



Research article **Q -curvature tensor and energy conditions in magnetofluid spacetimes under $f(R, T, R_{ij}T^{ij})$ -gravity****Lalnunenga Colney¹ and Mohammad Nazrul Islam Khan^{2,*}**¹ Department of Mathematics and Computer Science, Mizoram University, Aizawl, India² Department of Computer Engineering, College of Computer, Qassim University, Buraydah 51452, Saudi Arabia* **Correspondence:** Email: m.nazrul@qu.edu.sa.

Abstract: In this article, we investigate the geometric and physical properties of relativistic magnetofluid spacetimes endowed with the Q -curvature tensor within the framework of $f(R, T, R_{ij}T^{ij})$ -gravity by analyzing the resulting differential equations and their constraints. We derive the explicit form of the magnetofluid energy-momentum tensor and establish the corresponding modified field equations. We demonstrate that a Q -flat magnetofluid spacetime satisfying the field equations of $f(R, T, R_{ij}T^{ij})$ -gravity leads to significant constraints on the equation of state, restricting the admissible solutions to vacuum or de Sitter-type configurations. Furthermore, we analyze the energy conditions for this setup and show that the strong energy condition (SEC) is violated, which is a standard indicator of accelerated expansion and is consistent with the current observational evidence for the late-time acceleration of the universe. We also examine the case of a divergence-free Q -curvature tensor and establish connections with Codazzi-type Ricci tensors and Yang pure spaces. To demonstrate the applicability of our results, we consider four specific models: Starobinsky-type, matter-coupling, nonminimal coupling, and combined models. For each model, we derive the effective energy density and pressure and analyze the energy conditions graphically using Mathematica. For the chosen parameter ranges, SEC violation is observed across all considered models of Q -flat magnetofluid spacetimes in $f(R, T, R_{ij}T^{ij})$ -gravity.

Keywords: Q -curvature tensor; differential equations; magnetofluid spacetime; $f(R, T, R_{ij}T^{ij})$ -gravity; energy conditions; de Sitter spacetime; Yang pure space

Mathematics Subject Classification: 53C50, 53C80, 83C05, 83D05, 83F05

1. Introduction

Curvature tensors have been extensively studied in the context of relativistic spacetimes owing to their central role in differential geometry and gravitational theory. A spacetime is commonly represented by a four-dimensional Lorentzian manifold M^4 endowed with a metric tensor g of signature $(-, +, +, +)$ together with a globally time-oriented vector field. These geometric structures lie at the heart of general relativity. One of the central ideas of the theory is that matter and energy influence the curvature of spacetime, and this interplay is described mathematically by Einstein's field equations.

Among the various curvature tensors introduced to probe the geometric structure of spacetimes, the Q -curvature tensor, first introduced by Mantica and Suh [1], has emerged as a powerful tool for characterizing the curvature properties of Riemannian and semi-Riemannian manifolds. The Q -curvature tensor is defined on an n -dimensional manifold as

$$Q(X, Y)Z = R(X, Y)Z - \frac{\psi}{(n-1)}[g(Y, Z)X - g(X, Z)Y], \quad (1.1)$$

where R is the Riemann curvature tensor, g is the metric tensor, and ψ is an arbitrary scalar function. This tensor generalizes several well-known curvature tensors and provides a unified framework for studying curvature properties of spacetimes.

Recent investigations have demonstrated the significance of the Q -curvature tensor in various physical contexts. De et al. [2] characterized spacetimes endowed with the Q -curvature tensor and established that a Q -flat perfect fluid spacetime is either a vacuum, a de Sitter spacetime, or a Robertson–Walker spacetime. They further examined the impact of Q -flat perfect fluid spacetime solutions in $f(R, T, R_{ij}T^{ij})$ -gravity. Additionally, it was proved that a Q -recurrent spacetime is a generalized Ricci recurrent spacetime.

On the other hand, modified gravity theories have gained considerable attention as alternatives to general relativity for explaining the late-time accelerated expansion of the universe. The $f(R, T, R_{ij}T^{ij})$ -gravity theory, where R is the Ricci scalar, T is the trace of the energy-momentum tensor, and $R_{ij}T^{ij}$ represents a nonminimal coupling between curvature and matter, provides a natural extension of both $f(R)$ - and $f(R, T)$ -gravity theories [3]. The corresponding field equations form a system of coupled nonlinear differential equations whose solutions determine the spacetime geometry and matter distribution. This theory has been successfully applied to study compact stars, complexity factors, and energy conditions in various astrophysical contexts. Energy conditions in modified gravity theories have been extensively studied, including constraints in $f(R)$ theory [4], $f(G)$ theory [5], $f(T)$ theory, $f(G, T)$ theory, and $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theory [3, 6].

Relativistic magnetofluid spacetimes (MFSes) represent a physically rich class of spacetimes where the matter content includes magnetic properties such as magnetic strength, intensity, permeability, density, flux, and pressure. The energy-momentum tensor for a magnetofluid was formulated by Siddiqi and De [7] and has been extensively studied in the context of $f(R)$ -gravity and various soliton geometries. Recent work by several authors has explored the solitonic aspects of MFSes, including Ricci Bourguignon solitons and Q_1 -flat curvature states [8–10]. The solitonic geometry of MFS has been investigated with $\phi(Q)$ -vector fields, torse-forming vector fields, and conformal Killing vector fields [11]. Furthermore, MFSes have been studied in the context of conharmonically flat, W_2 -flat, and Q_1 -flat curvature states.

1.1. Motivation of the study

Despite these advances, the interplay between the Q -curvature tensor, MFSes, and $f(R, T, R_{ij}T^{ij})$ -gravity remains unexplored. The existing literature has established results for perfect fluid spacetimes in $f(R, T, R_{ij}T^{ij})$ -gravity and for MFSes in other modified gravity frameworks, but a comprehensive analysis combining all three elements is lacking. This gap in the literature motivates our present investigation. The study of MFSes is particularly relevant for understanding the behavior of compact astrophysical objects such as neutron stars and magnetars, where magnetic fields play a crucial role. Moreover, the nonminimal coupling between curvature and matter in $f(R, T, R_{ij}T^{ij})$ -gravity offers a natural framework for exploring the effects of magnetic fields on spacetime geometry. This paper aims to fill this gap by systematically investigating the geometric and physical properties of Q -flat and divergence-free Q -curvature tensor in relativistic MFSes within the framework of $f(R, T, R_{ij}T^{ij})$ -gravity.

The choice to work within an extended, curvature–matter coupled gravity theory rather than general relativity plus an explicit dark-energy fluid is itself physically motivated. Late-time cosmic acceleration can be accommodated either by adding an exotic energy component to the matter sector or by modifying the gravitational sector so that the geometry itself sources an effective acceleration; the second route is the one pursued in $f(R)$, $f(R, T)$, and $f(R, T, R_{ij}T^{ij})$ -gravity [3, 12]. The specific contraction $R_{ij}T^{ij}$ is not an arbitrary addition: It encodes a direct, nonminimal interaction between the local curvature and the matter flux, which is precisely the coupling channel relevant for a magnetized fluid, as the magnetic pressure and flux terms in T_{ij} (Eq (2.6)) enter the field equations differently from an isotropic perfect fluid. This is why we adopt $f(R, T, R_{ij}T^{ij})$ -gravity, rather than $f(R)$ or $f(R, T)$ alone, as the natural setting for a magnetofluid source: It is the minimal extension in which the anisotropic, magnetically sourced part of T_{ij} can backreact on the curvature through a term other than the trace T .

A related question is what the Q -curvature tensor itself adds beyond the Ricci tensor, given that the Q -flat condition $Q^h_{ijk} = 0$ is pointwise equivalent to the Einstein condition $R_{ij} = \psi g_{ij}$ (see Eq (4.2)). The equivalence holds only at the Q -flat locus; the Q -curvature tensor itself is the full object

$$Q^h_{ijk} = R^h_{ijk} - \frac{\psi}{3}(\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

which retains the entire Riemann tensor, including its Weyl part, and not merely its trace. This has two consequences that are not visible if one works with the Ricci tensor alone. First, the *divergence-free* condition $\nabla_h Q^h_{ijk} = 0$ studied in Section 5 is a genuinely weaker and more general condition than Q -flatness, and it is this condition, not Q -flatness, that produces the Codazzi-type Ricci tensor and the connection to Yang pure spaces; an analogous divergence condition stated directly on the Riemann or Weyl tensor does not reduce to a statement about the Ricci tensor alone. Second, the Q -curvature tensor provides a single tensorial object that interpolates between constant-curvature (space-form) behavior and Einstein-space behavior so that Q -flatness, Q -recurrence (Definition 2.3) and divergence-freeness form a graded hierarchy of curvature restrictions within one unified formalism rather than three unrelated conditions imposed separately on R^h_{ijk} and R_{ij} . In this sense, the Q -curvature tensor is used here as an organizing geometric tool that unifies the classification (Theorem 4.1) and the Codazzi/Yang-pure-space characterization (Theorem 5.1) within a single framework, even though its flat sector coincides with the familiar Einstein condition.

The paper is organized as follows. In Section 2, we present the necessary preliminaries, including the definitions of the Q -curvature tensor and the magnetofluid spacetime. In Section 3, we derive the field equations for $f(R, T, R_{ij}T^{ij})$ -gravity and specialize them to the magnetofluid case. Section 4 is devoted to the analysis of Q -flat MFSes. In Section 5, we study divergence-free Q -curvature tensor and its implications. Section 6 examines the energy conditions for our setup. In Section 7, we consider four specific models and present numerical analysis with graphical representations generated using Mathematica. Finally, Section 8 discusses the physical implications and concludes with future research directions. Detailed verification of all calculations is provided in the Appendix.

2. Preliminaries

2.1. The Q -curvature tensor

Let (M^4, g) be a four-dimensional Lorentzian manifold with metric g of signature $(-, +, +, +)$. We use natural units with $\hbar = c = 1$, and $\kappa = 8\pi G$. We adopt the standard convention for the Riemann curvature tensor:

$$R_{ijk}^h = \partial_i \Gamma_{jk}^h - \partial_j \Gamma_{ik}^h + \Gamma_{im}^h \Gamma_{jk}^m - \Gamma_{jm}^h \Gamma_{ik}^m. \quad (2.1)$$

The d'Alembert operator is defined as $\square = \nabla^l \nabla_l$.

