



Research article

Fractal-fractional Maclaurin-type inequalities via generalized harmonic convexity and ANN-based computational illustrations

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Abstract: This work developed a class of new integral inequalities within the framework of fractal-fractional calculus. The core contribution was the derivation of a new fractal-fractional auxiliary identity, which provides a unified basis for establishing several Maclaurin-type inequalities. The obtained results were formulated for functions whose local fractional derivatives satisfy generalized harmonic convexity. Furthermore, complementary upper bounds were derived by means of the generalized Hölder inequality and the power mean inequality, thereby extending the scope and flexibility of the proposed estimates. The validity of the theoretical results was illustrated through a representative example and graphical visualizations, which confirm the consistency of the developed inequalities. A feed-forward artificial neural network was also implemented to provide a complementary computational validation of one of the derived estimates by learning the dependence of the upper bound on the Hölder conjugate parameters. An application to selected special fractal means was also included to demonstrate the usefulness and practical significance of the proposed approach.

Keywords: fractal-fractional calculus; generalized harmonic convexity; Maclaurin's inequalities; generalized Hölder inequality; generalized power mean inequality

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1. Background and motivation

The approximation of definite integrals is one of the classical themes of numerical analysis. In many situations, exact evaluation is either impossible or impractical, and one must rely on quadrature formulas to obtain accurate numerical values. Standard examples include the trapezoidal rule, Simpson's rule, and the Newton–Cotes family, all of which approximate the integral through weighted combinations of function values at selected nodes.

Among open-type quadrature formulas, Maclaurin's rule occupies a distinguished place. Since

it avoids the endpoints of the interval, it is particularly appropriate when the integrand is singular, undefined, or difficult to evaluate at the boundary points. This feature makes the formula attractive not only from a theoretical perspective but also in concrete computational applications. For this reason, Maclaurin-type estimates have been the subject of sustained interest in recent years.

A classical result in this direction was obtained by Alomari and Dragomir in [1]. They showed that if h is four times continuously differentiable, then the following inequality holds:

$$\left| \frac{1}{8} \left(3h\left(\frac{5c+d}{6}\right) + 2h\left(\frac{c+d}{2}\right) + 3h\left(\frac{c+5d}{6}\right) \right) - \frac{1}{d-c} \int_c^d h(\varpi) d\varpi \right| \leq \frac{7(d-c)^2}{51,840} \|h^{(4)}\|_{\infty}.$$

The derivation of error bounds for quadrature formulas remains a central problem, especially when the smoothness assumptions imposed on the integrand are weakened. In this context, convexity arguments have proved highly effective. They often allow one to control the remainder term without resorting to higher-order derivatives, and they have therefore become a standard tool in the analysis of numerical integration. Early contributions linking convexity with quadrature error estimates may be found in [2, 3]. A further Maclaurin-type inequality was later established in [4], namely,

$$\left| \frac{1}{8} \left(3h\left(\frac{5c+d}{6}\right) + 2h\left(\frac{c+d}{2}\right) + 3h\left(\frac{c+5d}{6}\right) \right) - \frac{1}{d-c} \int_c^d h(\varpi) d\varpi \right| \leq \frac{25(d-c)}{576} (|h'(c)| + |h'(d)|),$$

provided that $|h'|$ is convex on $[c, d]$. For further studies on Maclaurin-type inequalities, we refer the reader to [5–7].

Although ordinary convexity is extremely useful, it is not always flexible enough to capture more refined geometric behaviors. This limitation has led to the introduction of several generalized convexity classes. One of the best known is harmonic convexity, introduced by İscan [8]. This notion is especially well-suited to problems in which reciprocal expressions or harmonic means arise naturally. Specifically, a function $h: \mathcal{I} \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is said to be harmonically convex on $\mathcal{I}$ if

$$h\left(\frac{ab}{\nu a + (1-\nu)b}\right) \leq (1-\nu)h(a) + \nu h(b)$$

for every $\nu \in [0, 1]$ and all $a, b \in \mathcal{I}$.

Remark 1.1. A function $h: (0, \infty) \rightarrow \mathbb{R}$ is harmonically convex on $\mathcal{I}$, if and only if the function $\tilde{h}(\varpi) = h\left(\frac{1}{\varpi}\right)$ is convex on $\mathcal{I}^{-1} = \left\{\frac{1}{\varpi} : \varpi \in \mathcal{I}\right\}$.

This idea has generated a substantial body of work on integral inequalities. In particular, İscan [9] developed a parametric approach that led to several inequalities of midpoint, trapezoid, Bullen, and Simpson type. These results demonstrated that harmonic convexity offers a natural setting in which many classical quadrature estimates can be reformulated and extended. For further developments and applications of harmonic convexity, the interested reader is referred to [10–13].

A second important line of development comes from fractional calculus. Because fractional operators are nonlocal, they are capable of describing memory and hereditary effects that cannot be modeled within the classical framework. Among the many operators used in this area, the

Riemann–Liouville fractional integrals are perhaps the most prominent. In combination with harmonic convexity, they have been used to derive fractional versions of Hermite–Hadamard, midpoint, and trapezium-type inequalities; see [14–18].

A broader class of fractional operators was introduced by Sarikaya and Ertuğral in [19]. Their generalized fractional integrals are recalled below.

Definition 1.2. ([19]) Let $h \in L[c, d]$. The left-sided and right-sided generalized fractional integrals associated with h are defined by

$$I_{c^+}^\phi h(\kappa) = \int_c^\kappa \frac{\phi(\kappa - \varpi)}{\kappa - \varpi} h(\varpi) d\varpi, \quad c < \kappa \quad (1.1)$$

and

$$I_{d^-}^\phi h(\kappa) = \int_\kappa^d \frac{\phi(\varpi - \kappa)}{\varpi - \kappa} h(\varpi) d\varpi, \quad \kappa < d, \quad (1.2)$$

where the kernel function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfies

$$\int_0^1 \frac{\phi(\varpi)}{\varpi} d\varpi < \infty.$$

Another source of motivation arises from the analysis of functions defined on irregular or fractal-type sets. Such objects frequently fall outside the reach of classical calculus because ordinary differentiability is no longer available. Local fractional calculus was introduced precisely to overcome this difficulty, providing differential and integral tools adapted to fractal structures [20, 21].

Within this framework, convexity itself had to be reformulated. One arrives in this way at the notion of generalized convexity.

Definition 1.3. ([22]) A mapping $h: \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}^\wp$ is said to be generalized convex on $\mathcal{I}$ if, for every $a, b \in \mathcal{I}$ and every $\nu \in [0, 1]$,

$$h(\nu a + (1 - \nu)b) \leq \nu^\wp h(a) + (1 - \nu)^\wp h(b).$$

Sun then proposed the corresponding harmonic version in the generalized setting.

Definition 1.4. ([23]) Let $h: \mathcal{I} \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^\wp$. The function h is called generalized harmonically convex on $\mathcal{I}$ if

$$h\left(\frac{ab}{\nu a + (1 - \nu)b}\right) \leq (1 - \nu)^\wp h(a) + \nu^\wp h(b)$$

for all $a, b \in \mathcal{I}$ and $\nu \in [0, 1]$.

Using this concept, Sun derived Hermite–Hadamard-type and midpoint-type inequalities in fractal environments.

A further extension of the theory is given by fractal-fractional integrals, which combine the nonlocal character of fractional integration with the geometric structure of fractal measures. These operators are defined as follows.

Definition 1.5. ([24]) For $h: [c, d] \rightarrow \mathbb{R}^{\varphi}$, the left-sided and right-sided fractal-fractional integral operators are given by

$${}^{\phi}_{c+}I^{(\varphi)}h(\kappa) = \frac{1}{\Gamma(1+\varphi)} \int_c^{\kappa} \frac{\phi(\kappa-\varpi)}{(\kappa-\varpi)^{\varphi}} h(\varpi)(d\varpi)^{\varphi}, \quad c < \kappa \quad (1.3)$$

and

$${}^{\phi}_{d-}I^{(\varphi)}h(\kappa) = \frac{1}{\Gamma(1+\varphi)} \int_{\kappa}^d \frac{\phi(\varpi-\kappa)}{(\varpi-\kappa)^{\varphi}} h(\varpi)(d\varpi)^{\varphi}, \quad \kappa < d. \quad (1.4)$$

The kernel $\phi: [0^{\varphi}, \infty^{\varphi}) \rightarrow [0^{\varphi}, \infty^{\varphi})$ is assumed to satisfy

$$\frac{1}{\Gamma(1+\varphi)} \int_0^1 \frac{\phi(\varpi)}{\varpi^{\varphi}} (d\varpi)^{\varphi} < \infty^{\varphi}.$$

These operators have recently played an important role in the derivation of various integral inequalities under diverse convexity frameworks. For example, Lakhdari et al. [25] established Hermite–Hadamard- and Milne-type inequalities for generalized s -convex functions, whereas Alsharari et al. [26] derived Simpson-type inequalities in the same generalized setting.

Additional advances were reported by Yuan et al. [27], who proposed the notion of generalized (P, m) -convexity and formulated a multiparameter identity associated with fractal-fractional operators. As a consequence, they obtained several new bounds for functions involving second-order local fractional derivatives. Subsequently, Gong et al. [28] investigated generalized (s, P) -convexity and established a number of quadrature-type inequalities through the development of suitable fractal-fractional integral identities.

The situation of higher-order local fractional differentiability has likewise attracted attention. In [24], midpoint-type inequalities were proved for twice locally fractionally differentiable functions under generalized convexity hypotheses. Inspired by these developments, Butt et al. derived parametric inequalities in [29] and Simpson-type inequalities in [30] for functions whose second local fractional derivatives fulfill generalized convexity assumptions.

Similarly, Bullen-type estimates have been investigated in the same framework. Butt et al. [31] employed fractal-fractional integrals to study functions with first local fractional derivatives and derived several new Bullen-type inequalities. Xu et al. [32] presented a comprehensive study of fractal-fractional reverse Minkowski-type inequalities. Moreover, Saleh et al. [33] proposed another approach based on an extended form of generalized convexity in the fractal-fractional setting.

More recently, He et al. [34] presented the first study to employ the notion of generalized harmonic convexity within the fractal-fractional calculus framework, using it to derive numerous Milne-type inequalities.

Following this pioneering direction, the main objective of the present work is to establish a set of Maclaurin-type inequalities for fractal-fractional integral operators within the setting of generalized harmonic convexity, representing only the second study of its kind in the literature. Unlike the closed quadrature rules studied in [34], the Maclaurin formula is an open quadrature rule. This feature makes it particularly well-suited and highly advantageous for analyzing functions that are not defined, are singular, or are computationally difficult to evaluate at the boundaries of the interval. By exploiting the interplay between Maclaurin's open formula and generalized harmonic convexity in the context

of fractal-fractional calculus, this paper bridges an important gap in the literature and offers a highly general framework that unifies several existing quadrature bounds.

The paper is arranged as follows. Section 2 reviews the preliminary material from local fractional calculus needed for the subsequent analysis. In Section 3, we establish a new fractal-fractional integral identity. Section 4 applies this identity to obtain Maclaurin-type inequalities for functions whose local fractional derivatives are generalized harmonically convex. In Section 5, we further extend the analysis by deriving additional estimates through generalized Hölder and power-mean inequalities. Section 6 provides a numerical illustration, a neural-network-based validation, and an application concerning special fractal means. The final section, Section 7, is devoted to the concluding remarks.

2. Essential concepts in local fractional calculus

The study of physical and geometric phenomena on irregular, self-similar domains, commonly referred to as fractal sets, demands mathematical tools that go beyond classical calculus, which implicitly relies on smooth Euclidean spaces. Because functions defined on these fractal sets are typically continuous but nowhere differentiable in the classical sense, local fractional calculus (also known as fractal calculus) has emerged as the natural framework to analyze them. Unlike non-local fractional derivatives, the local fractional derivative acts as a localized differential operator designed specifically to measure the scaling behavior and rate of change directly on Cantor-like sets and fractal spaces $\mathbb{R}^\varphi$ (where $0 < \varphi \leq 1$ represents the fractal dimension). This mathematical framework provides a rigorous, localized computational tool that preserves the intrinsic geometric symmetries of fractal sets, making it highly effective for modeling transport and diffusion in complex, non-smooth media.

To facilitate a clear and consistent understanding of the mathematical framework used throughout this study, we summarize the primary notations, fractal spaces, operators, and parameters in Table 1.

Table 1. Summary of mathematical notations and symbols used throughout the paper.

Symbol	Type	Description/mathematical meaning
φ	Parameter ($0 < \varphi \leq 1$)	Fractal dimension
$\mathbb{R}^\varphi$	Algebraic space	Set of real-line numbers mapped onto the fractal space of order φ
$d^\varphi \varpi$	Differential	The local fractional differential increment of the fractal variable ϖ
$\mathfrak{h}^{(\varphi)}(\varpi)$	Operator	The local fractional derivative of the function $\mathfrak{h}$ of order φ at the point ϖ
$\phi(t)$	Function	General auxiliary scaling function utilized for weighting
$\mathcal{M}_\varphi(\varpi)$	Function	Normalization term or scaling function associated with the weight function ϕ
$\mathfrak{c}, \mathfrak{d}$	Parameters	Left and right boundary limits ($\mathfrak{c} < \mathfrak{d}$) of the interval under study
$\Gamma(1 + \varphi)$	Mathematical constant	The Euler-gamma function
$\ln_\varphi(\varpi)$	Function	The local fractional logarithmic function
$\mathcal{C}_i, \mathcal{D}_i$	Constants	Explicit algebraic constants derived during the integration of the fractal kernel

After establishing these fundamental notations, we recall some basic definitions and properties of local fractional calculus.

