



*Research article***Iterative approach for simultaneously finding roots of nonlinear equations with multiplicities****Monika Panwar¹, Sonia Bhalla¹, Higinio Ramos^{2,3,*} and Ramandeep Behl⁴**¹ Department of Mathematics, Chandigarh University, Gharuan, Mohali, Punjab 140413, India² Scientific Computing Group, Universidad de Salamanca, Plaza de la Merced, 37008 Salamanca, Spain³ Escuela Polit'ecnica Superior, Universidad de Salamanca, Campus Viriato, 49029 Zamora, Spain⁴ Mathematical Modelling and Applied Computation Research Group (MMAC), Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia*** Correspondence:** Email: higra@usal.es.

Abstract: This paper presents a two-step iterative approach for simultaneously approximating a finite number of roots of a nonlinear equation, where the multiplicities are assumed to be known. A convergence analysis of the proposed scheme is also presented. The proposed method achieves an order of convergence $4p$ for polynomial equations and $2p$ for non-polynomial equations when combined with any iterative method of order p for finding multiple roots. Computational results and error analysis graphs demonstrate that the special cases of the proposed scheme exhibit superior performance in terms of CPU execution time, residual error, and convergence rate.

Keywords: nonlinear equations; iterative methods; simultaneous multiple roots; convergence analysis

Mathematics Subject Classification: 65B99, 65H05

1. Introduction

Consider the problem of solving a nonlinear equation $\mathcal{F}(z) = 0$, where $\mathcal{F} : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ is a continuously differentiable function on a domain in the complex plane. Nonlinear equations arise naturally in many scientific and engineering applications, including control system theory, nonlinear circuits, transfer function analysis, mathematical modeling, differential and discrete equations, chemical equilibrium, population dynamics, and eigenvalue problems [1]. Many practical problems require the computation of all roots of a nonlinear equation. This has motivated the development of simultaneous iterative methods, which determine all roots in a single iterative process and provide

an efficient alternative to computing each root separately. Existing iterative methods may be broadly classified into methods for finding single roots, methods for multiple roots, and methods for simultaneously finding all roots.

In recent years, simultaneous root-finding methods have received considerable attention because they approximate all roots in a single iterative process. These methods are computationally efficient, avoid repeated deflation procedures, are generally more stable, and can be implemented in parallel on computer systems. One of the best-known schemes for simultaneously approximating the simple roots ψ_μ , $\mu = 1, 2, \dots, n$, of a polynomial $p(z)$ of degree n is the Weierstrass method [2]. This method has second-order convergence and laid the foundation for the development of simultaneous iterative methods for polynomial root-finding. It is given by

$$z_\mu^{(\kappa+1)} = z_\mu^{(\kappa)} - W(z_\mu^{(\kappa)}) = z_\mu^{(\kappa)} - \frac{p(z_\mu^{(\kappa)})}{\prod_{\substack{\nu=1 \\ \nu \neq \mu}}^n (z_\mu^{(\kappa)} - z_\nu^{(\kappa)})}, \quad \kappa = 0, 1, \dots$$

In the 1960s, the Weierstrass method was independently rediscovered by Durand [3], Dochev [4], Kerner [5], and Presic [6], demonstrating its effectiveness for the simultaneous approximation of polynomial roots. To improve the convergence rate, Ehrlich [7] proposed a third-order method for simultaneously finding all simple roots of a polynomial, given by

$$z_\mu^{(\kappa+1)} = z_\mu^{(\kappa)} - \frac{1}{\frac{p'(z_\mu^{(\kappa)})}{p(z_\mu^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{1}{z_\mu^{(\kappa)} - z_\nu^{(\kappa)}}}, \quad \kappa = 0, 1, \dots$$

In 1977, Nourein's [8] further improved Ehrlich's method by constructing a fourth-order iterative scheme for simultaneously approximating all simple roots of a polynomial. The method is given by

$$z_\mu^{(\kappa+1)} = z_\mu^{(\kappa)} - \frac{1}{\frac{p'(z_\mu^{(\kappa)})}{p(z_\mu^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^n \frac{1}{z_\mu^{(\kappa)} - z_\nu^{(\kappa)} + \frac{p(z_\nu^{(\kappa)})}{p'(z_\nu^{(\kappa)})}}}, \quad \kappa = 0, 1, \dots$$

Subsequently, considerable attention has been devoted to the development of higher-order simultaneous root-finding methods. Petković et al. [9] proposed a four-step iterative scheme with tenth-order convergence by combining a three-step Kung–Traub family with the final step of the Ehrlich method for simultaneously approximating all simple roots of a polynomial. Cordero et al. [10–12] developed several high-order simultaneous root-finding methods, including derivative-free techniques and methods applicable to nonlinear equations. Higher-order multi-step iterative methods have also been developed for solving systems of nonlinear equations, with applications to nonlinear ordinary and partial differential equations [13]. Proinov et al. [14–16] established general convergence results and proposed efficient simultaneous root-finding iterative schemes with improved convergence properties. While most existing simultaneous root-finding methods are designed for simple roots, Cordero et al. [11] introduced a method for multiple roots with sixth-order convergence

for polynomial equations and fourth-order convergence for non-polynomial equations. Furthermore, Cordero et al. [10] proposed a general framework that transforms any iterative method of order p into a simultaneous root-finding method of order $2p$ for nonlinear equations with simple roots. More recently, Jerezano et al. [17] proposed a framework for computing multiple roots of nonlinear equations without prior knowledge of their multiplicities. Their approach is based on composing Schröder-type predictors with Ehrlich-type correction steps. In particular, they showed that combining Schröder's method with a single Ehrlich correction yields a fourth-order simultaneous method for multiple roots. More generally, they proved that if an iterative method of order p for multiple roots is used as a predictor, then the subsequent Ehrlich correction produces a simultaneous method of order $2p$. However, their approach requires the evaluation of the second derivative, which increases the computational cost and may limit its applicability when higher-order derivatives are unavailable or expensive to compute. In another recent contribution, Shams and Carpentieri [18] developed a high-order parallel framework for the simultaneous computation of multiple solutions of nonlinear systems. These developments highlight the continuous progress in the development of efficient simultaneous root-finding methods. However, there is still a need for more general frameworks that achieve higher convergence orders and are applicable to both polynomial and non-polynomial equations with multiple roots.

These developments motivate the construction of a general and efficient framework for simultaneous computation of multiple roots. In particular, there is a need for methods that provide higher convergence orders for both polynomial and non-polynomial equations while requiring only first-derivative evaluations.

The main contribution of this work is the development of a new general framework for the simultaneous computation of multiple roots of nonlinear equations with known multiplicities. The proposed framework can be combined with any iterative method of order p for multiple roots. It achieves convergence order $4p$ for polynomial equations and $2p$ for general non-polynomial equations. Moreover, the proposed framework requires only the evaluation of the first derivative, avoiding the computation of second and higher-order derivatives. Thus, it provides an efficient and general approach for constructing high-order simultaneous methods for nonlinear equations with multiple roots.

The construction of the proposed iterative approach is presented in Section 2. The convergence of the proposed approach is analyzed in Section 3. A comparison of the proposed methods with existing methods, along with the corresponding error analysis graphs and numerical results, is presented in Sections 4 and 5. Finally, Section 6 concludes the paper.

2. Construction of the proposed iterative approach

Let us consider $\mathcal{F}(z) = \prod_{v=1}^m (z - \xi_v)^{\alpha_v}$, a monic polynomial of degree n with real and complex roots $\xi_1, \xi_2, \dots, \xi_m$ having multiplicities $\alpha_1, \alpha_2, \dots, \alpha_m$. The well-known Newton method for approximating a root of multiplicity α_μ is given by

$$z_\mu^{(\kappa+1)} = z_\mu^{(\kappa)} - \alpha_\mu \frac{\mathcal{F}(z_\mu^{(\kappa)})}{\mathcal{F}'(z_\mu^{(\kappa)})}. \quad (2.1)$$

By taking the logarithmic derivative of $\mathcal{F}(z)$, we obtain

$$\frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})} = \frac{\alpha_\mu}{z_\mu^{(\kappa)} - \xi_\mu} + \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_\nu}{z_\mu^{(\kappa)} - \xi_\nu}, \quad (2.2)$$

where the summation explicitly excludes the index $\nu = \mu$ to avoid the singular term corresponding to the root ξ_μ .