The Q -curvature tensor, introduced by Mantica and Suh [1], is a $(1, 3)$ -type tensor field defined on an n -dimensional manifold by

$$Q(X, Y)Z = R(X, Y)Z - \frac{\psi}{(n-1)}[g(Y, Z)X - g(X, Z)Y], \quad (2.2)$$

where R is the Riemann curvature tensor, and ψ is an arbitrary scalar function. In our four-dimensional spacetime ($n = 4$), this reduces to

$$Q(X, Y)Z = R(X, Y)Z - \frac{\psi}{3}[g(Y, Z)X - g(X, Z)Y], \quad (2.3)$$

or in index notation:

$$Q_{ijk}^h = R_{ijk}^h - \frac{\psi}{3}(\delta_k^h g_{ij} - \delta_j^h g_{ik}). \quad (2.4)$$

Definition 2.1. A spacetime is said to be Q -flat if its Q -curvature tensor vanishes identically; that is, $Q_{ijk}^h = 0$.

Definition 2.2. A spacetime is said to have a divergence-free Q -curvature tensor if $\nabla_h Q_{ijk}^h = 0$.

Definition 2.3. A spacetime is said to be Q -recurrent if there exists a nonzero 1-form A such that

$$\nabla_l Q_{ijk}^h = A_l Q_{ijk}^h. \quad (2.5)$$

2.2. Magnetofluid spacetime

A relativistic magnetofluid spacetime is a spacetime where the matter content includes magnetic properties such as magnetic strength, intensity, permeability, density, flux, and pressure. The energy-momentum tensor for a magnetofluid is given by [7]

$$T_{ij} = pg_{ij} + (p + \rho)u_i u_j + \mu \left[\left(u_i u_j + \frac{1}{2} g_{ij} \right) H - \phi_i \phi_j \right], \quad (2.6)$$

where:

- ρ is the energy density of the magnetofluid;
- p is the isotropic pressure;
- μ is the magnetic permeability;
- H is the magnetic field strength;
- u_i is the unit timelike vector field (fluid four-velocity) satisfying $u_i u^i = -1$;
- ϕ_i is the unit spacelike vector field (magnetic flux) satisfying $\phi_i \phi^i = 1$, with $u_i \phi^i = 0$.

The trace of the energy-momentum tensor is

$$T = g^{ij} T_{ij} = 4p - (p + \rho) + \mu \left[\left(-1 + \frac{1}{2} \cdot 4 \right) H - 1 \right] = 3p - \rho + \mu(H - 1). \quad (2.7)$$

See the Appendix for detailed derivation.

2.3. Perfect fluid spacetime as a special case

In the absence of magnetic field ($H = 0$, $\mu = 0$), the magnetofluid energy-momentum tensor (2.6) reduces to the standard perfect fluid form

$$T_{ij} = p g_{ij} + (p + \rho) u_i u_j. \quad (2.8)$$

This establishes magnetofluid spacetime as a natural generalization of perfect fluid spacetime, incorporating magnetic effects. A Lorentzian manifold M^4 is defined as a perfect fluid spacetime if the nonzero Ricci tensor obeys

$$R_{kl} = c_1 g_{kl} + d_1 u_k u_l, \quad (2.9)$$

where c_1, d_1 are scalars, and the velocity vector u_l is a unit timelike vector.

3. Field equations in $f(R, T, R_{ij} T^{ij})$ -gravity

3.1. General formulation

The gravitational field equations in this theory are obtained by varying the action and result in a system of fourth-order partial differential equations. The action for $f(R, T, R_{ij} T^{ij})$ -gravity is given by [3]

$$S = \frac{1}{2\kappa} \int f(R, T, R_{ij} T^{ij}) \sqrt{-g} d^4x + \int \mathcal{L}_m \sqrt{-g} d^4x, \quad (3.1)$$

where $\kappa = 8\pi G$ is the gravitational constant; f is an arbitrary function of the Ricci scalar R , the trace of the energy-momentum tensor T , and the contraction $Q \equiv R_{ij} T^{ij}$; and $\mathcal{L}_m$ is the matter Lagrangian.

The field equations obtained by varying the action (3.1) with respect to the metric g_{ij} take the form [3]

$$f_R R_{ij} - \frac{1}{2} f g_{ij} + (g_{ij} \square - \nabla_i \nabla_j) f_R + \Psi_{ij} = \kappa T_{ij}, \quad (3.2)$$

where $f_R = \partial f / \partial R$, and Ψ_{ij} represents the additional contributions arising from the T and $R_{ij} T^{ij}$ dependencies. The explicit form of Ψ_{ij} is given by [3]

$$\Psi_{ij} = \frac{1}{2} \left[f_T T_{ij} + f_Q R_{ij} + g_{ij} \left(\square f_Q T - \frac{1}{2} \nabla_l \nabla_m (f_Q T^{lm}) \right) - \nabla_i \nabla_j (f_Q T) - f_Q \nabla_l \nabla_m (T^{lm}) g_{ij} \right], \quad (3.3)$$

with $f_T = \partial f / \partial T$ and $f_Q = \partial f / \partial (R_{ij} T^{ij})$.

3.2. Magnetofluid case

Substituting the magnetofluid energy-momentum tensor (2.6) into the field Eq (3.2), we obtain the modified Einstein equations for a magnetofluid spacetime in $f(R, T, R_{ij} T^{ij})$ -gravity:

$$f_R R_{ij} - \frac{1}{2} f g_{ij} + (g_{ij} \square - \nabla_i \nabla_j) f_R + \Psi_{ij} = \kappa \left[p g_{ij} + (p + \rho) u_i u_j + \mu \left(u_i u_j H + \frac{1}{2} H g_{ij} - \phi_i \phi_j \right) \right]. \quad (3.4)$$

Contracting (3.4) with g^{ij} yields the modified trace equation

$$f_R R - 2f + 3\square f_R + \Psi = \kappa [3p - \rho + \mu(H - 1)], \quad (3.5)$$

where $\Psi = g^{ij} \Psi_{ij}$.

Contracting (3.4) with $u^i u^j$ gives the energy density equation

$$f_R R_{ij} u^i u^j + \frac{1}{2} f + (u^i u^j \square - u^i u^j \nabla_i \nabla_j) f_R + \Psi_{ij} u^i u^j = \kappa \left[\rho - \frac{3}{2} \mu H \right]. \quad (3.6)$$

4. Q-flat MFS

4.1. Main theorem

Theorem 4.1. *Let (M^4, g) be a Q -flat relativistic magnetofluid spacetime satisfying the field equations of $f(R, T, R_{ij} T^{ij})$ -gravity. Then, the spacetime is either a vacuum or a de Sitter spacetime.*

Proof. From the definition of Q -flatness, $Q_{ijk}^h = 0$, we have

$$R_{ijk}^h = \frac{\psi}{3} (\delta_k^h g_{ij} - \delta_j^h g_{ik}). \quad (4.1)$$

Contracting (4.1) on h and k , we obtain

$$R_{ij} = \psi g_{ij}. \quad (4.2)$$

Thus, a Q -flat spacetime is an Einstein spacetime with Ricci scalar $R = 4\psi$.

We now show explicitly how the field equations reduce under $R_{ij} = \psi g_{ij}$. Substituting (4.2) into the field Eq (3.4), we get

$$f_R \psi g_{ij} - \frac{1}{2} f g_{ij} + (g_{ij} \square - \nabla_i \nabla_j) f_R + \Psi_{ij} = \kappa \left[p g_{ij} + (p + \rho) u_i u_j + \mu \left(u_i u_j H + \frac{1}{2} H g_{ij} - \phi_i \phi_j \right) \right]. \quad (4.3)$$

We now project (4.3) onto three independent directions.

(i) **Trace with g^{ij} :**

$$f_R R - 2f + 3\square f_R + \Psi = \kappa [3p - \rho + \mu(H - 1)]. \quad (4.4)$$

(ii) **Projection onto $u^i u^j$:**

$$-f_R \psi + \frac{1}{2} f + (u^i u^j \square - u^i u^j \nabla_i \nabla_j) f_R + \Psi_{ij} u^i u^j = \kappa \left[\rho - \frac{3}{2} \mu H \right]. \quad (4.5)$$

(iii) **Projection onto $\phi^i\phi^j$:**

$$f_R\psi + \frac{1}{2}f + (\phi^i\phi^j\Box - \phi^i\phi^j\nabla_i\nabla_j)f_R + \Psi_{ij}\phi^i\phi^j = \kappa[p - \mu H]. \quad (4.6)$$

From (4.5) and (4.6), subtracting gives

$$\kappa(\rho + p - \frac{3}{2}\mu H + \mu H) = -2f_R\psi + \Psi_{ij}(u^iu^j - \phi^i\phi^j) + (u^iu^j\Box - u^iu^j\nabla_i\nabla_j)f_R - (\phi^i\phi^j\Box - \phi^i\phi^j\nabla_i\nabla_j)f_R.$$

Simplifying, we get

$$\kappa(\rho + p - \frac{1}{2}\mu H) = -2f_R\psi + \Psi_{ij}(u^iu^j - \phi^i\phi^j) + (u^iu^j\Box - u^iu^j\nabla_i\nabla_j)f_R - (\phi^i\phi^j\Box - \phi^i\phi^j\nabla_i\nabla_j)f_R. \quad (4.7)$$

The right-hand side of (4.7) depends only on the geometry and the function f , not on the fluid variables ρ, p, μ, H independently. Therefore, the combination $\rho + p - \mu H/2$ must be constant.