For $0 < \varphi \leq 1$, we have the following φ -type sets:

- The $\wp$ -type set of integers is defined as:

$$\mathbb{Z}^\wp = \{0, \pm 1^\wp, \pm 2^\wp, \pm 3^\wp, \dots\}.$$

- The $\wp$ -type set of rational numbers is defined as:

$$\mathbb{Q}^\wp = \left\{ s^\wp = \left(\frac{p}{q}\right)^\wp : p, q \in \mathbb{Z} \text{ with } q \neq 0 \right\}.$$

- The $\wp$ -type set of irrational numbers is defined as:

$$\mathbb{J}^\wp = \left\{ s^\wp \neq \left(\frac{p}{q}\right)^\wp : p, q \in \mathbb{Z} \text{ with } q \neq 0 \right\}.$$

- The $\wp$ -type set of real numbers is defined as: $\mathbb{R}^\wp := \mathbb{Q}^\wp \cup \mathbb{J}^\wp$.

Let $s_1^\wp, s_2^\wp$, and $s_3^\wp$ belong to $\mathbb{R}^\wp$ with $\wp \in (0, 1]$. Then the subsequent algebraic properties hold:

- **Closure:** $s_1^\wp + s_2^\wp \in \mathbb{R}^\wp$ and $s_1^\wp s_2^\wp \in \mathbb{R}^\wp$.
- **Addition:**

$$\begin{aligned} s_1^\wp + s_2^\wp &= s_2^\wp + s_1^\wp = (s_1 + s_2)^\wp, \\ s_1^\wp + (s_2^\wp + s_3^\wp) &= (s_1 + s_2)^\wp + s_3^\wp. \end{aligned}$$

- **Multiplication:**

$$\begin{aligned} s_1^\wp s_2^\wp &= s_2^\wp s_1^\wp = (s_1 s_2)^\wp, \\ s_1^\wp (s_2^\wp s_3^\wp) &= (s_1^\wp s_2^\wp) s_3^\wp. \end{aligned}$$

- **Distributive law:**

$$s_1^\wp (s_2^\wp + s_3^\wp) = s_1^\wp s_2^\wp + s_1^\wp s_3^\wp.$$

- **Identity elements:**

$$s_1^\wp + 0^\wp = s_1^\wp, \quad s_1^\wp \cdot 1^\wp = s_1^\wp.$$

Definition 2.1. ([35]) We say that a mapping $h: \mathbb{R} \rightarrow \mathbb{R}^\wp$ is *local fractional continuous* at ϖ_0 whenever, for any $\varsigma > 0$, one can find $\varepsilon > 0$ satisfying

$$|h(\varpi) - h(\varpi_0)| < \varsigma^\wp \quad \text{whenever} \quad |\varpi - \varpi_0| < \varepsilon.$$

The collection of all such functions on (c, d) is denoted by $C_\wp(c, d)$.

Definition 2.2. ([35]) One defines the *local fractional derivative* of h at ϖ_0 by

$$h^{(\wp)}(\varpi_0) = \left. \frac{d^\wp h(\varpi)}{d\varpi^\wp} \right|_{\varpi=\varpi_0} = \lim_{\varpi \rightarrow \varpi_0} \frac{\Delta^\wp(h(\varpi) - h(\varpi_0))}{(\varpi - \varpi_0)^\wp},$$

where

$$\Delta^\wp(h(\varpi) - h(\varpi_0)) \cong \Gamma(\wp + 1)(h(\varpi) - h(\varpi_0)).$$

If

$$h^{(l\wp)}(\varpi) = \underbrace{D^\wp D^\wp \dots D^\wp}_{l \text{ times}} h(\varpi)$$

exists on an interval $I \subseteq \mathbb{R}$, then $h \in \mathcal{D}_{l\wp}(I)$.

Definition 2.3. ([35]) Let $h \in C_\varphi([c, d])$. Its *local fractional integral* over $[c, d]$ is expressed as

$${}_c I_d^{(\varphi)} h(\varpi) = \frac{1}{\Gamma(\varphi + 1)} \int_c^d h(\nu) (d\nu)^\varphi = \frac{1}{\Gamma(\varphi + 1)} \lim_{\Delta\nu \rightarrow 0} \sum_{l=0}^{M-1} h(\nu_l) (\Delta\nu_l)^\varphi,$$

where $c = \kappa_0 < \kappa_1 < \dots < \kappa_M = d$ and

$$\Delta\kappa_l = \kappa_{l+1} - \kappa_l, \quad \Delta\kappa = \max\{\Delta\kappa_0, \dots, \Delta\kappa_{M-1}\}.$$

In particular,

$${}_c I_d^{(\varphi)} h(\varpi) = 0 \quad \text{if} \quad c = d$$

and

$${}_c I_d^{(\varphi)} h(\varpi) = -{}_d I_c^{(\varphi)} h(\varpi) \quad \text{for} \quad c < d.$$

The set of functions for which this integral exists on $[c, d]$ is denoted by $I_\varpi^{(\varphi)}([c, d])$.

Lemma 2.4. ([35]) If $h(\varpi) = f^{(\varphi)}(\varpi)$ and $h \in C_\varphi([c, d])$, then

$${}_c I_d^{(\varphi)} h(\varpi) = f(d) - f(c).$$

Lemma 2.5. ([35]) For $h(\varpi) = \varpi^l$ with $l > 0$, one has

$$\frac{d\varpi^l}{d\varpi^\varphi} = \frac{\Gamma(1 + l\varphi)}{\Gamma(1 + (l - 1)\varphi)} \varpi^{(l-1)\varphi}$$

and

$$\frac{1}{\Gamma(1 + \varphi)} \int_c^d \varpi^l (d\varpi)^\varphi = \frac{\Gamma(1 + l\varphi)}{\Gamma(1 + (l + 1)\varphi)} (d^{(l+1)\varphi} - c^{(l+1)\varphi}).$$

We also recall Yang's local fractional version of Hölder's inequality [35].

Lemma 2.6. ([35]) Let $h, f \in C_\varphi([c, d])$, and assume that h^p and f^q are locally fractionally integrable on $[c, d]$, where $p, q > 1$ satisfy $p + q = pq$. Then

$$\frac{1}{\Gamma(1 + \varphi)} \int_c^d |h(\varpi)f(\varpi)| (d\varpi)^\varphi \leq \left(\frac{1}{\Gamma(1 + \varphi)} \int_c^d |h(\varpi)|^p (d\varpi)^\varphi \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1 + \varphi)} \int_c^d |f(\varpi)|^q (d\varpi)^\varphi \right)^{\frac{1}{q}}.$$

Finally, we record the generalized power-mean inequality on fractal domains due to Yu et al. [36].

Lemma 2.7. ([36]) Let $h, f \in C_\varphi([c, d])$. If both h and $h f^q$ are locally fractionally integrable on $[c, d]$ for some $q \geq 1$, then

$$\frac{1}{\Gamma(1 + \varphi)} \int_c^d |h(\varpi)f(\varpi)| (d\varpi)^\varphi \leq \left(\frac{1}{\Gamma(1 + \varphi)} \int_c^d |h(\varpi)| (d\varpi)^\varphi \right)^{1 - \frac{1}{q}} \left(\frac{1}{\Gamma(1 + \varphi)} \int_c^d |h(\varpi)| |f(\varpi)|^q (d\varpi)^\varphi \right)^{\frac{1}{q}}.$$

3. Fractal-fractional identity

This section presents a new fractal-fractional identity that serves as the main analytical tool for the derivation of our subsequent results.

Throughout the remainder of this paper, we assume that the function ϕ is not identically zero (i.e., $\phi \not\equiv 0^\varphi$ on $[0, +\infty)$). Consequently, it follows that

$$\frac{1}{\Gamma(1 + \varphi)} \int_0^1 \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\varpi\right)}{\varpi^\varphi} (d\varpi)^\varphi \neq 0,$$

for all $\mathfrak{c}, \mathfrak{d} \in \mathbb{R} \setminus \{0\}$.

Lemma 3.1. Consider a local fractional differentiable mapping $\mathfrak{h}: \mathcal{I} \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^\varphi$ on $\mathcal{I}^\circ$. If its local fractional derivative $\mathfrak{h}^{(\varphi)}$ belongs to $C_\varphi[\mathfrak{c}, \mathfrak{d}]$, where $\mathfrak{c}, \mathfrak{d} \in \mathcal{I}^\circ$ and $\mathfrak{c} < \mathfrak{d}$, then the following identity follows

$$\begin{aligned} & \left(\frac{1}{8}\right)^\varphi \left(3^\varphi \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+5\mathfrak{d}}\right) + 2^\varphi \mathfrak{h}\left(\frac{2\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+\mathfrak{d}}\right) + 3^\varphi \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c}+\mathfrak{d}}\right)\right) - \frac{\Gamma(\varphi+1)}{2^\varphi \mathcal{M}_\varphi(1)} \left(\left(\frac{1}{\mathfrak{d}}\right)^+ \phi \mathfrak{I}^{(\varphi)} \mathfrak{h}\left(g\left(\frac{1}{\mathfrak{c}}\right)\right) + \left(\frac{1}{\mathfrak{c}}\right)^- \phi \mathfrak{I}^{(\varphi)} \mathfrak{h}\left(g\left(\frac{1}{\mathfrak{d}}\right)\right)\right) \\ &= \frac{(\mathfrak{c}\mathfrak{d})^\varphi (\mathfrak{d}-\mathfrak{c})^\varphi}{\mathcal{M}_\varphi(1) \Gamma(\varphi+1)} \left(\int_0^1 \mathcal{K}(\nu, \varphi, \mathcal{M}_\varphi) \left(\frac{\mathfrak{h}^{(\varphi)}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c}+(1-\nu)\mathfrak{d}}\right)}{(\nu\mathfrak{c}+(1-\nu)\mathfrak{d})^{2\varphi}} - \frac{\mathfrak{h}^{(\varphi)}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c}+\nu\mathfrak{d}}\right)}{((1-\nu)\mathfrak{c}+\nu\mathfrak{d})^{2\varphi}} \right) (d\nu)^\varphi \right), \end{aligned}$$

where $g(u) = \frac{1}{u}$, and

$$\mathcal{M}_\varphi(\nu) = \frac{1}{\Gamma(\varphi+1)} \int_0^\nu \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\varpi\right)}{\varpi^\varphi} (d\varpi)^\varphi \quad (3.1)$$

and

$$\mathcal{K}(\nu, \varphi, \mathcal{M}_\varphi) = \begin{cases} \mathcal{M}_\varphi(\nu), & \text{if } \nu \in \left[0, \frac{1}{6}\right], \\ \mathcal{M}_\varphi(\nu) - \left(\frac{3}{8}\right)^\varphi \mathcal{M}_\varphi(1), & \text{if } \nu \in \left[\frac{1}{6}, \frac{1}{2}\right], \\ \mathcal{M}_\varphi(\nu) - \left(\frac{5}{8}\right)^\varphi \mathcal{M}_\varphi(1), & \text{if } \nu \in \left[\frac{1}{2}, \frac{5}{6}\right], \\ \mathcal{M}_\varphi(\nu) - \mathcal{M}_\varphi(1), & \text{if } \nu \in \left[\frac{5}{6}, 1\right]. \end{cases}$$

Proof. Let

$$\mathcal{J}_1 = \frac{1}{\Gamma(\varphi+1)} \int_0^{\frac{1}{6}} \mathcal{M}_\varphi(\nu) \left(\frac{\mathfrak{h}^{(\varphi)}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c}+(1-\nu)\mathfrak{d}}\right)}{(\nu\mathfrak{c}+(1-\nu)\mathfrak{d})^{2\varphi}} - \frac{\mathfrak{h}^{(\varphi)}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c}+\nu\mathfrak{d}}\right)}{((1-\nu)\mathfrak{c}+\nu\mathfrak{d})^{2\varphi}} \right) (d\nu)^\varphi,$$

$$\mathcal{J}_2 = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \left(\mathcal{M}_{\wp}(\nu) - \left(\frac{3}{8}\right)^{\wp} \mathcal{M}_{\wp}(1) \right) \left(\frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right)}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} - \frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right)}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (d\nu)^{\wp},$$

$$\mathcal{J}_3 = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\mathcal{M}_{\wp}(\nu) - \left(\frac{5}{8}\right)^{\wp} \mathcal{M}_{\wp}(1) \right) \left(\frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right)}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} - \frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right)}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (d\nu)^{\wp}$$

and

$$\mathcal{J}_4 = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 \left(\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1) \right) \left(\frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right)}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} - \frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right)}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (d\nu)^{\wp}.$$