Hence,

$$\frac{\alpha_\mu}{z_\mu^{(\kappa)} - \xi_\mu} = \frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_\nu}{z_\mu^{(\kappa)} - \xi_\nu}. \quad (2.3)$$

Taking the reciprocal of both sides of Eq (2.3), we obtain

$$\frac{z_\mu^{(\kappa)} - \xi_\mu}{\alpha_\mu} = \frac{1}{\frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_\nu}{z_\mu^{(\kappa)} - \xi_\nu}}, \quad (2.4)$$

from which it readily follows that

$$z_\mu^{(\kappa)} - \xi_\mu = \frac{\alpha_\mu}{\frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_\nu}{z_\mu^{(\kappa)} - \xi_\nu}}, \quad (2.5)$$

and, consequently,

$$\xi_\mu = z_\mu^{(\kappa)} - \frac{\alpha_\mu}{\frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_\nu}{z_\mu^{(\kappa)} - \xi_\nu}}. \quad (2.6)$$

Replacing the roots in (2.1) with the approximations provided by Newton's method,

$$\bar{z}_\nu^{(\kappa)} = z_\nu^{(\kappa)} - \alpha_\nu \frac{\mathcal{F}(z_\nu^{(\kappa)})}{\mathcal{F}'(z_\nu^{(\kappa)})},$$

the new approximation $z_\mu^{(\kappa+1)}$ to the root ξ_μ is given by

$$z_\mu^{(\kappa+1)} = z_\mu^{(\kappa)} - \frac{\alpha_\mu}{\frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_\nu}{z_\mu^{(\kappa)} - z_\nu^{(\kappa)} + \alpha_\nu \frac{\mathcal{F}(z_\nu^{(\kappa)})}{\mathcal{F}'(z_\nu^{(\kappa)})}}}, \quad (2.7)$$

which is, in fact, Nouredin's method [8], recently studied by Proinov and Vasileva [19]. On the other hand,

$$\begin{aligned} \frac{1}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)} + \alpha_{\nu} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})}} &= \frac{1}{(z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}) \left(1 + \frac{\alpha_{\nu}}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})} \right)} \\ &= \frac{1}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}} \left(1 + \frac{\alpha_{\nu}}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})} \right)^{-1}. \end{aligned} \quad (2.8)$$

Assuming $\left| \frac{\alpha_{\nu}}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})} \right| < 1$, we employ the first-order binomial expansion $(1+x)^{-1} = 1 - x + O(x^2)$, and neglecting the higher-order terms, Eq (2.8) becomes

$$\frac{1}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)} + \alpha_{\nu} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})}} = \frac{1}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}} \left(1 - \frac{\alpha_{\nu}}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})} \right). \quad (2.9)$$

Substituting Eq (2.9) into Eq (2.7), we obtain

$$\sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_{\nu}}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)} + \alpha_{\nu} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})}} = \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_{\nu}}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_{\nu}^2}{(z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)})^2} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})}.$$

Hence, provided that $z_{\mu}^{(\kappa)} \neq z_{\nu}^{(\kappa)}$ for $\mu \neq \nu$, Eq (2.7) reduces to

$$z_{\mu}^{(\kappa+1)} = z_{\mu}^{(\kappa)} - \frac{\alpha_{\mu}}{\frac{\mathcal{F}'(z_{\mu}^{(\kappa)})}{\mathcal{F}(z_{\mu}^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_{\nu}}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}} + \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_{\nu}^2}{(z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)})^2} \frac{\mathcal{F}(z_{\nu}^{(\kappa)})}{\mathcal{F}'(z_{\nu}^{(\kappa)})}}. \quad (2.10)$$

Now, based on the above, let us consider any differentiable function $\mathcal{F}(z)$, assuming that the multiplicity of each root is known. We construct a two-step iterative scheme to approximate all multiple roots of $\mathcal{F}(z)$ simultaneously, by taking as a first step any existing root-finding iterative method for multiple roots with order of convergence p having the form

$$\tilde{z}_{\mu}^{(\kappa)} = \psi(z_{\mu}^{(\kappa)}),$$

and considering as a second step the one provided by Eq (2.10). In the next section, we will show that the obtained method has order of convergence $4p$ for polynomial equations and order of convergence $2p$ for non-polynomial equations. The method is as follows:

$$\begin{aligned} \tilde{z}_{\mu}^{(\kappa)} &= \psi(z_{\mu}^{(\kappa)}), \\ z_{\mu}^{(\kappa+1)} &= \tilde{z}_{\mu}^{(\kappa)} - \frac{\alpha_{\mu}}{\frac{\mathcal{F}'(\tilde{z}_{\mu}^{(\kappa)})}{\mathcal{F}(\tilde{z}_{\mu}^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_{\nu}}{\tilde{z}_{\mu}^{(\kappa)} - \tilde{z}_{\nu}^{(\kappa)}} + \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{\alpha_{\nu}^2}{(\tilde{z}_{\mu}^{(\kappa)} - \tilde{z}_{\nu}^{(\kappa)})^2} \frac{\mathcal{F}(\tilde{z}_{\nu}^{(\kappa)})}{\mathcal{F}'(\tilde{z}_{\nu}^{(\kappa)})}}, \quad \kappa = 0, 1, \dots, \quad \mu = 1, 2, \dots, m, \end{aligned} \quad (2.11)$$

provided that $\tilde{z}_{\mu}^{(\kappa)} \neq \tilde{z}_{\nu}^{(\kappa)}$ for $\mu \neq \nu$.

2.1. Some special cases

Some special cases that will be used in the numerical section are described below.

- (1) Taking as the first step Newton's method for simple roots, we obtain the following iterative scheme with fourth-order convergence for polynomial equations and second-order convergence for non-polynomial equations, denoted by P_{4s} :

$$\begin{aligned}\tilde{z}_\mu^{(\kappa)} &= z_\mu^{(\kappa)} - \frac{\mathcal{F}(z_\mu^{(\kappa)})}{\mathcal{F}'(z_\mu^{(\kappa)})}, \\ z_\mu^{(\kappa+1)} &= \tilde{z}_\mu^{(\kappa)} - \frac{\alpha_\mu}{\frac{\mathcal{F}'(\tilde{z}_\mu^{(\kappa)})}{\mathcal{F}(\tilde{z}_\mu^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v^2}{(\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(\kappa)})}{\mathcal{F}'(\tilde{z}_v^{(\kappa)})}}, \quad \kappa = 0, 1, \dots, \quad \mu = 1, 2, \dots, m.\end{aligned}\tag{2.12}$$

- (2) Taking as the first step Newton's method for multiple roots, we obtain the following iterative scheme with eighth-order convergence for polynomial equations and fourth-order convergence for non-polynomial equations, denoted by PN_{4s} :

$$\begin{aligned}\tilde{z}_\mu^{(\kappa)} &= z_\mu^{(\kappa)} - \alpha_\mu \frac{\mathcal{F}(z_\mu^{(\kappa)})}{\mathcal{F}'(z_\mu^{(\kappa)})}, \\ z_\mu^{(\kappa+1)} &= \tilde{z}_\mu^{(\kappa)} - \frac{\alpha_\mu}{\frac{\mathcal{F}'(\tilde{z}_\mu^{(\kappa)})}{\mathcal{F}(\tilde{z}_\mu^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v^2}{(\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(\kappa)})}{\mathcal{F}'(\tilde{z}_v^{(\kappa)})}}, \quad \kappa = 0, 1, \dots, \quad \mu = 1, 2, \dots, m.\end{aligned}\tag{2.13}$$

- (3) Now, consider as a first step the Ehrlich method [7] for multiple roots, which has third-order convergence for polynomial equations and second-order convergence for non-polynomial equations. The resulting method has twelfth-order convergence for polynomial equations and fourth-order convergence for non-polynomial equations, denoted by PE_{4s} , and is given by

$$\begin{aligned}\tilde{z}_\mu^{(\kappa)} &= z_\mu^{(\kappa)} - \frac{\alpha_\mu}{\frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{z_\mu^{(\kappa)} - z_v^{(\kappa)}}}, \\ z_\mu^{(\kappa+1)} &= \tilde{z}_\mu^{(\kappa)} - \frac{\alpha_\mu}{\frac{\mathcal{F}'(\tilde{z}_\mu^{(\kappa)})}{\mathcal{F}(\tilde{z}_\mu^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v^2}{(\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(\kappa)})}{\mathcal{F}'(\tilde{z}_v^{(\kappa)})}}, \quad \kappa = 0, 1, \dots, \quad \mu = 1, 2, \dots, m.\end{aligned}\tag{2.14}$$

- (4) Consider as a first step the improved Ehrlich method [8] for multiple roots, having fourth-order convergence for polynomial equations and second-order convergence for non-polynomial equations. The resulting method has sixteenth-order convergence for polynomial equations and fourth-order convergence for non-polynomial equations, denoted by PEM_{4s} , and is given by

$$\begin{aligned} \tilde{z}_\mu^{(\kappa)} &= z_\mu^{(\kappa)} - \frac{\alpha_\mu}{\frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{z_\mu^{(\kappa)} - z_v^{(\kappa)} + \frac{\alpha_v \mathcal{F}(z_v^{(\kappa)})}{\mathcal{F}'(z_v^{(\kappa)})}}, \\ \kappa &= 0, 1, \dots, \quad \mu = 1, 2, \dots, m. \\ z_\mu^{(\kappa+1)} &= \tilde{z}_\mu^{(\kappa)} - \frac{\alpha_\mu}{\frac{\mathcal{F}'(\tilde{z}_\mu^{(\kappa)})}{\mathcal{F}(\tilde{z}_\mu^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)} + \sum_{v=1}^m \frac{\alpha_v^2}{(\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(\kappa)})}{\mathcal{F}'(\tilde{z}_v^{(\kappa)})}}. \end{aligned} \quad (2.15)$$

3. Convergence analysis

In this section, the convergence analysis of the proposed method for both polynomial and non-polynomial equations is provided via the following theorems.

Theorem 3.1. Let $\xi_1, \dots, \xi_m$ be real or complex roots with multiplicities $\alpha_1, \alpha_2, \dots, \alpha_m$ of the polynomial equation $\mathcal{F}(z) = 0$. Assume that $z_1^{(0)}, \dots, z_m^{(0)}$ are initial approximations sufficiently close to the corresponding roots. Then, the iterative method (2.11), obtained by taking as its first step any root-finding method for multiple roots of convergence order p , has convergence order $4p$.