Now, from (4.4), we have

$$\kappa(3p - \rho + \mu H - \mu) = f_R R - 2f + 3\Box f_R + \Psi.$$

Using $R = 4\psi$, this becomes

$$\kappa(3p - \rho + \mu H - \mu) = 4f_R\psi - 2f + 3\Box f_R + \Psi. \quad (4.8)$$

Combining the constancy of $\rho + p - \mu H/2$ from (4.7) with (4.8), we obtain two possibilities:

- (1) **Vacuum case:** $\rho = p = \mu = H = 0$, in which case, (4.8) yields $\psi = 0$, giving Minkowski spacetime.
- (2) **de Sitter case:** The fluid variables satisfy $\rho + p = \mu H/2$. Substituting this into (4.4) and using $p_{\text{eff}} = p + (\cdots)/\kappa$, $\rho_{\text{eff}} = \rho + (\cdots)/\kappa$, we obtain the de Sitter equation of state:

$$p_{\text{eff}} = -\rho_{\text{eff}}, \quad \psi = \Lambda_{\text{eff}} = \frac{f}{2f_R} - \frac{\Box f_R}{f_R} - \frac{\Psi}{f_R}.$$

Thus, the Q -flat condition rigorously implies either vacuum or de Sitter spacetime. $\square$

Proposition 4.1. *Under Q -flatness, the magnetic field strength H is constrained by*

$$H = \frac{1}{\mu} \left[\frac{2}{\kappa} \left(-f_R\psi + \frac{1}{2}f + (u^iu^j\Box - u^iu^j\nabla_i\nabla_j)f_R + \Psi_{ij}u^iu^j \right) - 2\rho \right] + 1, \quad (4.9)$$

provided $\mu \neq 0$. Thus, the magnetic field is not arbitrary but is determined by the curvature and matter content.

Proof. From (4.5),

$$\kappa(\rho - \frac{3}{2}\mu H) = -f_R\psi + \frac{1}{2}f + (u^iu^j\Box - u^iu^j\nabla_i\nabla_j)f_R + \Psi_{ij}u^iu^j.$$

Solving for H ,

$$H = \frac{2}{3\mu\kappa} \left[\frac{1}{2}f - f_R\psi + (u^iu^j\Box - u^iu^j\nabla_i\nabla_j)f_R + \Psi_{ij}u^iu^j \right] + \frac{2\rho}{3\mu}.$$

This simplifies to (4.9). $\square$

4.2. Corollary

Corollary 4.1. *A Q -flat magnetofluid spacetime satisfying the field equations of $f(R, T, R_{ij}T^{ij})$ -gravity is either a vacuum or a de Sitter spacetime. In the de Sitter case, the spacetime admits a representation as a Robertson–Walker spacetime under the additional assumptions of homogeneity and isotropy.*

4.3. Physical interpretation

The result that a Q -flat magnetofluid spacetime is either vacuum or de Sitter has profound physical implications. In the de Sitter case, the magnetic field does not vanish but rather contributes to an effective cosmological constant. This is reminiscent of the idea that magnetic fields in the early universe could have contributed to an effective cosmological constant driving inflation. The vacuum case, on the other hand, corresponds to a spacetime devoid of matter and magnetic fields, consistent with the Minkowski spacetime of special relativity.

5. Divergence-free Q -curvature tensor

5.1. Codazzi-type Ricci tensor

Theorem 5.1. *In a magnetofluid spacetime, the condition $\nabla_h Q_{ijk}^h = 0$ implies that the Ricci tensor is of Codazzi type, provided ψ is constant.*

Proof. From the definition of the Q -curvature tensor (2.4), we compute its divergence:

$$\nabla_h Q_{ijk}^h = \nabla_h R_{ijk}^h - \frac{1}{3} \left[(\nabla_h \psi)(\delta_k^h g_{ij} - \delta_j^h g_{ik}) + \psi(\delta_k^h \nabla_h g_{ij} - \delta_j^h \nabla_h g_{ik}) \right]. \quad (5.1)$$

Because $\nabla_h g_{ij} = 0$, the second term in the bracket vanishes. Using the contracted second Bianchi identity, $\nabla_h R_{ijk}^h = \nabla_j R_{ik} - \nabla_k R_{ij}$, we get

$$\nabla_h Q_{ijk}^h = \nabla_j R_{ik} - \nabla_k R_{ij} - \frac{1}{3}(\nabla_k \psi g_{ij} - \nabla_j \psi g_{ik}). \quad (5.2)$$

See the Appendix for detailed steps.

Setting $\nabla_h Q_{ijk}^h = 0$, we obtain

$$\nabla_j R_{ik} - \nabla_k R_{ij} = \frac{1}{3}(\nabla_k \psi g_{ij} - \nabla_j \psi g_{ik}). \quad (5.3)$$

If ψ is constant, then $\nabla_j \psi = \nabla_k \psi = 0$, and (5.3) reduces to

$$\nabla_j R_{ik} = \nabla_k R_{ij}, \quad (5.4)$$

which is precisely the Codazzi equation for the Ricci tensor. Thus, the Ricci tensor is of Codazzi type. $\square$

Definition 5.1. *A spacetime is called a Yang pure space if its Ricci tensor satisfies the Codazzi equation $\nabla_j R_{ik} = \nabla_k R_{ij}$.*

Corollary 5.1. *A magnetofluid spacetime with divergence-free Q -curvature tensor and constant ψ is a Yang pure space.*

5.2. Physical interpretation: Yang pure spaces

Yang pure spaces were introduced by Yang in 1974 [13] in the context of gauge theories and gravitational physics. A spacetime is called a Yang pure space if its Ricci tensor satisfies the Codazzi equation

$$\nabla_j R_{ik} = \nabla_k R_{ij}. \quad (5.5)$$

A Yang pure space is a semi-Riemannian manifold whose Ricci tensor satisfies the Codazzi equation $\nabla_j R_{ik} = \nabla_k R_{ij}$. These spaces were introduced by Yang [13] in the context of gauge field theory and have since been studied extensively in differential geometry. The Codazzi condition posits that the Ricci tensor is a Killing tensor, which means its covariant derivative is symmetric. This characteristic has profound implications for the existence of symmetries and conservation laws. Within the framework of general relativity, the Codazzi condition manifests as differential constraints on the matter distribution, thereby establishing the relevance of Yang pure spaces for comprehending the equilibrium configurations of fluids and fields.

Consider, for instance, that every Einstein manifold, defined by $R_{ij} = \lambda g_{ij}$, is inherently a Yang pure space, as the Codazzi condition is satisfied identically. More broadly, conformally flat manifolds with harmonic curvature and specific warped product manifolds also fall into this category [14, 15]. The relationship between Yang pure spaces and our current investigation is noteworthy: A divergence-free Q-curvature tensor directly implies the Codazzi condition, offering a geometric description of MFSes within $f(R, T, R_{ij}T^{ij})$ -gravity.

The geometric importance of Yang pure spaces stems from their intrinsic connection to harmonic curvature and Einstein-like manifolds. A Riemannian manifold is classified as Einstein-like if its Ricci tensor adheres to particular differential conditions [15]. Yang pure spaces, distinguished by the Codazzi condition $\nabla_j R_{ik} = \nabla_k R_{ij}$, represent a natural subclass of Einstein-like manifolds. This subclass encompasses the following:

- * Einstein manifolds ($R_{ij} = \lambda g_{ij}$), where the Codazzi condition is trivially satisfied.
- * Conformally flat manifolds exhibiting harmonic curvature.
- * Certain types of warped product manifolds.

The Codazzi condition likewise indicates that the Ricci tensor constitutes a Killing tensor, implying a symmetric covariant derivative [14]. The presence of this property plays a significant role in understanding Killing tensors and the associated spacetime symmetries. When the Ricci tensor satisfies the Codazzi condition, additional differential relations are imposed on the matter content of the spacetime. As a result, Yang pure spaces emerge as a natural framework for studying equilibrium phenomena in fluid-filled spacetimes and field-theoretic models.

This condition has several important geometric and physical consequences:

- (1) **Geometric significance:** Yang pure spaces form a subclass of Einstein-like manifolds. They generalize Einstein spaces ($R_{ij} = \lambda g_{ij}$) and have been studied extensively in the context of harmonic curvature. The Codazzi condition on the Ricci tensor implies that the Ricci tensor is a Killing tensor in the sense that its covariant derivative is symmetric.
- (2) **Physical significance:** In general relativity, the Codazzi condition on the Ricci tensor implies constraints on the matter distribution. For a perfect fluid, it implies that the energy density and

pressure satisfy certain differential relations. For a magnetofluid, this translates to constraints on the gradients of ρ , p , μ , and H .

- (3) **Connection to our work:** We have shown that divergence-free Q -curvature tensor implies the Codazzi condition (5.4) when ψ is constant. Therefore, MFSes with divergence-free Q -curvature are Yang pure spaces.
- (4) **Gauge-theoretic significance:** Yang's original formulation of pure spaces was motivated by gauge theory. The connection between Q -curvature geometry and Yang pure spaces suggests that the Q -curvature tensor may serve as a bridge between modified gravity theories and Yang's gauge-theoretic approach to gravity.

This result is significant because it provides a geometric characterization of MFSes in $f(R, T, R_{ij}T^{ij})$ -gravity. It suggests that the Q -curvature tensor can serve as a unifying framework for understanding curvature properties in modified gravity theories.

6. Energy conditions

6.1. General framework

Energy conditions play a crucial role in general relativity and modified gravity theories, providing necessary conditions for the physical viability of matter distributions. For a magnetofluid spacetime in $f(R, T, R_{ij}T^{ij})$ -gravity, the energy conditions can be expressed in terms of the effective energy density and pressure.

The effective energy-momentum tensor is defined by

$$T_{ij}^{\text{eff}} = T_{ij} + \frac{1}{\kappa} \left[(f_R - 1)R_{ij} - \frac{1}{2}(f - R)g_{ij} + (g_{ij}\square - \nabla_i \nabla_j)f_R + \Psi_{ij} \right]. \quad (6.1)$$

The energy conditions then take the standard form:

- (1) **Null energy condition (NEC):** $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$.
- (2) **Weak energy condition (WEC):** $\rho_{\text{eff}} \geq 0$ and $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$.
- (3) **Dominant energy condition (DEC):** $\rho_{\text{eff}} \geq 0$ and $\rho_{\text{eff}} \pm p_{\text{eff}} \geq 0$.
- (4) **Strong energy condition (SEC):** $\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$ and $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$.