Applying the local fractional integration by parts, $\mathcal{J}_1$ gives

$$\begin{aligned} \mathcal{J}_1 &= \frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \mathcal{M}_{\wp}(\nu) \left(\frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right)}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} - \frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right)}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (d\nu)^{\wp} \\ &= \frac{1^{\wp}}{(\mathfrak{c}\mathfrak{d})^{\wp} (\mathfrak{d} - \mathfrak{c})^{\wp}} \mathcal{M}_{\wp}(\nu) \left(\mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right) \right) \Bigg|_0^{\frac{1}{6}} \\ &\quad - \frac{1^{\wp}}{(\mathfrak{c}\mathfrak{d})^{\wp} (\mathfrak{d} - \mathfrak{c})^{\wp}} \left(\frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\nu\right)}{\nu^{\wp}} \left(\mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right) \right) (d\nu)^{\wp} \right) \\ &= \frac{1^{\wp}}{(\mathfrak{c}\mathfrak{d})^{\wp} (\mathfrak{d} - \mathfrak{c})^{\wp}} \mathcal{M}_{\wp}\left(\frac{1}{6}\right) \left(\mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + 5\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c} + \mathfrak{d}}\right) \right) - \frac{1^{\wp}}{(\mathfrak{c}\mathfrak{d})^{\wp} (\mathfrak{d} - \mathfrak{c})^{\wp}} \left(\frac{1}{\Gamma(\wp + 1)} \right. \\ &\quad \left. \int_0^{\frac{1}{6}} \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\nu\right)}{\nu^{\wp}} \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right) (d\nu)^{\wp} + \frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\nu\right)}{\nu^{\wp}} \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right) (d\nu)^{\wp} \right). \quad (3.2) \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \mathcal{J}_2 &= \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \left(\mathcal{M}_{\wp}(\nu) - \left(\frac{3}{8}\right)^{\wp} \mathcal{M}_{\wp}(1) \right) \left(\frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right)}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} - \frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right)}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (d\nu)^{\wp} \\ &= \frac{1^{\wp}}{(\mathfrak{c}\mathfrak{d})^{\wp} (\mathfrak{d} - \mathfrak{c})^{\wp}} \left(\mathcal{M}_{\wp}(\nu) - \left(\frac{3}{8}\right)^{\wp} \mathcal{M}_{\wp}(1) \right) \left(\mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}}\right) \right) \Bigg|_{\frac{1}{6}}^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& - \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(\frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\phi\left(\frac{\mathfrak{d}-c}{c\mathfrak{d}}\nu\right)}{\nu^\varphi} \left(\mathfrak{h}\left(\frac{c\mathfrak{d}}{\nu c + (1-\nu)\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{c\mathfrak{d}}{(1-\nu)c + \nu\mathfrak{d}}\right) \right) (d\nu)^\varphi \right) \\
& = \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(2^\varphi \mathcal{M}_\varphi\left(\frac{1}{2}\right) - \left(\frac{3}{4}\right)^\varphi \mathcal{M}_\varphi(1) \right) \mathfrak{h}\left(\frac{2c\mathfrak{d}}{c + \mathfrak{d}}\right) \\
& - \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(\mathcal{M}_\varphi\left(\frac{1}{6}\right) - \left(\frac{3}{8}\right)^\varphi \mathcal{M}_\varphi(1) \right) \left(\mathfrak{h}\left(\frac{6c\mathfrak{d}}{c + 5\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{6c\mathfrak{d}}{5c + \mathfrak{d}}\right) \right) \\
& - \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(\frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\phi\left(\frac{\mathfrak{d}-c}{c\mathfrak{d}}\nu\right)}{\nu^\varphi} \mathfrak{h}\left(\frac{c\mathfrak{d}}{\nu c + (1-\nu)\mathfrak{d}}\right) (d\nu)^\varphi \right. \\
& \left. + \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\phi\left(\frac{\mathfrak{d}-c}{c\mathfrak{d}}\nu\right)}{\nu^\varphi} \mathfrak{h}\left(\frac{c\mathfrak{d}}{(1-\nu)c + \nu\mathfrak{d}}\right) (d\nu)^\varphi \right) \tag{3.3}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{J}_3 & = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\mathcal{M}_\varphi(\nu) - \left(\frac{5}{8}\right)^\varphi \mathcal{M}_\varphi(1) \right) \left(\frac{\mathfrak{h}^{(\varphi)}\left(\frac{c\mathfrak{d}}{\nu c + (1-\nu)\mathfrak{d}}\right)}{(\nu c + (1-\nu)\mathfrak{d})^{2\varphi}} - \frac{\mathfrak{h}^{(\varphi)}\left(\frac{c\mathfrak{d}}{(1-\nu)c + \nu\mathfrak{d}}\right)}{((1-\nu)c + \nu\mathfrak{d})^{2\varphi}} \right) (d\nu)^\varphi \\
& = \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(\mathcal{M}_\varphi(\nu) - \left(\frac{5}{8}\right)^\varphi \mathcal{M}_\varphi(1) \right) \left(\mathfrak{h}\left(\frac{c\mathfrak{d}}{\nu c + (1-\nu)\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{c\mathfrak{d}}{(1-\nu)c + \nu\mathfrak{d}}\right) \right) \Big|_{\frac{1}{2}}^{\frac{5}{6}} \\
& - \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(\frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\phi\left(\frac{\mathfrak{d}-c}{c\mathfrak{d}}\nu\right)}{\nu^\varphi} \left(\mathfrak{h}\left(\frac{c\mathfrak{d}}{\nu c + (1-\nu)\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{c\mathfrak{d}}{(1-\nu)c + \nu\mathfrak{d}}\right) \right) (d\nu)^\varphi \right) \\
& = \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(\mathcal{M}_\varphi\left(\frac{5}{6}\right) - \left(\frac{5}{8}\right)^\varphi \mathcal{M}_\varphi(1) \right) \left(\mathfrak{h}\left(\frac{6c\mathfrak{d}}{5c + \mathfrak{d}}\right) + \mathfrak{h}\left(\frac{6c\mathfrak{d}}{c + 5\mathfrak{d}}\right) \right) \\
& - \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(2^\varphi \mathcal{M}_\varphi\left(\frac{1}{2}\right) - \left(\frac{5}{4}\right)^\varphi \mathcal{M}_\varphi(1) \right) \mathfrak{h}\left(\frac{2c\mathfrak{d}}{c + \mathfrak{d}}\right) \\
& - \frac{1^\varphi}{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi} \left(\frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\phi\left(\frac{\mathfrak{d}-c}{c\mathfrak{d}}\nu\right)}{\nu^\varphi} \mathfrak{h}\left(\frac{c\mathfrak{d}}{\nu c + (1-\nu)\mathfrak{d}}\right) (d\nu)^\varphi \right. \\
& \left. + \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\phi\left(\frac{\mathfrak{d}-c}{c\mathfrak{d}}\nu\right)}{\nu^\varphi} \mathfrak{h}\left(\frac{c\mathfrak{d}}{(1-\nu)c + \nu\mathfrak{d}}\right) (d\nu)^\varphi \right) \tag{3.4}
\end{aligned}$$

and

$$\begin{aligned}
 \mathcal{J}_4 &= \frac{1}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 (\mathcal{M}_\wp(\nu) - \mathcal{M}_\wp(1)) \left(\frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c}+(1-\nu)\mathfrak{d}}\right)}{(\nu\mathfrak{c}+(1-\nu)\mathfrak{d})^{2\wp}} - \frac{\mathfrak{h}^{(\wp)}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c}+\nu\mathfrak{d}}\right)}{((1-\nu)\mathfrak{c}+\nu\mathfrak{d})^{2\wp}} \right) (d\nu)^\wp \\
 &= \frac{1^\wp}{(\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} (\mathcal{M}_\wp(\nu) - \mathcal{M}_\wp(1)) \left(\mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c}+(1-\nu)\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c}+\nu\mathfrak{d}}\right) \right) \Bigg|_{\frac{5}{6}}^1 \\
 &\quad - \frac{1^\wp}{(\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\nu\right)}{\nu^\wp} \left(\mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c}+(1-\nu)\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c}+\nu\mathfrak{d}}\right) \right) (d\nu)^\wp \right) \\
 &= \frac{1^\wp}{(\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} \left(\mathcal{M}_\wp(1) - \mathcal{M}_\wp\left(\frac{5}{6}\right) \right) \left(\mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c}+\mathfrak{d}}\right) + \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+5\mathfrak{d}}\right) \right) \\
 &\quad - \frac{1^\wp}{(\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\nu\right)}{\nu^\wp} \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c}+(1-\nu)\mathfrak{d}}\right) (d\nu)^\wp \right. \\
 &\quad \left. + \frac{1}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\nu\right)}{\nu^\wp} \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c}+\nu\mathfrak{d}}\right) (d\nu)^\wp \right). \tag{3.5}
 \end{aligned}$$

Adding (3.2)–(3.5), and then using the changes of variable $\frac{1}{u} = \frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c}+(1-\nu)\mathfrak{d}}$ and $\frac{1}{v} = \frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c}+\nu\mathfrak{d}}$, we get

$$\begin{aligned}
 \sum_{k=1}^4 \mathcal{J}_k &= \frac{\mathcal{M}_\wp(1)}{4^\wp (\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} \left(3^\wp \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+5\mathfrak{d}}\right) + 2^\wp \mathfrak{h}\left(\frac{2\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+\mathfrak{d}}\right) + 3^\wp \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c}+\mathfrak{d}}\right) \right) \\
 &\quad - \frac{1^\wp}{(\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} \left(\frac{1}{\Gamma(\wp+1)} \int_0^1 \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\nu\right)}{\nu^\wp} \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c}+(1-\nu)\mathfrak{d}}\right) (d\nu)^\wp \right. \\
 &\quad \left. + \frac{1}{\Gamma(\wp+1)} \int_0^1 \frac{\phi\left(\frac{\mathfrak{d}-\mathfrak{c}}{\mathfrak{c}\mathfrak{d}}\nu\right)}{\nu^\wp} \mathfrak{h}\left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c}+\nu\mathfrak{d}}\right) (d\nu)^\wp \right) \\
 &= \frac{\mathcal{M}_\wp(1)}{4^\wp (\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} \left(3^\wp \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+5\mathfrak{d}}\right) + 2^\wp \mathfrak{h}\left(\frac{2\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+\mathfrak{d}}\right) + 3^\wp \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c}+\mathfrak{d}}\right) \right) \\
 &\quad - \frac{1^\wp}{(\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} \left(\int_{\frac{1}{\mathfrak{d}}}^{\frac{1}{\mathfrak{c}}} \frac{\phi\left(\frac{1}{\mathfrak{c}}-u\right)}{\left(\frac{1}{\mathfrak{c}}-u\right)^\wp} \mathfrak{h}\left(\frac{1}{u}\right) (du)^\wp + \int_{\frac{1}{\mathfrak{d}}}^{\frac{1}{\mathfrak{c}}} \frac{\phi\left(v-\frac{1}{\mathfrak{d}}\right)}{\left(v-\frac{1}{\mathfrak{d}}\right)^\wp} \mathfrak{h}\left(\frac{1}{v}\right) (dv)^\wp \right) \\
 &= \frac{\mathcal{M}_\wp(1)}{4^\wp (\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d}-\mathfrak{c})^\wp} \left(3^\wp \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+5\mathfrak{d}}\right) + 2^\wp \mathfrak{h}\left(\frac{2\mathfrak{c}\mathfrak{d}}{\mathfrak{c}+\mathfrak{d}}\right) + 3^\wp \mathfrak{h}\left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c}+\mathfrak{d}}\right) \right)
 \end{aligned}$$

$$- \frac{\Gamma(\varphi + 1)}{(cd)^\varphi (d - c)^\varphi} \left(\left(\frac{1}{d}\right)^+ {}^\phi I^{(\varphi)} h \left(g \left(\frac{1}{c} \right) \right) + \left(\frac{1}{c}\right)^- {}^\phi I^{(\varphi)} h \left(g \left(\frac{1}{d} \right) \right) \right). \quad (3.6)$$

Multiplying both sides of (3.6) by $\frac{(cd)^\varphi (d-c)^\varphi}{2^\varphi \mathcal{M}_\varphi(1)}$, we get the desired result. $\square$

Corollary 3.2. By setting $\phi(\varpi) = \varpi$ along with $\varphi = 1$, Lemma 3.1 reduces to the following identity for the classical Riemann integral:

$$\begin{aligned} & \frac{1}{8} \left(3h \left(\frac{6cd}{c+5d} \right) + 2h \left(\frac{2cd}{c+d} \right) + 3h \left(\frac{6cd}{5c+d} \right) \right) - \frac{cd}{d-c} \int_c^d \frac{h(\varpi)}{\varpi^2} d\varpi \\ &= cd(d-c) \left(\int_0^1 \tilde{\mathcal{K}}(\nu) \left(\frac{h' \left(\frac{cd}{\nu c + (1-\nu)d} \right)}{(\nu c + (1-\nu)d)^2} - \frac{h' \left(\frac{cd}{(1-\nu)c + \nu d} \right)}{((1-\nu)c + \nu d)^2} \right) d\nu \right), \end{aligned}$$

where

$$\tilde{\mathcal{K}}(\nu) = \begin{cases} \nu, & \text{if } \nu \in \left[0, \frac{1}{6}\right], \\ \nu - \frac{3}{8}, & \text{if } \nu \in \left[\frac{1}{6}, \frac{1}{2}\right], \\ \nu - \frac{5}{8}, & \text{if } \nu \in \left[\frac{1}{2}, \frac{5}{6}\right], \\ \nu - 1, & \text{if } \nu \in \left[\frac{5}{6}, 1\right]. \end{cases}$$

4. Fractal-fractional Maclaurin-type inequalities via generalized harmonic convexity

This section is devoted to the derivation of new Maclaurin-type inequalities for functions with generalized harmonically convex local fractional derivatives.

Theorem 4.1. Consider a local fractional differentiable mapping $h: [c, d] \rightarrow \mathbb{R}^\varphi$, such that $h \in D_\varphi[c, d]$ and $h^{(\varphi)} \in C_\varphi[c, d]$ with $0 < c < d$. If $|h^{(\varphi)}|$ is generalized harmonically convex on this interval, then the following estimate is valid:

$$\begin{aligned} & \left| \left(\frac{1}{8}\right)^\varphi \left(3^\varphi h \left(\frac{6cd}{c+5d} \right) + 2^\varphi h \left(\frac{2cd}{c+d} \right) + 3^\varphi h \left(\frac{6cd}{5c+d} \right) \right) \right. \\ & \quad \left. - \frac{\Gamma(\varphi + 1)}{2^\varphi \mathcal{M}_\varphi(1)} \left(\left(\frac{1}{d}\right)^+ {}^\phi I^{(\varphi)} h \left(g \left(\frac{1}{c} \right) \right) + \left(\frac{1}{c}\right)^- {}^\phi I^{(\varphi)} h \left(g \left(\frac{1}{d} \right) \right) \right) \right| \\ & \leq \frac{(cd)^\varphi (d-c)^\varphi}{\mathcal{M}_\varphi(1)} \left(\left(\sum_{i=1}^8 \mathcal{C}_i(c, d) \right) |h^{(\varphi)}(c)| + \left(\sum_{i=9}^{16} \mathcal{C}_i(c, d) \right) |h^{(\varphi)}(d)| \right), \end{aligned}$$

where $\mathcal{M}_\varphi(\nu)$ is defined as in (3.1) and $\mathcal{C}_i(c, d)$ for $i = 1$ to 16 are defined as in (4.2) to (4.17), respectively.