Proof. Let

$$e_{\mu,\kappa} = z_\mu^{(\kappa)} - \xi_\mu, \quad (3.1)$$

$$\tilde{e}_{\mu,\kappa} = \tilde{z}_\mu^{(\kappa)} - \xi_\mu, \quad (3.2)$$

$$e_{\mu,\kappa+1} = z_\mu^{(\kappa+1)} - \xi_\mu, \quad (3.3)$$

be the errors in the approximations $z_\mu^{(\kappa)}$, $\tilde{z}_\mu^{(\kappa)}$, and $z_\mu^{(\kappa+1)}$, respectively. In view of (2.2), the term $\frac{\mathcal{F}'(z_\mu^{(\kappa)})}{\mathcal{F}(z_\mu^{(\kappa)})}$ in the second step of (2.11) can be written as

$$\frac{\mathcal{F}'(\tilde{z}_\mu^{(\kappa)})}{\mathcal{F}(\tilde{z}_\mu^{(\kappa)})} = \frac{\alpha_\mu}{\tilde{z}_\mu^{(\kappa)} - \xi_\mu} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{\tilde{z}_\mu^{(\kappa)} - \xi_v}. \quad (3.4)$$

Using this value of $\frac{\mathcal{F}'(\tilde{z}_\mu^{(k)})}{\mathcal{F}(\tilde{z}_\mu^{(k)})}$ in the second step of (2.11) results in

$$\begin{aligned}
 z_\mu^{(k+1)} &= \tilde{z}_\mu^{(k)} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{z}_\mu^{(k)} - \xi_\mu} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{\tilde{z}_\mu^{(k)} - \xi_v} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v^2}{(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})}} \\
 &= \tilde{z}_\mu^{(k)} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{z}_\mu^{(k)} - \xi_\mu} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v \left(\frac{1}{\tilde{z}_\mu^{(k)} - \xi_v} - \frac{1}{\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)}} + \frac{\alpha_v}{(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})} \right)} \\
 &= \tilde{z}_\mu^{(k)} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{z}_\mu^{(k)} - \xi_\mu} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v \left(\frac{-(\tilde{z}_v^{(k)} - \xi_v)}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})} + \frac{\alpha_v}{(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})} \right)}.
 \end{aligned} \tag{3.5}$$

Subtracting ξ_μ from both sides of the above formula gives

$$\begin{aligned}
 z_\mu^{(k+1)} - \xi_\mu &= \tilde{z}_\mu^{(k)} - \xi_\mu - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{z}_\mu^{(k)} - \xi_\mu} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v \left(\frac{-(\tilde{z}_v^{(k)} - \xi_v)}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})} + \frac{\alpha_v}{(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})} \right)}.
 \end{aligned} \tag{3.6}$$

Using the definition of the intermediate error $\tilde{e}_{\mu,k} = \tilde{z}_\mu^{(k)} - \xi_\mu$ from (3.2) and the new error $e_{\mu,k+1} = z_\mu^{(k+1)} - \xi_\mu$ from (3.3), Eq (3.6) can be written as

$$\begin{aligned}
 e_{\mu,k+1} &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{e}_{\mu,k}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v \left(\frac{-\tilde{e}_{v,k}}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})} + \frac{\alpha_v}{(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})} \right)}.
 \end{aligned} \tag{3.7}$$

Now, using the Taylor expansions of $\mathcal{F}(\tilde{z}_v^{(k)})$ and $\mathcal{F}'(\tilde{z}_v^{(k)})$ about ξ_v , where

$$C_{v,1} = \frac{\alpha_v!}{(\alpha_v + 1)!} \frac{\mathcal{F}^{(\alpha_v+1)}(\xi_v)}{\mathcal{F}^{(\alpha_v)}(\xi_v)},$$

we obtain

$$\mathcal{F}(\tilde{z}_v^{(k)}) = \frac{\mathcal{F}^{(\alpha_v)}(\xi_v)}{\alpha_v!} \left((\tilde{e}_{v,k})^{\alpha_v} + C_{v,1}(\tilde{e}_{v,k})^{\alpha_v+1} \right) + O((\tilde{e}_{v,k})^{\alpha_v+2}), \tag{3.8}$$

$$\mathcal{F}'(\tilde{z}_v^{(k)}) = \frac{\mathcal{F}^{(\alpha_v)}(\xi_v)}{\alpha_v!} \left(\alpha_v(\tilde{e}_{v,k})^{\alpha_v-1} + (\alpha_v + 1)C_{v,1}(\tilde{e}_{v,k})^{\alpha_v} \right) + O((\tilde{e}_{v,k})^{\alpha_v+1}). \tag{3.9}$$

We get

$$\begin{aligned}
 \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})} &= \frac{(\tilde{e}_{v,k})^{\alpha_v} + C_{v,1}(\tilde{e}_{v,k})^{\alpha_v+1} + O((\tilde{e}_{v,k})^{\alpha_v+2})}{\alpha_v(\tilde{e}_{v,k})^{\alpha_v-1} + (\alpha_v + 1)C_{v,1}(\tilde{e}_{v,k})^{\alpha_v} + O((\tilde{e}_{v,k})^{\alpha_v+1})} \\
 &= \frac{(\tilde{e}_{v,k})^{\alpha_v} + C_{v,1}(\tilde{e}_{v,k})^{\alpha_v+1} + O((\tilde{e}_{v,k})^{\alpha_v+2})}{\alpha_v(\tilde{e}_{v,k})^{\alpha_v-1} \left(1 + \frac{\alpha_v+1}{\alpha_v}C_{v,1}\tilde{e}_{v,k} + O((\tilde{e}_{v,k})^2)\right)} \\
 &= \frac{\tilde{e}_{v,k} + C_{v,1}(\tilde{e}_{v,k})^2 + O((\tilde{e}_{v,k})^3)}{\alpha_v \left(1 + \frac{\alpha_v+1}{\alpha_v}C_{v,1}\tilde{e}_{v,k} + O((\tilde{e}_{v,k})^2)\right)} \\
 &= \frac{1}{\alpha_v} \left(\tilde{e}_{v,k} + C_{v,1}(\tilde{e}_{v,k})^2 + O((\tilde{e}_{v,k})^3) \right) \left(1 + \frac{\alpha_v+1}{\alpha_v}C_{v,1}\tilde{e}_{v,k} + O((\tilde{e}_{v,k})^2) \right)^{-1}.
 \end{aligned} \tag{3.10}$$

Using the first-order binomial expansion $(1+x)^{-1} = 1 - x + O(x^2)$, $|x| < 1$, where $x = \frac{\alpha_v+1}{\alpha_v}C_{v,1}\tilde{e}_{v,k} + O((\tilde{e}_{v,k})^2)$, we obtain

$$\begin{aligned}
 \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})} &= \frac{1}{\alpha_v} \left(\tilde{e}_{v,k} + C_{v,1}(\tilde{e}_{v,k})^2 + O((\tilde{e}_{v,k})^3) \right) \\
 &\quad \times \left(1 - \frac{\alpha_v+1}{\alpha_v}C_{v,1}\tilde{e}_{v,k} + O((\tilde{e}_{v,k})^2) \right).
 \end{aligned}$$

Multiplying the above series and collecting terms of the same order yields

$$\begin{aligned}
 \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})} &= \frac{1}{\alpha_v} \left(\tilde{e}_{v,k} - \frac{C_{v,1}}{\alpha_v}(\tilde{e}_{v,k})^2 + O((\tilde{e}_{v,k})^3) \right) \\
 &= \frac{1}{\alpha_v} \left(\tilde{e}_{v,k} + O((\tilde{e}_{v,k})^2) \right).
 \end{aligned} \tag{3.11}$$

Using relation (3.11) in (3.7), we obtain

$$\begin{aligned}
 e_{\mu,k+1} &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{e}_{\mu,k}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v \left(\frac{-\tilde{e}_{v,k}}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})} + \frac{\alpha_v}{(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \frac{\tilde{e}_{v,k}}{\alpha_v} \right)} \\
 &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{e}_{\mu,k}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v \tilde{e}_{v,k} \left(\frac{-1}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})} + \frac{1}{(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \right)} \\
 &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{e}_{\mu,k}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v \tilde{e}_{v,k} \left(\frac{\tilde{z}_v^{(k)} - \xi_v}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \right)} \\
 &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{e}_{\mu,k}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v \tilde{e}_{v,k} \left(\frac{\tilde{e}_{v,k}}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \right)} \\
 &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu}{\frac{\alpha_\mu}{\tilde{e}_{\mu,k}} + \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v (\tilde{e}_{v,k})^2}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2}} \\
 &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu \tilde{e}_{\mu,k}}{\alpha_\mu + \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v (\tilde{e}_{v,k})^2 \tilde{e}_{\mu,k}}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2}}.
 \end{aligned} \tag{3.12}$$

Denoting $E_{\mu,v} = \frac{1}{(\tilde{z}_\mu^{(k)} - \xi_v)(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2}$, and noting that the initial approximations are sufficiently close to the corresponding distinct roots, the quantities $\tilde{z}_\mu^{(k)} - \xi_v$ and $\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)}$ remain bounded away from zero for $v \neq \mu$. Consequently, $E_{\mu,v}$ is bounded, and (3.12) can be rewritten as

$$\begin{aligned}
 e_{\mu,k+1} &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu \tilde{e}_{\mu,k}}{\alpha_\mu + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v (\tilde{e}_{v,k})^2 \tilde{e}_{\mu,k} E_{\mu,v}} \\
 &= \frac{\sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v (\tilde{e}_{v,k})^2 (\tilde{e}_{\mu,k})^2 E_{\mu,v}}{\alpha_\mu + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v (\tilde{e}_{v,k})^2 \tilde{e}_{\mu,k} E_{\mu,v}}.
 \end{aligned} \tag{3.13}$$

If the initial approximations are sufficiently close to the corresponding distinct roots, then the

intermediate approximations remain in their respective neighborhoods of the roots. Consequently, the intermediate errors [20] satisfy $|\tilde{e}_{\mu,k}| = O(\tilde{e}_k)$, $\mu = 1, 2, \dots, m$, where $\tilde{e}_k = \max_{1 \leq \mu \leq m} |\tilde{e}_{\mu,k}|$.

