcal{M}(\varrho, y, t) = \frac{t}{t + d(\varrho, y)},$$

we will verify assumptions (i) and (ii). For (i), let

$$f_{\varrho, y}(t) = \frac{t}{t + d(\varrho, y)}.$$

Then

$$f'_{\varrho, y}(t) = \frac{d(\varrho, y)}{(t + d(\varrho, y))^2}.$$

Since $(t + d(\varrho, y))^2 \geq 4td(\varrho, y)$, so $|f'_{\varrho, y}(t)| \leq \frac{1}{4t}$. Now fix $t_0 > 0$. Then if $|t - t_0| < \frac{t_0}{2}$ that is $t \geq \frac{t_0}{2}$, $|f'_{\varrho, y}(t)| \leq \frac{1}{2t_0}$. Hence, by the mean value theorem, we have

$$|\mathcal{M}(\varrho, y, t) - \mathcal{M}(\varrho, y, t_0)| \leq \frac{|t - t_0|}{2t_0}.$$

Thus for a given $\epsilon > 0$, choosing $\delta = \min\{\frac{t_0}{2}, 2t_0\epsilon\}$, we obtain $|t - t_0| < \delta$ implies

$$|\mathcal{M}(\varrho, y, t) - \mathcal{M}(\varrho, y, t_0)| < \epsilon,$$

uniformly for all $\varrho, y \in \mathcal{U}$. Thus the family $\{t \rightarrow \mathcal{M}(\varrho_k, y_k, t) : k \in \mathbb{N}\}$ is equicontinuous at every $t_0 > 0$.

For (ii), suppose that $0 < t < t'$ and $\mathcal{M}(\varrho, y, t') > 1 - r$. Then

$$\frac{t'}{t' + d(\varrho, y)} > 1 - r \implies d(\varrho, y) < \frac{rt'}{1 - r}.$$

Therefore,

$$\mathcal{M}(\varrho, y, t) = \frac{t}{t + d(\varrho, y)} > \frac{t}{t + \frac{r'}{1-r}}.$$

Consequently, $\mathcal{M}(\varrho, y, t) > 1 - \phi_{t,r'}(r)$ where $\phi_{t,r'}(r) = \frac{r'}{(1-r)t + r'}$. As $0 < r < 1$, $0 < \phi_{t,r'}(r) < 1$.

Therefore, standard metric-induced fuzzy structures naturally satisfy the additional assumptions; hence, they are not formal restrictions introduced for the proof. Now we present the proof of the theorem.

Proof of the Theorem 4.3. Let $\Upsilon_0 \in \mathcal{U}$ be a fixed element and define $\Upsilon_{n+1} = T\Upsilon_n$, $n = 0, 1, 2, \dots$.

If $\Upsilon_{k+1} = \Upsilon_k$ for some $k \in \mathbb{N}$, then T has a fixed point in $\mathcal{U}$. So we assume that $\Upsilon_{n+1} \neq \Upsilon_n$, $\forall n \in \mathbb{N}$.

We claim that

$$\lim_{n \rightarrow \infty} \mathcal{M}_F(\Upsilon_{n+1}, \Upsilon_n, t) = 1, \quad \forall t > 0.$$

Applying the contraction condition (4.1) repeatedly, for all $t > 0$ and for all $i \geq 1$, we have

$$\left(\frac{1}{\mathcal{M}_F(\Upsilon_{n+i}, \Upsilon_n, t)} - 1 \right) \leq a \left(\frac{1}{\mathcal{M}_F(\Upsilon_{n+i-1}, \Upsilon_{n-1}, t)} - 1 \right) \dots \leq a^n \left(\frac{1}{\mathcal{M}_F(\Upsilon_i, \Upsilon_0, t)} - 1 \right). \quad (4.2)$$

In particular, for $i = 1$, (4.2) yields

$$\left(\frac{1}{\mathcal{M}_F(\Upsilon_{n+1}, \Upsilon_n, t)} - 1 \right) \leq a^n \left(\frac{1}{\mathcal{M}_F(\Upsilon_1, \Upsilon_0, t)} - 1 \right).$$

Since, $0 < a < 1$, letting $n \rightarrow \infty$ in the above inequality gives

$$\lim_{n \rightarrow \infty} \mathcal{M}_F(\Upsilon_{n+1}, \Upsilon_n, t) = 1, \quad \forall t > 0. \quad (4.3)$$

We assert that the sequence $\{\Upsilon_n\}$ is $EF\mathcal{F}$ -bounded. If possible, suppose that $\{\Upsilon_n\}$ is $EF\mathcal{F}$ -unbounded and fix an arbitrary $0 < \epsilon < 1$.

We first construct a subsequence $\{\Upsilon_{n_k}\}$. Set $n_1 = 1$. Since $\{\Upsilon_n\}$ is $EF\mathcal{F}$ -unbounded, every tail $\{\Upsilon_n : n \geq N\}$ is also $EF\mathcal{F}$ -unbounded. Indeed, if a tail $\{\Upsilon_n : n \geq N\}$ were $EF\mathcal{F}$ -bounded then, the whole sequence would be $EF\mathcal{F}$ -bounded, as it is the union of this $EF\mathcal{F}$ -bounded tail and the $EF\mathcal{F}$ -bounded finite set $\{\Upsilon_0, \Upsilon_1, \dots, \Upsilon_{N-1}\}$, which is a contradiction.

Now fix $k \in \mathbb{N}$. Since the tail $\{\Upsilon_m : m > n_k\}$ is also $EF\mathcal{F}$ -unbounded, $m > n_k$ exists such that

$$\mathcal{M}_F(\Upsilon_m, \Upsilon_{n_k}, t_0) \leq (1 - \epsilon).$$

Otherwise, we have

$$\mathcal{M}_F(\Upsilon_m, \Upsilon_{n_k}, t_0) > (1 - \epsilon)$$

and $(EF\mathcal{F}-\mathcal{M}_F4)$ gives

$$\mathcal{M}_F(\Upsilon_i, \Upsilon_j, 2t_0) \geq F(\mathcal{M}_F(\Upsilon_i, \Upsilon_{n_k}, t_0) \star \mathcal{M}_F(\Upsilon_j, \Upsilon_{n_k}, t_0)) \geq F((1 - \epsilon) \star (1 - \epsilon))$$

for any $i, j > n_k$.

Since $F(u) > 0$ for every $u \in (0, 1)$, we have $0 < F((1 - \epsilon) \star (1 - \epsilon)) < 1$. Assume

$$F((1 - \epsilon) \star (1 - \epsilon)) = 1 - r',$$

for some $r' \in (0, 1)$.

Hence the tail $\{\Upsilon_m : m > n_k\}$ would be $EF\mathcal{F}$ -bounded, which is a contradiction. Therefore, the set

$$\{m > n_k : \mathcal{M}_F(\Upsilon_m, \Upsilon_{n_k}, t_0) \leq (1 - \epsilon)\} \neq \emptyset.$$

Suppose that n_{k+1} is minimal in the sense

$$\mathcal{M}_F(\Upsilon_{n_{k+1}}, \Upsilon_{n_k}, t_0) \leq (1 - \epsilon) \quad (4.4)$$

and hence

$$\mathcal{M}_F(\Upsilon_m, \Upsilon_{n_k}, t_0) > (1 - \epsilon) \quad (4.5)$$

holds for all $m \in \{n_k + 1, n_k + 2, \dots, n_{k+1} - 1\}$.

From $(EF\mathcal{F}-\mathcal{M}_F4)$, for any $h > 0$, we get

$$\mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0 + h) \geq F(\mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k}, t_0) \star \mathcal{M}_F(\Upsilon_{n_k}, \Upsilon_{n_k-1}, h)).$$

Application of (4.3) and (4.5) together with the continuity of $\star$ gives

$$\liminf_{k \rightarrow \infty} \left(\mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k}, t_0) \star \mathcal{M}_F(\Upsilon_{n_k}, \Upsilon_{n_k-1}, h) \right) \geq (1 - \epsilon) \star 1 = (1 - \epsilon).$$

Then, because F is nondecreasing and continuous, we have

$$\liminf_{k \rightarrow \infty} \mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0 + h) \geq F(1 - \epsilon). \quad (4.6)$$

Next, we claim that (4.6) implies

$$\liminf_{k \rightarrow \infty} \mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0) \geq F(1 - \epsilon). \quad (4.7)$$

First, we consider a sequence $\{h_m\} \in (0, \infty)$ such that $h_m \rightarrow 0$ as $m \rightarrow \infty$. Define

$$f_k(s) = \mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, s), \quad \forall s > 0.$$

By (i), the family $\{f_k\}$ is equicontinuous at t_0 . Therefore, we have

$$\sup_{k \in \mathbb{N}} |f_k(t_0 + h_m) - f_k(t_0)| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Again,

$$|\liminf_{k \rightarrow \infty} f_k(t_0 + h_m) - \liminf_{k \rightarrow \infty} f_k(t_0)| \leq \sup_{k \in \mathbb{N}} |f_k(t_0 + h_m) - f_k(t_0)|.$$

Consequently,

$$\lim_{m \rightarrow \infty} \liminf_{k \rightarrow \infty} f_k(t_0 + h_m) = \liminf_{k \rightarrow \infty} f_k(t_0). \quad (4.8)$$

Again (4.6) gives, the following. For every m

$$\liminf_{k \rightarrow \infty} f_k(t_0 + h_m) \geq F(1 - \epsilon).$$

Hence, by (4.8), we have

$$\liminf_{k \rightarrow \infty} f_k(t_0) \geq F(1 - \epsilon).$$

Therefore

$$\liminf_{k \rightarrow \infty} \mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0) \geq F(1 - \epsilon).$$

Hence, claim (4.7) is established.

Since (4.7) holds for all $\epsilon \in (0, 1)$, letting $\epsilon \rightarrow 0^+$, by $(\mathcal{F}^*2)$, $F(1 - \epsilon) \rightarrow F(1) = 1$. Consequently

$$\liminf_{k \rightarrow \infty} \mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0) = 1. \quad (4.9)$$

Because $\mathcal{M}_F$ belongs to $[0, 1]$, (4.9) implies

$$\lim_{k \rightarrow \infty} \mathcal{M}_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0) = 1. \quad (4.10)$$

Now combined application of (4.4) and (4.1) yields

$$\left(\frac{1}{1 - \epsilon} - 1 \right) \leq \left(\frac{1}{M_F(\Upsilon_{n_{k+1}}, \Upsilon_{n_k}, t_0)} - 1 \right) \leq a \left(\frac{1}{M_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0)} - 1 \right).$$

From (4.10), we have

$$\left(\frac{1}{M_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0)} - 1 \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore

$$\limsup_{k \in \mathbb{N}} \left(\frac{1}{M_F(\Upsilon_{n_{k+1}-1}, \Upsilon_{n_k-1}, t_0)} - 1 \right) = 0,$$

which implies that

$$\left(\frac{1}{1 - \epsilon} - 1 \right) \leq 0 \quad \text{i.e.,} \quad \frac{\epsilon}{1 - \epsilon} \leq 0,$$

which is an absurd conclusion. Hence, $\{\Upsilon_n\}$ is bounded.