Proof. In view of Lemma 3.1 combined with the properties of the modulus, we deduce that

$$\begin{aligned}
& \left| \left(\frac{1}{8} \right)^{\wp} \left(3^{\wp} \mathfrak{h} \left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + 5\mathfrak{d}} \right) + 2^{\wp} \mathfrak{h} \left(\frac{2\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + \mathfrak{d}} \right) + 3^{\wp} \mathfrak{h} \left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c} + \mathfrak{d}} \right) \right) - \frac{\Gamma(\wp + 1)}{2^{\wp} \mathcal{M}_{\wp}(1)} \left(\left(\frac{1}{\mathfrak{d}} \right)^{+} \mathfrak{I}^{(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{c}} \right) \right) + \left(\frac{1}{\mathfrak{c}} \right)^{-} \mathfrak{I}^{(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{d}} \right) \right) \right) \right| \\
& \leq \frac{(\mathfrak{c}\mathfrak{d})^{\wp} (\mathfrak{d} - \mathfrak{c})^{\wp}}{\mathcal{M}_{\wp}(1)} \left[\frac{1}{\Gamma(\wp+1)} \int_0^{\frac{1}{6}} |\mathcal{M}_{\wp}(\nu)| \left(\frac{\left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}} \right) \right|}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} + \frac{\left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}} \right) \right|}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (\mathrm{d}\nu)^{\wp} \right. \\
& \quad + \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \left| \mathcal{M}_{\wp}(\nu) - \left(\frac{3}{8} \right)^{\wp} \mathcal{M}_{\wp}(1) \right| \left(\frac{\left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}} \right) \right|}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} \frac{\left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}} \right) \right|}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (\mathrm{d}\nu)^{\wp} \\
& \quad + \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \left| \mathcal{M}_{\wp}(\nu) - \left(\frac{5}{8} \right)^{\wp} \mathcal{M}_{\wp}(1) \right| \left(\frac{\left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}} \right) \right|}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} + \frac{\left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}} \right) \right|}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (\mathrm{d}\nu)^{\wp} \\
& \quad \left. + \frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 |\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)| \left(\frac{\left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1-\nu)\mathfrak{d}} \right) \right|}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} + \frac{\left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{(1-\nu)\mathfrak{c} + \nu\mathfrak{d}} \right) \right|}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} \right) (\mathrm{d}\nu)^{\wp} \right]. \quad (4.1)
\end{aligned}$$

By using the generalized harmonic convexity of $|\mathfrak{h}^{(\wp)}|$ in inequality (4.1), it yields

$$\begin{aligned}
& \left| \left(\frac{1}{8} \right)^{\wp} \left(3^{\wp} \mathfrak{h} \left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + 5\mathfrak{d}} \right) + 2^{\wp} \mathfrak{h} \left(\frac{2\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + \mathfrak{d}} \right) + 3^{\wp} \mathfrak{h} \left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c} + \mathfrak{d}} \right) \right) - \frac{\Gamma(\wp + 1)}{2^{\wp} \mathcal{M}_{\wp}(1)} \left(\left(\frac{1}{\mathfrak{d}} \right)^{+} \mathfrak{I}^{(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{c}} \right) \right) + \left(\frac{1}{\mathfrak{c}} \right)^{-} \mathfrak{I}^{(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{d}} \right) \right) \right) \right| \\
& \leq \frac{(\mathfrak{c}\mathfrak{d})^{\wp} (\mathfrak{d} - \mathfrak{c})^{\wp}}{\mathcal{M}_{\wp}(1)} \left[\frac{1}{\Gamma(\wp+1)} \int_0^{\frac{1}{6}} |\mathcal{M}_{\wp}(\nu)| \left(\frac{1^{\wp}}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} ((1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{c})| + \nu^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|) \right. \right. \\
& \quad \left. \left. + \frac{1^{\wp}}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} (\nu^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{c})| + (1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|) \right) (\mathrm{d}\nu)^{\wp} \right. \\
& \quad + \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \left| \mathcal{M}_{\wp}(\nu) - \left(\frac{3}{8} \right)^{\wp} \mathcal{M}_{\wp}(1) \right| \left(\frac{1^{\wp}}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} ((1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{c})| + \nu^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|) \right. \\
& \quad \left. \left. + \frac{1^{\wp}}{((1-\nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp}} (\nu^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{c})| + (1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|) \right) (\mathrm{d}\nu)^{\wp} \right. \\
& \quad \left. + \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \left| \mathcal{M}_{\wp}(\nu) - \left(\frac{5}{8} \right)^{\wp} \mathcal{M}_{\wp}(1) \right| \left(\frac{1^{\wp}}{(\nu\mathfrak{c} + (1-\nu)\mathfrak{d})^{2\wp}} ((1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{c})| + \nu^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} \left(\nu^\varphi |\mathfrak{h}^{(\varphi)}(c)| + (1-\nu)^\varphi |\mathfrak{h}^{(\varphi)}(d)| \right) (d\nu)^\varphi \\
& + \frac{1}{\Gamma(\varphi+1)} \int_{\frac{5}{6}}^1 |\mathcal{M}_\varphi(\nu) - \mathcal{M}_\varphi(1)| \left(\frac{1^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} \left((1-\nu)^\varphi |\mathfrak{h}^{(\varphi)}(c)| + \nu^\varphi |\mathfrak{h}^{(\varphi)}(d)| \right) \right. \\
& + \left. \frac{1^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} \left(\nu^\varphi |\mathfrak{h}^{(\varphi)}(c)| + (1-\nu)^\varphi |\mathfrak{h}^{(\varphi)}(d)| \right) \right) (d\nu)^\varphi \\
& = \frac{(cd)^\varphi (d-c)^\varphi}{\mathcal{M}_\varphi(1)} \left(\left(\frac{1}{\Gamma(\varphi+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_\varphi(\nu)|(1-\nu)^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\frac{1}{\Gamma(\varphi+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_\varphi(\nu)|\nu^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(d)| \right. \\
& + \left(\frac{1}{\Gamma(\varphi+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_\varphi(\nu)|\nu^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\frac{1}{\Gamma(\varphi+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_\varphi(\nu)|(1-\nu)^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(d)| \\
& + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_\varphi(\nu) - (\frac{3}{8})^\varphi \mathcal{M}_\varphi(1)|(1-\nu)^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_\varphi(\nu) - (\frac{3}{8})^\varphi \mathcal{M}_\varphi(1)|\nu^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(d)| \\
& + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_\varphi(\nu) - (\frac{3}{8})^\varphi \mathcal{M}_\varphi(1)|\nu^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_\varphi(\nu) - (\frac{3}{8})^\varphi \mathcal{M}_\varphi(1)|(1-\nu)^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(d)| \\
& + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_\varphi(\nu) - (\frac{5}{8})^\varphi \mathcal{M}_\varphi(1)|(1-\nu)^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_\varphi(\nu) - (\frac{5}{8})^\varphi \mathcal{M}_\varphi(1)|\nu^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(d)| \\
& + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_\varphi(\nu) - (\frac{5}{8})^\varphi \mathcal{M}_\varphi(1)|\nu^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_\varphi(\nu) - (\frac{5}{8})^\varphi \mathcal{M}_\varphi(1)|(1-\nu)^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(d)| \\
& + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_\varphi(\nu) - \mathcal{M}_\varphi(1)|(1-\nu)^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_\varphi(\nu) - \mathcal{M}_\varphi(1)|\nu^\varphi}{(\nu c + (1-\nu)d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(d)| \\
& + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_\varphi(\nu) - \mathcal{M}_\varphi(1)|\nu^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\frac{1}{\Gamma(\varphi+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_\varphi(\nu) - \mathcal{M}_\varphi(1)|(1-\nu)^\varphi}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^\varphi \right) |\mathfrak{h}^{(\varphi)}(d)|
\end{aligned}$$

$$= \frac{(c d)^{\varphi} (d - c)^{\varphi}}{\mathcal{M}_{\varphi}(1)} \left(\left(\sum_{i=1}^8 \mathcal{C}_i(c, d) \right) |\mathfrak{h}^{(\varphi)}(c)| + \left(\sum_{i=9}^{16} \mathcal{C}_i(c, d) \right) |\mathfrak{h}^{(\varphi)}(d)| \right),$$

where

$$\mathcal{C}_1(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\varphi}(\nu)| (1 - \nu)^{\varphi}}{(\nu c + (1 - \nu) d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.2)$$

$$\mathcal{C}_2(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\varphi}(\nu)| \nu^{\varphi}}{((1 - \nu) c + \nu d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.3)$$

$$\mathcal{C}_3(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\left| \mathcal{M}_{\varphi}(\nu) - \left(\frac{3}{8}\right)^{\varphi} \mathcal{M}_{\varphi}(1) \right| (1 - \nu)^{\varphi}}{(\nu c + (1 - \nu) d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.4)$$

$$\mathcal{C}_4(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\left| \mathcal{M}_{\varphi}(\nu) - \left(\frac{3}{8}\right)^{\varphi} \mathcal{M}_{\varphi}(1) \right| \nu^{\varphi}}{((1 - \nu) c + \nu d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.5)$$

$$\mathcal{C}_5(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\left| \mathcal{M}_{\varphi}(\nu) - \left(\frac{5}{8}\right)^{\varphi} \mathcal{M}_{\varphi}(1) \right| (1 - \nu)^{\varphi}}{(\nu c + (1 - \nu) d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.6)$$

$$\mathcal{C}_6(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\left| \mathcal{M}_{\varphi}(\nu) - \left(\frac{5}{8}\right)^{\varphi} \mathcal{M}_{\varphi}(1) \right| \nu^{\varphi}}{((1 - \nu) c + \nu d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.7)$$

$$\mathcal{C}_7(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{5}{6}}^1 \frac{\left| \mathcal{M}_{\varphi}(\nu) - \mathcal{M}_{\varphi}(1) \right| (1 - \nu)^{\varphi}}{(\nu c + (1 - \nu) d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.8)$$

$$\mathcal{C}_8(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{5}{6}}^1 \frac{\left| \mathcal{M}_{\varphi}(\nu) - \mathcal{M}_{\varphi}(1) \right| \nu^{\varphi}}{((1 - \nu) c + \nu d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.9)$$

$$\mathcal{C}_9(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\varphi}(\nu)| \nu^{\varphi}}{(\nu c + (1 - \nu) d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.10)$$

$$\mathcal{C}_{10}(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\varphi}(\nu)| (1 - \nu)^{\varphi}}{((1 - \nu) c + \nu d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.11)$$

$$\mathcal{C}_{11}(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\left| \mathcal{M}_{\varphi}(\nu) - \left(\frac{3}{8}\right)^{\varphi} \mathcal{M}_{\varphi}(1) \right| \nu^{\varphi}}{(c\nu + (1-\nu)d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.12)$$

$$\mathcal{C}_{12}(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\left| \mathcal{M}_{\varphi}(\nu) - \left(\frac{3}{8}\right)^{\varphi} \mathcal{M}_{\varphi}(1) \right| (1-\nu)^{\varphi}}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.13)$$

$$\mathcal{C}_{13}(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\left| \mathcal{M}_{\varphi}(\nu) - \left(\frac{5}{8}\right)^{\varphi} \mathcal{M}_{\varphi}(1) \right| \nu^{\varphi}}{(c\nu + (1-\nu)d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.14)$$

$$\mathcal{C}_{14}(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\left| \mathcal{M}_{\varphi}(\nu) - \left(\frac{5}{8}\right)^{\varphi} \mathcal{M}_{\varphi}(1) \right| (1-\nu)^{\varphi}}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^{\varphi}, \quad (4.15)$$

$$\mathcal{C}_{15}(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{5}{6}}^1 \frac{\left| \mathcal{M}_{\varphi}(\nu) - \mathcal{M}_{\varphi}(1) \right| \nu^{\varphi}}{(c\nu + (1-\nu)d)^{2\varphi}} (d\nu)^{\varphi} \quad (4.16)$$

and

$$\mathcal{C}_{16}(c, d) = \frac{1}{\Gamma(\varphi + 1)} \int_{\frac{5}{6}}^1 \frac{\left| \mathcal{M}_{\varphi}(\nu) - \mathcal{M}_{\varphi}(1) \right| (1-\nu)^{\varphi}}{((1-\nu)c + \nu d)^{2\varphi}} (d\nu)^{\varphi}. \quad (4.17)$$