Furthermore, if the chosen root-finding iterative method for multiple roots has convergence order p , i.e.,

$$\tilde{e}_k = O(e_{\mu,k}^p),$$

then we obtain

$$e_{\mu,k+1} = \frac{\sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v (\tilde{e}_k)^4 E_{\mu,v}}{\alpha_\mu + \sum_{\substack{v=1 \\ v \neq \mu}}^m \alpha_v (\tilde{e}_k)^3 E_{\mu,v}} = O(e_{\mu,k}^{4p}). \quad (3.14)$$

Therefore, the proposed method (2.11) achieves convergence order $4p$ for polynomial equations. $\square$

Let us now consider the case of non-polynomial equations. The convergence analysis of the proposed method (2.11) in this case is established in Theorem 3.2.

Theorem 3.2. Suppose that $\xi_1, \xi_2, \dots, \xi_m$ are the real or complex zeros of the nonlinear equation $\mathcal{F}(z) = 0$ with known multiplicities $\alpha_1, \alpha_2, \dots, \alpha_m$. Let $z_1^{(0)}, z_2^{(0)}, \dots, z_m^{(0)}$ be initial approximations sufficiently close to the corresponding roots. Then, if the predictor step is any iterative method for multiple roots of convergence order p , the iterative scheme defined by (2.11) possesses a convergence order of $2p$.

Proof. Let $e_{\mu,k} = z_\mu^{(k)} - \xi_\mu$ denote the error associated with the approximation of the μ^{th} root at the k^{th} iteration. Similarly, define $\tilde{e}_{\mu,k} = \tilde{z}_\mu^{(k)} - \xi_\mu$ as the error corresponding to the predictor approximation. Since ψ is an iterative method of convergence order p , it follows that $\tilde{e}_{\mu,k} \sim e_{\mu,k}^p$.

Next, by applying the Taylor expansion to $\mathcal{F}(\tilde{z}_\mu^{(k)})$ and $\mathcal{F}'(\tilde{z}_\mu^{(k)})$ about ξ_μ , we obtain

$$\mathcal{F}(\tilde{z}_\mu^{(k)}) = \frac{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}{\alpha_\mu!} \left((\tilde{e}_{\mu,k})^{\alpha_\mu} + C_{\mu,1}(\tilde{e}_{\mu,k})^{\alpha_\mu+1} + C_{\mu,2}(\tilde{e}_{\mu,k})^{\alpha_\mu+2} \right) + O((\tilde{e}_{\mu,k})^{\alpha_\mu+3}), \quad (3.15)$$

$$\mathcal{F}'(\tilde{z}_\mu^{(k)}) = \frac{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}{\alpha_\mu!} \left(\alpha_\mu (\tilde{e}_{\mu,k})^{\alpha_\mu-1} + (\alpha_\mu + 1)C_{\mu,1}(\tilde{e}_{\mu,k})^{\alpha_\mu} + (\alpha_\mu + 2)C_{\mu,2}(\tilde{e}_{\mu,k})^{\alpha_\mu+1} \right) + O((\tilde{e}_{\mu,k})^{\alpha_\mu+2}), \quad (3.16)$$

where

$$C_{\mu,1} = \frac{\alpha_\mu!}{(\alpha_\mu + 1)!} \frac{\mathcal{F}^{(\alpha_\mu+1)}(\xi_\mu)}{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}, \quad C_{\mu,2} = \frac{\alpha_\mu!}{(\alpha_\mu + 2)!} \frac{\mathcal{F}^{(\alpha_\mu+2)}(\xi_\mu)}{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}.$$

More generally, we find

$$C_{\mu,v} = \frac{\alpha_\mu!}{(\alpha_\mu + v)!} \frac{\mathcal{F}^{(\alpha_\mu+v)}(\xi_\mu)}{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}, \quad v \geq 1.$$

The second step in (2.11) can be written as

$$z_\mu^{(k+1)} = \tilde{z}_\mu^{(k)} - \frac{\alpha_\mu \mathcal{F}(\tilde{z}_\mu^{(k)})}{\mathcal{F}'(\tilde{z}_\mu^{(k)}) - \mathcal{F}(\tilde{z}_\mu^{(k)}) \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)}} + \mathcal{F}(\tilde{z}_\mu^{(k)}) \sum_{v=1, v \neq \mu}^m \frac{\alpha_v^2}{(\tilde{z}_\mu^{(k)} - \tilde{z}_v^{(k)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(k)})}{\mathcal{F}'(\tilde{z}_v^{(k)})}}. \quad (3.17)$$

Substituting (3.15), (3.16), and (3.11) into the denominator of the second step and truncating the expansion after the $O((\tilde{e}_{\mu,k})^{\alpha_\mu+1})$ term, we obtain

$$\begin{aligned}
 & \mathcal{F}'(\tilde{z}_\mu^{(\kappa)}) - \mathcal{F}(\tilde{z}_\mu^{(\kappa)}) \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)}} + \mathcal{F}(\tilde{z}_\mu^{(\kappa)}) \sum_{v=1, v \neq \mu}^m \frac{\alpha_v^2}{(\tilde{z}_\mu^{(\kappa)} - \tilde{z}_v^{(\kappa)})^2} \frac{\mathcal{F}(\tilde{z}_v^{(\kappa)})}{\mathcal{F}'(\tilde{z}_v^{(\kappa)})} \\
 &= \frac{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}{\alpha_\mu!} \left(\alpha_\mu (\tilde{e}_{\mu,k})^{\alpha_\mu-1} + \left((\alpha_\mu + 1)C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu} \right. \\
 & \quad \left. + \left((\alpha_\mu + 2)C_{\mu,2} - C_{\mu,1} \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} + \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{(\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k})^2} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu+1} \right) \\
 & \quad + O((\tilde{e}_{\mu,k})^{\alpha_\mu+2}) \\
 &= \frac{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}{\alpha_\mu!} \left(\alpha_\mu (\tilde{e}_{\mu,k})^{\alpha_\mu-1} + \left((\alpha_\mu + 1)C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu} + O((\tilde{e}_{\mu,k})^{\alpha_\mu+1}) \right).
 \end{aligned} \tag{3.18}$$

Thus, using (3.18) and (3.15) in (3.17) gives

$$\begin{aligned}
 & \frac{\alpha_\mu \frac{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}{\alpha_\mu!} ((\tilde{e}_{\mu,k})^{\alpha_\mu} + C_{\mu,1}(\tilde{e}_{\mu,k})^{\alpha_\mu+1}) + O((\tilde{e}_{\mu,k})^{\alpha_\mu+2})}{\frac{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}{\alpha_\mu!} \left(\alpha_\mu (\tilde{e}_{\mu,k})^{\alpha_\mu-1} + \left((\alpha_\mu + 1)C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu} + O((\tilde{e}_{\mu,k})^{\alpha_\mu+1}) \right)} \\
 & z_\mu^{(\kappa+1)} = \tilde{z}_\mu^{(\kappa)} - \frac{\alpha_\mu \frac{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}{\alpha_\mu!} ((\tilde{e}_{\mu,k})^{\alpha_\mu} + C_{\mu,1}(\tilde{e}_{\mu,k})^{\alpha_\mu+1}) + O((\tilde{e}_{\mu,k})^{\alpha_\mu+2})}{\frac{\mathcal{F}^{(\alpha_\mu)}(\xi_\mu)}{\alpha_\mu!} \left(\alpha_\mu (\tilde{e}_{\mu,k})^{\alpha_\mu-1} + \left((\alpha_\mu + 1)C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu} + O((\tilde{e}_{\mu,k})^{\alpha_\mu+1}) \right)}.
 \end{aligned} \tag{3.19}$$

Subtracting ξ_μ from both sides in (3.19) gives

$$\begin{aligned}
 & z_\mu^{(\kappa+1)} - \xi_\mu = \tilde{z}_\mu^{(\kappa)} - \xi_\mu \\
 & \quad - \frac{\alpha_\mu ((\tilde{e}_{\mu,k})^{\alpha_\mu} + C_{\mu,1}(\tilde{e}_{\mu,k})^{\alpha_\mu+1}) + O((\tilde{e}_{\mu,k})^{\alpha_\mu+2})}{\alpha_\mu (\tilde{e}_{\mu,k})^{\alpha_\mu-1} + \left((\alpha_\mu + 1)C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu} + O((\tilde{e}_{\mu,k})^{\alpha_\mu+1})}.
 \end{aligned} \tag{3.20}$$