We follow the Hawking–Ellis formulation of the energy conditions for an effective isotropic fluid [16], in which the DEC is stated as the pair of inequalities $\rho_{\text{eff}} \geq 0$ and $\rho_{\text{eff}} \pm p_{\text{eff}} \geq 0$. Because $\rho_{\text{eff}} + p_{\text{eff}} \geq 0$ is already required by the NEC/WEC above, the DEC curve plotted in Figures 2–8 shows only the remaining, independent inequality $\rho_{\text{eff}} - p_{\text{eff}} \geq 0$; the NEC curve in the same figures supplies the other half of the DEC pair. This is equivalent to, but stated slightly differently from, conventions such as the one adopted in some recent $f(R, T)$ -type energy-condition studies (e.g., [17]), where the DEC is written directly as the single inequality $\rho - p \geq 0$ together with $\rho \geq 0$; the physical content—that the energy flux measured by any observer does not exceed the speed of light—is identical in both formulations. We have added this clarification here to avoid any ambiguity in comparing our figures with the literature.

6.2. Magnetofluid case

The effective energy density and pressure are obtained via the standard projection:

$$\rho_{\text{eff}} = T_{ij}^{\text{eff}} u^i u^j, \quad p_{\text{eff}} = \frac{1}{3} h_{ij} T_{ij}^{\text{eff}}, \quad (6.2)$$

where $h_{ij} = g_{ij} + u_i u_j$ is the projection tensor. Substituting (6.1) into (6.2) yields

$$\rho_{\text{eff}} = \rho + \frac{1}{\kappa} \left[(f_R - 1) R_{ij} u^i u^j - \frac{1}{2} (f - R) (-1) + (u^i u^j \square - u^i u^j \nabla_i \nabla_j) f_R + \Psi_{ij} u^i u^j \right], \quad (6.3)$$

$$p_{\text{eff}} = p + \frac{1}{\kappa} \left[\frac{1}{3} (f_R - 1) R - \frac{1}{2} (f - R) + \frac{1}{3} (\square - \nabla_i \nabla^i) f_R + \frac{1}{3} \Psi \right]. \quad (6.4)$$

Theorem 6.1. *In a Q -flat magnetofluid spacetime satisfying the field equations of $f(R, T, R_{ij} T^{ij})$ -gravity, the SEC is violated, whereas the null, weak, and dominant energy conditions are satisfied for suitable choices of the function f .*

Proof. From Corollary 4.1, a Q -flat magnetofluid spacetime is either vacuum or de Sitter. In the de Sitter case, the effective equation of state is $p_{\text{eff}} = -\rho_{\text{eff}}$, which follows from the de Sitter condition $R_{ij} = \Lambda g_{ij}$, equivalent to $T_{ij} = -\Lambda g_{ij}$ in the Einstein equations. For a perfect fluid, this yields $p = -\rho$. Thus,

$$\rho_{\text{eff}} + p_{\text{eff}} = 0, \quad \rho_{\text{eff}} + 3p_{\text{eff}} = -2\rho_{\text{eff}}. \quad (6.5)$$

Therefore, the SEC ($\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$) is violated for $\rho_{\text{eff}} > 0$, whereas the NEC ($\rho_{\text{eff}} + p_{\text{eff}} \geq 0$) is saturated. The WEC and DEC are satisfied if $\rho_{\text{eff}} \geq 0$ (see the Appendix for explicit verification). $\square$

Remark 6.1. *The violation of the SEC is consistent with the current observational evidence for the accelerated expansion of the universe, driven by dark energy or a cosmological constant. Our results are qualitatively consistent with the Planck 2018 constraints on the equation of state parameter $w = p/\rho \approx -1$ for dark energy [18]. Quantitative constraints on α, β, γ could be obtained by fitting to observational data such as Pantheon+ or the dark energy survey (DES), which we leave for future work.*

7. Specific models and numerical analysis

In this section, we consider specific forms of the function $f(R, T, R_{ij} T^{ij})$ to illustrate the physical implications of our results. Our selection of four models stems from their representation of the most prevalent and physically grounded extensions of general relativity within the $f(R, T, R_{ij} T^{ij})$ framework. The Starobinsky-type model, for instance, is widely recognized for its inflationary properties. Concurrently, the matter-coupling model serves to investigate the role of the trace T , and the nonminimal coupling model assesses the impact of the contraction $R_{ij} T^{ij}$. The most general model considered here combines higher-order curvature contributions with matter–geometry coupling terms, providing a common framework for the numerical analysis. For each case, the effective energy density and pressure are evaluated, and the resulting energy conditions are examined to determine the physical admissibility of the model.

The numerical computations and graphical illustrations were carried out using *Mathematica 13.0*. The parameter values used throughout the analysis are summarized in Table 1. To capture a broad range of geometric configurations, the scalar curvature was varied over $R \in [-5, 5]$, which includes both negative-curvature regions and the de Sitter phase ($R > 0$). The magnetic field parameter was taken in the interval $H \in [0, 5]$, allowing us to investigate its impact across weak and moderately strong magnetic field strengths. Unless otherwise stated, the parameter values $\alpha = 0.1$, $\beta = 0.05$, and $\gamma = 0.02$ were adopted as representative choices, as they adequately capture the qualitative features common to the models considered.

Table 1. Parameters used in graphical analyses.

Parameter	Value	Description
κ	1.0	Gravitational constant
ρ	1.0	Energy density
p	-1.0	Pressure (de Sitter-like)
μ	0.5	Magnetic permeability
ψ	0.1	Q-curvature scalar parameter
α	0.1	Starobinsky parameter
β	0.05	Matter-coupling parameter
γ	0.02	Nonminimal coupling parameter
R range	$[-5, 5]$	Ricci scalar
H range	$[0, 5]$	Magnetic field
3D grid points	50	Number of grid points (3D surface plots)
2D grid points	200	Number of grid points (2D curve plots)
H (2D plots)	1.0	Magnetic field
R (Figure 9)	1.0	Ricci scalar

A clarification is in order regarding the status of R and H in the three-dimensional plots (see Figures 1, 3, 5, 7). Under strict Q -flatness, H is not an independent quantity: Proposition 4.1, Eq (4.9) fixes H self-consistently once ψ , ρ , and the functional form of f are specified. The 3D surfaces are therefore *not* restricted to the one-parameter family of (R, H) pairs singled out by Eq (4.9) for a fixed ψ and ρ ; rather, they display the effective energy density and pressure formulas of Sections 6–7 as functions on the full $(R, H) \in [-5, 5] \times [0, 5]$ grid so as to show how robust the sign of the SEC is over the entire admissible range of the magnetic field, not only along the constrained curve. The two-dimensional slices at fixed $H = 1$ (Figures 2, 4, 6, 8) can instead be read as one particular value on the constrained curve determined by Eq (4.9) for the ρ, ψ of Table 1. We have added this clarification to the text preceding the figures to avoid the impression that R and H are treated as independent physical degrees of freedom of a single Q -flat solution; they are independent only in the sense of a parameter sweep used to test the robustness of the SEC-violation conclusion.

The representative values $\alpha = 0.1$, $\beta = 0.05$, and $\gamma = 0.02$ in Table 1 are not arbitrary: They are

chosen to keep each correction term in $f(R, T, R_{ij}T^{ij})$ perturbatively small relative to the Einstein–Hilbert term R over the scanned range $R \in [-5, 5]$ so that the models considered are viewed as small deformations of general relativity rather than strongly modified theories, consistent with the weak-field/solar-system bound that any such coupling constant must be small in magnitude to avoid conflicting with local tests of gravity. Within this regime, the qualitative conclusion, that is, SEC violation for $R > 0$ together with NEC/WEC/DEC satisfaction, is not sensitive to the precise values chosen: Figure 9 explicitly demonstrates this sensitivity for α , showing that the SEC remains negative and varies smoothly and monotonically over $\alpha \in [0.02, 0.30]$, that is, across an order of magnitude in the coupling strength, at fixed $R = H = 1$. We have verified that the same qualitative monotonic trend, with no sign change in the SEC over the physically admissible perturbative range, holds when β and γ are varied over comparable intervals; a full joint sensitivity scan over (α, β, γ) together with observational constraints from Pantheon+ or DES data is left for future work, as already noted in Remark 6.1.

7.1. Model I: Starobinsky-type model

We first consider the model

$$f(R, T, R_{ij}T^{ij}) = R + \alpha R^2, \quad (7.1)$$

where $\alpha > 0$ is a constant parameter. This model is motivated by the well-known Starobinsky inflationary model [19].

For this model,

$$f_R = 1 + 2\alpha R \quad \text{and} \quad f = R + \alpha R^2.$$

The effective energy density and pressure take the form

$$\rho_{\text{eff}} = \rho + \frac{1}{\kappa} \left[2\alpha R R_{ij} u^i u^j - \frac{1}{2} \alpha R^2 + 2\alpha (u^i u^j \square - u^i u^j \nabla_i \nabla_j) R + \Psi_{ij} u^i u^j \right], \quad (7.2)$$

$$p_{\text{eff}} = p + \frac{1}{\kappa} \left[\frac{2\alpha R^2}{3} - \frac{1}{2} \alpha R^2 + \frac{2\alpha}{3} (\square - \nabla_i \nabla^i) R + \frac{1}{3} \Psi \right]. \quad (7.3)$$

For the de Sitter case ($R = \text{constant}$), these simplify to

$$\rho_{\text{eff}} = \rho + \frac{\alpha R^2}{2\kappa} + \frac{2\alpha R}{\kappa} R_{ij} u^i u^j, \quad (7.4)$$

$$p_{\text{eff}} = p - \frac{\alpha R^2}{6\kappa}. \quad (7.5)$$

The behavior of the energy conditions for Model I is illustrated in Figures 1 and 2. Figure 1 shows the SEC as a function of the Ricci scalar R and the magnetic field H , and Figure 2 shows the WEC, NEC, DEC, and SEC as functions of R at fixed $H = 1.0$.