This completes the proof. $\square$

Corollary 4.2. Setting $\phi(\nu) = \nu^{\varphi}$, Theorem 4.1 yields

$$\begin{aligned} & \left| \left(\frac{1}{8} \right)^{\varphi} \left(3^{\varphi} \mathfrak{h} \left(\frac{6cd}{c+5d} \right) + 2^{\varphi} \mathfrak{h} \left(\frac{2cd}{c+d} \right) + 3^{\varphi} \mathfrak{h} \left(\frac{6cd}{5c+d} \right) \right) \right. \\ & \quad \left. - \frac{(cd)^{\varphi} \Gamma(\varphi + 1)}{(d-c)^{\varphi} 2^{\varphi}} \left(\left(\frac{1}{d} \right)^{+} I^{(\varphi)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{c} \right) \right) + \left(\frac{1}{c} \right)^{-} I^{(\varphi)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{d} \right) \right) \right) \right| \\ & \leq \frac{(cd)^{\varphi} (d-c)^{\varphi}}{\Gamma(\varphi + 1)} \left(\widetilde{\mathcal{C}}_1(c, d) \left| \mathfrak{h}^{(\varphi)}(c) \right| + \widetilde{\mathcal{C}}_2(c, d) \left| \mathfrak{h}^{(\varphi)}(d) \right| \right), \end{aligned}$$

where

$$\begin{aligned} \widetilde{\mathcal{C}}_1(c, d) = & \frac{1^{\varphi}}{(d-c)^{\varphi}} \left(-\frac{2c}{(c+5d)(5c+d)} \right)^{\varphi} + \frac{2^{\varphi} (c+d)^{\varphi}}{(d-c)^{3\varphi}} \left(\ln_{\varphi} d - \ln_{\varphi} \frac{c+5d}{6} \right) \\ & + \frac{4^{\varphi} c^{\varphi}}{(d-c)^{3\varphi}} \left(\ln_{\varphi} c - \ln_{\varphi} \frac{5c+d}{6} \right) + \frac{(11c+5d)^{\varphi}}{4^{\varphi} (d-c)^{3\varphi}} \left(2^{\varphi} \ln_{\varphi} \frac{3c+5d}{8} - \ln_{\varphi} \frac{c+d}{2} - \ln_{\varphi} \frac{c+5d}{6} \right) \\ & + \frac{(13c+3d)^{\varphi}}{4^{\varphi} (d-c)^{3\varphi}} \left(2^{\varphi} \ln_{\varphi} \frac{5c+3d}{8} - \ln_{\varphi} \frac{c+d}{2} - \ln_{\varphi} \frac{5c+d}{6} \right) \end{aligned} \quad (4.18)$$

and

$$\begin{aligned}\widetilde{\mathcal{E}}_2(c, d) = & \frac{1^\varphi}{(d-c)^\varphi} \left(\frac{2d}{(5c+d)(c+5d)} \right)^\varphi + \frac{2^\varphi (c+d)^\varphi}{(d-c)^{3\varphi}} \left(\ln_\varphi \frac{5c+d}{6} - \ln_\varphi c \right) \\ & + \frac{4^\varphi d^\varphi}{(d-c)^{3\varphi}} \left(\ln_\varphi \frac{c+5d}{6} - \ln_\varphi d \right) + \frac{(3c+13d)^\varphi}{4^\varphi (d-c)^{3\varphi}} \left(\ln_\varphi \frac{c+d}{2} + \ln_\varphi \frac{c+5d}{6} - 2^\varphi \ln_\varphi \frac{3c+5d}{8} \right) \\ & + \frac{(5c+11d)^\varphi}{4^\varphi (d-c)^{3\varphi}} \left(\ln_\varphi \frac{5c+d}{6} + \ln_\varphi \frac{c+d}{2} - 2^\varphi \ln_\varphi \frac{5c+3d}{8} \right),\end{aligned}\quad (4.19)$$

where $\ln_\varphi$ represents the inverse function of the Mittag-Leffler function.

Corollary 4.3. By choosing $\varphi = 1$, Corollary 4.2 yields

$$\left| \frac{1}{8} \left(3h \left(\frac{6cd}{c+5d} \right) + 2h \left(\frac{2cd}{c+d} \right) + 3h \left(\frac{6cd}{5c+d} \right) \right) - \frac{cd}{d-c} \int_c^d \frac{h(\varpi)}{\varpi^2} d\varpi \right| \leq \mathcal{E}_1(c, d) |h'(c)| + \mathcal{E}_2(c, d) |h'(d)|,$$

where

$$\begin{aligned}\mathcal{E}_1(c, d) = & -\frac{2c^2d}{(c+5d)(5c+d)} + \frac{2cd(c+d)}{(d-c)^2} \ln \frac{6d}{c+5d} + \frac{4c^2d}{(d-c)^2} \ln \frac{6c}{5c+d} \\ & + \frac{cd(11c+5d)}{4(d-c)^2} \ln \frac{3(3c+5d)^2}{16(c+d)(c+5d)} + \frac{cd(13c+3d)}{4(d-c)^2} \ln \frac{3(5c+3d)^2}{16(c+d)(c+5d)}\end{aligned}\quad (4.20)$$

and

$$\begin{aligned}\mathcal{E}_2(c, d) = & \frac{2cd^2}{(5c+d)(c+5d)} + \frac{2cd(c+d)}{(d-c)^{2\varphi}} \ln \frac{5c+d}{6c} + \frac{4cd^2}{(d-c)^2} \ln \frac{c+5d}{6d} \\ & + \frac{cd(3c+13d)}{4(d-c)^2} \ln \frac{16(c+d)(c+5d)}{3(3c+5d)^2} + \frac{cd(5c+11d)}{4(d-c)^2} \ln \frac{16(c+d)(5c+d)}{3(5c+3d)^2}.\end{aligned}\quad (4.21)$$

5. Additional outcomes

In this section, we derive several further estimates by applying the generalized Hölder and power-mean inequalities within the fractal-fractional setting.

Theorem 5.1. Consider a local fractional differentiable mapping $h: [c, d] \rightarrow \mathbb{R}^\varphi$, such that $h^{(\varphi)} \in C_\varphi[c, d]$ with $0 < c < d$. If $|h^{(\varphi)}|^q$ is generalized harmonically convex on this interval, then the following estimate is valid:

$$\begin{aligned}& \left| \left(\frac{1}{8} \right)^\varphi \left(3^\varphi h \left(\frac{6cd}{c+5d} \right) + 2^\varphi h \left(\frac{2cd}{c+d} \right) + 3^\varphi h \left(\frac{6cd}{5c+d} \right) \right) - \frac{\Gamma(\varphi+1)}{2^\varphi \mathcal{M}_\varphi(1)} \left(\left(\frac{1}{d} \right)^+ \phi_{\Gamma(\varphi)} h \left(g \left(\frac{1}{c} \right) \right) + \left(\frac{1}{c} \right)^- \phi_{\Gamma(\varphi)} h \left(g \left(\frac{1}{d} \right) \right) \right) \right| \\ & \leq \frac{(cd)^\varphi (d-c)^\varphi}{\mathcal{M}_\varphi(1)} \left(\frac{\Gamma(1+\varphi)}{3^\varphi \Gamma(1+2\varphi)} \right)^{\frac{1}{q}} \left(\left((\mathcal{D}_1(c, d, p))^{\frac{1}{p}} + (\mathcal{D}_8(c, d, p))^{\frac{1}{p}} \right) \left(\frac{11^\varphi |h^{(\varphi)}(c)|^q + |h^{(\varphi)}(d)|^q}{12^\varphi} \right)^{\frac{1}{q}} \right)\end{aligned}$$

$$\begin{aligned}
& + \left((\mathcal{D}_2(c, d, p))^{\frac{1}{p}} + (\mathcal{D}_7(c, d, p))^{\frac{1}{p}} \right) \left(\frac{|\mathfrak{h}^{(\wp)}(c)|^q + 11^\wp |\mathfrak{h}^{(\wp)}(d)|^q}{12^\wp} \right)^{\frac{1}{q}} \\
& + (2^\wp)^{\frac{1}{q}} \left((\mathcal{D}_3(c, d, p))^{\frac{1}{p}} + (\mathcal{D}_6(c, d, p))^{\frac{1}{p}} \right) \left(\frac{2^\wp |\mathfrak{h}^{(\wp)}(c)|^q + |\mathfrak{h}^{(\wp)}(d)|^q}{3^\wp} \right)^{\frac{1}{q}} \\
& + (2^\wp)^{\frac{1}{q}} \left((\mathcal{D}_4(c, d, p))^{\frac{1}{p}} + (\mathcal{D}_5(c, d, p))^{\frac{1}{p}} \right) \left(\frac{|\mathfrak{h}^{(\wp)}(c)|^q + 2^\wp |\mathfrak{h}^{(\wp)}(d)|^q}{3^\wp} \right)^{\frac{1}{q}},
\end{aligned}$$

where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\mathcal{D}_1(c, d, p) = \frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_\wp(v)|^p}{(\nu c + (1 - \nu)d)^{2\wp p}} (dv)^\wp, \quad (5.1)$$

$$\mathcal{D}_2(c, d, p) = \frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_\wp(v)|^p}{((1 - \nu)c + \nu d)^{2\wp p}} (dv)^\wp, \quad (5.2)$$

$$\mathcal{D}_3(c, d, p) = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\left| \mathcal{M}_\wp(v) - \left(\frac{3}{8}\right)^\wp \mathcal{M}_\wp(1) \right|^p}{(\nu c + (1 - \nu)d)^{2\wp p}} (dv)^\wp, \quad (5.3)$$

$$\mathcal{D}_4(c, d, p) = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\left| \mathcal{M}_\wp(v) - \left(\frac{3}{8}\right)^\wp \mathcal{M}_\wp(1) \right|^p}{((1 - \nu)c + \nu d)^{2\wp p}} (dv)^\wp, \quad (5.4)$$

$$\mathcal{D}_5(c, d, p) = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\left| \mathcal{M}_\wp(v) - \left(\frac{5}{8}\right)^\wp \mathcal{M}_\wp(1) \right|^p}{(\nu c + (1 - \nu)d)^{2\wp p}} (dv)^\wp, \quad (5.5)$$

$$\mathcal{D}_6(c, d, p) = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{\left| \mathcal{M}_\wp(v) - \left(\frac{5}{8}\right)^\wp \mathcal{M}_\wp(1) \right|^p}{((1 - \nu)c + \nu d)^{2\wp p}} (dv)^\wp, \quad (5.6)$$

$$\mathcal{D}_7(c, d, p) = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_\wp(v) - \mathcal{M}_\wp(1)|^p}{(\nu c + (1 - \nu)d)^{2\wp p}} (dv)^\wp, \quad (5.7)$$

$$\mathcal{D}_8(c, d, p) = \frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_\wp(v) - \mathcal{M}_\wp(1)|^p}{((1 - \nu)c + \nu d)^{2\wp p}} (dv)^\wp. \quad (5.8)$$

Proof. In view of Lemma 3.1, together with the modulus properties and the generalized Hölder inequality, we deduce that

$$\begin{aligned}
& \left| \left(\frac{1}{8} \right)^{\wp} \left(3^{\wp} \mathfrak{h} \left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + 5\mathfrak{d}} \right) + 2^{\wp} \mathfrak{h} \left(\frac{2\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + \mathfrak{d}} \right) + 3^{\wp} \mathfrak{h} \left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c} + \mathfrak{d}} \right) \right) - \frac{\Gamma(\wp + 1)}{2^{\wp} \mathcal{M}_{\wp}(1)} \left(\left(\frac{1}{\mathfrak{d}} \right)^{+ \phi \Gamma(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{c}} \right) \right) + \left(\frac{1}{\mathfrak{c}} \right)^{- \phi \Gamma(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{d}} \right) \right) \right) \right| \\
& \leq \frac{(\mathfrak{c}\mathfrak{d})^{\wp} (\mathfrak{d} - \mathfrak{c})^{\wp}}{\mathcal{M}_{\wp}(1)} \left(\left(\frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\wp}(\nu)|^p}{(\nu\mathfrak{c} + (1 - \nu)\mathfrak{d})^{2\wp p}} (d\nu)^{\wp} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1 - \nu)\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \right. \\
& + \left(\frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\wp}(\nu)|^p}{((1 - \nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp p}} (d\nu)^{\wp} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} \left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{(1 - \nu)\mathfrak{c} + \nu\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_{\wp}(\nu) - \left(\frac{3}{8} \right)^{\wp} \mathcal{M}_{\wp}(1)|^p}{(\nu\mathfrak{c} + (1 - \nu)\mathfrak{d})^{2\wp p}} (d\nu)^{\wp} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1 - \nu)\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_{\wp}(\nu) - \left(\frac{3}{8} \right)^{\wp} \mathcal{M}_{\wp}(1)|^p}{((1 - \nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp p}} (d\nu)^{\wp} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{(1 - \nu)\mathfrak{c} + \nu\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_{\wp}(\nu) - \left(\frac{5}{8} \right)^{\wp} \mathcal{M}_{\wp}(1)|^p}{(\nu\mathfrak{c} + (1 - \nu)\mathfrak{d})^{2\wp p}} (d\nu)^{\wp} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1 - \nu)\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_{\wp}(\nu) - \left(\frac{5}{8} \right)^{\wp} \mathcal{M}_{\wp}(1)|^p}{((1 - \nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp p}} (d\nu)^{\wp} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{(1 - \nu)\mathfrak{c} + \nu\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)|^p}{(\nu\mathfrak{c} + (1 - \nu)\mathfrak{d})^{2\wp p}} (d\nu)^{\wp} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 \left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{\nu\mathfrak{c} + (1 - \nu)\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left. \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)|^p}{((1 - \nu)\mathfrak{c} + \nu\mathfrak{d})^{2\wp p}} (d\nu)^{\wp} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 \left| \mathfrak{h}^{(\wp)} \left(\frac{\mathfrak{c}\mathfrak{d}}{(1 - \nu)\mathfrak{c} + \nu\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \right). \quad (5.9)
\end{aligned}$$