By using errors on both sides, (3.20) can be written as

$$\begin{aligned}
 e_{\mu,k+1} &= \tilde{e}_{\mu,k} - \frac{\alpha_\mu \left((\tilde{e}_{\mu,k})^{\alpha_\mu} + C_{\mu,1} (\tilde{e}_{\mu,k})^{\alpha_\mu+1} \right) + O\left((\tilde{e}_{\mu,k})^{\alpha_\mu+2}\right)}{\alpha_\mu (\tilde{e}_{\mu,k})^{\alpha_\mu-1} + \left((\alpha_\mu + 1) C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu} + O\left((\tilde{e}_{\mu,k})^{\alpha_\mu+1}\right)} \\
 &= \frac{\left((\alpha_\mu + 1) C_{\mu,1} - \alpha_\mu C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu+1} + O\left((\tilde{e}_{\mu,k})^{\alpha_\mu+2}\right)}{\alpha_\mu (\tilde{e}_{\mu,k})^{\alpha_\mu-1} + \left((\alpha_\mu + 1) C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu} + O\left((\tilde{e}_{\mu,k})^{\alpha_\mu+1}\right)} \\
 &= \frac{\left((\alpha_\mu + 1) C_{\mu,1} - \alpha_\mu C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^{\alpha_\mu+1} + O\left((\tilde{e}_{\mu,k})^{\alpha_\mu+2}\right)}{(\tilde{e}_{\mu,k})^{\alpha_\mu-1} \left(\alpha_\mu + \left((\alpha_\mu + 1) C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) \tilde{e}_{\mu,k} \right) + O\left((\tilde{e}_{\mu,k})^2\right)} \\
 &= \frac{\left(C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) (\tilde{e}_{\mu,k})^2 + O\left((\tilde{e}_{\mu,k})^3\right)}{\alpha_\mu + \left((\alpha_\mu + 1) C_{\mu,1} - \sum_{v=1, v \neq \mu}^m \frac{\alpha_v}{\tilde{e}_{\mu,k} + \xi_\mu - \xi_v - \tilde{e}_{v,k}} \right) \tilde{e}_{\mu,k} + O\left((\tilde{e}_{\mu,k})^2\right)}. \tag{3.21}
 \end{aligned}$$

Thus, from Eq (3.21) and the fact that $\tilde{e}_{\mu,k} = O(e_{\mu,k}^p)$, we obtain

$$e_{\mu,k+1} = O\left((\tilde{e}_{\mu,k})^2\right) = O\left((e_{\mu,k}^p)^2\right) = O\left(e_{\mu,k}^{2p}\right). \tag{3.22}$$

Therefore, the order of convergence of method (2.11) is $2p$ for non-polynomial equations. $\square$

Remark 3.3 (Extension to real multiplicities). *The proposed iterative framework is not restricted to integer multiplicities. In the non-polynomial case, suppose that, in a neighborhood of a root ξ_μ , the function $\mathcal{F}$ admits the local factorization*

$$\mathcal{F}(z) = (z - \xi_\mu)^{\alpha_\mu} G_\mu(z), \quad \alpha_\mu \in \mathbb{R}, \quad G_\mu(\xi_\mu) \neq 0.$$

For non-integer α_μ , a consistent local branch of $(z - \xi_\mu)^{\alpha_\mu}$ is assumed. Then,

$$\frac{\mathcal{F}'(z)}{\mathcal{F}(z)} = \frac{\alpha_\mu}{z - \xi_\mu} + \frac{G'_\mu(z)}{G_\mu(z)}.$$

Thus, the proposed framework can also be applied to nonlinear equations with known real (non-integer) multiplicities. The local convergence analysis for the non-polynomial case remains applicable, and the resulting method retains the convergence order $2p$.

4. Numerical experiments

In this section, several numerical experiments are performed to illustrate the behavior of the proposed schemes P_{4s} , PN_{4s} , PE_{4s} , PEM_{4s} , as given in (2.12), (2.13), (2.14), and (2.15), respectively. We have compared these methods with the scheme by Mir et al. [21] (denoted by MMN_8) for simultaneously finding multiple roots, which has eighth-order convergence for nonlinear equations, given by

$$\begin{aligned}y_{\mu}^{(\kappa)} &= z_{\mu}^{(\kappa)} - \alpha_{\mu} \frac{\mathcal{F}(z_{\mu}^{(\kappa)})}{\mathcal{F}'(z_{\mu}^{(\kappa)})}, \\t_{\mu}^{(\kappa)} &= z_{\mu}^{(\kappa)} - \frac{\alpha_{\mu}}{\frac{\mathcal{F}'(z_{\mu}^{(\kappa)})}{\mathcal{F}(z_{\mu}^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{z_{\mu}^{(\kappa)} - y_v^{(\kappa)}}}, \\z_{\mu}^{(\kappa+1)} &= t_{\mu}^{(\kappa)} - \frac{\alpha_{\mu}}{\frac{\mathcal{F}'(t_{\mu}^{(\kappa)})}{\mathcal{F}(t_{\mu}^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{t_{\mu}^{(\kappa)} - t_v^{(\kappa)}} - a}\end{aligned}$$

where $a \in \mathbb{R}$, $\kappa = 0, 1, \dots$, and $\mu = 1, 2, \dots, m$.

Additionally, we compare the proposed schemes with the fourth-order multiple root finding method of Jerezano et al. [17] (denoted by $SEYY_4$), which is given by

$$\begin{aligned}y_{\mu}^{(\kappa)} &= z_{\mu}^{(\kappa)} - u(z_{\mu}^{(\kappa)}), \\z_{\mu}^{(\kappa+1)} &= y_{\mu}^{(\kappa)} - \frac{1}{\frac{1}{u(y_{\mu}^{(\kappa)})} - \sigma_{\mu}^{(\kappa)}},\end{aligned}$$

where $u(x) = \frac{g(x)}{g'(x)}$, $g(x) = \frac{\mathcal{F}(x)}{\mathcal{F}'(x)}$, and $\sigma_{\mu}^{(\kappa)} = \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{1}{y_{\mu}^{(\kappa)} - y_v^{(\kappa)}}$. Here, $\kappa = 0, 1, \dots$ and $\mu = 1, 2, \dots, m$.

We also compare these schemes with Ehrlich–Aberth's method, which incorporates Newton's corrections and is also known as Nouredin's method [8] (denoted by N_4). This method presents fourth-order convergence for polynomial equations and second-order convergence for non-polynomial equations and is given by

$$z_{\mu}^{(\kappa+1)} = z_{\mu}^{(\kappa)} - \frac{\alpha_{\mu}}{\frac{\mathcal{F}'(z_{\mu}^{(\kappa)})}{\mathcal{F}(z_{\mu}^{(\kappa)})} - \sum_{\substack{v=1 \\ v \neq \mu}}^m \frac{\alpha_v}{z_{\mu}^{(\kappa)} - z_v^{(\kappa)} + \frac{\alpha_v \mathcal{F}(z_v^{(\kappa)})}{\mathcal{F}'(z_v^{(\kappa)})}}}.$$

Here, $\kappa = 0, 1, \dots$ and $\mu = 1, 2, \dots, m$.

For comparison, we also consider Aberth's method [22] (denoted by BI_2), which has second-order

convergence and is given by

$$z_{\mu}^{(\kappa+1)} = z_{\mu}^{(\kappa)} - \frac{1}{\frac{\mathcal{F}'(z_{\mu}^{(\kappa)})}{\mathcal{F}(z_{\mu}^{(\kappa)})} - \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^m \frac{1}{z_{\mu}^{(\kappa)} - z_{\nu}^{(\kappa)}}}.$$

Here, $\kappa = 0, 1, \dots$ and $\mu = 1, 2, \dots, m$. All numerical experiments were carried out using Mathematica Version 11.1 on a desktop computer equipped with an 11th Gen Intel(R) Core(TM) i3-1125G4 processor (2.00 GHz, 64-bit architecture), 8 GB RAM, Intel(R) UHD Graphics, and running the Microsoft Windows 10 operating system. In Tables 1–5, $p(\pm n)$ denotes $p \times 10^{\pm n}$. The Euclidean norm of the difference between two successive approximations is represented by $\|z^{(\kappa+1)} - z^{(\kappa)}\|_2$, where $z^{(\kappa)} = (z_1^{(\kappa)}, z_2^{(\kappa)}, \dots, z_m^{(\kappa)})$ denotes the vector of approximations to the m roots at the κ th iteration. Likewise, the Euclidean norm of the function values at the $(\kappa+1)$ th iteration is expressed as $\|\mathcal{F}(z^{(\kappa+1)})\|_2$, where

$$\mathcal{F}(z^{(\kappa+1)}) = (\mathcal{F}(z_1^{(\kappa+1)}), \mathcal{F}(z_2^{(\kappa+1)}), \dots, \mathcal{F}(z_m^{(\kappa+1)})).$$

For the numerical experiments, the computations were performed using variable-precision arithmetic with 100,000 digits. The tolerance was fixed at $Tol = 10^{-300}$. The iterative process was terminated when the following stopping criterion was satisfied:

$$\|z^{(\kappa+1)} - z^{(\kappa)}\|_2 + \|\mathcal{F}(z^{(\kappa)})\|_2 < Tol.$$

The number of iterations required by the different iterative methods to satisfy this stopping criterion is reported in Table 8. The approximate computational order of convergence (ACOC) [23], which estimates the theoretical order of convergence ρ , was computed using

$$\rho \approx ACO C = \frac{\ln\left(\frac{\|z^{(\kappa+2)} - z^{(\kappa+1)}\|_2}{\|z^{(\kappa+1)} - z^{(\kappa)}\|_2}\right)}{\ln\left(\frac{\|z^{(\kappa+1)} - z^{(\kappa)}\|_2}{\|z^{(\kappa)} - z^{(\kappa-1)}\|_2}\right)}, \quad \kappa = 1, 2, \dots$$

Furthermore, the computational performance in terms of the iteration process and residual error reduction for Examples 4.1–4.5 is illustrated graphically in Figures 1–5. In these figures, ξ_{μ} denotes the μ th root of the nonlinear equation. The solid curves correspond to the proposed methods, whereas the dotted curves represent the existing iterative methods used for comparison.