Now we show that $\{\Upsilon_n\}$ is a Cauchy sequence in $\mathcal{U}$. Since $\{\Upsilon_n\}$ is a bounded sequence in $\mathcal{U}$, $t' > 0$ and $0 < r < 1$ exist such that

$$\mathcal{M}_F(\Upsilon_i, \Upsilon_0, t') > 1 - r, \quad \forall i \geq 1.$$

If $t \geq t'$, by monotonicity, we have

$$\mathcal{M}_F(\Upsilon_i, \Upsilon_0, t) \geq \mathcal{M}_F(\Upsilon_i, \Upsilon_0, t') > 1 - r, \quad \forall i \geq 1.$$

If $0 < t < t'$, by (ii), $\phi_{t,t'}(r) \in (0, 1)$ exists such that

$$\mathcal{M}_F(\Upsilon_i, \Upsilon_0, t) > 1 - \phi_{t,t'}(r), \quad \forall i \geq 1.$$

Thus, by defining

$$r_t = \begin{cases} \phi_{t,t'}(r), & \text{if } 0 < t < t', \\ r, & \text{if } t \geq t', \end{cases}$$

we get

$$\mathcal{M}_F(\Upsilon_i, \Upsilon_0, t) > 1 - r_t, \quad \forall i \geq 1. \quad (4.11)$$

Hence, for all natural numbers $n, i \geq 1$, from (4.2), we have

$$\left(\frac{1}{\mathcal{M}_F(\Upsilon_{n+i}, \Upsilon_n, t)} - 1\right) \leq a^n \left(\frac{1}{\mathcal{M}_F(\Upsilon_i, \Upsilon_0, t)} - 1\right) < a^n \frac{r_t}{1 - r_t},$$

which implies

$$\inf_{i \geq 1} \mathcal{M}_F(\Upsilon_{n+i}, \Upsilon_n, t) \geq \frac{1}{1 + a^n C_t},$$

where $C_t = \frac{r_t}{1-r_t}$. Since $0 < a < 1$ and $\mathcal{M}_F \in [0, 1]$, we have

$$\liminf_{n \rightarrow \infty} \inf_{i \geq 1} \mathcal{M}_F(\Upsilon_{n+i}, \Upsilon_n, t) = 1.$$

Now take the arbitrary natural numbers $m > n \geq 1$ and set $i = m - n$. Then

$$\mathcal{M}_F(\Upsilon_m, \Upsilon_n, t) \geq \inf_{i \geq 1} \mathcal{M}_F(\Upsilon_{n+i}, \Upsilon_n, t)$$

implies

$$\liminf_{\substack{n \rightarrow \infty \\ m > n}} \mathcal{M}_F(\Upsilon_m, \Upsilon_n, t) \geq 1.$$

Since all the values of $\mathcal{M}_F$ lie in $[0, 1]$,

$$\lim_{m, n \rightarrow \infty} \mathcal{M}_F(\Upsilon_m, \Upsilon_n, t) = 1, \quad \forall t > 0.$$

Therefore, $\{\Upsilon_n\}$ is a Cauchy sequence in $\mathcal{U}$

Since $\mathcal{U}$ is complete, some $y \in \mathcal{U}$ exists such that $\lim_{n \rightarrow \infty} \Upsilon_n = y$. Hence

$$\lim_{n \rightarrow \infty} \mathcal{M}_F(\Upsilon_n, y, s) = 1, \quad \forall s > 0. \quad (4.12)$$

Now, we claim that y is a fixed point of T . We first show that, for all $t > 0$, $\mathcal{M}_F(\Upsilon_{n+1}, Ty, t) \rightarrow 1$ as $n \rightarrow \infty$.

By the contraction condition (4.1), for every $t > 0$

$$\left(\frac{1}{\mathcal{M}_F(Ty, \Upsilon_{n+1}, t)} - 1\right) \leq a \left(\frac{1}{\mathcal{M}_F(y, \Upsilon_n, t)} - 1\right).$$

Then letting $n \rightarrow \infty$ on the inequality above and using (4.12), we get

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\mathcal{M}_F(Ty, \Upsilon_{n+1}, t)} - 1\right) \leq \lim_{n \rightarrow \infty} a \left(\frac{1}{\mathcal{M}_F(y, \Upsilon_n, t)} - 1\right) = 0 \quad \text{or} \quad \lim_{n \rightarrow \infty} \mathcal{M}_F(Ty, \Upsilon_{n+1}, t) = 1. \quad (4.13)$$

On the contrary, assume that $Ty \neq y$. Then $q > 0$ exists such that

$$\mathcal{M}_F(Ty, y, q) < 1. \quad (4.14)$$

Now, by using the inequality $(EF\mathcal{F}\text{-}\mathcal{M}_F4)$, we get

$$\mathcal{M}_F(y, Ty, q) \geq F\left(\mathcal{M}_F(y, \Upsilon_{n+1}, \frac{q}{2}) \star \mathcal{M}_F(Ty, \Upsilon_{n+1}, \frac{q}{2})\right).$$

Through the combined application of the continuity of $\star$ and (4.12) and (4.13), we obtain

$$\lim_{n \rightarrow \infty} \left(\mathcal{M}_F(y, \Upsilon_{n+1}, \frac{q}{2}) \star \mathcal{M}_F(Ty, \Upsilon_{n+1}, \frac{q}{2}) \right) = 1 \star 1 = 1.$$

Hence from $(\mathcal{F}^*2)$, we have

$$\lim_{n \rightarrow \infty} F \left(\mathcal{M}_F(y, \Upsilon_{n+1}, \frac{q}{2}) \star \mathcal{M}_F(Ty, \Upsilon_{n+1}, \frac{q}{2}) \right) = 1,$$

which yields

$$\mathcal{M}_F(y, Ty, q) = 1.$$

This contradicts our assumption in (4.14).

Finally, suppose that $\varrho \in \mathcal{U}$ is another fixed point of T . Then by $T\varrho = \varrho$ and applying (4.1), we get

$$\left(\frac{1}{\mathcal{M}_F(y, \varrho, t)} - 1 \right) = \left(\frac{1}{\mathcal{M}_F(Ty, T\varrho, t)} - 1 \right) \leq a \left(\frac{1}{\mathcal{M}_F(y, \varrho, t)} - 1 \right), \quad \forall t > 0,$$

which implies

$$\mathcal{M}_F(y, \varrho, t) = 1, \quad \forall t > 0,$$

since $0 < a < 1$. Consequently, $y = \varrho$. Hence, the proof is completed. $\square$

Example 4.4. Let $\mathcal{U} = [-2, 2]$ and define a function $\mathcal{M}_F: \mathcal{U}^2 \times (0, \infty) \rightarrow [0, 1]$ by

$$\mathcal{M}_F(\varrho, y, t) = \exp \left(- \frac{|\varrho - y|^3}{t} \right),$$

for all $\varrho, y \in \mathcal{U}$, $t > 0$. Then by the Example 3.4, we can conclude that $\mathcal{M}_F$ is an EF $\mathcal{F}$ -MS on $\mathcal{U}$. Moreover, $\mathcal{U}$ is complete in the EF $\mathcal{F}$ -sense.

Now, we verify Assumptions (i) and (ii) of Theorem 4.3. Let $f_{\varrho, y}(t) = e^{-\frac{|\varrho - y|^3}{t}}$, for all $t > 0$. Then

$$f'_{\varrho, y}(t) = \frac{|\varrho - y|^3}{t^2} e^{-\frac{|\varrho - y|^3}{t}},$$

for all $t > 0$.

We set $z = \frac{|\varrho - y|^3}{t}$. Then $z \geq 0$ and

$$\sup_{z \geq 0} z e^{-z} = \frac{1}{e}.$$

Therefore, we obtain $|f'_{\varrho, y}(t)| \leq \frac{1}{et}$.

This bound is independent of ϱ, y . Now fix $t_0 > 0$ and take $0 < \delta < \frac{t_0}{2}$. Then

$$|t - t_0| < \delta \implies \frac{t_0}{2} < t_0 - \delta < t.$$

Hence, for every ξ lying between t and t_0 , we have

$$|f_{\varrho, y}(t) - f_{\varrho, y}(t_0)| = |f'_{\varrho, y}(\xi)| |t - t_0| \leq \frac{1}{e\xi} |t - t_0| < \frac{\delta}{e(t_0 - \delta)},$$

because $\xi \geq \min\{t, t_0\} > t_0 - \delta$.

Therefore, for arbitrarily chosen $\epsilon > 0$, we can choose $\delta > 0$ such that

$$|t - t_0| < \delta \implies |\mathcal{M}_F(x, y, t) - \mathcal{M}_F(x, y, t_0)| < \epsilon$$

holds, where $\delta < \frac{t_0}{2}$ and $\frac{\delta}{e(t_0 - \delta)} < \epsilon$. Consequently, for any sequence $\{\varrho_k\}, \{y_k\} \subseteq \mathcal{U}$ the family $\{t \rightarrow \mathcal{M}_F(\varrho_k, y_k, t) : k \in \mathbb{N}\}$ is equicontinuous at every $t_0 > 0$.

Next, to verify Assumption (ii) at $t_0 > 0$, suppose $0 < t < t_0$ and $\mathcal{M}_F(\varrho, y, t_0) > (1 - r)$, for some $r \in (0, 1)$. Then

$$\exp\left(-\frac{|\varrho - y|^3}{t_0}\right) > (1 - r) \implies -|\varrho - y|^3 > t_0 \ln(1 - r).$$

Therefore,

$$\mathcal{M}_F(\varrho, y, t) = \exp\left(-\frac{|\varrho - y|^3}{t}\right) > \exp\left(\frac{t_0}{t} \ln(1 - r)\right) = (1 - r)^{\frac{t_0}{t}}.$$

Thus we can choose $\phi_{t,t_0}(r) = 1 - (1 - r)^{\frac{t_0}{t}}$ and because $0 < t < t_0$, $0 < \phi_{t,t_0}(r) < 1$. Moreover

$$\lim_{r \rightarrow 0^+} \phi_{t,t_0}(r) = 0.$$

Now, define a mapping $T: \mathcal{U} \rightarrow \mathcal{U}$ by $T(\varrho) = 0.6\varrho + 0.05 \sin(5\varrho)$, for all $\varrho \in \mathcal{U}$. Note that $T \in C^1$ and for all $\varrho \in \mathcal{U}$, $|T'(\varrho)| \leq 0.6 + 0.25 = 0.85 = L$ (say). Thus T is a Lipschitz function with a constant $L \in (0, 1)$.

For an arbitrary $\varrho, y \in \mathbb{R}$ and each $t > 0$ set $u = \frac{|\varrho - y|^3}{t} > 0$. Then

$$\frac{e^{|T\varrho - Ty|^3/t} - 1}{e^{|\varrho - y|^3/t} - 1} = \frac{e^{(L^3)u} - 1}{e^u - 1} = L^3 \cdot \frac{(e^{L^3u} - 1)/(L^3u)}{(e^u - 1)/u} \leq L^3,$$

because $s \mapsto (e^s - 1)/s$ is increasing on $(0, \infty)$ and $L^3u \leq u$. This implies that

$$\left(\frac{1}{\mathcal{M}_F(T\varrho, Ty, t)} - 1\right) \leq a \left(\frac{1}{\mathcal{M}_F(\varrho, y, t)} - 1\right)$$

holds where $0 < L^3 \leq a < 1$, which is exactly the EF $\mathcal{F}$ -Banach type contraction inequality (4.1) with a constant $a \in (0, 1)$. Hence T satisfies the hypothesis of Theorem 4.3; therefore, by Theorem 4.3, T has exactly one fixed point in $\mathcal{U}$. Since here $T(0) = 0$, the point 0 is a fixed point of T , and hence it is the unique fixed point of T in $\mathcal{U}$.

To illustrate the theoretical analysis, we present a graph (Figure 1a) that illustrates the behavior of T and demonstrates that it has exactly one fixed point in $\mathbb{R}$. We present a MATLAB simulation of the Banach-type contraction inequality that generates the surfaces corresponding to both sides of the inequality and compares them graphically. The plot (see Figure 1b) shows that the surface corresponding to the right-hand side (RHS) lies entirely above the surface of the left-hand side (LHS) over the chosen domain of (ϱ, y) . This demonstrates that the inequality holds for the chosen parameter. In addition, Figure 1c displays the surface of the difference LHS-RHS for the same parameter values $a = 0.9$ and $t = 100000$, where

$$LHS = \left(\frac{1}{\mathcal{M}_F(T\varrho, Ty, t)} - 1\right) \quad \text{and} \quad RHS = a \left(\frac{1}{\mathcal{M}_F(\varrho, y, t)} - 1\right).$$

The difference surface remains strictly non-positive over the entire domain, indicating that the contractive inequality holds at all sampled points. Values closer to zero indicate points where the inequality is nearly achieved.

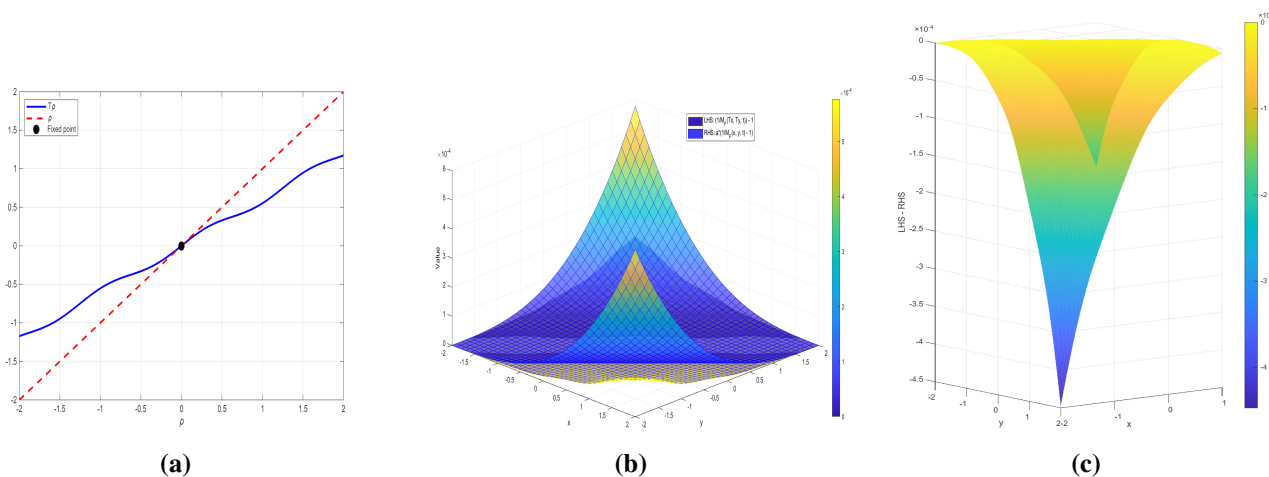


Figure 1. (a) Graph of $T\rho$ and ρ illustrating the existence of unique fixed point; (b) 3D surface plot comparing the RHS and LHS of the Banach-type contraction inequality for $a = 0.9$ and $t = 100000$; (c) Surface plot of the difference LHS-RHS for the Banach-type contraction inequality with $a = 0.9$ and $t = 100000$ for the defined map T .