By utilizing the generalized harmonic convexity of $|\mathfrak{h}^{(\wp)}|^q$ in inequality (5.9), it yields

$$\begin{aligned}
& \left| \left(\frac{1}{8} \right)^\wp \left(3^\wp \mathfrak{h} \left(\frac{6\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + 5\mathfrak{d}} \right) + 2^\wp \mathfrak{h} \left(\frac{2\mathfrak{c}\mathfrak{d}}{\mathfrak{c} + \mathfrak{d}} \right) + 3^\wp \mathfrak{h} \left(\frac{6\mathfrak{c}\mathfrak{d}}{5\mathfrak{c} + \mathfrak{d}} \right) \right) - \frac{\Gamma(\wp + 1)}{2^\wp \mathcal{M}_\wp(1)} \left(\left(\frac{1}{\mathfrak{d}} \right)^+ \phi \mathbf{I}^{(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{c}} \right) \right) + \left(\frac{1}{\mathfrak{c}} \right)^- \phi \mathbf{I}^{(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{d}} \right) \right) \right) \right| \\
& \leq \frac{(\mathfrak{c}\mathfrak{d})^\wp (\mathfrak{d} - \mathfrak{c})^\wp}{\mathcal{M}_\wp(1)} \left((\mathcal{D}_1(\mathfrak{c}, \mathfrak{d}, p))^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} ((1 - \nu)^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{c})|^q + \nu^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^\wp \right)^{\frac{1}{q}} \right. \\
& \quad + (\mathcal{D}_2(\mathfrak{c}, \mathfrak{d}, p))^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_0^{\frac{1}{6}} (\nu^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{c})|^q + (1 - \nu)^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^\wp \right)^{\frac{1}{q}} \\
& \quad + (\mathcal{D}_3(\mathfrak{c}, \mathfrak{d}, p))^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} ((1 - \nu)^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{c})|^q + \nu^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^\wp \right)^{\frac{1}{q}} \\
& \quad + (\mathcal{D}_4(\mathfrak{c}, \mathfrak{d}, p))^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{6}}^{\frac{1}{2}} (\nu^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{c})|^q + (1 - \nu)^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^\wp \right)^{\frac{1}{q}} \\
& \quad + (\mathcal{D}_5(\mathfrak{c}, \mathfrak{d}, p))^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} ((1 - \nu)^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{c})|^q + \nu^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^\wp \right)^{\frac{1}{q}} \\
& \quad + (\mathcal{D}_6(\mathfrak{c}, \mathfrak{d}, p))^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{1}{2}}^{\frac{5}{6}} (\nu^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{c})|^q + (1 - \nu)^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^\wp \right)^{\frac{1}{q}} \\
& \quad + (\mathcal{D}_7(\mathfrak{c}, \mathfrak{d}, p))^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 ((1 - \nu)^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{c})|^q + \nu^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^\wp \right)^{\frac{1}{q}} \\
& \quad \left. + (\mathcal{D}_8(\mathfrak{c}, \mathfrak{d}, p))^{\frac{1}{p}} \left(\frac{1}{\Gamma(\wp + 1)} \int_{\frac{5}{6}}^1 (\nu^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{c})|^q + (1 - \nu)^\wp |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^\wp \right)^{\frac{1}{q}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi}{\mathcal{M}_\varphi(1)} \left(\frac{\Gamma(1 + \varphi)}{\Gamma(1 + 2\varphi)} \right)^{\frac{1}{q}} \left((\mathcal{D}_1(c, \mathfrak{d}, p))^{\frac{1}{p}} \left(\left(\frac{11}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + \left(\frac{1}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q \right)^{\frac{1}{q}} \right. \\
&\quad + (\mathcal{D}_2(c, \mathfrak{d}, p))^{\frac{1}{p}} \left(\left(\frac{1}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + \left(\frac{11}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q \right)^{\frac{1}{q}} \\
&\quad + (\mathcal{D}_3(c, \mathfrak{d}, p))^{\frac{1}{p}} \left(\left(\frac{16}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + \left(\frac{8}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q \right)^{\frac{1}{q}} \\
&\quad + (\mathcal{D}_4(c, \mathfrak{d}, p))^{\frac{1}{p}} \left(\left(\frac{8}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + \left(\frac{16}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q \right)^{\frac{1}{q}} \\
&\quad + (\mathcal{D}_5(c, \mathfrak{d}, p))^{\frac{1}{p}} \left(\left(\frac{8}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + \left(\frac{16}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q \right)^{\frac{1}{q}} \\
&\quad + (\mathcal{D}_6(c, \mathfrak{d}, p))^{\frac{1}{p}} \left(\left(\frac{16}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + \left(\frac{8}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q \right)^{\frac{1}{q}} \\
&\quad + (\mathcal{D}_7(c, \mathfrak{d}, p))^{\frac{1}{p}} \left(\left(\frac{1}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + \left(\frac{11}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q \right)^{\frac{1}{q}} \\
&\quad \left. + (\mathcal{D}_8(c, \mathfrak{d}, p))^{\frac{1}{p}} \left(\left(\frac{11}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + \left(\frac{1}{36} \right)^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q \right)^{\frac{1}{q}} \right) \\
&= \frac{(c\mathfrak{d})^\varphi (\mathfrak{d} - c)^\varphi}{\mathcal{M}_\varphi(1)} \left(\frac{\Gamma(1 + \varphi)}{3^\varphi \Gamma(1 + 2\varphi)} \right)^{\frac{1}{q}} \left(\left((\mathcal{D}_1(c, \mathfrak{d}, p))^{\frac{1}{p}} + (\mathcal{D}_8(c, \mathfrak{d}, p))^{\frac{1}{p}} \right) \left(\frac{11^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q}{12^\varphi} \right)^{\frac{1}{q}} \right. \\
&\quad + \left((\mathcal{D}_2(c, \mathfrak{d}, p))^{\frac{1}{p}} + (\mathcal{D}_7(c, \mathfrak{d}, p))^{\frac{1}{p}} \right) \left(\frac{|\mathfrak{h}^{(\varphi)}(c)|^q + 11^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q}{12^\varphi} \right)^{\frac{1}{q}} \\
&\quad + (2^\varphi)^{\frac{1}{q}} \left((\mathcal{D}_3(c, \mathfrak{d}, p))^{\frac{1}{p}} + (\mathcal{D}_6(c, \mathfrak{d}, p))^{\frac{1}{p}} \right) \left(\frac{2^\varphi |\mathfrak{h}^{(\varphi)}(c)|^q + |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q}{3^\varphi} \right)^{\frac{1}{q}} \\
&\quad \left. + (2^\varphi)^{\frac{1}{q}} \left((\mathcal{D}_4(c, \mathfrak{d}, p))^{\frac{1}{p}} + (\mathcal{D}_5(c, \mathfrak{d}, p))^{\frac{1}{p}} \right) \left(\frac{|\mathfrak{h}^{(\varphi)}(c)|^q + 2^\varphi |\mathfrak{h}^{(\varphi)}(\mathfrak{d})|^q}{3^\varphi} \right)^{\frac{1}{q}} \right).
\end{aligned}$$

This completes the proof. $\square$

Theorem 5.2. Assume that $\mathfrak{h}: [c, \mathfrak{d}] \rightarrow \mathbb{R}^\varphi$ is local fractional differentiable on $[c, \mathfrak{d}]$, where $0 < c < \mathfrak{d}$, and that $\mathfrak{h}^{(\varphi)} \in \mathcal{C}_\varphi[c, \mathfrak{d}]$. If the function $|\mathfrak{h}^{(\varphi)}|^q$ is generalized harmonically convex on $[c, \mathfrak{d}]$ for some $q > 1$, then one obtains

$$\left| \left(\frac{1}{8} \right)^\varphi \left(3^\varphi \mathfrak{h} \left(\frac{6c\mathfrak{d}}{c + 5\mathfrak{d}} \right) + 2^\varphi \mathfrak{h} \left(\frac{2c\mathfrak{d}}{c + \mathfrak{d}} \right) + 3^\varphi \mathfrak{h} \left(\frac{6c\mathfrak{d}}{5c + \mathfrak{d}} \right) \right) - \frac{\Gamma(\varphi + 1)}{2^\varphi \mathcal{M}_\varphi(1)} \left(\left(\frac{1}{\mathfrak{d}} \right)^+ \phi \mathbf{I}^{(\varphi)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{c} \right) \right) + \left(\frac{1}{c} \right)^- \phi \mathbf{I}^{(\varphi)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{d}} \right) \right) \right) \right|$$

$$\begin{aligned}
&\leq \frac{(cd)^{\wp} (d-c)^{\wp}}{\mathcal{M}_{\wp}(1)} \left((\mathcal{D}_1(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{C}_1(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{C}_9(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \right. \\
&+ (\mathcal{D}_2(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{C}_2(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{C}_{10}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
&+ (\mathcal{D}_3(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{C}_3(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{C}_{11}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
&+ (\mathcal{D}_4(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{C}_4(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{C}_{12}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
&+ (\mathcal{D}_5(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{C}_5(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{C}_{13}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
&+ (\mathcal{D}_6(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{C}_6(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{C}_{14}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
&+ (\mathcal{D}_7(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{C}_7(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{C}_{15}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
&\left. + (\mathcal{D}_8(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{C}_8(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{C}_{16}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \right),
\end{aligned}$$

where $\mathcal{C}_i(c, d)$ for $i = 1$ to 16 are defined as in (4.2) to (4.17), and $\mathcal{D}_j(c, d)$ for $j = 1$ to 8 are defined as in (5.1) to (5.8), respectively.

Proof. Combining Lemma 3.1 with the properties of the modulus and the generalized power mean inequality, it follows that

$$\begin{aligned}
&\left| \left(\frac{1}{8} \right)^{\wp} \left(3^{\wp} \mathfrak{h} \left(\frac{6cd}{c+5d} \right) + 2^{\wp} \mathfrak{h} \left(\frac{2cd}{c+d} \right) + 3^{\wp} \mathfrak{h} \left(\frac{6cd}{5c+d} \right) \right) - \frac{\Gamma(\wp+1)}{2^{\wp} \mathcal{M}_{\wp}(1)} \left(\left(\frac{1}{d} \right)^{+} {}^{\phi} \mathbf{I}^{(\wp)} \mathfrak{h} \left(g \left(\frac{1}{c} \right) \right) + \left(\frac{1}{c} \right)^{-} {}^{\phi} \mathbf{I}^{(\wp)} \mathfrak{h} \left(g \left(\frac{1}{d} \right) \right) \right) \right| \\
&\leq \frac{(cd)^{\wp} (d-c)^{\wp}}{\mathcal{M}_{\wp}(1)} \left[\left(\frac{1}{\Gamma(\wp+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\wp}(\nu)|}{(\nu c + (1-\nu)d)^{2\wp}} (d\nu)^{\wp} \right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\wp}(\nu)|}{(\nu c + (1-\nu)d)^{2\wp}} \left| \mathfrak{h}^{(\wp)} \left(\frac{cd}{\nu c + (1-\nu)d} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \right. \\
&+ \left(\frac{1}{\Gamma(\wp+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\wp}(\nu)|}{((1-\nu)c + \nu d)^{2\wp}} (d\nu)^{\wp} \right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\wp}(\nu)|}{((1-\nu)c + \nu d)^{2\wp}} \left| \mathfrak{h}^{(\wp)} \left(\frac{cd}{(1-\nu)c + \nu d} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
&+ \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{3}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)d)^{2\wp}} (d\nu)^{\wp} \right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{3}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)d)^{2\wp}} \left| \mathfrak{h}^{(\wp)} \left(\frac{cd}{\nu c + (1-\nu)d} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
&\left. + \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{3}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu d)^{2\wp}} (d\nu)^{\wp} \right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{3}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu d)^{2\wp}} \left| \mathfrak{h}^{(\wp)} \left(\frac{cd}{(1-\nu)c + \nu d} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{5}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)\mathfrak{d})^{2\wp}} (d\nu)^{\wp} \right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{5}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)\mathfrak{d})^{2\wp}} \left| \mathfrak{h}^{(\wp)} \left(\frac{c\mathfrak{d}}{\nu c + (1-\nu)\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{5}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu\mathfrak{d})^{2\wp}} (d\nu)^{\wp} \right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{5}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu\mathfrak{d})^{2\wp}} \left| \mathfrak{h}^{(\wp)} \left(\frac{c\mathfrak{d}}{(1-\nu)c + \nu\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left(\frac{1^{\wp}}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)\mathfrak{d})^{2\wp}} (d\nu)^{\wp} \right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)\mathfrak{d})^{2\wp}} \left| \mathfrak{h}^{(\wp)} \left(\frac{c\mathfrak{d}}{\nu c + (1-\nu)\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + \left(\frac{1^{\wp}}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu\mathfrak{d})^{2\wp}} (d\nu)^{\wp} \right)^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu\mathfrak{d})^{2\wp}} \left| \mathfrak{h}^{(\wp)} \left(\frac{c\mathfrak{d}}{(1-\nu)c + \nu\mathfrak{d}} \right) \right|^q (d\nu)^{\wp} \right)^{\frac{1}{q}}. \tag{5.10}
\end{aligned}$$