Example 4.1. The Van der Waals equation of state [24] offers an explanation for the behavior of real gases by incorporating two specific parameters, b_1 and b_2 , into the ideal gas equations. This equation, which introduces corrections to the ideal gas law, is given by

$$\left(P + \frac{b_1 n^2}{V^2}\right)(V - nb_2) = nRT,$$

where P , V , n , and T represent pressure, volume, amount of substance, and temperature, respectively, while R is the gas constant. The equation describes how the pressure, volume, and temperature of a gas are interrelated, considering deviations from ideal gas behavior.

The determination of the gas volume, V , involves solving the nonlinear equation

$$PV^3 - (na_2P + nRT)V^2 + b_1b_2n^2 = 0.$$

For a particular gas with known constants b_1 and b_2 , values of n , P , and T can be found in such a way that this equation possesses three real roots. Upon substitution of specific values, the nonlinear equation takes the form:

$$\mathcal{F}(z) = z^3 - 5.22z^2 + 9.0825z - 5.2675 = 0.$$

This equation exhibits three zeros, with one being a repeated root at $\xi_1 = 1.75$ of multiplicity 2, and the other being a simple root at $\xi_2 = 1.72$. The initial approximations have been taken as $z_1^{(0)} = 1.5$ and $z_2^{(0)} = 1.9$.

Table 1. Numerical comparison of different iterative methods for Example 4.1.

Method	$\ z^{(2)} - z^{(1)}\ $	$\ z^{(3)} - z^{(2)}\ $	$\ z^{(4)} - z^{(3)}\ $	$\ \mathcal{F}(z^{(4)})\ $	ACOC	CPU Time
P_{4s}	3.6(-2)	4.2(-4)	7.2(-14)	2.0(-75)	6.0915	0.296
PN_{4s}	1.7(-2)	8.1(-7)	4.9(-40)	1.1(-308)	8.0327	0.297
PE_{4s}	8.5(-3)	5.0(-12)	1.8(-121)	1.2(-1437)	12.0152	0.234
PEM_{4s}	4.6(-3)	2.9(-19)	5.9(-277)	3.0(-4403)	16.0088	0.406
N_4	4.8(-2)	3.0(-3)	6.5(-7)	1.5(-24)	4.1209	0.310
MMN_8	8.0(-2)	9.9(-5)	2.6(-26)	1.4(-199)	8.1319	0.325
$SEYY_4$	6.7(-3)	3.4(-3)	1.1(-4)	6.1(-14)	4.1587	0.921

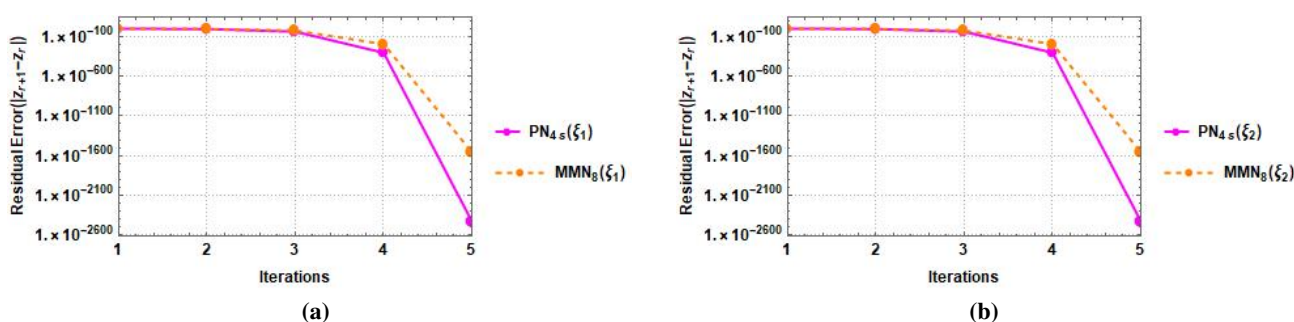


Figure 1. Convergence behavior of PN_{4s} and MMN_8 for the roots ξ_1 and ξ_2 in Example 4.1.

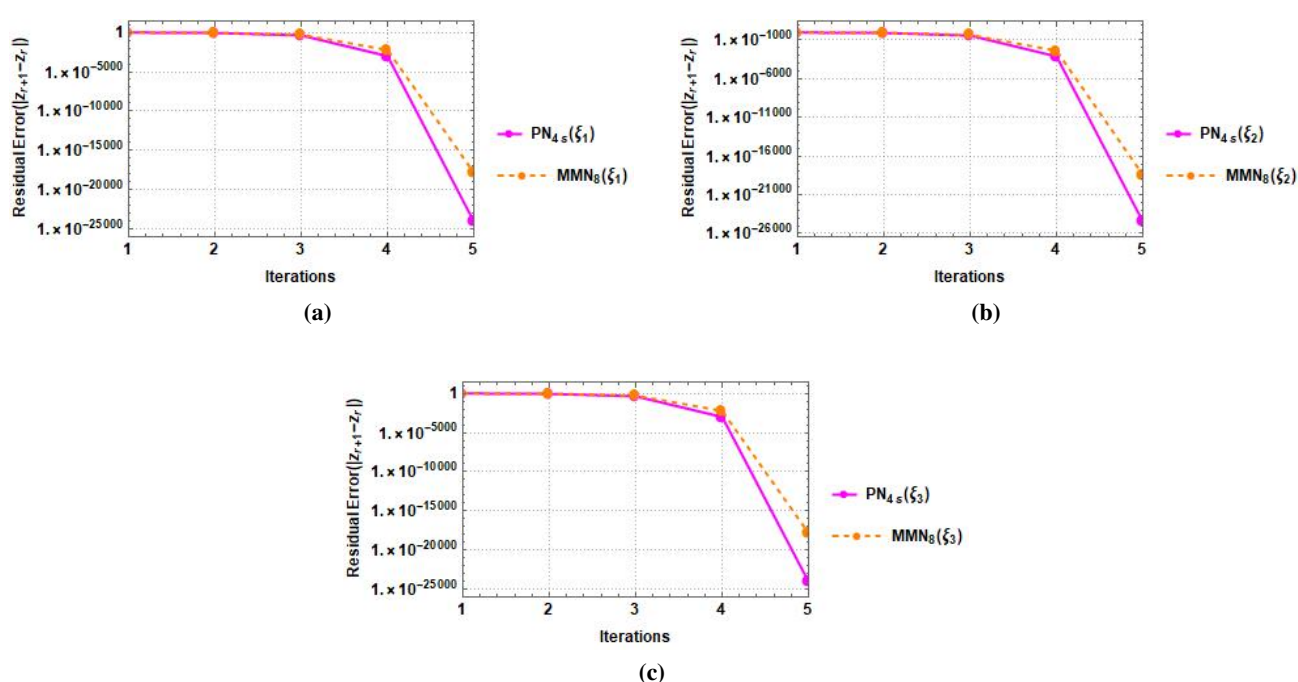
Example 4.2. A beam of length r units is placed against one edge of a cubical box with side lengths of 1 unit. Here, the beam is positioned in such a way that one of its ends strikes against the wall and the other touches the floor. The problem is to evaluate the distance between the base of the wall and the bottom of the beam, denoted by y . Let z represent the distance from the bottom of the cubical box to the bottom of the beam. Given a specific value for r , the equation

$$z^4 + 4z^3 - 24z^2 + 16z + 16 = 0$$

arises. This equation possesses four roots. One of these roots is a repeated root located at $\xi_1 = 2$ with a multiplicity of 2, while the other roots are distinct and occur at $\xi_2 = -2(\sqrt{3} + 2)$ and $\xi_3 = 2(\sqrt{3} - 2)$. The initial estimates for finding the roots are taken as $z_1^{(0)} = 1.5$, $z_2^{(0)} = -7.9$, and $z_3^{(0)} = -0.9$.

Table 2. Numerical comparison of different iterative methods for Example 4.2.