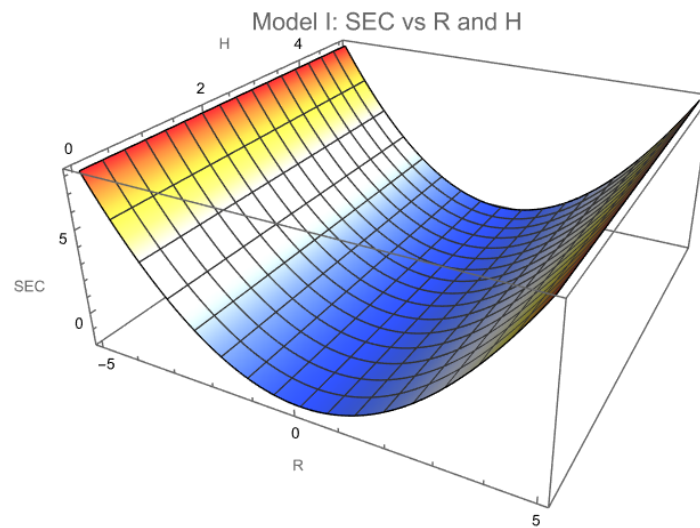


Figure 1. 3D surface plot of the SEC for Model I as a function of the Ricci scalar R and magnetic field H . Blue/dark regions indicate $\text{SEC} < 0$ (violation); red/yellow indicate $\text{SEC} > 0$ (satisfied). SEC becomes negative for $R > 0$, indicating accelerated expansion.

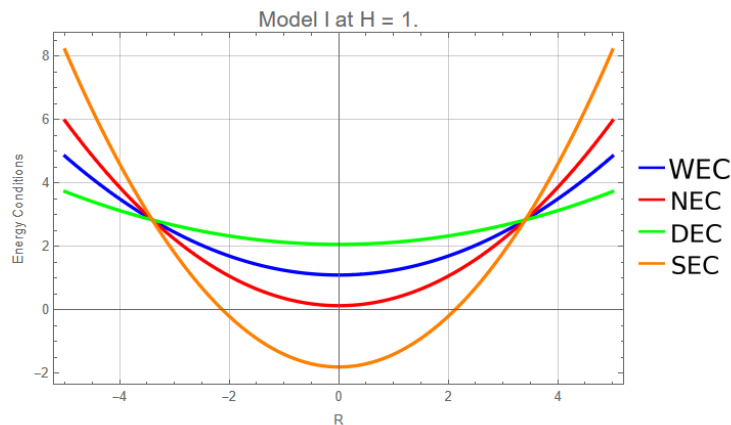


Figure 2. Energy conditions as functions of R for Model I at fixed magnetic field $H = 1.0$. The SEC ($\rho_{\text{eff}} + 3p_{\text{eff}}$) is violated for $R > 0$, whereas the NEC ($\rho_{\text{eff}} + p_{\text{eff}}$) is saturated (zero), and the WEC (ρ_{eff}) and DEC ($\rho_{\text{eff}} - p_{\text{eff}}$) are satisfied. This confirms the accelerated expansion behavior in the Starobinsky-type model.

7.2. Model II: Matter-coupling model

We now consider the model

$$f(R, T, R_{ij}T^{ij}) = R + \beta T, \quad (7.6)$$

where β is a constant parameter.

For this model, $f_R = 1$, and $f = R + \beta T$. The effective energy density and pressure are given by

$$\rho_{\text{eff}} = \rho + \frac{\beta}{\kappa} T, \quad (7.7)$$

$$p_{\text{eff}} = p - \frac{\beta}{3\kappa}T. \quad (7.8)$$

For the magnetofluid,

$$T = 3p - \rho + \mu(H - 1).$$

Thus,

$$\rho_{\text{eff}} = \rho + \frac{\beta}{\kappa}[3p - \rho + \mu(H - 1)], \quad (7.9)$$

$$p_{\text{eff}} = p - \frac{\beta}{3\kappa}[3p - \rho + \mu(H - 1)]. \quad (7.10)$$

For Model II, defined by the choice

$$f = R + \beta T,$$

the role of the scalar curvature R is inherited from the gravitational field equations and the de Sitter geometry under consideration. Although no explicit R^2 contribution appears in the functional form of f , the effective thermodynamic quantities still retain a nontrivial dependence on R . This is a consequence of the de Sitter condition, which imposes the relation

$$R_{ij} = \frac{R}{4}g_{ij},$$

thereby establishing a direct connection between spacetime curvature and the energy–momentum distribution. As a result, the curvature scalar continues to affect the physical behavior of the model despite the absence of higher-order curvature terms. Thus, the SEC plots versus R are well-defined.

The magnetic field H modifies the effective energy density through the trace term

$$T = 3p - \rho + \mu(H - 1).$$

This magnetic contribution affects the energy conditions, particularly the DEC and SEC, as shown in Figures 3 and 4.

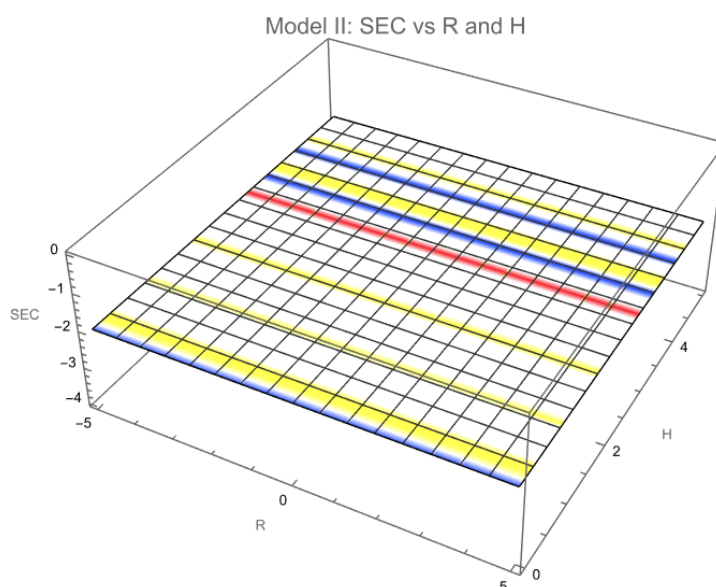


Figure 3. 3D surface plot of the SEC for Model II as a function of R and H . Similar to Model I, the SEC is violated for $R > 0$. The violation region depends on the value of β .

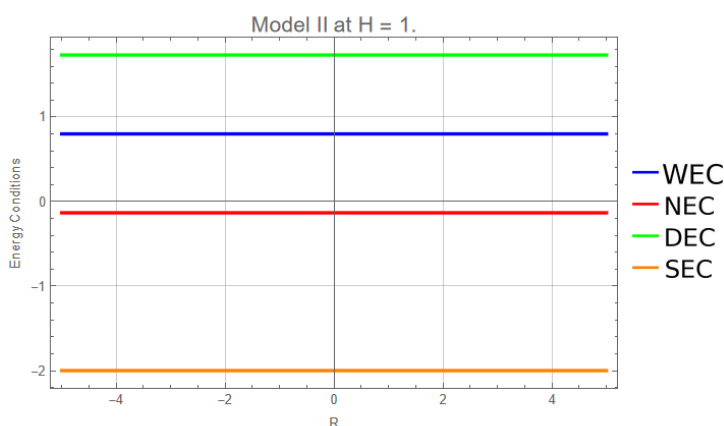


Figure 4. Energy conditions as functions of R for Model II at $H = 1.0$. The SEC violation is observed for $R > 0$, whereas other energy conditions are satisfied. The matter-coupling term βT modifies the magnitude of the violation compared to Model I.

7.3. Model III: Nonminimal coupling model

We consider the model

$$f(R, T, R_{ij}T^{ij}) = R + \gamma R_{ij}T^{ij}, \quad (7.11)$$

where γ is a constant parameter.

For this model, $f_R = 1$, and $f = R + \gamma R_{ij}T^{ij}$. The effective energy density and pressure take the form

$$\rho_{\text{eff}} = \rho + \frac{\gamma}{\kappa} R_{ij}T^{ij}, \quad (7.12)$$

$$p_{\text{eff}} = p + \frac{\gamma}{3\kappa} R_{ij}T^{ij}. \quad (7.13)$$

Using

$$R_{ij} = (R/4)g_{ij}$$

in the de Sitter case and

$$T = 3p - \rho + \mu(H - 1),$$

we get

$$R_{ij}T^{ij} = \frac{R}{4}T = \frac{R}{4}[3p - \rho + \mu(H - 1)]. \quad (7.14)$$

Thus,

$$\rho_{\text{eff}} = \rho + \frac{\gamma R}{4\kappa}[3p - \rho + \mu(H - 1)], \quad (7.15)$$

$$p_{\text{eff}} = p + \frac{\gamma R}{12\kappa}[3p - \rho + \mu(H - 1)]. \quad (7.16)$$

For Model III ($f = R + \gamma R_{ij}T^{ij}$), the R -dependence is explicit through the contraction

$$R_{ij}T^{ij} = (R/4)T.$$

Hence, the SEC shows a linear dependence on R , as illustrated in Figure 5 (the SEC as a function of R and H) and Figure 6 (the energy conditions as functions of R at $H = 1.0$).

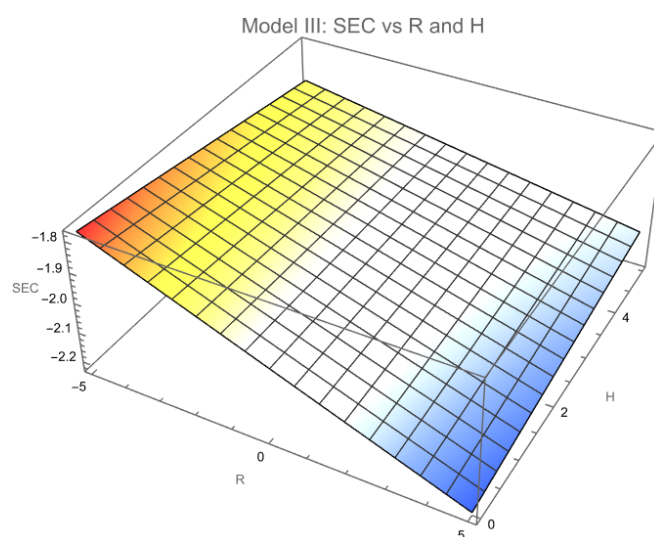


Figure 5. 3D surface plot of the SEC for Model III as a function of R and H . The nonminimal coupling term $\gamma R_{ij}T^{ij}$ introduces a linear dependence on R , leading to SEC violation for $R > 0$.