By utilizing the generalized harmonic convexity of $|\mathfrak{h}^{(\wp)}|^q$ in inequality (5.10), it yields

$$\begin{aligned}
& \left| \left(\frac{1}{8} \right)^{\wp} \left(3^{\wp} \mathfrak{h} \left(\frac{6c\mathfrak{d}}{c+5\mathfrak{d}} \right) + 2^{\wp} \mathfrak{h} \left(\frac{2c\mathfrak{d}}{c+\mathfrak{d}} \right) + 3^{\wp} \mathfrak{h} \left(\frac{6c\mathfrak{d}}{5c+\mathfrak{d}} \right) \right) - \frac{\Gamma(\wp+1)}{2^{\wp} \mathcal{M}_{\wp}(1)} \left(\left(\frac{1}{\mathfrak{d}} \right)^{+} \Gamma^{(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{c} \right) \right) + \left(\frac{1}{c} \right)^{-} \Gamma^{(\wp)} \mathfrak{h} \left(\mathfrak{g} \left(\frac{1}{\mathfrak{d}} \right) \right) \right) \right| \\
& \leq \frac{(c\mathfrak{d})^{\wp}(\mathfrak{d}-c)^{\wp}}{\mathcal{M}_{\wp}(1)} \left((\mathcal{D}_1(c, \mathfrak{d}, 1))^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\wp}(\nu)|}{(\nu c + (1-\nu)\mathfrak{d})^{2\wp}} ((1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(c)|^q + \nu^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^{\wp} \right)^{\frac{1}{q}} \right. \\
& + (\mathcal{D}_2(c, \mathfrak{d}, 1))^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_0^{\frac{1}{6}} \frac{|\mathcal{M}_{\wp}(\nu)|}{((1-\nu)c + \nu\mathfrak{d})^{2\wp}} (\nu^{\wp} |\mathfrak{h}^{(\wp)}(c)|^q + (1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_3(c, \mathfrak{d}, 1))^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{3}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)\mathfrak{d})^{2\wp}} ((1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(c)|^q + \nu^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_4(c, \mathfrak{d}, 1))^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{3}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu\mathfrak{d})^{2\wp}} (\nu^{\wp} |\mathfrak{h}^{(\wp)}(c)|^q + (1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(\mathfrak{d})|^q) (d\nu)^{\wp} \right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
& + (\mathcal{D}_5(c, d, 1))^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{5}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)d)^{2\wp}} \left((1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(c)|^q + \nu^{\wp} |\mathfrak{h}^{(\wp)}(d)|^q \right) (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_6(c, d, 1))^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{1}{2}}^{\frac{5}{6}} \frac{|\mathcal{M}_{\wp}(\nu) - (\frac{5}{8})^{\wp} \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu d)^{2\wp}} \left(\nu^{\wp} |\mathfrak{h}^{(\wp)}(c)|^q + (1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(d)|^q \right) (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_7(c, d, 1))^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)|}{(\nu c + (1-\nu)d)^{2\wp}} \left((1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(c)|^q + \nu^{\wp} |\mathfrak{h}^{(\wp)}(d)|^q \right) (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_8(c, d, 1))^{1-\frac{1}{q}} \left(\frac{1}{\Gamma(\wp+1)} \int_{\frac{5}{6}}^1 \frac{|\mathcal{M}_{\wp}(\nu) - \mathcal{M}_{\wp}(1)|}{((1-\nu)c + \nu d)^{2\wp}} \left(\nu^{\wp} |\mathfrak{h}^{(\wp)}(c)|^q + (1-\nu)^{\wp} |\mathfrak{h}^{(\wp)}(d)|^q \right) (d\nu)^{\wp} \right)^{\frac{1}{q}} \\
& = \frac{(cd)^{\wp} (d-c)^{\wp}}{\mathcal{M}_{\wp}(1)} \left((\mathcal{D}_1(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{E}_1(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{E}_9(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right) (d\nu)^{\frac{1}{q}} \right. \\
& + (\mathcal{D}_2(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{E}_2(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{E}_{10}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_3(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{E}_3(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{E}_{11}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_4(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{E}_4(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{E}_{12}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_5(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{E}_5(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{E}_{13}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_6(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{E}_6(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{E}_{14}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
& + (\mathcal{D}_7(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{E}_7(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{E}_{15}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \\
& \left. + (\mathcal{D}_8(c, d, 1))^{1-\frac{1}{q}} \left(\mathcal{E}_8(c, d) |\mathfrak{h}^{(\wp)}(c)|^q + \mathcal{E}_{16}(c, d) |\mathfrak{h}^{(\wp)}(d)|^q \right)^{\frac{1}{q}} \right),
\end{aligned}$$

where $\mathcal{E}_i(c, d)$ for $i = 1$ to 16 are defined as in (4.2) to (4.17), respectively.

This completes the proof. $\square$

6. Numerical investigations, computational surrogate-modeling, and applications

This section presents an illustrative example and graphical interpretations in order to demonstrate the validity and applicability of the obtained results, together with an application to certain special fractal means.

6.1. Numerical illustration

We now provide a concrete example in order to demonstrate the applicability of the derived inequality for a particular harmonically convex function.

Example 6.1. Let $\wp = 1$ and $c, d \in [2, 9]$ with $c < d$, and consider the function $h: [c, d] \rightarrow \mathbb{R}$ defined by

$$h(\varpi) = \frac{1}{\varpi^2}.$$

Then, the first-order derivative is given by

$$h'(\varpi) = -\frac{2}{\varpi^3}, \quad |h'(\varpi)| = \frac{2}{\varpi^3}.$$

Hence, by Remark 1.1, the function $|h^{(\wp)}|$ is harmonically convex on $[c, d]$.

Moreover,

$$h(c) = \frac{1}{c^2}, \quad h(d) = \frac{1}{d^2}, \quad h\left(\frac{2cd}{c+d}\right) = \frac{(c+d)^2}{4c^2d^2}.$$

Also,

$$\frac{cd}{d-c} \int_c^d \frac{h(\varpi)}{\varpi^2} d\varpi = \frac{cd}{d-c} \int_c^d \frac{1}{\varpi^4} d\varpi = \frac{c^2 + cd + d^2}{3c^2d^2}.$$

Therefore, the expression on the left-hand side of the inequality presented in Corollary 4.3 takes the form

$$\begin{aligned} & \left| \frac{1}{8} \left(3h(c) + 2h\left(\frac{2cd}{c+d}\right) + 3h(d) \right) - \frac{cd}{d-c} \int_c^d \frac{h(\varpi)}{\varpi^2} d\varpi \right| \\ &= \left| \frac{1}{8} \left(\frac{3}{c^2} + \frac{(c+d)^2}{2c^2d^2} + \frac{3}{d^2} \right) - \frac{c^2 + cd + d^2}{3c^2d^2} \right| \\ &= \frac{5(d-c)^2}{48c^2d^2}. \end{aligned}$$

On the other hand,

$$|h'(c)| = \frac{2}{c^3}, \quad |h'(d)| = \frac{2}{d^3}.$$

Accordingly, the right-hand side can be written as

$$\mathcal{C}_1(c, d) |h'(c)| + \mathcal{C}_2(c, d) |h'(d)| = 2 \left(\frac{\mathcal{C}_1(c, d)}{c^3} + \frac{\mathcal{C}_2(c, d)}{d^3} \right).$$

Hence, inequality (6.1) reduces to

$$\frac{5(d-c)^2}{48c^2d^2} \leq \frac{2\mathcal{C}_1(c, d)}{c^3} + \frac{2\mathcal{C}_2(c, d)}{d^3}. \quad (6.1)$$

To visualize this estimate, we take

$$d = c + 1, \quad c \in [2, 8].$$

It follows that the left-hand side may be represented as

$$\frac{5}{48c^2(c+1)^2},$$

while the right-hand side is expressed as

$$2\left(\frac{\mathcal{C}_1(c, c+1)}{c^3} + \frac{\mathcal{C}_2(c, c+1)}{(c+1)^3}\right).$$

According to Figure 1, the graph of the left-hand side remains below that of the right-hand side throughout the interval $c \in [2, 8]$. This numerical observation confirms the validity of inequality (6.1) and further supports the correctness of our result.

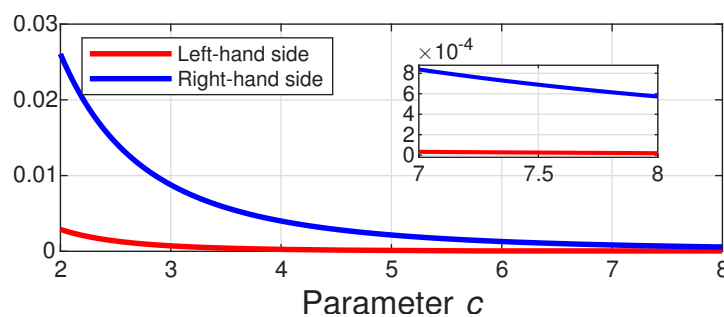


Figure 1. Comparison of the two sides of inequality (6.1) for $c \in [2, 8]$ with $d = c + 1$.

6.2. Computational illustration via an artificial neural network

To complement the analytical upper bounds established in Theorem 5.1, this section presents a machine-learning-based sensitivity analysis. It is crucial to emphasize that the neural network does not serve as an independent mathematical validation of the proof, as the training targets are deterministically derived from the analytical formula under the Hölder conjugacy constraint

$$1/p + 1/q = 1.$$

Instead, the computational illustration via an artificial neural network (ANN) is deployed as a surrogate model to illustrate the numerical behavior, stability, and sensitivity of the fractional upper bound across varying parametric ranges of p and q .

We fix the interval endpoints as

$$c_0 = 2, \quad d_0 = 5,$$

and we consider the ordinary case $\varphi = 1$ together with the normalized kernel choice

$$\mathcal{M}_1(v) = v, \quad \mathcal{M}_1(1) = 1.$$

The conjugate parameters satisfy

$$q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

or equivalently,

$$p = \frac{q}{q-1}.$$

For this numerical illustration, we select the test function

$$\mathfrak{h}(\varpi) = \ln(\varpi), \quad \varpi > 0.$$

Then,

$$|\mathfrak{h}'(\varpi)| = \frac{1}{\varpi}.$$

Consequently,

$$|\mathfrak{h}'(\varpi)|^q = \frac{1}{\varpi^q}, \quad q > 1,$$

which is harmonically convex on $(0, \infty)$. Hence, the hypotheses of Theorem 5.1 are fulfilled in the ordinary case.

The exact left-hand side is given by

$$\mathcal{L} = \left| \frac{1}{8} \left[3\mathfrak{h}\left(\frac{6c_0\mathfrak{d}_0}{c_0 + 5\mathfrak{d}_0}\right) + 2\mathfrak{h}\left(\frac{2c_0\mathfrak{d}_0}{c_0 + \mathfrak{d}_0}\right) + 3\mathfrak{h}\left(\frac{6c_0\mathfrak{d}_0}{5c_0 + \mathfrak{d}_0}\right) \right] - \frac{c_0\mathfrak{d}_0}{\mathfrak{d}_0 - c_0} \int_{c_0}^{\mathfrak{d}_0} \frac{\mathfrak{h}(\varpi)}{\varpi^2} d\varpi \right|.$$

For $\mathfrak{h}(\varpi) = \ln(\varpi)$, the integral can be evaluated explicitly as

$$\int_{c_0}^{\mathfrak{d}_0} \frac{\ln(\varpi)}{\varpi^2} d\varpi = \frac{\ln(c_0) + 1}{c_0} - \frac{\ln(\mathfrak{d}_0) + 1}{\mathfrak{d}_0}.$$

On the other hand, Theorem 5.1 gives an upper bound depending on p and q , denoted by $\mathcal{R}_{5.1}(p, q)$. Thus, the inequality under consideration is

$$\mathcal{L} \leq \mathcal{R}_{5.1}(p, q), \quad q > 1, \quad p = \frac{q}{q-1}.$$

Since the endpoints c_0 and $\mathfrak{d}_0$ are fixed, the quantity $\mathcal{L}$ is constant. Therefore, the main learning task concerns the nonlinear dependence of the upper bound $\mathcal{R}_{5.1}(p, q)$ on the Hölder conjugate parameters. For this reason, the neural network is trained as a regression model of the form

$$(p, q) \mapsto \widehat{\mathcal{R}}_{5.1}(p, q),$$

where $\widehat{\mathcal{R}}_{5.1}$ denotes the ANN-predicted upper bound.

6.2.1. Network architecture and reproducibility specifications

To evaluate the sensitivity of the analytical upper bound $\mathcal{R}_{5.1}(p, q)$ formulated in Theorem 5.1, a multilayer feed-forward ANN is constructed as a computational surrogate model. To guarantee full scientific reproducibility of the computational experiments, all mathematical, architectural, and optimization parameters are strictly defined in accordance with the underlying implementation:

- **Dataset and sampling:** A synthetic dataset comprising 1600 distinct sample points was generated by varying the parameter q uniformly in the range $[1.10, 10.0]$ under the ordinary case ($\wp = 1.0$), with the parameter p dynamically constrained by the Hölder conjugacy relationship $p = \frac{q}{q-1}$. For each pair (p, q) , the target output $\mathcal{R}_{5.1}(p, q)$ was calculated analytically.

- **Data partitioning:** To ensure unbiased learning and evaluation, the generated dataset was randomly partitioned into a training set (70%, 1120 samples), a validation set (15%, 240 samples), and an independent testing set (15%, 240 samples).
- **Preprocessing and reproducibility:** Input features (p, q) and target values y were standardized using a z-score normalization scheme (standardscaler) based on training set statistics. To ensure computational stability and identical weight initializations across runs, the random seeds for both NumPy and TensorFlow were fixed to a constant value (SEED = 42).
- **Network topology:** The proposed multilayer perceptron architecture features (see Figure 2):
 - An input layer accepting 2 inputs (p and q).
 - Two consecutive hidden layers containing 64 neurons each, followed by a third hidden layer with 32 neurons. All hidden layers utilize the hyperbolic tangent (tanh) activation function.
 - A single-neuron output layer with a linear activation function to predict the estimated upper bound $\widehat{\mathcal{R}}_{5,1}$.
- **Training and optimization protocol:** The model was trained using the Adam optimizer with a starting learning rate of 10^{-3} and a batch size of 32 over a maximum of 1200 epochs. The loss function was minimized using the mean squared error (MSE) metric. To avoid overfitting, an early stopping mechanism was enforced with a patience of 120 epochs, automatically restoring the best performing model weights upon termination.