Method	$\ z^{(2)} - z^{(1)}\ $	$\ z^{(3)} - z^{(2)}\ $	$\ z^{(4)} - z^{(3)}\ $	$\ \mathcal{F}(z^{(4)})\ $	ACOC	CPU Time
P_{4s}	3.7(-5)	6.5(-30)	1.8(-178)	3.6(-1068)	6.0050	0.609
PN_{4s}	5.1(-6)	4.1(-47)	7.1(-376)	2.1(-3004)	8.0032	0.469
PE_{4s}	1.0(-8)	3.8(-103)	2.2(-1236)	1.7(-14833)	12.0015	0.432
PEM_{4s}	4.6(-12)	3.1(-191)	5.2(-3058)	8.5(-48925)	16.0008	0.813
N_4	3.1(-3)	1.9(-12)	2.9(-49)	5.7(-195)	4.0123	0.669
MMN_8	1.3(-4)	4.8(-35)	6.3(-279)	2.1(-2228)	8.0043	0.589
$SEYY_4$	6.9(-3)	6.1(-11)	5.0(-43)	1.3(-169)	4.0114	1.844

**Figure 2.** Convergence behavior of PN_{4s} and MMN_8 for the roots (ξ_1, ξ_2, ξ_3) for Example 4.2.

Example 4.3. Consider the following nonlinear equation having two real roots, each with a multiplicity of three:

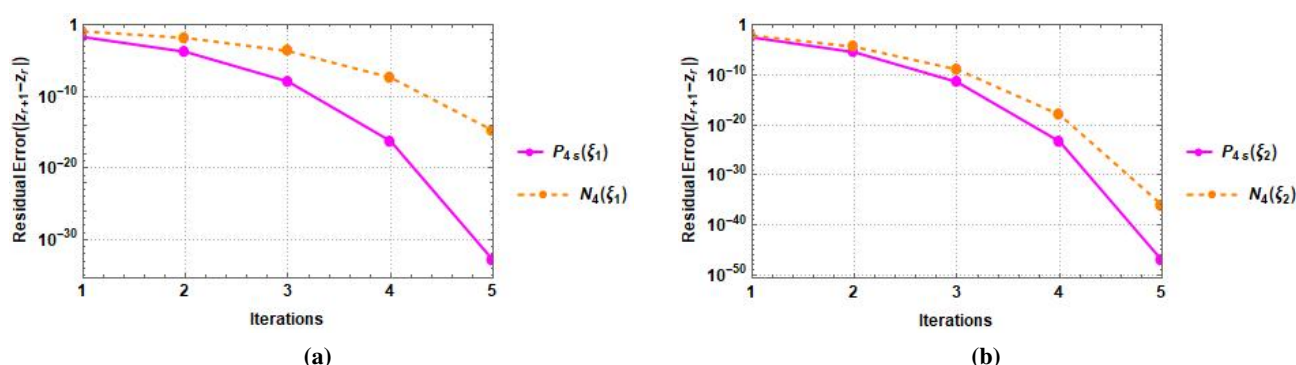
$$\mathcal{F}(z) = ze^{-3z} - \frac{3}{10}ze^{-2z} + \frac{3}{10}ze^{-z} - \frac{1}{1000} = 0.$$

The roots are $\xi_1 \approx 0.1118326$ and $\xi_2 \approx 3.577152$, and the initial approximations have been taken as $z_1^{(0)} = 0.5$, $z_2^{(0)} = 3.7$.

Table 3. Numerical comparison of different iterative methods for Example 4.3.

Method	$\ z^{(2)} - z^{(1)}\ $	$\ z^{(3)} - z^{(2)}\ $	$\ z^{(4)} - z^{(3)}\ $	$\ \mathcal{F}(z^{(4)})\ $	ACOC	CPU Time
P_{4s}	2.3(-2)	1.9(-4)	1.3(-8)	1.2(-49)	2.0005	1.406
PN_{4s}	5.1(-2)	5.3(-6)	7.1(-22)	5.7(-255)	4.0000	1.296
PE_{4s}	1.9(-2)	5.9(-8)	5.7(-30)	6.9(-353)	4.0000	1.548
PEM_{4s}	1.8(-2)	4.9(-8)	2.7(-30)	1.4(-356)	4.0000	2.031
N_4	1.4(-1)	1.7(-2)	2.5(-4)	6.0(-23)	2.0034	1.032
MMN_8	dvgt	dvgt	dvgt	dvgt	dvgt	dvgt
$SEYY_4$	3.4(-2)	2.2(-6)	4.1(-23)	4.9(-269)	4.0000	3.969

(dvgt: stands for divergent.)

**Figure 3.** Convergence behavior of P_{4s} and N_4 for the roots (ξ_1, ξ_2) for Example 4.3.**Example 4.4.** Consider the monic polynomial of degree $n = 12$ having complex roots given by

$$\mathcal{F}(z) = z^{12} - 6z^{11} + 9z^{10} - z^2 + 6z + 9 = 0.$$

This polynomial possesses twelve roots. One of these roots is a repeated root located at $\xi_1 = 3$ with a multiplicity of 2, while the other roots are simple. The initial approximations are taken as $z_1^{(0)} = -2$, $z_2^{(0)} = 2$, $z_3^{(0)} = 2.5$, $z_4^{(0)} = 0.5 + i$, $z_5^{(0)} = 0.5 - i$, $z_6^{(0)} = -0.5 + i$, $z_7^{(0)} = -0.5 - i$, $z_8^{(0)} = -1 + 0.5i$, $z_9^{(0)} = -1 - 0.5i$, $z_{10}^{(0)} = 1 + 0.5i$, and $z_{11}^{(0)} = 1 - 0.5i$.

Table 4. Numerical comparison of different iterative methods for Example 4.4.

Method	$\ z^{(2)} - z^{(1)}\ $	$\ z^{(3)} - z^{(2)}\ $	$\ z^{(4)} - z^{(3)}\ $	$\ \mathcal{F}(z^{(4)})\ $	ACOC	CPU Time
P_{4s}	4.8(-1)	1.7(-3)	1.9(-19)	6.6(-112)	6.2194	5.031
PN_{4s}	4.8(-1)	5.7(-5)	4.6(-37)	9.2(-288)	8.2119	5.109
PE_{4s}	2.4(-3)	1.8(-31)	4.4(-370)	1.8(-4430)	12.0054	5.266
PEM_{4s}	2.8(-2)	8.6(-28)	1.7(-431)	9.7(-6889)	16.0092	6.031
N_4	22.88	2.8(-2)	1.6(-7)	2.8(-26)	4.5482	7.968
MMN_8	22.81	5.3(-4)	8.3(-26)	1.7(-198)	8.0500	5.343
$SEYY_4$	dvgt	dvgt	dvgt	dvgt	dvgt	dvgt

(dvgt: stands for divergent.)

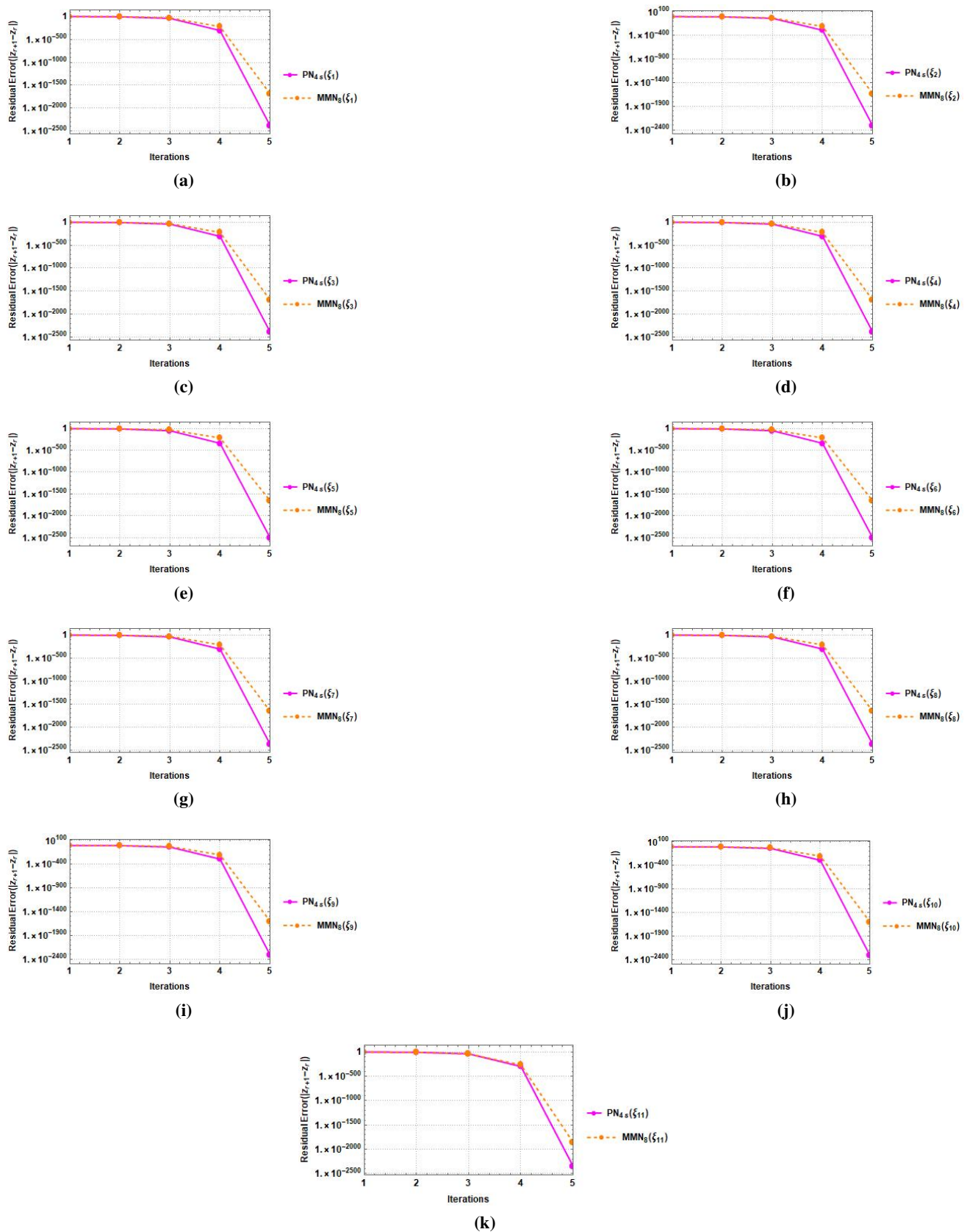


Figure 4. Convergence behavior of PN_{4s} and MMN_8 for the eleven roots $(\xi_1, \dots, \xi_{11})$ for Example 4.4.