From the proof above, the contraction constant satisfies $a \geq L^3$, where $L \leq 0.85$. Since the contraction inequality remains valid for any larger constant in $(0, 1)$, we select $a = 0.9$ as a representative admissible value to provide a clearer visualization of the inequality. Moreover, the parameter t controls the scale of the fuzzy metric $\mathcal{M}_F(\rho, y, t)$. We choose $t = 100000$ in the numerical simulation to obtain a graphical representation of both sides of the contraction inequality and to clearly visualize their relation.

5. Applications over fractals

Fractal constructions often rely on IFS and contractive mappings defined on suitable metric frameworks. In this section, we extend these ideas to the setting of $EF\mathcal{F}$ -MSs, where proximity between points and sets is governed by the fuzzy metric $\mathcal{M}_F$. We introduce fuzzy analogues of Hausdorff distance, proximity measures, and Hutchinson–Barnsley operators, and establish a complete framework for constructing fuzzy fractals through $EF\mathcal{F}$ -IFS. These theoretical foundations provide the basis for later applications, where we use the fuzzy Banach contraction principle to generate fractal attractors, along with the corresponding numerical demonstrations.

To formalize this approach, we now introduce the necessary definitions and operators in the $EF\mathcal{F}$ -MS setting, which serve as the groundwork for constructing Hutchinson–Barnsley-type operators and IFS in this fuzzy framework.

Definition 5.1. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS and we define

$$CB(\mathcal{U}) = \{A \subset \mathcal{U} : A \text{ is nonempty, } EF\mathcal{F}\text{-bounded and, } EF\mathcal{F}\text{-closed}\}.$$

We call it an $EF\mathcal{F}$ -closed bounded hyperspace. For $x \in \mathcal{U}$ and $A \in CB(\mathcal{U})$ and $t > 0$, define the point to set proximity as

$$\mathcal{M}_F(x, A, t) = \sup_{a \in A} \mathcal{M}_F(x, a, t).$$

For $A, B \in CB(\mathcal{U})$ and $t > 0$, define set-to-set proximity and Hausdorff $EF\mathcal{F}$ -distance as

$$\mathcal{M}_F(A, B, t) = \inf_{a \in A} \mathcal{M}_F(a, B, t) \quad \& \quad \mathcal{H}_F(A, B, t) = \min \{ \mathcal{M}_F(A, B, t), \mathcal{M}_F(B, A, t) \}.$$

Based on this, we now define a natural operator on compact subsets that generalizes the classical Hutchinson–Barnsley operator in the $EF\mathcal{F}$ -metric setting, allowing us to study IFS in this fuzzy framework.

Definition 5.2. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS and $\{T_i : \mathcal{U} \rightarrow \mathcal{U} : i = 1, 2, \dots, l\}$ be a finite collection of $EF\mathcal{F}$ -Banach contractions. An operator Ψ_F over $CB(\mathcal{U})$ defined by

$$\Psi_F(A) = T_1(A) \cup T_2(A) \cup \dots \cup T_l(A), \quad \forall A \in CB(\mathcal{U})$$

is called an $EF\mathcal{F}$ -Hutchinson–Barnsley-type operator ($EF\mathcal{F}$ -HBO).

With the defined Hutchinson–Barnsley-type operator, we can now formalize the notion of an IFS in the $EF\mathcal{F}$ -metric context, which serves as the central object for constructing fuzzy fractal attractors.

Definition 5.3. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS. If $\{T_i : \mathcal{U} \rightarrow \mathcal{U} ; i = 1, 2, \dots, l\}$ is a finite collection of $EF\mathcal{F}$ -Banach contractions with some constant $a \in (0, 1)$ then $(\mathcal{U} : T_1, T_2, \dots, T_l)$ is called an $EF\mathcal{F}$ -iterated function system ($EF\mathcal{F}$ -IFS).

These constructions enable the formulation of a Banach-type fixed-point theorem in $CB(\mathcal{U})$ endowed with the Hausdorff $EF\mathcal{F}$ -distance. The following result establishes the completeness of this space, the contractivity of the Hutchinson–Barnsley-type operator, and the existence of a unique attractor with a geometric convergence rate, providing the theoretical foundation for generating fuzzy fractals.

Theorem 5.4. Let $(\mathcal{U}, \mathcal{M}_F, F, \star)$ be an $EF\mathcal{F}$ -MS satisfying all the assumptions of the Theorem 4.3. Additionally, suppose that $\mathcal{M}_F$ is sequentially continuous in both the first and second variables and it satisfies: For every $t > 0$ and $\epsilon > 0$, $\delta \in (0, 1)$ exists such that

$$\mathcal{M}_F(y, z, t) > 1 - \delta \implies |\mathcal{M}_F(q, y, t) - \mathcal{M}_F(q, z, t)| < \epsilon \quad (5.1)$$

for all $q, y, z \in \mathcal{U}$.

Let $\Psi_F = \cup_{i=1}^l T_i$ be the $EF\mathcal{F}$ -HBO on $CB(\mathcal{U})$ with respect to the $EF\mathcal{F}$ -IFS $(\mathcal{U} : T_1, T_2, \dots, T_l)$. Then we have the following:

- (i) $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$ is an $EF\mathcal{F}$ -MS.
- (ii) $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$ is $EF\mathcal{F}$ -complete if $(\mathcal{U}, \mathcal{M}_F, F, \star)$ is $EF\mathcal{F}$ -complete.
- (iii) Ψ_F is an $EF\mathcal{F}$ -Banach type contraction on $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$ with a constant ‘ a ’.
- (iv) Ψ_F has a unique fixed point $K^* \in CB(\mathcal{U})$.

Proof.

(i) First, we will prove that $\mathcal{H}_F$ satisfies the axioms $(EF\mathcal{F}\text{-}\mathcal{M}_F1)$ – $(EF\mathcal{F}\text{-}\mathcal{M}_F4)$.

Let $A, B \in \mathcal{CB}(\mathcal{U})$. Fix $t > 0$ and choose $a_0 \in A$, $b_0 \in B$. Since both A, B are $EF\mathcal{F}$ -bounded, $t_A, t_B > 0$ and $r_A, r_B \in (0, 1)$ exist such that

$$\mathcal{M}_F(a, a_0, t_A) > 1 - r_A, \quad \forall a \in A \quad \text{and} \quad \mathcal{M}_F(b, b_0, t_B) > 1 - r_B, \quad \forall b \in B. \quad (5.2)$$

Now choose $s > 0$ such that $s + t_A + t_B = t'$. By $(EF\mathcal{F}\text{-}\mathcal{M}_F1)$, $\mathcal{M}_F(a_0, b_0, s) > 0$. Then for any $a \in A$, $b \in B$, applying $(EF\mathcal{F}\text{-}\mathcal{M}_F4)$ twice

$$\begin{aligned} \mathcal{M}_F(a, b, t_A + t_B + s) &\geq F(\mathcal{M}_F(a, a_0, t_A) \star \mathcal{M}_F(a_0, b, t_B + s)) \\ &\geq F(\mathcal{M}_F(a, a_0, t_A) \star F(\mathcal{M}_F(a_0, b_0, s) \star \mathcal{M}_F(b_0, b, t_B))) \\ &\geq F((1 - r_A) \star F(\mathcal{M}_F(a_0, b_0, s) \star (1 - r_B))). \end{aligned}$$

Therefore, applying (5.2) and $F(u) > 0$, for all $u \in (0, 1)$, we can find $r_2 \in (0, 1)$ such that

$$F(\mathcal{M}_F(a, a_0, t_A) \star F(\mathcal{M}_F(a_0, b_0, s) \star \mathcal{M}_F(b_0, b, t_B))) > 1 - r_2.$$

Hence, we have

$$\mathcal{M}_F(a, b, t') > 1 - r_2.$$

If $t \geq t'$, then by $(EF\mathcal{F}\text{-}\mathcal{M}_F5)$,

$$\mathcal{M}_F(a, b, t) > 1 - r_2.$$

If $0 < t < t'$, then by Assumption (ii) of Theorem 4.3, $\phi_{t,t'}(r_2) \in (0, 1)$ exists such that

$$\mathcal{M}_F(a, b, t) > 1 - \phi_{t,t'}(r_2).$$

Therefore

$$\mathcal{M}_F(a, b, t) > 1 - r'_t,$$

where

$$r'_t = \begin{cases} \phi_{t,t'}(r_2), & \text{if } 0 < t < t', \\ r_2, & \text{if } t' \leq t. \end{cases}$$

This gives

$$\mathcal{M}_F(A, B, t) = \inf_{\varrho \in A} \sup_{y \in B} \mathcal{M}_F(\varrho, y, t) \geq 1 - r'_t.$$

Likewise, $\mathcal{M}_F(B, A, t) \geq 1 - r'_t$. Hence, $\mathcal{H}_F(A, B, t) \geq 1 - r'_t > 0$. Since, clearly, $\mathcal{H}_F(A, B, t) \leq 1$, we obtain $0 < \mathcal{H}_F(A, B, t) \leq 1$. Hence, $(EF\mathcal{F}\text{-}\mathcal{M}_F1)$ holds for $\mathcal{H}_F$.

If $A = B$, then

$$\mathcal{M}_F(A, A, t) = \inf_{\varrho \in A} \sup_{y \in A} \mathcal{M}_F(\varrho, y, t),$$

for all $t > 0$.

Now for every fixed $\varrho \in A$, choosing $y = \varrho$ gives $\mathcal{M}_F(\varrho, y, t) = 1$ which implies

$$\sup_{y \in A} \mathcal{M}_F(\varrho, y, t) = 1.$$

Hence,

$$\mathcal{M}_F(A, A, t) = \inf_{\varrho \in A} \sup_{y \in A} \mathcal{M}_F(\varrho, y, t) = 1.$$

Consequently, $\mathcal{H}_F(A, A, t) = 1$, for all $t > 0$.

Next suppose that, $\mathcal{H}_F(A, B, t) = 1$ for all $t > 0$. Then $\min\{\mathcal{M}_F(A, B, t), \mathcal{M}_F(B, A, t)\} = 1$, for all $t > 0$, and hence $\mathcal{M}_F(A, B, t) = 1 = \mathcal{M}_F(B, A, t)$ for all $t > 0$.

Fix $\varrho \in A$ and an arbitrary $t' > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < t'$. Then, for every $m \geq n$, we have

$$\frac{1}{m} \leq \frac{1}{n} < t'.$$

Since $\mathcal{M}_F(A, B, \frac{1}{m}) = 1$, we have

$$\inf_{\varrho \in A} \sup_{y \in B} \mathcal{M}_F(\varrho, y, \frac{1}{m}) = 1.$$

Consequently, for a fixed $\varrho \in A$, we have

$$\sup_{y \in B} \mathcal{M}_F\left(\varrho, y, \frac{1}{m}\right) = 1.$$

Therefore, for each $m \geq n$, $y_m \in B$ exists such that

$$1 - \frac{1}{m} < \mathcal{M}_F(\varrho, y_m, \frac{1}{m}).$$

Using $(EF\mathcal{F}-\mathcal{M}_F5)$,

$$\mathcal{M}_F(\varrho, y_m, t') \geq \mathcal{M}_F(\varrho, y_m, \frac{1}{m}) > 1 - \frac{1}{m}, \quad \forall m \geq n,$$

which implies that

$$\lim_{m \rightarrow \infty} \mathcal{M}_F(\varrho, y_m, t') = 1.$$

Since $t' > 0$ is arbitrary, it follows that $\{y_n\} \subseteq B$ converges to ϱ in $EF\mathcal{F}$ -sense. As B is $\mathcal{M}_F$ -closed, so $\varrho \in B$. Thus $A \subseteq B$. Using $\mathcal{M}_F(B, A, t) = 1$ for every $t > 0$, the reverse inclusion $B \subseteq A$ follows similarly. Consequently, $A = B$. Therefore, $(EF\mathcal{F}-\mathcal{M}_F2)$ holds. $(EF\mathcal{F}-\mathcal{M}_F3)$ follows directly from the definition of $\mathcal{H}_F$. Hence, it remains to verify only the inequality axiom $(EF\mathcal{F}-\mathcal{M}_F4)$. Let $A, B, C \in \mathcal{CB}(\mathcal{U})$ and $s, t > 0$. We first need to show

$$\mathcal{M}_F(A, B, s + t) \geq F(\mathcal{M}_F(A, C, s) \star \mathcal{M}_F(C, B, t)).$$

Fix $\varrho \in A$. For every $y \in B$ and every $z \in C$, by $(EF\mathcal{F}-\mathcal{M}_F4)$, we have

$$\mathcal{M}_F(\varrho, y, s + t) \geq F(\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, y, t)).$$

Taking the supremum over $y \in B$, we obtain

$$\sup_{y \in B} \mathcal{M}_F(\varrho, y, s + t) \geq \sup_{y \in B} F(\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, y, t)) = F\left(\sup_{y \in B} (\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, y, t))\right),$$

where the equality follows from the continuity together with the nondecreasing property of F .