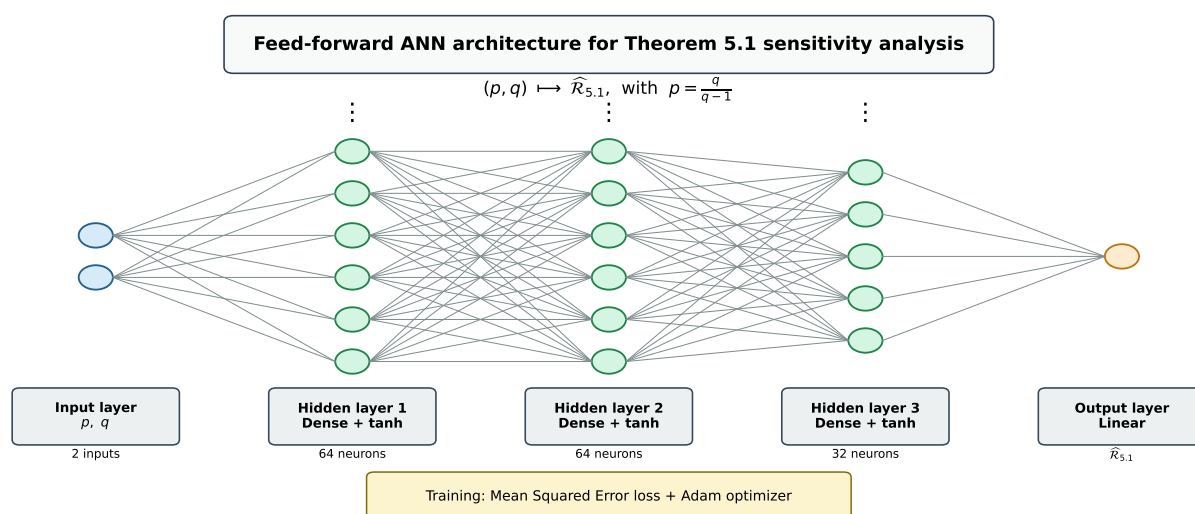


Figure 2. Structure of the multilayer feed-forward neural network utilized as a computational surrogate model for Theorem 5.1.

6.2.2. Surrogate performance and evaluation metrics

The generalization capability of the trained surrogate model was evaluated on the independent 15% testing subset using three statistical criteria: mean absolute error (MAE), root mean squared error (RMSE), and the coefficient of determination (R^2). The training and validation loss curves are displayed in Figure 3, while the performance metrics achieved by the network are

structured in Table 2 below.

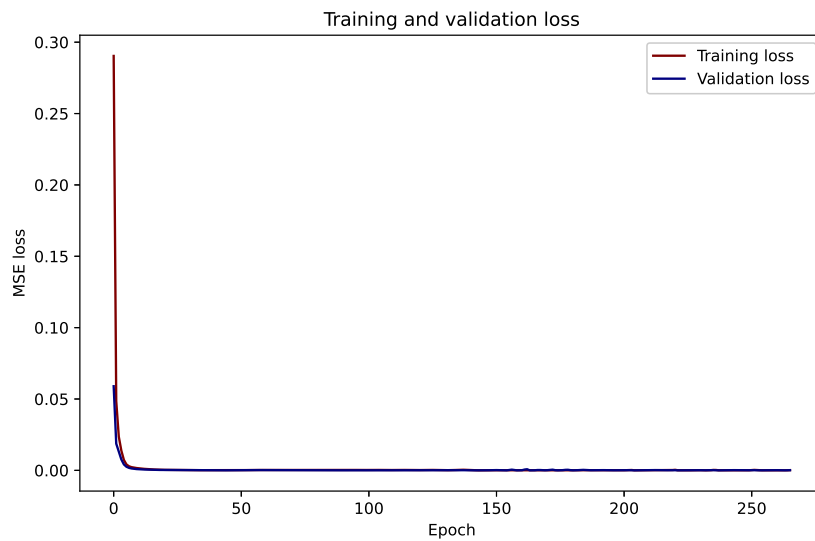


Figure 3. Evolution of training and validation MSE losses showing model convergence without overfitting.

Table 2. Surrogate ANN prediction performance metrics on the independent test set.

Target quantity	MAE	RMSE	R^2
$\mathcal{R}_{5,1}(p, q)$	1.760×10^{-6}	1.996×10^{-6}	0.999995

The distribution of prediction errors is shown in Figure 4. The errors are concentrated around zero, which indicates that the trained network approximates the theoretical upper bound with high accuracy.

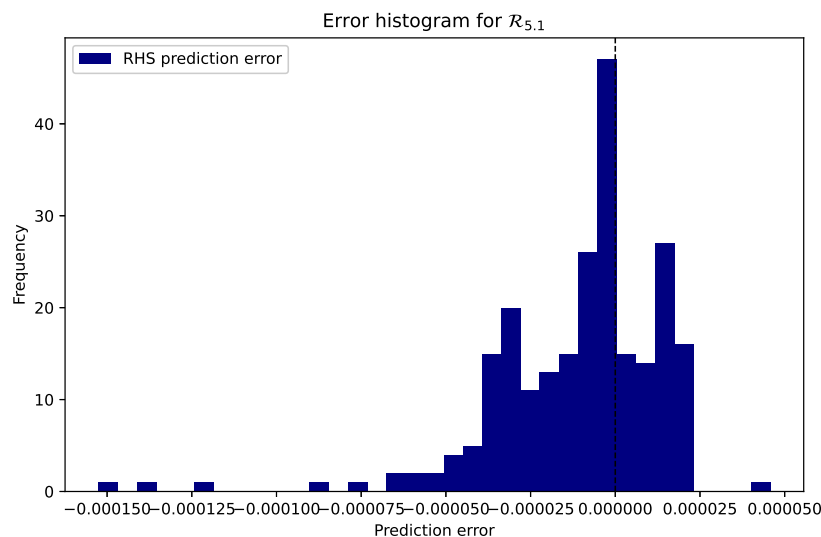


Figure 4. Distribution of prediction errors ($\mathcal{R}_{5,1} - \widehat{\mathcal{R}}_{5,1}$) on the independent test set, illustrating the approximation capability of the surrogate model.

Figure 5 compares the exact upper bound provided by Theorem 5.1 with the ANN-predicted values as functions of q , where $p = q/(q - 1)$. The exact left-hand side is also displayed as a horizontal reference line. The predicted curve closely follows the exact upper bound over the whole considered range of q .

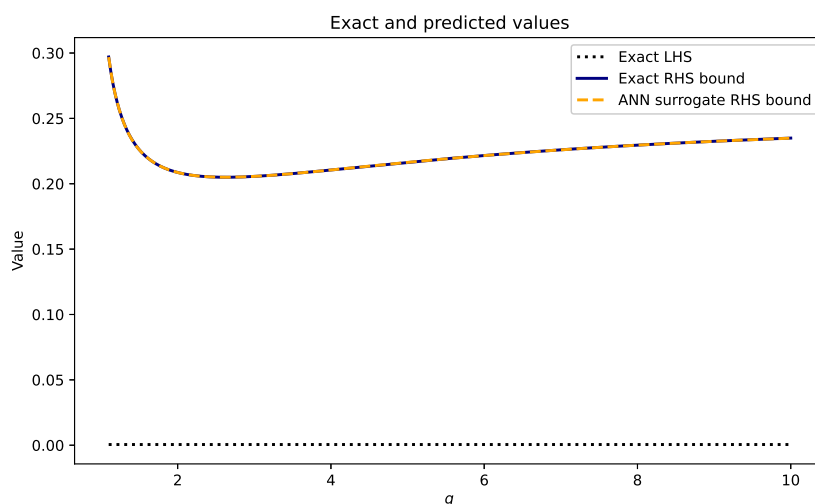


Figure 5. Comparison of the exact left-hand side, analytical right-hand side bound, and ANN-predicted surrogate bound over the range of $q \in [1.10, 10.0]$ under the ordinary case ($\varphi = 1.0$).

The absolute prediction error is plotted in Figure 6. The small magnitude of the error confirms the ability of the neural network to reproduce the analytical bound.

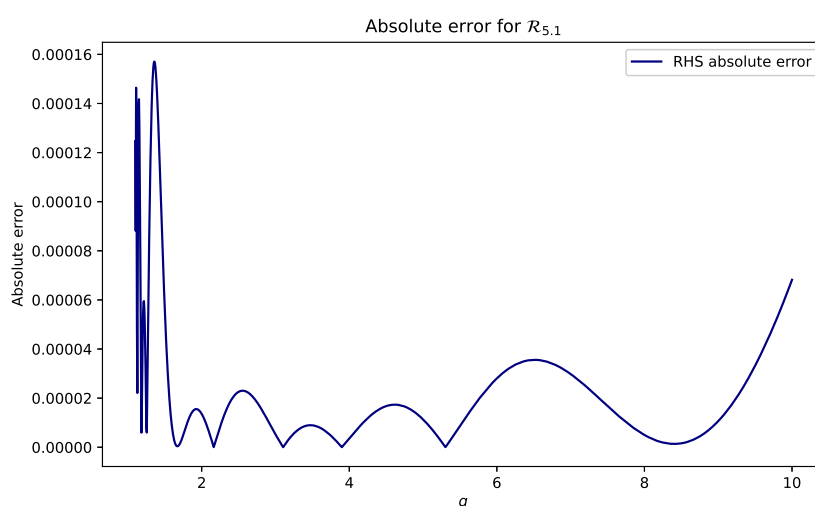


Figure 6. Absolute error between the exact and ANN-predicted values of $\mathcal{R}_{5,1}(p, q)$.

Finally, Figure 7 gives a three-dimensional comparison in the (p, q, value) -space. Since p and q are conjugate, the admissible points lie on the curve $p = q/(q - 1)$. The exact and predicted curves are

nearly superposed, which confirms that the ANN captures the nonlinear dependence of the bound on the Hölder parameters.

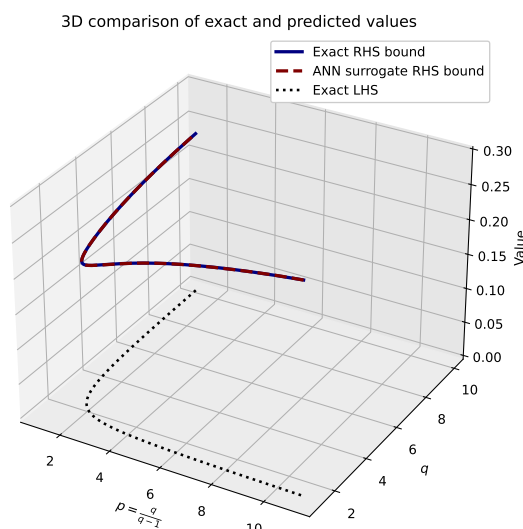


Figure 7. Three-dimensional comparison between the exact and ANN-predicted upper bounds in the (p, q, value) -space.

These numerical experiments provide an additional computational confirmation of Theorem 5.1. In particular, the ANN predictions remain very close to the analytical values, and the predicted upper bound stays above the exact left-hand side throughout the tested range of the conjugate parameters.

6.3. Application

The definitions of the $\wp$ -type arithmetic mean, $\wp$ -type harmonic mean, and $\wp$ -type p -logarithmic mean in the context of fractal spaces are as follows:

$$\mathcal{H}_{\wp}(c, d) = \frac{2^{\wp} c^{\wp} d^{\wp}}{c^{\wp} + d^{\wp}},$$

$$\mathcal{H}_{\wp}(m, n, c, d) = \frac{(m+n)^{\wp} c^{\wp} d^{\wp}}{(nc)^{\wp} + (md)^{\wp}}$$

and

$$\mathcal{L}_p(c, d) = \left[\frac{\Gamma(1+p\wp)}{\Gamma(1+(p+1)\wp)} \left(\frac{d^{(p+1)\wp} - c^{(p+1)\wp}}{d^{\wp} - c^{\wp}} \right) \right]^{\frac{1}{p}}, \quad p \in \mathbb{Z} \setminus \{-1, 0\} \text{ and } c \neq d.$$

Proposition 1. Let $c, d \in \mathbb{R}$ with $0 < c < d$, and $n \geq 2$. Then

$$\begin{aligned} & \left| 3^{\wp} \mathcal{H}_{\wp}(5, 1, c, d) + 2^{\wp} \mathcal{H}_{\wp}(c, d) + 3^{\wp} \mathcal{H}_{\wp}(1, 5, c, d) - (8cd)^{\wp} \Gamma(\wp + 1) \mathcal{L}_{2-n}^{2-n}(c, d) \right| \\ & \leq \frac{8^{\wp} (cd)^{\wp} (d-c)^{\wp}}{\Gamma(\wp + 1)} \left(\widetilde{\mathcal{E}}_1(c, d) \left| \mathfrak{h}^{(\wp)}(c) \right| + \widetilde{\mathcal{E}}_2(c, d) \left| \mathfrak{h}^{(\wp)}(d) \right| \right), \end{aligned}$$

where $\widetilde{\mathcal{E}}_1(c, d)$ and $\widetilde{\mathcal{E}}_2(c, d)$ are defined as in (4.18) and (4.19), respectively.

Proof. This claim is an immediate consequence of Corollary 4.2 when it is applied to $\mathfrak{h}(\varpi) = \varpi^{n\wp}$. $\square$

7. Conclusions

In this study, we successfully developed and analyzed new integral inequalities within the context of local fractional calculus. Our approach was founded on the establishment of a novel fractal-fractional identity, which served as a fundamental auxiliary tool. Utilizing this identity, we derived a series of Maclaurin-type inequalities for functions possessing generalized harmonically convex local fractional derivatives. We further expanded our results by incorporating the generalized Hölder and power mean inequalities to obtain additional fractional estimates.

To ensure the reliability of our theoretical framework, we presented an illustrative example accompanied by graphical representations, which successfully confirmed the consistency and exactness of our mathematical derivations. Moreover, the practical significance of our research was highlighted through an application to special fractal means. We believe that the novel identity and the associated inequalities presented in this work significantly contribute to the field of fractal-fractional calculus and offer valuable mathematical tools that can inspire future investigations in generalized convexity and fractional operators.

Looking ahead, several promising theoretical directions emerge from this study. On the one hand, the mathematical framework developed here could be extended to explore other classes of generalized convexity, such as coordinated or strongly convex functions on fractal spaces. On the other hand, these Maclaurin-type inequalities could be investigated in the context of other fractional integral operators or adapted to study the stability and error bounds of advanced fractional differential equations.

Use of Generative-AI tools declaration

During the preparation of this work, the author used Google Gemini in order to improve the English language, grammar, and phrasing in the Introduction section. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Conflict of interest

The author reports no competing interest to declare.

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