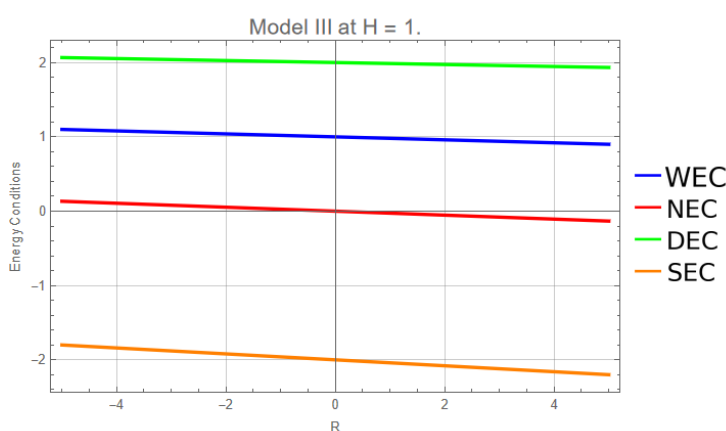


Figure 6. Energy conditions as functions of R for Model III at $H = 1.0$. The SEC is violated for $R > 0$. The DEC and WEC remain positive, whereas the NEC is saturated at zero.

7.4. Model IV: combined model

Finally, we consider the combined model

$$f(R, T, R_{ij}T^{ij}) = R + \alpha R^2 + \beta T + \gamma R_{ij}T^{ij}, \quad (7.17)$$

where α, β, γ are constant parameters.

For this model,

$$f_R = 1 + 2\alpha R$$

and

$$f = R + \alpha R^2 + \beta T + \gamma R_{ij} T^{ij}.$$

The effective energy density and pressure are given by

$$\rho_{\text{eff}} = \rho + \frac{1}{\kappa} \left[2\alpha R R_{ij} u^i u^j - \frac{1}{2} \alpha R^2 + 2\alpha (u^i u^j \square - u^i u^j \nabla_i \nabla_j) R + \beta T + \gamma R_{ij} T^{ij} \right], \quad (7.18)$$

$$p_{\text{eff}} = p + \frac{1}{\kappa} \left[\frac{2\alpha R^2}{3} - \frac{1}{2} \alpha R^2 + \frac{2\alpha}{3} (\square - \nabla_i \nabla^i) R + \frac{1}{3} \beta T + \frac{1}{3} \gamma R_{ij} T^{ij} \right]. \quad (7.19)$$

In the de Sitter case, these simplify to

$$\rho_{\text{eff}} = \rho + \frac{1}{\kappa} \left[\frac{\alpha R^2}{2} + \frac{\beta}{4} T + \frac{\gamma R}{4} T \right], \quad (7.20)$$

$$p_{\text{eff}} = p + \frac{1}{\kappa} \left[-\frac{\alpha R^2}{6} + \frac{\beta}{12} T + \frac{\gamma R}{12} T \right]. \quad (7.21)$$

The SEC for Model IV is shown as a function of R and H in Figure 7, and the energy conditions as functions of R at $H = 1.0$ are shown in Figure 8. The dependence of the SEC on the parameter α at fixed $R = H = 1.0$ is shown in Figure 9.

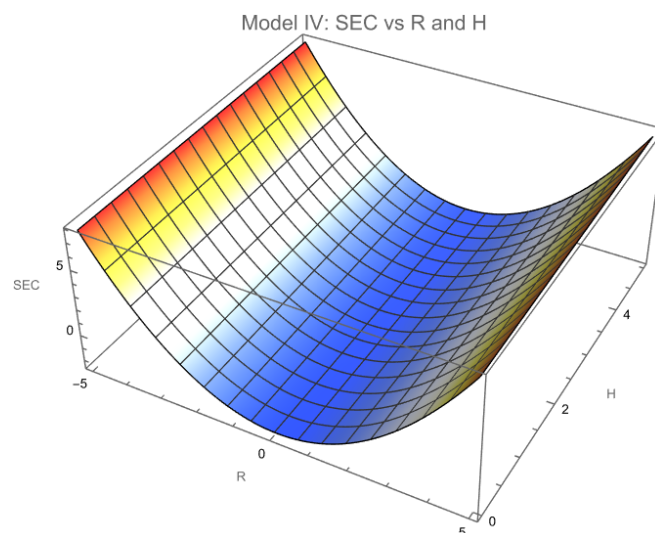


Figure 7. 3D surface plot of the SEC for Model IV (combined model) as a function of R and H . This model combines all previous contributions. The SEC violation is robust and occurs for $R > 0$ across a wide range of parameter values, indicating that the accelerated expansion is a generic feature of Q -flat v in $f(R, T, R_{ij} T^{ij})$ -gravity.

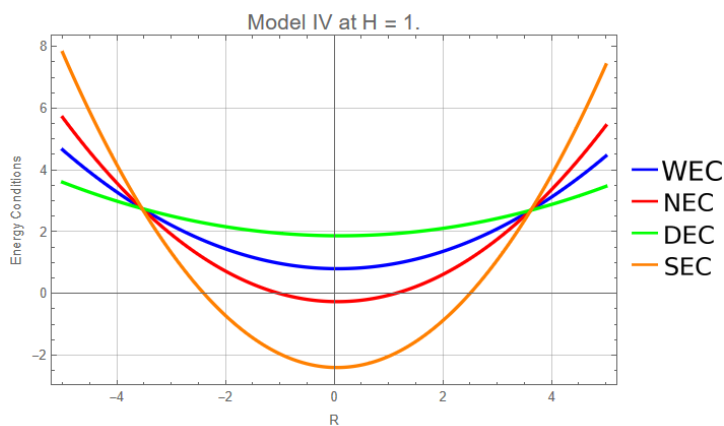


Figure 8. Energy conditions as functions of R for Model IV at $H = 1.0$. The combined model shows the most robust SEC violation. The WEC, DEC, and NEC are satisfied, confirming the physical viability of the model.

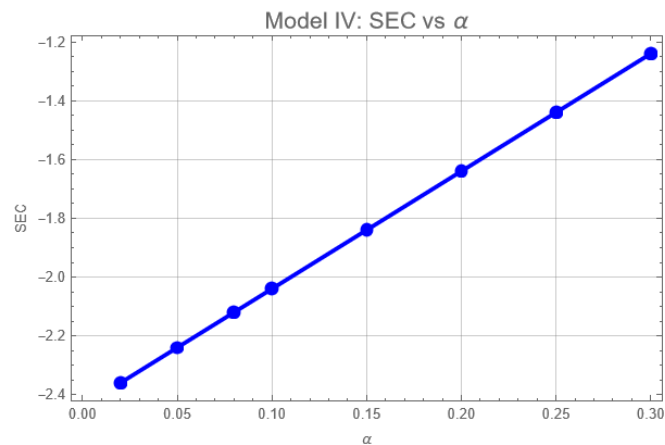


Figure 9. SEC versus the parameter α for Model IV at fixed $R = H = 1.0$. Larger α gives a more negative SEC, showing that higher-order curvature corrections strengthen the accelerated-expansion signal and that the SEC violation is sensitive to the chosen model parameters.

7.5. Summary of models

The present study examines four modified gravity models, summarized in Table 2, and investigates their implications for the SEC.

Table 2. Summary of specific models.

Model	Function	Key feature	SEC Violation
I	$R + \alpha R^2$	Starobinsky-type	Yes (for $R > 0$)
II	$R + \beta T$	Matter-coupling	Depends on β
III	$R + \gamma R_{ij}T^{ij}$	Nonminimal coupling	Depends on γ
IV	$R + \alpha R^2 + \beta T + \gamma R_{ij}T^{ij}$	Combined	Yes (for suitable parameters)

Model I is based on the Starobinsky-type function $f(R) = R + \alpha R^2$, where the quadratic curvature correction leads to SEC violation in the positive-curvature regime. Model II extends the Einstein–Hilbert action by including the matter trace through the term $R + \beta T$, with the occurrence of SEC violation depending on the coupling parameter β . In Model III, the effects of nonminimal matter–geometry coupling are examined through the function $f = R + \gamma R_{ij}T^{ij}$. In this case, the behavior of the SEC depends on the coupling parameter γ . Model IV extends this framework by incorporating all of the terms considered previously, namely

$$f = R + \alpha R^2 + \beta T + \gamma R_{ij}T^{ij},$$

thereby representing the most general model analyzed in the present study.

The numerical results displayed in Figures 1–9 reveal several important features. One observation common to all four models is the violation of the SEC for positive values of the scalar curvature R . Since positive curvature corresponds to the de Sitter regime, this behavior is consistent with the accelerated expansion associated with de Sitter cosmology.

The three-dimensional plots in Figures 1–7 show that the magnetic field strength H influences the extent of the SEC violation. Although the violation remains present over the entire range of magnetic field values considered, its magnitude changes with H . This dependence agrees with the magnetic field constraint obtained in Proposition 4.1, where the magnetic field is linked to both the curvature of spacetime and the matter content.

The role of the model parameters can be seen more clearly in Figure 9. As the parameter α increases, the SEC becomes increasingly negative, indicating a stronger deviation from the attractive gravitational behavior predicted by standard general relativity. This suggests that higher-order curvature terms can have a substantial impact on the acceleration mechanism.

Additional support for these conclusions is provided by two-dimensional plots in Figures 2–8. Although the SEC is violated, the null, weak, and dominant energy conditions remain satisfied throughout the parameter ranges examined. Consequently, the matter distribution retains the basic physical properties required for a viable model.

Among the four models considered, Model IV offers the broadest framework by combining higher-order curvature corrections with matter–geometry coupling effects. It also exhibits the strongest departure from the SEC while preserving the remaining energy conditions. These results suggest that Model IV provides a particularly suitable setting for investigating accelerated expansion in the context of $f(R, T, R_{ij}T^{ij})$ gravity.