Since $\star$ is continuous and nondecreasing, we have

$$\sup_{y \in B} (\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, y, t)) = \mathcal{M}_F(\varrho, z, s) \star \sup_{y \in B} \mathcal{M}_F(z, y, t) = (\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, B, t)),$$

since $\mathcal{M}_F(z, B, t) = \sup_{y \in B} \mathcal{M}_F(z, y, t)$. Therefore

$$\sup_{y \in B} \mathcal{M}_F(\varrho, y, s + t) \geq F(\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, B, t)).$$

Now, by taking the supremum over $z \in C$, we get

$$\sup_{y \in B} \mathcal{M}_F(\varrho, y, s + t) \geq \sup_{z \in C} F(\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, B, t)) = F(\sup_{z \in C} (\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, B, t))), \quad (5.3)$$

since F is nondecreasing and continuous. Since

$$\inf_{z \in C} \mathcal{M}_F(z, B, t) \leq \mathcal{M}_F(z, B, t)$$

for every $z \in C$, and $\star$ is nondecreasing, we have

$$\mathcal{M}_F(\varrho, z, s) \star \inf_{z \in C} \mathcal{M}_F(z, B, t) \leq \mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, B, t).$$

Taking the supremum over $z \in C$ yields

$$\sup_{z \in C} (\mathcal{M}_F(\varrho, z, s) \star \inf_{z \in C} \mathcal{M}_F(z, B, t)) \leq \sup_{z \in C} (\mathcal{M}_F(\varrho, z, s) \star \mathcal{M}_F(z, B, t)).$$

Since $\star$ is continuous

$$\begin{aligned} \sup_{z \in C} (\mathcal{M}_F(\varrho, z, s) \star \inf_{z \in C} \mathcal{M}_F(z, B, t)) &= \sup_{z \in C} \mathcal{M}_F(\varrho, z, s) \star \inf_{z \in C} \mathcal{M}_F(z, B, t) \\ &= \mathcal{M}_F(\varrho, C, s) \star \mathcal{M}_F(C, B, t), \end{aligned}$$

which together with (5.3) yields

$$\sup_{y \in B} \mathcal{M}_F(\varrho, y, s + t) \geq F(\mathcal{M}_F(\varrho, C, s) \star \mathcal{M}_F(C, B, t)).$$

Finally, taking the infimum over $\varrho \in A$, we obtain

$$\mathcal{M}_F(A, B, s + t) = \inf_{\varrho \in A} \sup_{y \in B} \mathcal{M}_F(\varrho, y, s + t) \geq \inf_{\varrho \in A} F(\mathcal{M}_F(\varrho, C, s) \star \mathcal{M}_F(C, B, t)).$$

Using once again the continuity and non-decreasing property of both $\star$ and F , we conclude that

$$\begin{aligned} \inf_{\varrho \in A} F(\mathcal{M}_F(\varrho, C, s) \star \mathcal{M}_F(C, B, t)) &= F(\inf_{\varrho \in A} \mathcal{M}_F(\varrho, C, s) \star \mathcal{M}_F(C, B, t)) \\ &= F(\mathcal{M}_F(A, C, s) \star \mathcal{M}_F(C, B, t)), \end{aligned}$$

and therefore

$$\mathcal{M}_F(A, B, s+t) \geq F(\mathcal{M}_F(A, C, s) \star \mathcal{M}_F(C, B, t)).$$

Similarly, we can show

$$\mathcal{M}_F(B, A, s+t) \geq F(\mathcal{M}_F(B, C, t) \star \mathcal{M}_F(C, A, s)).$$

By definition, we get

$$\mathcal{H}_F(A, C, s) = \min\{\mathcal{M}_F(A, C, s), \mathcal{M}_F(C, A, s)\} \quad \text{and} \quad \mathcal{H}_F(C, B, t) = \min\{\mathcal{M}_F(C, B, t), \mathcal{M}_F(B, C, t)\}.$$

Therefore, for all $A, B, C \in \mathcal{CB}(\mathcal{U})$ and $s, t > 0$, we have

$$\mathcal{H}_F(A, C, s) \star \mathcal{H}_F(C, B, t) \leq \min\left\{\mathcal{M}_F(A, C, s) \star \mathcal{M}_F(C, B, t), \mathcal{M}_F(B, C, t) \star \mathcal{M}_F(C, A, s)\right\}.$$

Applying the monotonicity of F , we have

$$F(\mathcal{H}_F(A, C, s) \star \mathcal{H}_F(C, B, t)) \leq F(\mathcal{M}_F(A, C, s) \star \mathcal{M}_F(C, B, t))$$

and

$$F(\mathcal{H}_F(A, C, s) \star \mathcal{H}_F(B, C, t)) \leq F(\mathcal{M}_F(B, C, t) \star \mathcal{M}_F(C, A, s)).$$

Therefore,

$$\begin{aligned} \mathcal{H}_F(A, B, s+t) &= \min(\mathcal{M}_F(A, B, s+t), \mathcal{M}_F(B, A, s+t)) \\ &\geq \min(F(\mathcal{M}_F(A, C, s) \star \mathcal{M}_F(C, B, t)), F(\mathcal{M}_F(B, C, t) \star \mathcal{M}_F(C, A, s))) \\ &\geq F(\mathcal{H}_F(A, C, s) \star \mathcal{H}_F(C, B, t)). \end{aligned}$$

Hence, $(\mathcal{CB}(\mathcal{U}), \mathcal{H}_F, F, \star)$ is an $EF\mathcal{F}$ -MS.

(ii) Now we prove the $EF\mathcal{F}$ -completeness of $(\mathcal{CB}(\mathcal{U}), \mathcal{H}_F, F, \star)$. For this, suppose $\{A_n\}$ is an $EF\mathcal{F}$ -Cauchy sequence in $\mathcal{CB}(\mathcal{U})$. Define

$$A = \{x \in \mathcal{U} : \exists \text{ a strictly increasing sequence } \{m_k\} \subset \mathbb{N} \text{ and points } x_{m_k} \in A_{m_k} \text{ such that } x_{m_k} \longrightarrow x\}.$$

(a) We shall first show that $A \neq \emptyset$. Fix an arbitrary $t > 0$ and $r \in (0, 1)$. Choose a sequence $\{t_i\} \subset (0, \infty)$ such that

$$\sum_{i=1}^{\infty} t_i < t.$$

Thus, for every $l > k$, we have

$$\sum_{i=k}^{l-1} t_i < t.$$

From $(\mathcal{F}^*2)$, for the considered $r \in (0, 1)$, $\delta_1 \in (0, 1)$ exists such that

$$1 - \delta_1 < u \leq 1 \implies 1 - r < F(u) \leq 1. \quad (5.4)$$

By Remark 2.4, for the obtained $\delta_1 \in (0, 1)$, $\delta_2 \in (0, 1)$ exists such that

$$(1 - \delta_2) \star (1 - \delta_2) \geq (1 - \delta_1). \quad (5.5)$$

For each $i \geq 2$, by $(\mathcal{F}^*2)$, the continuity of $\star$ with $1 \star 1 = 1$, having chosen $\delta_i \in (0, 1)$, $\delta_{i+1} \in (0, 1)$ exists such that

$$v_1, v_2 > 1 - \delta_{i+1} \implies F(v_1 \star v_2) > 1 - \delta_i. \quad (5.6)$$

Thus we obtain a sequence $\{\delta_i\} \subset (0, 1)$. Now choose a decreasing sequence $\{r_j\} \subset (0, 1)$ with $r_j \rightarrow 0$ such that

$$r_j < \min\{\delta_2, \delta_3, \dots, \delta_{j+1}\}, \quad j \geq 1. \quad (5.7)$$

By the choice of $\{r_i\}$, for every $j \geq k \geq K$, we then have

$$0 < r_j < \delta_{j-k+2}, \quad (5.8)$$

because $2 \leq j - k + 2 \leq j + 1$.

Now, because $\{A_n\}$ is $EF\mathcal{F}$ -Cauchy, for each $i \in \mathbb{N}$, we can choose $n_i \in \mathbb{N}$ inductively such that $n_1 < n_2 < \dots$ and

$$\mathcal{H}_F(A_n, A_m, t_i) > 1 - r_i, \quad \forall n, m \geq n_i.$$

Since,

$$\mathcal{H}_F(A_n, A_m, t_i) = \min\{\mathcal{M}_F(A_n, A_m, t_i), \mathcal{M}_F(A_m, A_n, t_i)\},$$

we get

$$\mathcal{M}_F(A_n, A_m, t_i) > 1 - r_i, \quad \forall n, m \geq n_i.$$

In particular

$$\mathcal{M}_F(A_{n_i}, A_{n_{i+1}}, t_i) > 1 - r_i.$$

By the definition of $\mathcal{M}_F(A_{n_i}, A_{n_{i+1}}, t_i)$, we have

$$\inf_{a \in A_{n_i}} \sup_{b \in A_{n_{i+1}}} \mathcal{M}_F(a, b, t_i) > 1 - r_i \implies \sup_{b \in A_{n_{i+1}}} \mathcal{M}_F(a, b, t_i) > 1 - r_i, \quad \forall a \in A_{n_i}. \quad (5.9)$$

First, we choose a $x_{n_1} \in A_{n_1}$. Then, by (5.9), $x_{n_2} \in A_{n_2}$ exists such that

$$\mathcal{M}_F(x_{n_1}, x_{n_2}, t_1) > 1 - r_1.$$

In particular, for the previously chosen point $x_{n_i} \in A_{n_i}$, $x_{n_{i+1}} \in A_{n_{i+1}}$ exists such that

$$\mathcal{M}_F(x_{n_i}, x_{n_{i+1}}, t_i) > 1 - r_i.$$

Thus, inductively, we obtain a sequence $\{x_{n_i}\} \subseteq A_{n_i}$ satisfying

$$\mathcal{M}_F(x_{n_i}, x_{n_{i+1}}, t_i) > 1 - r_i. \quad (5.10)$$

Now for $l \geq k$, by repeated application of $(EF\mathcal{F}-\mathcal{M}_F4)$, we have

$$\mathcal{M}_F(x_{n_l}, x_{n_k}, \sum_{i=k}^{l-1} t_i) \geq F(\mathcal{M}_F(x_{n_l}, x_{n_{k+1}}, \sum_{i=k+1}^{l-1} t_i) \star \mathcal{M}_F(x_{n_{k+1}}, x_{n_k}, t_k))$$

$$\begin{aligned}
&\geq F(F(\mathcal{M}_F(x_{n_l}, x_{n_{k+2}}, \sum_{i=k+2}^{l-1} t_i) \star \mathcal{M}_F(x_{n_{k+2}}, x_{n_{k+1}}, t_{k+1})) \star \mathcal{M}_F(x_{n_k}, x_{n_{k+1}}, t_k)) \\
&\quad \dots \\
&\geq F(F \cdots (F(\mathcal{M}_F(x_{n_l}, x_{n_{l-1}}, t_{l-1}) \star \mathcal{M}_F(x_{n_{l-1}}, x_{n_{l-2}}, t_{l-2})) \star \cdots) \star \mathcal{M}_F(x_{n_{k+1}}, x_{n_k}, t_k))
\end{aligned}$$

Now,

$$\mathcal{M}_F(x_{n_l}, x_{n_{l-1}}, t_{l-1}) > (1 - r_{l-1}) \geq (1 - r_{l-2}) > (1 - \delta_{l-k}), \forall l \geq k \geq K,$$

where the first comparison uses the monotonicity of $\{r_i\}$ and the second uses (5.8), and hence

$$\mathcal{M}_F(x_{n_l}, x_{n_{l-1}}, t_{l-1}), \mathcal{M}_F(x_{n_{l-1}}, x_{n_{l-2}}, t_{l-2}) > (1 - \delta_{l-k}), \forall l \geq k \geq K,$$

which yields

$$F(\mathcal{M}_F(x_{n_l}, x_{n_{l-1}}, t_{l-1}) \star \mathcal{M}_F(x_{n_{l-1}}, x_{n_{l-2}}, t_{l-2})) > (1 - \delta_{l-k-1}), \forall l \geq k \geq K.$$

Again,

$$\mathcal{M}_F(x_{n_{l-2}}, x_{n_{l-3}}, t_{l-3}) > (1 - r_{l-3}) > (1 - \delta_{l-k-1}), \forall l \geq k \geq K$$

implies

$$F(F(\mathcal{M}_F(x_{n_l}, x_{n_{l-1}}, t_{l-1}) \star \mathcal{M}_F(x_{n_{l-1}}, x_{n_{l-2}}, t_{l-2})) \star \mathcal{M}_F(x_{n_{l-2}}, x_{n_{l-3}}, t_{l-3})) > (1 - \delta_{l-k-2}),$$

whenever $l \geq k \geq K$.

Evaluating in this way and, applying (5.4), on arrival to at $\mathcal{M}_F(x_{n_{k+1}}, x_{n_k}, t_k)$, we get

$$F(\cdots (F(\mathcal{M}_F(x_{n_l}, x_{n_{l-1}}, t_{l-1}) \star \mathcal{M}_F(x_{n_{l-1}}, x_{n_{l-2}}, t_{l-2})) \star \cdots \star \mathcal{M}_F(x_{n_{k+2}}, x_{n_{k+1}}, t_{k+1}))) > (1 - \delta_2)$$

and

$$\mathcal{M}_F(x_{n_{k+1}}, x_{n_k}, t_k) > (1 - r_k) > (1 - \delta_2),$$

whenever $l \geq k \geq K$, which gives

$$\begin{aligned}
&F \cdots (F(\mathcal{M}_F(x_{n_l}, x_{n_{l-1}}, t_{l-1}) \star \mathcal{M}_F(x_{n_{l-1}}, x_{n_{l-2}}, t_{l-2})) \star \cdots \star \mathcal{M}_F(x_{n_{k+2}}, x_{n_{k+1}}, t_{k+1})) \star \mathcal{M}_F(x_{n_{k+1}}, x_{n_k}, t_k) \\
&> (1 - \delta_2) \star (1 - \delta_2), \forall l > k \geq K \\
&\geq (1 - \delta_1), \forall l \geq k \geq K.
\end{aligned}$$

Hence,

$$\begin{aligned}
&F(F \cdots (F(\mathcal{M}_F(x_{n_l}, x_{n_{l-1}}, t_{l-1}) \star \mathcal{M}_F(x_{n_{l-1}}, x_{n_{l-2}}, t_{l-2})) \\
&\quad \star \cdots \star \mathcal{M}_F(x_{n_{k+2}}, x_{n_{k+1}}, t_{k+1})), \mathcal{M}_F(x_{n_{k+1}}, x_{n_k}, t_k))) > (1 - r), \forall l \geq k \geq K.
\end{aligned}$$

Consequently,

$$\mathcal{M}_F(x_{n_l}, x_{n_k}, \sum_{i=k}^{l-1} t_i) > (1 - r), \forall l \geq k \geq K.$$

Since $\mathcal{M}_F$ satisfies $(EF\mathcal{F}-\mathcal{M}_F5)$,

$$\mathcal{M}_F(x_{n_l}, x_{n_k}, t) \geq \mathcal{M}_F(x_{n_l}, x_{n_k}, \sum_{i=k}^{l-1} t_i) > (1-r), \quad \forall l \geq k \geq K.$$

Since $t > 0$ and $r \in (0, 1)$ were arbitrary, it follows that $\{x_{n_k}\}$ is an $EF\mathcal{F}$ -Cauchy sequence in $\mathcal{U}$. By the $EF\mathcal{F}$ -completeness of $(\mathcal{U}, \mathcal{M}_F, F, \star)$, $x \in \mathcal{U}$ exists such that $x_{n_k} \rightarrow x$ in the $EF\mathcal{F}$ -sense. Since $x_{n_k} \in A_{n_k}$, for every k , the definition of A immediately gives $x \in A$. Therefore, $A \neq \emptyset$.

(b) Now we will prove that $A \in CB(\mathcal{U})$. We first show that A is $EF\mathcal{F}$ -closed.

Let $\{y_m\} \subseteq A$ be an $EF\mathcal{F}$ -convergent sequence such that $y_m \rightarrow y$ for some $y \in \mathcal{U}$. We shall prove that $y \in A$.

Since $y_m \in A$ for every $m \in \mathbb{N}$, by the definition of A , for each m , there is a strictly increasing sequence $\{p_{m,j}\}$, $j \in \mathbb{N}$ and points $z_{m,j} \in A_{p_{m,j}}$, $j \in \mathbb{N}$ such that $z_{m,j} \rightarrow y_m$ as $j \rightarrow \infty$.

We now construct a single sequence $\{z_m\} \subseteq \mathcal{U}$ such that $z_m \in A_{q_m}$ with $q_1 < q_2 < \dots$ and $z_m \rightarrow y$. Choose a sequence $\{\vartheta_m\} \subset (0, 1)$ such that $\vartheta_m \rightarrow 0$.

For each $m \in \mathbb{N}$, since $z_{m,j} \rightarrow y_m$, $J_m \in \mathbb{N}$ exists such that

$$\mathcal{M}_F(z_{m,j}, y_m, \frac{1}{m}) > 1 - \vartheta_m, \quad \forall j \geq J_m.$$

We now choose j_m inductively. Choose $j_1 \geq J_1$. Having chosen j_{m-1} , since $\{p_{m,j}\}_{j \in \mathbb{N}}$ is strictly increasing and hence unbounded, we can choose $j_m \geq J_m$ to be sufficiently large such that

$$p_{m,j_m} > p_{m-1,j_{m-1}}.$$

Put $q_m = p_{m,j_m}$ and $z_m = z_{m,j_m}$. Then $z_m \in A_{q_m}$ where $q_1 < q_2 < \dots$ and

$$\mathcal{M}_F(z_m, y_m, \frac{1}{m}) > 1 - \vartheta_m. \quad (5.11)$$

We claim that $z_m \rightarrow y$ in the $EF\mathcal{F}$ -sense. Indeed, fix an arbitrary $t > 0$ and $r \in (0, 1)$. By the property $(\mathcal{F}^*2)$, $\delta' \in (0, 1)$ exists such that

$$1 - \delta' < u \leq 1 \implies 1 - r < F(u) \leq 1. \quad (5.12)$$

By the Remark 1.1, $\delta \in (0, 1)$ exists such that

$$(1 - \delta) \star (1 - \delta) \geq 1 - \delta'. \quad (5.13)$$

Since $y_m \rightarrow y$ in the $EF\mathcal{F}$ -sense, $M_1 \in \mathbb{N}$ exists such that

$$\mathcal{M}_F(y_m, y, \frac{t}{2}) > 1 - \delta, \quad m \geq M_1. \quad (5.14)$$

Moreover, since $\vartheta_m \rightarrow 0$ and $\frac{1}{m} \rightarrow 0$, $M_2, M_3 \in \mathbb{N}$ exist such that

$$\vartheta_m < \delta, \quad \forall m \geq M_2 \quad \text{and} \quad \frac{1}{m} < \frac{t}{2}, \quad \forall m \geq M_3.$$

Let $M = \max\{M_1, M_2, M_3\}$. Then, by $(EF\mathcal{F}\text{-}\mathcal{M}_F5)$ and (5.11), we have

$$\mathcal{M}_F(z_m, y_m, \frac{t}{2}) \geq \mathcal{M}_F(z_m, y_m, \frac{1}{m}) > 1 - \vartheta_m > 1 - \delta, \quad \forall m \geq M. \quad (5.15)$$

Again, by $(EF\mathcal{F}\text{-}\mathcal{M}_F4)$

$$\mathcal{M}_F(z_m, y, t) \geq F\left(\mathcal{M}_F(z_m, y_m, \frac{t}{2}) \star \mathcal{M}_F(y_m, y, \frac{t}{2})\right).$$

Now by (5.14) and (5.15) together with (5.13), for all $m \geq M$, we have

$$\mathcal{M}_F(z_m, y_m, \frac{t}{2}) \star \mathcal{M}_F(y_m, y, \frac{t}{2}) > (1 - \delta) \star (1 - \delta) \geq (1 - \delta'), \quad \forall m \geq M,$$

which, from (5.12), yields

$$\mathcal{M}_F(z_m, y, t) \geq F(\mathcal{M}_F(z_m, y_m, \frac{t}{2}) \star \mathcal{M}_F(y_m, y, \frac{t}{2})) > 1 - r, \quad \forall m \geq M.$$

Since $t > 0$ and $r \in (0, 1)$ are arbitrary, it follows that $z_m \rightarrow y$ in the $EF\mathcal{F}$ -sense.

As $\{q_m\}$ is strictly increasing and $z_m \rightarrow y$, the defining property of A yields $y \in A$. Thus A is $EF\mathcal{F}$ -closed.

Now, we will prove that A is $EF\mathcal{F}$ -bounded. Take arbitrary $x, y \in A$.

By the definition of A , strictly increasing sequences of indices $\{m_k\}$, $\{n_k\}$ and points $x_{m_k} \in A_{m_k}$, $y_{n_k} \in A_{n_k}$ exist such that $x_{m_k} \rightarrow x$ and $y_{n_k} \rightarrow y$ in $EF\mathcal{F}$ -sense.

Take arbitrary $s > 0$ and $r_1 \in (0, 1)$. Since $\{A_n\}$ is $EF\mathcal{F}$ -Cauchy, $N \in \mathbb{N}$ exists such that

$$\mathcal{H}_F(A_n, A_m, s) > 1 - r_1, \quad \forall n, m \geq N.$$

Consequently

$$\mathcal{M}_F(A_n, A_m, s) > 1 - r_1 \quad \text{and} \quad \mathcal{M}_F(A_m, A_n, s) > 1 - r_1, \quad (5.16)$$

for all $n, m \geq N$.

Now fix this N . Since $A_N \in \mathcal{CB}(\mathcal{U})$, in particular, A_N is $EF\mathcal{F}$ -bounded. Hence, $t_0 > 0$ and $r_0 \in (0, 1)$ exist such that

$$\mathcal{M}_F(u, v, t_0) > 1 - r_0, \quad \forall u, v \in A_N. \quad (5.17)$$

Since $m_k, n_k \rightarrow \infty$, so for all sufficiently large values of k , $m_k, n_k \geq N$.

For such k , from (5.16) and the definition $\mathcal{M}_F(A_{m_k}, A_N, s)$, we obtain

$$\sup_{a \in A_N} \mathcal{M}_F(x_{m_k}, a, s) > 1 - r_1,$$

because $x_{m_k} \in A_{m_k}$. Therefore, we can choose $u_k \in A_N$ such that

$$\mathcal{M}_F(x_{m_k}, u_k, s) > 1 - r_1. \quad (5.18)$$

Similarly, from

$$\mathcal{M}_F(A_{n_k}, A_N, s) > 1 - r_1$$

and $y_{n_k} \in A_{n_k}$, $v_k \in A_N$ exists such that

$$\mathcal{M}_F(y_{n_k}, v_k, s) > 1 - r_1. \quad (5.19)$$

Moreover, by (5.17), we have

$$\mathcal{M}_F(u_k, v_k, t_0) > 1 - r_0. \quad (5.20)$$

Now by applying $(EF\mathcal{F}\text{-}\mathcal{M}_F4)$ twice, we obtain

$$\begin{aligned} \mathcal{M}_F(x_{m_k}, y_{n_k}, 2s + t_0) &\geq F(\mathcal{M}_F(x_{m_k}, v_k, s + t_0) \star \mathcal{M}_F(v_k, y_{n_k}, s)) \\ &\geq F(F(\mathcal{M}_F(x_{m_k}, u_k, s) \star \mathcal{M}_F(u_k, v_k, t_0)) \star \mathcal{M}_F(u_k, y_{n_k}, s)) \\ &\geq F(F((1 - r_1) \star (1 - r_0)) \star (1 - r_1)), \end{aligned}$$

where the last inequality follows using (5.18)–(5.20) and the nondecreasing property of F .