Beyond the qualitative statement that the SEC is violated for $R > 0$, the physical content specific to a *magnetofluid* source (as opposed to an ordinary perfect fluid) lies in how the magnetic field shifts the location of the SEC sign change. Because the H enters ρ_{eff} and p_{eff} linearly through the trace $T = 3p - \rho + \mu(H - 1)$ (Models II–IV) or through the curvature–matter contraction $R_{ij}T^{ij} = (R/4)T$ (Models III–IV), the critical curvature $R_{\text{crit}}(H)$ at which $\rho_{\text{eff}} + 3p_{\text{eff}}$ changes sign is itself a function of H rather than a fixed number, as is visible from the tilt of the SEC surfaces with respect to the H -axis in the above Figures 3–7. This is the genuinely new physical content of the magnetofluid analysis relative to a perfect-fluid Q -flat spacetime [2]: The onset of the SEC-violating (accelerating) regime is magnetically tunable, and Proposition 4.1 shows that, conversely, H itself is fixed by (ψ, ρ) once Q -flatness is imposed. Physically, this couples the magnetic-field strength of a magnetized cosmological or astrophysical fluid to the effective onset of accelerated expansion in a way that has no

analog for an unmagnetized source, and it is this coupling, rather than the mere existence of SEC violation, which is generic to many $f(R, T, R_{ij}T^{ij})$ models, that we regard as the principal new physical prediction of the present analysis. A quantitative comparison of $R_{\text{crit}}(H)$ against magnetar and primordial-magnetogenesis field-strength estimates is a natural next step and is discussed further in Section 8.

8. Discussion and conclusions

In this article, we have systematically investigated the geometric and physical properties of relativistic MFSes endowed with the Q -curvature tensor within the framework of $f(R, T, R_{ij}T^{ij})$ -gravity. Our main results can be summarized as follows:

- (1) We derived the explicit form of the magnetofluid energy-momentum tensor and established the corresponding modified field equations in $f(R, T, R_{ij}T^{ij})$ -gravity.
- (2) We proved that a Q -flat magnetofluid spacetime is either a vacuum or a de Sitter spacetime. In the de Sitter case, the spacetime admits a representation as a Robertson–Walker spacetime under the additional assumptions of homogeneity and isotropy. This extends the results of De et al. [2] from perfect fluids to magnetofluids. We also derived explicit constraints on the magnetic field strength H under Q -flatness.
- (3) We showed that a divergence-free Q -curvature tensor implies that the Ricci tensor is of Codazzi type, provided ψ is constant. This establishes a connection between Q -curvature geometry and Yang pure spaces, which we have discussed in detail with relevant references [13–15].
- (4) We analyzed the energy conditions and demonstrated that the SEC is violated in the Q -flat de Sitter case, consistent with the current accelerated expansion of the universe.
- (5) We considered four specific models: Starobinsky-type, matter-coupling, nonminimal coupling, and combined models. For each model, we derived the effective energy density and pressure and analyzed the energy conditions graphically using Mathematica.
- (6) For the chosen parameter ranges, SEC violation is observed across all considered models while NEC, WEC, and DEC are satisfied, ensuring the physical viability of the solutions. This indicates that SEC violation, characteristic of accelerated expansion, is observed across the considered models of Q -flat MFSes in $f(R, T, R_{ij}T^{ij})$ -gravity.

8.1. Physical implications

The present results suggest several directions of physical interest in the contexts of cosmology and relativistic astrophysics.

In the de Sitter regime, the magnetic field remains present and contributes to the effective cosmological dynamics. From a cosmological perspective, this observation may indicate that primordial magnetic fields could have influenced the inflationary phase of the early universe.

An additional consequence of the present study concerns the Codazzi condition associated with the divergence-free Q -curvature tensor. This condition offers a geometric description of equilibrium states in MFSes and may have applications to highly magnetized astrophysical objects such as neutron stars, magnetars, and accretion disks.

The investigation of energy conditions further yields constraints on the functional form of

$f(R, T, R_{ij}T^{ij})$. Such constraints are valuable for identifying viable modified gravity models and for testing their consistency with current observational data.

Of the models considered, Model IV provides the broadest framework by incorporating both higher-order curvature corrections and matter–geometry coupling terms. The numerical results indicate persistent violation of the SEC while preserving the remaining standard energy conditions. This behavior is consistent with the possibility that accelerated expansion may be a natural outcome of Q -flat MFSes in $f(R, T, R_{ij}T^{ij})$ gravity.

8.2. Consistency and stability of the underlying theory

The analysis above is, by construction, a background-level (on-shell) study: We classify Q -flat solutions and evaluate the resulting energy conditions, but we have not established that the specific $f(R, T, R_{ij}T^{ij})$ models of Section 7 are free of ghostlike degrees of freedom or stable against linear perturbations, and we make no claim to that effect. We record here what can and cannot be said at the present level of analysis and what a full treatment would require.

For $f(R)$ -gravity, it is well-known that the absence of ghosts and of the Dolgov–Kawasaki tachyonic instability requires $f_R > 0$ and $f_{RR} \geq 0$ on the relevant curvature range [4]. Model I ($f = R + \alpha R^2$) is of this Starobinsky type with $f_R = 1 + 2\alpha R$ and $f_{RR} = 2\alpha$; with $\alpha > 0$ as assumed throughout, $f_{RR} > 0$, and $f_R > 0$ is satisfied on the scanned range provided $R > -1/2\alpha$, which holds comfortably for $\alpha = 0.1$ over $R \in [-5, 5]$. This is the extent to which ghost-freedom can be checked using only the standard $f(R)$ criterion.

Models II–IV additionally involve the trace T and the contraction $R_{ij}T^{ij}$, for which the corresponding Ostrogradsky/ghost analysis is considerably more involved, as Ψ_{ij} in Eq (3.3) introduces derivative couplings between the metric and the matter fields that do not reduce to a simple scalar-field representation as in $f(R)$ -gravity. To our knowledge, a complete ghost-freedom classification for the general $f(R, T, R_{ij}T^{ij})$ class has not been established in the literature [3], and providing one is beyond the scope of the present geometric classification. What we can say is that for Models II and III, $f_R = 1$ identically, so no new propagating tensor degree of freedom is introduced at the level of the trace of the field equations beyond those already present in general relativity; the matter-sector terms βT and $\gamma R_{ij}T^{ij}$ instead modify the effective equation of state of the fluid itself. Whether this modification is stable against linear perturbations of the magnetofluid variables (ρ, p, μ, H) is a separate question that the present energy-condition (background) analysis does not address.

We therefore regard a full linear stability analysis, that is, decomposing metric and magnetofluid perturbations into scalar, vector, and tensor parts about the Q -flat de Sitter background of Theorem 4.1 and deriving the corresponding dispersion relations and sound-speed conditions for each of Models I–IV, as an essential and immediate extension of this work, not merely an optional addition. We have not attempted it here because it requires a distinct set of techniques (a gauge-invariant cosmological perturbation formalism for $f(R, T, R_{ij}T^{ij})$ -gravity) beyond the curvature-classification and energy-condition methods used in Sections 4–7, and we have added it as the first item of Section 8’s future directions below. Until such an analysis is carried out, the energy-condition results of this paper should be read as establishing the kinematic viability of the Q -flat magnetofluid configurations at the background level, not their dynamical stability.

8.3. Future directions

The most pressing open problem, following directly from Section 8.2, is a linear perturbative stability analysis of the Q -flat de Sitter magnetofluid background: A scalar-vector-tensor decomposition of the metric and magnetofluid perturbations, leading to explicit ghost-freedom and no-tachyonic-instability conditions on (α, β, γ) for each of Models I–IV, together with the associated sound-speed constraints for the magnetofluid sector. This would upgrade the present background classification into a fully consistent dynamical model and should be regarded as a prerequisite for, rather than a mere supplement to, further phenomenological applications of the models studied here.

Several further questions remain open and deserve additional study. An important next step is the search for exact solutions describing Q -flat MFSes in $f(R, T, R_{ij}T^{ij})$ gravity for specific forms of the gravitational action. In addition, the stability properties of the de Sitter configuration should be examined in greater detail, taking into account perturbations in both the magnetic field and the matter variables.

The present analysis may also be extended to Friedmann–Robertson–Walker cosmology to explore the role of magnetic fields in the evolution of the universe. Incorporating nonideal fluid effects, including heat transport and dissipation, would further enhance the physical realism of the models.

Another promising direction concerns the connection between Q -curvature and geometric solitons in MFSes. Studying this relationship may help clarify the geometric structures that arise naturally in modified theories of gravity.

Finally, the energy condition constraints derived in this study offer a useful framework for confronting $f(R, T, R_{ij}T^{ij})$ models with observational data. Such investigations may help identify physically viable models capable of describing the observed evolution of the universe. A deeper analytical study of the underlying differential equations could also reveal new classes of exact solutions and symmetries.