Since $(1 - r_1) \star (1 - r_0) \in (0, 1)$ and $0 < F(u) \leq 1$, for all $u \in (0, 1)$, $F((1 - r_1) \star (1 - r_0)) \in (0, 1)$. Then choose $r_2 \in (0, 1)$ such that

$$1 - r_2 < F(F((1 - r_1) \star (1 - r_0)) \star (1 - r_1)).$$

This gives

$$\mathcal{M}_F(x_{m_k}, y_{n_k}, 2s + t_0) > 1 - r_2.$$

Then, letting $k \rightarrow \infty$, since $x_{m_k} \rightarrow x$ and $y_{n_k} \rightarrow y$ and $\mathcal{M}_F$ is sequentially continuous in its first and second variables, we obtain

$$\mathcal{M}_F(x, y, 2s + t_0) \geq 1 - r_2.$$

Now choose $r_2 < r_3 < 1$, then

$$\mathcal{M}_F(x, y, 2s + t_0) > 1 - r_3.$$

Since $x, y \in A$ are arbitrary, by the definition of $EF\mathcal{F}$ -boundedness, A is $EF\mathcal{F}$ -bounded.

(c) Final claim is $A_n \rightarrow A$ with respect to $\mathcal{H}_F$ in $CB(\mathcal{U})$. Let $t > 0$ and $r \in (0, 1)$ be arbitrary. Then take $r_0 \in (0, r)$. Now choose $\eta \in (0, r - r_0)$. By (5.1), for $\eta, \delta \in (0, 1)$ exists such that

$$\mathcal{M}_F(\varrho_2, \varrho_3, t) > 1 - \delta \implies |\mathcal{M}_F(\varrho_1, \varrho_2, t) - \mathcal{M}_F(\varrho_1, \varrho_3, t)| < \eta, \quad \forall \varrho_1, \varrho_2, \varrho_3 \in \mathcal{U}. \quad (5.21)$$

Since $\{A_n\}$ is $EF\mathcal{F}$ -Cauchy, $N = N(t, r)$ exists such that $\mathcal{H}_F(A_n, A_m, t) > 1 - r_0$, $\forall n, m \geq N$.

Hence, we have

$$\mathcal{M}_F(A_n, A_m, t) > 1 - r_0 \quad \text{and} \quad \mathcal{M}_F(A_m, A_n, t) > 1 - r_0,$$

for all $n, m \geq N$. We first prove that $\mathcal{M}_F(A_n, A, t) > 1 - r$, for all $n \geq N$.

Fix $n \geq N$ and $x \in A_n$. By the definition of

$$\mathcal{M}_F(A_n, A_m, t) = \inf_{a \in A_n} \sup_{b \in A_m} \mathcal{M}_F(a, b, t),$$

we obtain

$$\sup_{b \in A_m} \mathcal{M}_F(x, b, t) > 1 - r_0, \quad \forall m \geq N. \quad (5.22)$$

Choose $m_1 \geq N$. By (5.22), $y_{m_1} \in A_{m_1}$ exists such that

$$\mathcal{M}_F(x, y_{m_1}, t) > 1 - r_0.$$

Having chosen m_k and $y_{m_k} \in A_{m_k}$, choose $m_{k+1} > m_k$ to be sufficiently large so that $m_{k+1} \geq N$ and

$$\mathcal{H}_F(A_{m_k}, A_{m_{k+1}}, t_k) > 1 - r_k,$$

where $\sum_{j=0}^{\infty} t_j < t$ and $r_j \rightarrow 0$ as chosen in Part (a). Consequently

$$\mathcal{M}_F(A_{m_k}, A_{m_{k+1}}, t_k) > 1 - r_k.$$

Since $y_{m_k} \in A_{m_k}$, $y_{m_{k+1}} \in A_{m_{k+1}}$ exists such that

$$\mathcal{M}_F(y_{m_k}, y_{m_{k+1}}, t_k) > 1 - r_k.$$

Since $\{A_n\}$ is Cauchy, by choosing $y_{m_1} \in A_{m_1}$, we can choose points recursively such that

$$\mathcal{M}_F(y_{m_k}, y_{m_{k+1}}, t_k) > 1 - r_k,$$

Thus, by the same chain argument used in Part (a), the sequence $\{y_{m_k}\}$ is an $EF\mathcal{F}$ -Cauchy sequence. By the $EF\mathcal{F}$ -completeness of $\mathcal{U}$, $y \in \mathcal{U}$ exists such that $y_{m_k} \rightarrow y$.

Since $y_{m_k} \in A_{m_k}$ and $\{m_i\}$ is strictly increasing, by the definition of A , $y \in A$. Since $y_{m_k} \rightarrow y$, there exists $K_0 \in \mathbb{N}$ such that

$$\mathcal{M}_F(y_{m_k}, y, t) > 1 - \delta, \quad \forall k \geq K_0.$$

For each $k \geq K_0$, applying (5.21), we get

$$|\mathcal{M}_F(x, y_{m_k}, t) - \mathcal{M}_F(x, y, t)| < \eta,$$

which implies

$$\mathcal{M}_F(x, y, t) > \mathcal{M}_F(x, y_{m_k}, t) - \eta > 1 - r_0 - \eta > 1 - r,$$

where the last inequality follows from $\eta < r - r_0$. Since $y \in A$, we have

$$\mathcal{M}_F(x, A, t) = \sup_{b \in A} \mathcal{M}_F(x, b, t) \geq \mathcal{M}_F(x, y, t) > 1 - r.$$

As $x \in A_n$ was arbitrary, it follows that

$$\mathcal{M}_F(A_n, A, t) = \inf_{x \in A_n} \mathcal{M}_F(x, A, t) > 1 - r.$$

In fact, since the above strict inequality holds for every A_n with $n \geq N$, we obtain

$$\mathcal{M}_F(A_n, A, t) > 1 - r, \quad \forall n \geq N.$$

We next prove the estimate $\mathcal{M}_F(A, A_n, t) > 1 - r$, $\forall n \geq N$. Fix $n \geq N$ and let $x \in A$ be arbitrary. By definition of A , a strictly increasing sequence of indices $\{m_k\}$ and point $x_{m_k} \in A_{m_k}$ exists such that $x_{m_k} \rightarrow x$.

Now because $m_k \rightarrow \infty$, there is K_1 such that $m_k \geq N$, for $k \geq K_1$.

Since $x_{m_k} \longrightarrow x$, $K_2 \in \mathbb{N}$ exists such that

$$\mathcal{M}_F(x_{m_k}, x, t) > 1 - \delta, \quad \forall m_k \geq K_2.$$

Take $k \geq K = \max\{K_1, K_2\}$. Since A_n and A_{m_k} belong to the $EF\mathcal{F}$ -Cauchy sequence, for $n, m_k \geq N$, we have

$$\mathcal{H}_F(A_{m_k}, A_n, t) > 1 - r_0.$$

Therefore

$$\mathcal{M}_F(A_{m_k}, A_n, t) > 1 - r_0.$$

Therefore, for a particular $x_{m_k} \in A_{m_k}$

$$\sup_{b \in A_n} \mathcal{M}_F(x_{m_k}, b, t) > 1 - r_0, \quad \forall n \geq N.$$

Thus for sufficiently large $k \geq N$, $b_k \in A_n$ exists such that

$$\mathcal{M}_F(x_{m_k}, b_k, t) > 1 - r_0.$$

Since $\mathcal{M}_F(x, x_{m_k}, t) > 1 - \delta$, for $b_k \in A_n$, from (5.21), we get

$$|\mathcal{M}_F(b_k, x_{m_k}, t) - \mathcal{M}_F(b_k, x, t)| < \eta,$$

which gives

$$\mathcal{M}_F(b_k, x, t) > \mathcal{M}_F(b_k, x_{m_k}, t) - \eta > 1 - r_0 - \eta.$$

From which we can write

$$\mathcal{M}_F(x, b_k, t) > 1 - r.$$

Since $b_k \in A_n$, we have

$$\mathcal{M}_F(x, A_n, t) = \sup_{b \in A_n} \mathcal{M}_F(x, b, t) > 1 - r$$

for every $x \in A$. Hence

$$\mathcal{M}_F(A, A_n, t) = \inf_{a \in A} \sup_{b \in A_n} \mathcal{M}_F(a, b, t) > 1 - r, \quad \forall n \geq N.$$

This yields

$$\mathcal{H}_F(A_n, A, t) = \min\{\mathcal{M}_F(A_n, A, t), \mathcal{M}_F(A, A_n, t)\} > 1 - r, \quad \forall n \geq N, \quad \forall t > 0.$$

Since $r \in (0, 1)$ is arbitrary, we conclude that $A_n \longrightarrow A$ in $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$.

Thus every $EF\mathcal{F}$ -Cauchy sequence in $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$ converges to a limit belonging to $CB(\mathcal{U})$. Consequently, $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$ is $EF\mathcal{F}$ -complete.

(iii) First note that under the assumption of Theorem 5.4 and construction of Ψ_F , $\Psi_F(A) \in CB(\mathcal{U})$, for all $A \in CB(\mathcal{U})$. Hence, Ψ_F is a self-map on $CB(\mathcal{U})$. We prove the claim for $l = 2$. Let $A, B \in CB(\mathcal{U})$ and $t > 0$. We first show that

$$\mathcal{H}_F(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B), t) \geq \min\{\mathcal{H}_F(T_1(A), T_1(B), t), \mathcal{H}_F(T_2(A), T_2(B), t)\}, \quad \forall t > 0.$$

Now for all $t > 0$

$$\begin{aligned}\mathcal{M}_F(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B), t) &= \inf_{\varrho \in T_1(A) \cup T_2(A)} \sup_{y \in T_1(B) \cup T_2(B)} \mathcal{M}_F(\varrho, y, t) \\ &= \min\left\{ \inf_{\varrho \in T_1(A)} \sup_{y \in T_1(B) \cup T_2(B)} \mathcal{M}_F(\varrho, y, t), \inf_{\varrho \in T_2(A)} \sup_{y \in T_1(B) \cup T_2(B)} \mathcal{M}_F(\varrho, y, t) \right\}.\end{aligned}$$

Again, we have

$$\begin{aligned}\sup_{y \in T_1(B) \cup T_2(B)} \mathcal{M}_F(\varrho, y, t) &\geq \sup_{y \in T_1(B)} \mathcal{M}_F(\varrho, y, t) \\ \implies \inf_{\varrho \in T_1(A)} \sup_{y \in T_1(B) \cup T_2(B)} \mathcal{M}_F(\varrho, y, t) &\geq \inf_{\varrho \in T_1(A)} \sup_{y \in T_1(B)} \mathcal{M}_F(\varrho, y, t) \\ &= \mathcal{M}_F(T_1(A), T_1(B), t).\end{aligned}$$

Similarly, we can show that

$$\inf_{\varrho \in T_2(A)} \sup_{y \in T_1(B) \cup T_2(B)} \mathcal{M}_F(\varrho, y, t) \geq \mathcal{M}_F(T_2(A), T_2(B), t).$$

Therefore

$$\mathcal{M}_F(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B), t) \geq \min\{\mathcal{M}_F(T_1(A), T_1(B), t), \mathcal{M}_F(T_2(A), T_2(B), t)\}.$$

Again, for all $t > 0$, we have

$$\begin{aligned}&\min\{\mathcal{M}_F(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B), t), \mathcal{M}_F(T_1(B) \cup T_2(B), T_1(A) \cup T_2(A), t)\} \\ &\geq \min\{\min\{\mathcal{M}_F(T_1(A), T_1(B), t), \mathcal{M}_F(T_2(A), T_2(B), t)\}, \min\{\mathcal{M}_F(T_1(B), T_1(A), t), \mathcal{M}_F(T_2(B), T_2(A), t)\}\} \\ &= \min\{\mathcal{M}_F(T_1(A), T_1(B), t), \mathcal{M}_F(T_1(B), T_1(A), t), \mathcal{M}_F(T_2(A), T_2(B), t), \mathcal{M}_F(T_2(B), T_2(A), t)\} \\ &= \min\{\min\{\mathcal{M}_F(T_1(A), T_1(B), t), \mathcal{M}_F(T_1(B), T_1(A), t)\}, \min\{\mathcal{M}_F(T_2(A), T_2(B), t), \mathcal{M}_F(T_2(B), T_2(A), t)\}\} \\ &= \min\{\mathcal{H}_F(T_1(A), T_1(B), t), \mathcal{H}_F(T_2(A), T_2(B), t)\},\end{aligned}$$

which concludes

$$\mathcal{H}_F(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B), t) \geq \min\{\mathcal{H}_F(T_1(A), T_1(B), t), \mathcal{H}_F(T_2(A), T_2(B), t)\}. \quad (5.23)$$

Now applying the pointwise contraction condition in (4.1) on each T_i , $i = 1, 2$, we have

$$\begin{aligned}\left(\frac{1}{\mathcal{M}_F(T_i\varrho, T_iy, t)} - 1\right) &\leq a\left(\frac{1}{\mathcal{M}_F(\varrho, y, t)} - 1\right), \quad \forall \varrho, y \in \mathcal{U}, \forall t > 0 \\ \text{or } \frac{1}{\mathcal{M}_F(T_i\varrho, T_iy, t)} &\leq \frac{a}{\mathcal{M}_F(\varrho, y, t)} + 1 - a, \quad \forall \varrho, y \in \mathcal{U}, \forall t > 0.\end{aligned}$$

Let us define a function $\phi_a: [0, 1] \rightarrow [0, 1]$ by $\phi_a(u) = \frac{u}{a+u(1-a)}$. Then ϕ_a is an increasing and continuous function on $[0, 1]$. Hence

$$\frac{1}{\mathcal{M}_F(T_i\varrho, T_iy, t)} \leq \frac{1}{\phi_a(\mathcal{M}_F(\varrho, y, t))} \quad \text{or} \quad \mathcal{M}_F(T_i\varrho, T_iy, t) \geq \phi_a(\mathcal{M}_F(\varrho, y, t)).$$

On the application of continuity and the increasing property of ϕ_a , the inequality above yields

$$\sup_{y \in B} \mathcal{M}_F(T_i \varrho, T_i y, t) \geq \sup_{y \in B} \phi_a(\mathcal{M}_F(\varrho, y, t)) = \phi_a(\sup_{y \in B} \mathcal{M}_F(\varrho, y, t))$$

or

$$\inf_{\varrho \in A} \sup_{y \in B} \mathcal{M}_F(T_i \varrho, T_i y, t) \geq \inf_{\varrho \in A} \phi_a(\mathcal{M}_F(\varrho, B, t)) = \phi_a(\inf_{\varrho \in A} \mathcal{M}_F(\varrho, B, t))$$

or

$$\mathcal{M}_F(T_i A, T_i B, t) \geq \phi_a(\mathcal{M}_F(A, B, t)).$$

Therefore, for all $t > 0$ and for $i = 1, 2$, we have

$$\begin{aligned} \mathcal{H}_F(T_i A, T_i B, t) &= \min \{ \mathcal{M}_F(T_i A, T_i B, t), \mathcal{M}_F(T_i B, T_i A, t) \} \\ &\geq \min \{ \phi_a(\mathcal{M}_F(A, B, t)), \phi_a(\mathcal{M}_F(B, A, t)) \} \\ &= \phi_a(\min \{ \mathcal{M}_F(A, B, t), \mathcal{M}_F(B, A, t) \}) \quad (\text{since } \phi_a \text{ is increasing}) \\ &= \phi_a(\mathcal{H}_F(A, B, t)). \end{aligned}$$

Hence, from (5.23), we have

$$\begin{aligned} \mathcal{H}_F(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B), t) &\geq \min \{ \phi_a(\mathcal{H}_F(A, B, t)), \phi_a(\mathcal{H}_F(A, B, t)) \} \\ &= \phi_a(\mathcal{H}_F(A, B, t)) = \frac{\mathcal{H}_F(A, B, t)}{a + (1 - a)\mathcal{H}_F(A, B, t)}, \end{aligned}$$

which implies

$$\left(\frac{1}{\mathcal{H}_F(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B), t)} - 1 \right) \leq a \left(\frac{1}{\mathcal{H}_F(A, B, t)} - 1 \right).$$

This establishes that the operator $\Psi_F = T_1 \cup T_2$ is an $EF\mathcal{F}$ -Banach type contraction in $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$ with respect to the constant coefficient ‘ a ’.

(iv) Let $K_0 \in CB(\mathcal{U})$ be an arbitrary initial set and define the iterative sequence as:

$$K_{n+1} = \Psi_F(K_n) = \Psi^n(K_0), \quad n = 0, 1, \dots \quad (5.24)$$

Since Ψ_F satisfies the $EF\mathcal{F}$ -Banach contraction condition, it follows that

$$\left(\frac{1}{\mathcal{H}_F(K_{n+1}, K_n, t)} - 1 \right) \leq a^n \left(\frac{1}{\mathcal{H}_F(K_1, K_0, t)} - 1 \right), \quad n = 0, 1, \dots \quad (5.25)$$

We next verify that the additional assumptions required in Theorem 4.3 are inherited by the induced function $\mathcal{H}_F$. First, since $\mathcal{M}_F(\varrho, y, \cdot)$ is nondecreasing on $(0, \infty)$, for every $\varrho, y \in \mathcal{U}$, it follows that $\mathcal{M}_F(A, B, \cdot)$ is nondecreasing for every $A, B \in CB(\mathcal{U})$. Consequently, $\mathcal{H}_F(A, B, \cdot)$ is also nondecreasing on $(0, \infty)$. Thus, $(EF\mathcal{F}-\mathcal{M}_F5)$ holds for $\mathcal{H}_F$.

Next, to verify that the equicontinuity property of $\mathcal{M}_F$ is inherited by $\mathcal{H}_F$, fix $t' > 0$, $\epsilon > 0$, and let $\{A_k\}, \{B_k\} \subseteq CB(\mathcal{U})$ be arbitrary sequences. By the assumed equicontinuity property of $\mathcal{M}_F$, $\delta > 0$ exists such that

$$|t - t'| < \delta \implies |\mathcal{M}_F(\varrho, y, t) - \mathcal{M}_F(\varrho, y, t')| < \frac{\epsilon}{2},$$

for all $\varrho, y \in \mathcal{U}$. Hence, for every $k \in \mathbb{N}$, $a \in A_k$, $b \in B_k$, we have

$$|\mathcal{M}_F(a, b, t) - \mathcal{M}_F(a, b, t')| < \frac{\epsilon}{2} \implies \left| \sup_{b \in B_k} \mathcal{M}_F(a, b, t) - \sup_{b \in B_k} \mathcal{M}_F(a, b, t') \right| \leq \frac{\epsilon}{2}.$$

Taking the infimum over $a \in A_k$, we obtain

$$|\mathcal{M}_F(A_k, B_k, t) - \mathcal{M}_F(A_k, B_k, t')| \leq \frac{\epsilon}{2}.$$

Similarly,

$$|\mathcal{M}_F(B_k, A_k, t) - \mathcal{M}_F(B_k, A_k, t')| \leq \frac{\epsilon}{2}.$$

Then the property of the minimum function and the definition of $\mathcal{H}_F$, we obtain

$$\begin{aligned} |\mathcal{H}_F(A_k, B_k, t) - \mathcal{H}_F(A_k, B_k, t')| &= |\min \{ \mathcal{M}_F(A_k, B_k, t), \mathcal{M}_F(B_k, A_k, t) \}, \min \{ \mathcal{M}_F(A_k, B_k, t'), \mathcal{M}_F(B_k, A_k, t') \}| \\ &\leq \max \{ |\mathcal{M}_F(A_k, B_k, t) - \mathcal{M}_F(A_k, B_k, t')|, |\mathcal{M}_F(B_k, A_k, t) - \mathcal{M}_F(B_k, A_k, t')| \} \\ &\leq \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

for every $k \in \mathbb{N}$. Therefore, the family $\{t \mapsto \mathcal{H}_F(A_k, B_k, t) : k \in \mathbb{N}\}$ is equicontinuous at t' . Finally, we verify condition (ii) for $\mathcal{H}_F$. Let $0 < t < t'$ and $0 < r < 1$. Suppose that $\mathcal{H}_F(A, B, t') > 1 - r$. Then

$$\mathcal{M}_F(A, B, t') > 1 - r \quad \text{and} \quad \mathcal{M}_F(B, A, t') > 1 - r.$$

Now, $\mathcal{M}_F(A, B, t') > 1 - r$ implies

$$\inf_{a \in A} \sup_{b \in B} \mathcal{M}_F(a, b, t') > 1 - r.$$

Thus for every $a \in A$, we have

$$\sup_{b \in B} \mathcal{M}_F(a, b, t') > 1 - r.$$

Hence for each $a \in A$, $b_a \in B$ exists such that $\mathcal{M}_F(a, b_a, t') > 1 - r$.

Using condition (ii) on Theorem 4.3, for every $0 < t < t'$, $\phi_{t,t'}(r) \in (0, 1)$ exists such that

$$\mathcal{M}_F(a, b_a, t') > 1 - \phi_{t,t'}(r) \implies \sup_{b \in B} \mathcal{M}_F(a, b, t') \geq 1 - \phi_{t,t'}(r).$$

Since it holds for every $a \in A$, we have

$$\mathcal{M}_F(A, B, t) = \inf_{a \in A} \sup_{b \in B} \mathcal{M}_F(a, b, t') \geq 1 - \phi_{t,t'}(r).$$

By same argument, $\mathcal{M}_F(B, A, t) \geq 1 - \phi_{t,t'}(r)$. Consequently, $\mathcal{H}_F(A, B, t) \geq 1 - \phi_{t,t'}(r)$.

As $\phi_{t,t'}(r) \in (0, 1)$, we can choose $0 < \phi_{t,t'}(r) < \psi_{t,t'}(r) < 1$, which gives $\mathcal{H}_F(A, B, t) > 1 - \psi_{t,t'}(r)$. Thus, condition (ii) is also satisfied by $\mathcal{H}_F$.

Hence, $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$ satisfies all the hypotheses of Theorem 4.3. Therefore, applying Theorem 4.3 to $\{K_n\}$, we see that $\{K_n\}$ is a Cauchy sequence in $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$. Since $CB(\mathcal{U})$ is $EF\mathcal{F}$ -complete, $\{K_n\}$ converges to some $K^* \in CB(\mathcal{U})$. Moreover, $K^* \in CB(\mathcal{U})$ is a unique fixed point of Ψ_F . $\square$

By the preceding theorem, the iterates of the $EF\mathcal{F}$ -HBO Ψ_F form a Cauchy sequence in $(CB(\mathcal{U}), \mathcal{H}_F, F, \star)$ and converge to a unique attractor. We now make this convergence explicit by introducing a sequence that measures the distance between consecutive iterates.

5.1. Iterate decay, convergence, and approximation error estimation

The inequality (5.25) shows that the consecutive iterates K_n and K_{n+1} become arbitrarily close in the $\mathcal{H}_F$ -sense, with geometric decay. The convergence result above also provides an explicit estimate of the approximation error for the generated fractal attractor. We obtained that the iterative sequence defined in (5.24) converges to K^* , which is the unique fixed point of Ψ_F . Applying the contraction inequality (4.1) repeatedly on Ψ_F and using (5.24), we obtain

$$\left(\frac{1}{\mathcal{H}_F(K_n, K^*, t)} - 1 \right) \leq a \left(\frac{1}{\mathcal{H}_F(K_{n-1}, K^*, t)} - 1 \right) \leq \cdots \leq a^n \left(\frac{1}{\mathcal{H}_F(K_0, K^*, t)} - 1 \right).$$

Therefore, the error of the n -th approximation with respect to the attractor decreases geometrically with the contraction constant $a \in (0, 1)$ and the error $\left(\frac{1}{\mathcal{H}_F(K_n, K^*, t)} - 1 \right)$ decays geometrically with a rate a^n . In particular, a smaller value of a guarantees a faster convergence of the iterative construction toward the unique attractor K^* .

5.2. Discussion on the convergence rate

In this example, the smaller contraction constant a implies that the error is reduced by at least a constant factor at each iteration, leading to fast stabilization of the approximating sets. Consequently, the contractive estimate obtained in Theorem 4.3 predicts a rapid geometric convergence of the iterates toward the unique attractor. This behavior is also reflected in Figure 2b, where the Hausdorff $EF\mathcal{F}$ -distance $\mathcal{H}_F$ between successive iterates decreases rapidly as the iteration number increases. Compared with the classical fuzzy IFS, convergence remains geometric, while the $EF\mathcal{F}$ -metric framework permits a broader class of auxiliary functions from $\mathcal{F}^*$. Thus, the extension enlarges the admissible framework without altering the fundamental geometric convergence behavior of the iterates.

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Remark 5.5. The additional condition (5.1) is a uniform continuity-type condition for the fuzzy distance with respect to its first or second variable. It provides the uniform control needed to pass from approximating points to their $EF\mathcal{F}$ -limits. The sequential continuity and (5.1) serve different purposes: Sequential continuity permits passage to the limit for a fixed pair of convergent sequences, whereas (5.1) provides a uniform estimate with respect to the arbitrary point $q \in \mathcal{U}$ which is required when taking the infimum and supremum over sets.