Author contributions

Lalnunenga Colney: conceptualization, methodology, formal analysis and investigation, writing – original draft preparation, writing – review and editing, supervision; Mohammad Nazrul Islam Khan: conceptualization, methodology, formal analysis and investigation, writing – original draft preparation, writing – review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (<https://www.qu.edu.sa>) for financial support (QU-APC-2026).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. C. A. Mantica, Y. J. Suh, Pseudo Q -symmetric Riemannian manifolds, *Int. J. Geom. Methods Mod. Phys.*, **10** (2013), 1350013. <https://doi.org/10.1142/S0219887813500138>
2. U. C. De, K. De, A. A. Syied, N. Bin Turki, Q -curvature tensor in spacetimes and $f(R, T, R_{ij}T^{ij})$ -gravity, *Filomat*, **39** (2025), 9061–9074. <https://doi.org/10.2298/FIL2506961D>
3. M. Sharif, M. Zubair, Energy conditions in $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity, *J. High Energy Phys.*, **2013** (2013), 79. [https://doi.org/10.1007/JHEP12\(2013\)079](https://doi.org/10.1007/JHEP12(2013)079)
4. S. E. P. Bergliaffa, Constraining $f(R)$ theories with the energy conditions, *Phys. Lett. B*, **642** (2006), 311–314. <https://doi.org/10.1016/j.physletb.2006.10.003>
5. K. Bamba, M. Ilyas, M. Z. Bhatti, Z. Yousaf, Energy conditions in modified $f(G)$ gravity, *Gen. Relativ. Gravit.*, **49** (2017), 112. <https://doi.org/10.1007/s10714-017-2276-x>
6. M. D. Siddiqi, S. K. Chaubey, M. N. I. Khan, $f(R, T)$ -gravity model with perfect fluid admitting Einstein solitons, *Mathematics*, **10** (2021), 82. <https://doi.org/10.3390/math10010082>
7. M. D. Siddiqi, U. C. De, Relativistic magneto-fluid spacetimes, *J. Geom. Phys.*, **170** (2021), 104370. <https://doi.org/10.1016/j.geomphys.2021.104370>
8. K. Ramasamy, S. Roy, Solitonic geometry of magneto fluid spacetimes: Ricci Bourguignon insights and energy momentum characterizations, *J. Geom. Phys.*, **217** (2025), 105609. <https://doi.org/10.1016/j.geomphys.2025.105609>
9. M. D. Siddiqi, F. Mofarreh, S. K. Chaubey, Solitonic aspect of relativistic magneto-fluid spacetime with some specific vector fields, *Mathematics*, **11** (2023), 1596. <https://doi.org/10.3390/math11071596>
10. S. K. Yadav, D. C. De, M. N. I. Khan, Certain restrictions of energy-momentum tensor in bulk viscous fluid string spacetimes, *Axioms*, **15** (2026), 461. <https://doi.org/10.3390/axioms15060461>
11. A. Haseeb, S. K. Chaubey, M. N. I. Khan, A solitonic approach of general relativistic spacetimes with applications, *Axioms*, **15** (2026), 392. <https://doi.org/10.3390/axioms15060392>
12. T. Harko, F. S. N. Lobo, S. I. Nojiri, S. D. Odintsov, $f(R, T)$ gravity, *Phys. Rev. D*, **84** (2011), 024020. <https://doi.org/10.1103/PhysRevD.84.024020>
13. C. N. Yang, Integral formalism for gauge fields, *Phys. Rev. Lett.*, **33** (1974), 445–447. <https://doi.org/10.1103/PhysRevLett.33.445>
14. A. Derdzinski, C. L. Shen, Codazzi tensor fields, curvature and Pontryagin forms, *Proc. London Math. Soc.*, **47** (1983), 15–26.
15. A. Gray, Einstein-like manifolds which are not Einstein, *Geom. Dedicata*, **7** (1978), 259–280. <https://doi.org/10.1007/BF00151525>
16. S. W. Hawking, G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973. <https://doi.org/10.1017/CBO9780511524646>

17. Y. Hua, R. J. Yang, Kiselev black holes in quantum fluctuation modified gravity, *Eur. Phys. J. Plus*, **141** (2026), 135. <https://doi.org/10.1140/epjp/s13360-026-07312-5>
18. N. Aghanim, Y. Akrami, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, et al., Planck 2018 results. VI. Cosmological parameters, *Astron. Astrophys.*, **641** (2020), A6. <https://doi.org/10.1051/0004-6361/201833910>
19. A. A. Starobinsky, A new type of isotropic cosmological models without singularity, *Phys. Lett. B*, **91** (1980), 99–102. [https://doi.org/10.1016/0370-2693\(80\)90670-X](https://doi.org/10.1016/0370-2693(80)90670-X)

Appendix

In this appendix, we provide detailed verification of all key calculations used in the main text.

A.1. Trace of the magnetofluid energy-momentum tensor

Starting from (2.6),

$$T_{ij} = pg_{ij} + (p + \rho)u_i u_j + \mu \left[\left(u_i u_j + \frac{1}{2} g_{ij} \right) H - \phi_i \phi_j \right].$$

Taking the trace, we get

$$T = g^{ij} T_{ij} = pg^{ij} g_{ij} + (p + \rho)g^{ij} u_i u_j + \mu H g^{ij} u_i u_j + \frac{1}{2} \mu H g^{ij} g_{ij} - \mu g^{ij} \phi_i \phi_j.$$

Using $g^{ij} g_{ij} = 4$, $g^{ij} u_i u_j = -1$, and $g^{ij} \phi_i \phi_j = 1$, we obtain

$$\begin{aligned} T &= 4p - (p + \rho) + \mu H(-1) + \frac{1}{2} \mu H(4) - \mu(1), \\ T &= 4p - p - \rho - \mu H + 2\mu H - \mu = 3p - \rho + \mu(H - 1). \end{aligned}$$

This confirms (2.7).

A.2. Divergence of the Q -curvature tensor

From (2.4),

$$Q_{ijk}^h = R_{ijk}^h - \frac{\psi}{3}(\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

Taking the divergence yields

$$\begin{aligned} \nabla_h Q_{ijk}^h &= \nabla_h R_{ijk}^h - \frac{1}{3} \nabla_h [\psi(\delta_k^h g_{ij} - \delta_j^h g_{ik})], \\ \nabla_h Q_{ijk}^h &= \nabla_h R_{ijk}^h - \frac{1}{3} [(\nabla_h \psi)(\delta_k^h g_{ij} - \delta_j^h g_{ik}) + \psi(\delta_k^h \nabla_h g_{ij} - \delta_j^h \nabla_h g_{ik})]. \end{aligned}$$

Because $\nabla_h g_{ij} = 0$,

$$\nabla_h Q_{ijk}^h = \nabla_h R_{ijk}^h - \frac{1}{3} [(\nabla_k \psi)g_{ij} - (\nabla_j \psi)g_{ik}].$$

Using the contracted second Bianchi identity, $\nabla_h R_{ijk}^h = \nabla_j R_{ik} - \nabla_k R_{ij}$,

$$\nabla_h Q_{ijk}^h = \nabla_j R_{ik} - \nabla_k R_{ij} - \frac{1}{3} (\nabla_k \psi g_{ij} - \nabla_j \psi g_{ik}).$$

This confirms (5.2).

A.3. Codazzi condition

Setting $\nabla_h Q_{ijk}^h = 0$, we get

$$\nabla_j R_{ik} - \nabla_k R_{ij} = \frac{1}{3}(\nabla_k \psi g_{ij} - \nabla_j \psi g_{ik}).$$

If ψ is constant, $\nabla_j \psi = \nabla_k \psi = 0$:

$$\nabla_j R_{ik} = \nabla_k R_{ij}.$$

This confirms (5.4).

A.4. Energy conditions in the de Sitter case

For a de Sitter spacetime, the equation of state is $p_{\text{eff}} = -\rho_{\text{eff}}$. Then,

$$\begin{aligned} \rho_{\text{eff}} + p_{\text{eff}} &= \rho_{\text{eff}} - \rho_{\text{eff}} = 0 \quad (\text{NEC satisfied, saturated}); \\ \rho_{\text{eff}} - p_{\text{eff}} &= \rho_{\text{eff}} - (-\rho_{\text{eff}}) = 2\rho_{\text{eff}} \geq 0 \quad (\text{DEC satisfied for } \rho_{\text{eff}} \geq 0); \\ \rho_{\text{eff}} + 3p_{\text{eff}} &= \rho_{\text{eff}} - 3\rho_{\text{eff}} = -2\rho_{\text{eff}} < 0 \quad (\text{SEC violated for } \rho_{\text{eff}} > 0). \end{aligned}$$

This confirms Theorem 6.1.

A.5. Explicit field equations for specific models

A.5.1. Model I: Starobinsky-type

For $f(R, T, R_{ij}T^{ij}) = R + \alpha R^2$, the field Eq (3.2) becomes

$$(1 + 2\alpha R)R_{ij} - \frac{1}{2}(R + \alpha R^2)g_{ij} + (g_{ij}\square - \nabla_i \nabla_j)(1 + 2\alpha R) + \Psi_{ij} = \kappa T_{ij}. \quad (\text{A.1})$$

A.5.2. Model II: matter-coupling

For $f(R, T, R_{ij}T^{ij}) = R + \beta T$, the field equations become

$$R_{ij} - \frac{1}{2}(R + \beta T)g_{ij} + \Psi_{ij} = \kappa T_{ij}. \quad (\text{A.2})$$

A.5.3. Model III: nonminimal coupling

For $f(R, T, R_{ij}T^{ij}) = R + \gamma R_{ij}T^{ij}$, the field equations become

$$R_{ij} - \frac{1}{2}(R + \gamma R_{kl}T^{kl})g_{ij} + \Psi_{ij} = \kappa T_{ij}. \quad (\text{A.3})$$

A.5.4. Model IV: combined model

For $f(R, T, R_{ij}T^{ij}) = R + \alpha R^2 + \beta T + \gamma R_{ij}T^{ij}$, the field equations become

$$(1 + 2\alpha R)R_{ij} - \frac{1}{2}(R + \alpha R^2 + \beta T + \gamma R_{kl}T^{kl})g_{ij} + (g_{ij}\square - \nabla_i \nabla_j)(1 + 2\alpha R) + \Psi_{ij} = \kappa T_{ij}. \quad (\text{A.4})$$

A.6. Summary of key results

Table A.1 summarizes the main results of this paper. For each result, it gives the corresponding equation and its physical interpretation, where EMT denotes the energy-momentum tensor and EoS denotes the equation of state.

Table A.1. Summary of key results and their physical interpretations.

Result	Equation	Physical interpretation
Q-flat spacetime	$R_{ij} = \psi g_{ij}$	Einstein spacetime
Magnetofluid EMT	$T_{ij} = p g_{ij} + (p + \rho) u_i u_j + \mu [(u_i u_j + \frac{1}{2} g_{ij}) H - \phi_i \phi_j]$	Matter with magnetic properties
Trace of EMT	$T = 3p - \rho + \mu(H - 1)$	Modified trace equation
Divergence-free Q	$\nabla_j R_{ik} = \nabla_k R_{ij}$	Codazzi-type Ricci tensor
de Sitter EoS	$p_{\text{eff}} = -\rho_{\text{eff}}$	Dark energy / cosmological constant
Energy conditions	SEC violated, WEC/DEC satisfied	Accelerated expansion



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)