5.3. Concrete numerical illustration for fractal applications

Consider the usual metric d on $\mathcal{U} = [0, 1]$ and we define

$$\mathcal{M}_F(q, y, t) = \exp \left(\frac{-d(q, y)^3}{t} \right),$$

for all $\varrho, y \in \mathcal{U}$, and $t > 0$. Then from Example 3.4, we get $\mathcal{M}_F$ is an $EF\mathcal{F}$ -metric over $\mathcal{U}$ with respect to the chosen t -norm and $F \in \mathcal{F}^*$. The considered function F satisfies $F(u) > 0$, for all $u \in (0, 1)$, and $\mathcal{M}_F$ is sequentially continuous on both the first and second variables. Lastly, to verify (5.1), fix $t > 0$ and $\epsilon > 0$. We need a $\delta \in (0, 1)$ that is, independent of $\varrho, y, z \in \mathcal{U}$. Suppose that $\mathcal{M}_F(y, z, t) > 1 - \delta$. Then

$$\exp\left(-\frac{|y-z|^3}{t}\right) > 1 - \delta \implies |y-z| < (-t \ln(1 - \delta))^{\frac{1}{3}}.$$

Now define $\phi_t(s) = e^{-\frac{s^3}{t}}$, $s \geq 0$. Then $\phi'_t(s) = -\frac{3s^2}{t} e^{-\frac{s^3}{t}}$ and hence $\sup_{s \geq 0} |\phi'_t(s)| < \infty$ which implies ϕ_t is globally Lipschitz on $[0, \infty)$. Let $L_t > 0$ be the corresponding Lipschitz constant. Using

$$||\varrho - y| - |\varrho - z|| \leq |y - z|,$$

we have

$$|\mathcal{M}_F(\varrho, y, t) - \mathcal{M}_F(\varrho, z, t)| \leq L_t ||\varrho - y| - |\varrho - z|| \leq L_t |y - z| < L_t (-t \ln(1 - \delta))^{\frac{1}{3}}.$$

Since the last expression tends to 0 as $\delta \rightarrow 0^+$, we can choose $\delta \in (0, 1)$ sufficiently small so that

$$(-t \ln(1 - \delta))^{\frac{1}{3}} L_t < \epsilon.$$

Therefore, condition (5.1) holds.

For nonempty closed and bounded sets $A, B \subset [0, 1]$, we have

$$\mathcal{M}_F(A, B, t) = \exp\left(-\frac{d(A, B)^3}{t}\right) \quad \& \quad \mathcal{H}_F(A, B, t) = \exp\left(-\frac{d(A, B)^3}{t}\right), \quad \forall t > 0,$$

where $d(A, B) = \{d(a, b) : a \in A, b \in B\}$. If in $(\mathcal{U}, d)$, a mapping $T : \mathcal{U} \rightarrow \mathcal{U}$ satisfies $d(T\varrho, Ty) \leq L d(\varrho, y)$ ($0 < L < 1$), then in the $EF\mathcal{F}$ -metric setting, we obtain

$$\left(\frac{1}{\mathcal{M}_F(T\varrho, Ty, t)} - 1\right) \leq a \left(\frac{1}{\mathcal{M}_F(\varrho, y, t)} - 1\right),$$

where $0 < L^3 \leq a < 1$, so that T is an $EF\mathcal{F}$ -Banach type contraction with a constant $a \in (0, 1)$.

Consequently, for a family $\{T_i\}_{i=1}^l$ sharing the same L , the $EF\mathcal{F}$ -HBO operator $\Psi_F(K) = \cup_{i=1}^l T_i(K)$ satisfies

$$\left(\frac{1}{\mathcal{H}_F(\Psi_F(A), \Psi_F(B), t)} - 1\right) \leq a \left(\frac{1}{\mathcal{H}_F(A, B, t)} - 1\right), \quad \forall A, B \in \mathcal{CB}(\mathcal{U})$$

and the iterates $K_{n+1} = \Psi_F(K_n)$ converge to the unique attractor K^* in $(\mathcal{CB}(\mathcal{U}), \mathcal{H}_F, F, \star)$.

5.4. Application to the standard Cantor IFS on $[0, 1]$

We consider the maps $T_1(\varrho) = \frac{\varrho}{3}$, and $T_2(\varrho) = \frac{\varrho}{3} + \frac{2}{3}$, for all $\varrho \in [0, 1]$. Both satisfy

$$|T_i(\varrho) - T_i(y)| = \frac{1}{3} |\varrho - y|, \quad \forall \varrho, y \in [0, 1], \quad i = 1, 2.$$

So the common Lipschitz constant is $L = \frac{1}{3}$.

With the $EF\mathcal{F}$ -metric, we have

$$\mathcal{M}_F(\varrho, y, t) = \exp\left(\frac{-|\varrho - y|^3}{t}\right),$$

the corresponding contraction constant for the defined T is

$$a = L^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}.$$

Hence, the $EF\mathcal{F}$ -HBO, $\Psi_F(K) = T_1(K) \cup T_2(K)$ satisfies

$$\left(\frac{1}{\mathcal{H}_F(\Psi_F(A), \Psi_F(B), t)} - 1\right) \leq \frac{1}{27} \left(\frac{1}{\mathcal{H}_F(A, B, t)} - 1\right), \quad \forall A, B \in \mathcal{CB}([0, 1]).$$

Therefore, the iterates $K_{n+1} = \Psi_F(K_n)$ converge in $(\mathcal{CB}([0, 1]), \mathcal{H}_F, F, \star)$ to the unique Cantor attractor K^* in the hyperspace $\mathcal{CB}([0, 1])$ and the convergence rate is geometric with the ratio $1/27$ in the $EF\mathcal{F}$ -metric sense.

5.5. Numerical visualization

To complement the analytical discussion, we illustrate the generation of the Cantor set through successive iterations of Ψ_F .

The iterative construction of the Cantor set can be described mathematically as follows:

$$K_0 = [0, 1], \quad K_1 = \Psi_F(K_0) = T_1(K_0) \cup T_2(K_0) = \frac{1}{3}K_0 \cup \left(\frac{2}{3} + \frac{1}{3}K_0\right), \dots,$$

$$K_{n+1} = \Psi_F(K_n) = T_1(K_n) \cup T_2(K_n) = \frac{1}{3}K_n \cup \left(\frac{2}{3} + \frac{1}{3}K_n\right), \dots.$$

Therefore, each iteration K_{n+1} is obtained from K_n by removing the open middle third of every subinterval. For example, we have

$$K_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \quad K_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \text{ etc.}$$

The limiting Cantor set is defined by: $K = \bigcap_{n=0}^{\infty} K_n$, which is a totally disconnected, closed, bounded, and self-similar fractal set.

The MATLAB simulation reproduces this iterative construction numerically. At each step, the number of intervals doubles while their lengths shrink by a factor of three. Figure 2 illustrates these iterates and the convergence behavior. In the numerical simulation, we set $t = 1$ for simplicity and normalization. Since the simulation demonstrates the convergence of the iterated function system in the Hausdorff $EF\mathcal{F}$ -distance $\mathcal{H}_F$, fixing the fuzzy parameter $t = 1$ provides a clear graphical representation of the iterative construction and convergence of the Cantor set. We choose $a = 1/27$ as an admissible contraction constant that satisfies the theoretical requirement. This value is chosen to visualize a clear numerical comparison between the two sides of the contraction inequality.

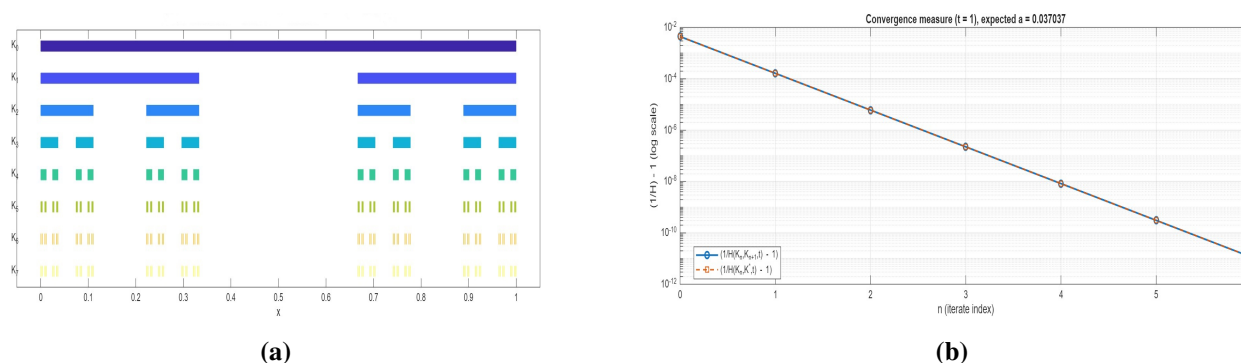


Figure 2. (a) Step-by-step construction of the Cantor set for the first seven iterations (top K_0 , bottom K_7): successive removal of the middle third of each subinterval in iterations; (b) numerical decay of the Hausdorff distance between consecutive iterates K_n and between K_n and the limit set K for $a = 1/27$ and $t = 1$.

6. Conclusions

In this work, we introduce and develop the framework of $EF\mathcal{F}$ -MSs, motivated to accommodate proximity structures governed by auxiliary functions that are nondecreasing but not necessarily strictly increasing or continuous. We analyze the basic topological properties of these spaces, including their convergence, Cauchy sequences, and the induced topology, and provide several examples illustrating the scope of the proposed framework. In this setting, we establish Banach-type fixed-point theorems for mappings satisfying adapted contractive conditions under the $EF\mathcal{F}$ -metric. These results not only generalize the classical Banach contraction principle but also unify and extend a wide class of fixed-point theorems in fuzzy metric and metric-type frameworks. To demonstrate the applicability of the proposed theory, we examine Hutchinson–Barnsley operators associated with IFS and establish that the $EF\mathcal{F}$ -metric provides an analytical framework for describing the convergence behavior of IFS-generated fractals. To validate the theoretical analysis, we provide numerical visualizations of the fractal construction on the Cantor set, implemented in MATLAB. Through iterative simulations and graphical plots, we illustrate how the $EF\mathcal{F}$ -Hutchinson–Barnsley operator converges to its unique fractal attractor, thereby validating the effectiveness of the extended framework.

Overall, $EF\mathcal{F}$ -MSs form a versatile foundation for further research in fixed point theory, nonlinear analysis, fractal geometry, fuzzy dynamic systems, and may provide a useful mathematical model for systems involving complex data structures, irregular patterns, and graded uncertainty in environmental systems, energy management, and essential resource allocation. One possible extension is to develop stochastic $EF\mathcal{F}$ -metric spaces and study stochastic fuzzy IFS with contraction mapping influenced by probabilistic perturbations or random selection mechanisms. Such a framework may give new existence and convergence results for random fractals and stochastic dynamical systems. Another promising direction concerns applications in image processing, particularly in image compression and segmentation based on fuzzy IFS. The present work is restricted to finite IFS consisting of a finite family of self-mappings. Extending the proposed framework to infinite families of mappings presents a challenging problem, since one must first establish that the induced Hutchinson operator remains well-defined on the hyperspace of closed and bounded subsets and preserves it. Moreover, deriving the inequalities for the Hausdorff

$EF\mathcal{F}$ -operator and analyzing the convergence of successive approximations may require additional assumptions. Consequently, the study of infinite $EF\mathcal{F}$ -IFS and the existence and uniqueness of attractors in this setting constitute promising topics for future investigation. Furthermore, the proposed framework may be extended to fractional fuzzy systems, where memory effects, uncertainty, and nonlocal interactions require more flexible mathematical structures. The generalized auxiliary function class introduced in this work, while preserving the essential convergence structure, may offer additional flexibility for analyzing fractional fuzzy differential and integral systems under relaxed contractive assumptions.

Author contributions

Abhishikta Das: conceptualization, methodology, formal analysis, investigation, writing - original draft preparation, writing – review and editing; Dipti Barman: methodology, formal analysis, investigation, writing – review and editing; Mohammad Sajid: conceptualization, methodology, supervision, writing – review and editing, funding acquisition; Tarapada Bag: conceptualization, methodology, formal analysis, supervision, writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

The Article Processing Charges (APC) was supported by the Deanship of Graduate Studies and Scientific Research at Qassim University (QU-APC-2026).

Acknowledgments

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (<https://www.qu.edu.sa>) for financial support (QU-APC-2026).

Conflict of interest

The authors affirm that there are no conflicts of interest to disclose.

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