Example 4.5. Consider the nonlinear equation with infinitely many roots, each having a multiplicity of six, except for one root, which has a multiplicity of twelve, given by

$$\mathcal{F}(z) = ((z - \pi) \sin(z - \pi))^6 = 0.$$

This function has infinitely many roots, but we will approximate the roots that lie within the interval $\left[\frac{\pi}{2}, \frac{15\pi}{2}\right]$. In this interval, there are seven roots: six roots with multiplicity six, at $\xi_1 = 2\pi$, $\xi_2 = 3\pi$, $\xi_3 = 4\pi$, $\xi_4 = 5\pi$, $\xi_5 = 6\pi$, and $\xi_6 = 7\pi$, and one root with multiplicity twelve, at $\xi_7 = \pi$. The initial approximations are taken as $z_1^{(0)} = 3$, $z_2^{(0)} = 6$, $z_3^{(0)} = 9$, $z_4^{(0)} = 12$, $z_5^{(0)} = 15$, $z_6^{(0)} = 18$, and $z_7^{(0)} = 22$.

Table 5. Numerical comparison of different iterative methods for Example 4.5.

Method	$\ z^{(2)} - z^{(1)}\ $	$\ z^{(3)} - z^{(2)}\ $	$\ z^{(4)} - z^{(3)}\ $	$\ \mathcal{F}(z^{(4)})\ $	ACOC	CPU Time
P_{4s}	1.3(-1)	3.8(-3)	4.1(-6)	2.0(-61)	2.0005	0.954
PN_{4s}	7.7(-2)	1.1(-7)	3.2(-31)	7.9(-742)	4.0000	0.797
PE_{4s}	6.5(-3)	1.1(-10)	1.1(-41)	1.6(-983)	4.0000	0.704
PEM_{4s}	1.6(-2)	4.6(-9)	3.1(-35)	1.4(-828)	4.0000	1.188
N_4	2.1(-1)	1.6(-2)	1.0(-4)	1.5(-43)	2.0011	0.938
MMN_8	3.4(-1)	1.5(-2)	2.8(-8)	2.4(-176)	4.0015	0.922
$SEYY_4$	1.0(-1)	6.9(-7)	4.4(-28)	3.4(-666)	4.0000	8.734

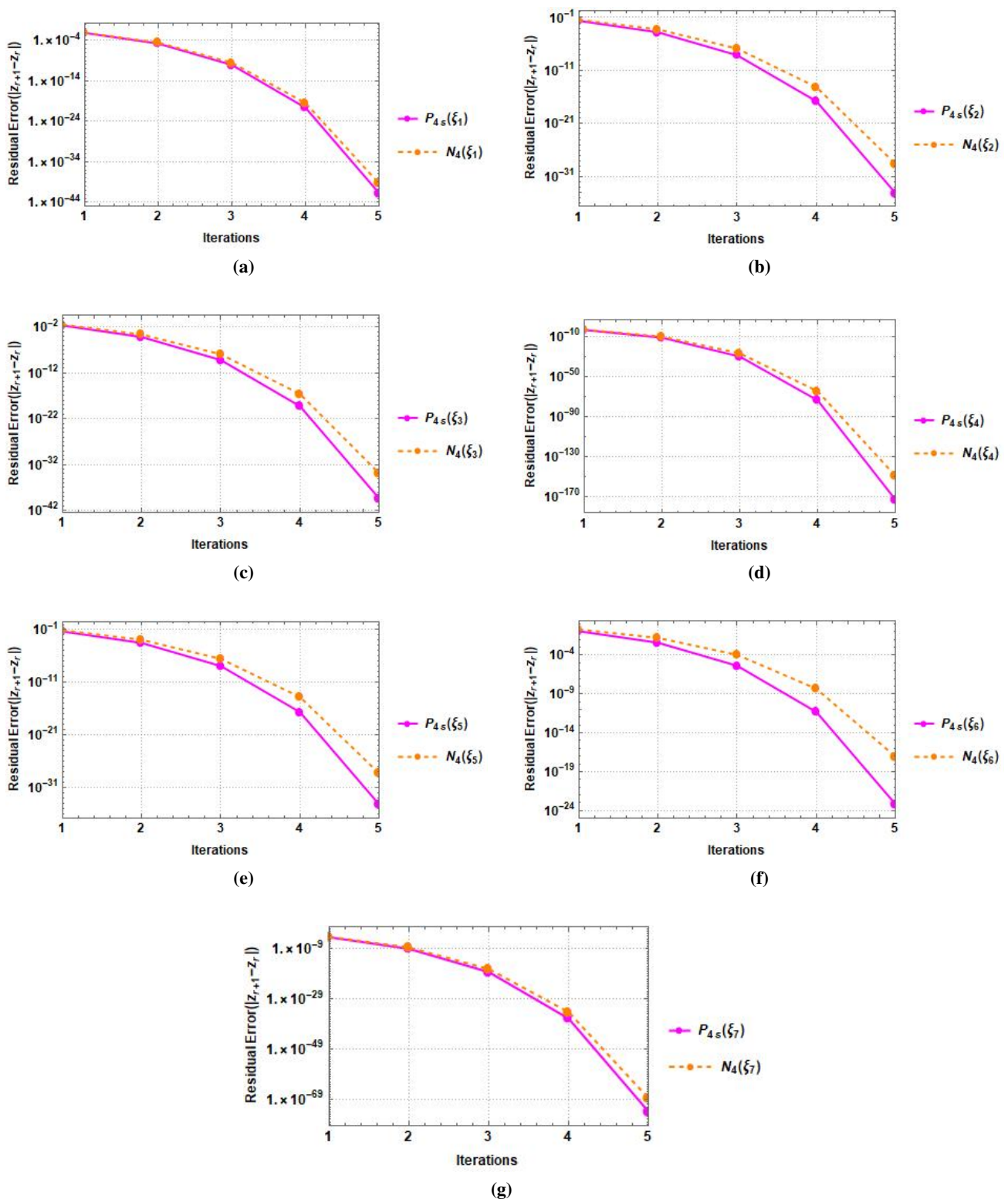


Figure 5. Convergence behavior of P_{4s} and N_4 for the roots $(\xi_1, \xi_2, \dots, \xi_7)$ for Example 4.5.

Example 4.6. Consider the nonlinear equation [22]

$$\mathcal{F}(z) = (z + 1)^{20} - (1 - i)z^{20} = 0.$$

The equation is a polynomial equation of degree 20 and has 20 distinct complex roots. The approximate roots are

$$\begin{aligned} \xi_1 &\approx -0.495666 - 0.009818i, & \xi_2 &\approx -0.495585 + 0.0691531i, & \xi_3 &\approx -0.495530 - 0.0892818i, \\ \xi_4 &\approx -0.495269 + 0.151661i, & \xi_5 &\approx -0.495147 - 0.173370i, & \xi_6 &\approx -0.494649 + 0.242495i, \\ \xi_7 &\approx -0.494430 - 0.267230i, & \xi_8 &\approx -0.493564 + 0.348442i, & \xi_9 &\approx -0.493186 - 0.378445i, \\ \xi_{10} &\approx -0.491664 + 0.480671i, & \xi_{11} &\approx -0.490983 - 0.519950i, & \xi_{12} &\approx -0.488109 + 0.660383i, \\ \xi_{13} &\approx -0.486752 - 0.717235i, & \xi_{14} &\approx -0.480510 + 0.935118i, & \xi_{15} &\approx -0.477261 - 1.030450i, \\ \xi_{16} &\approx -0.459667 + 1.440870i, & \xi_{17} &\approx -0.448632 - 1.646810i, & \xi_{18} &\approx -0.360153 + 2.793120i, \\ \xi_{19} &\approx -0.270134 - 3.600490i, & \xi_{20} &\approx 8.906890 + 21.311200i. \end{aligned}$$

All the roots are simple, i.e., each root has multiplicity one. The initial approximations are taken as

$$\begin{aligned} z_1^{(0)} &= -0.5 - 0.007i, & z_2^{(0)} &= -0.5 + 0.05i, & z_3^{(0)} &= -0.5 - 0.1i, & z_4^{(0)} &= -0.45 + 0.15i, \\ z_5^{(0)} &= -0.45 - 0.15i, & z_6^{(0)} &= -0.45 + 0.25i, & z_7^{(0)} &= -0.45 - 0.25i, & z_8^{(0)} &= -0.45 + 0.35i, \\ z_9^{(0)} &= -0.45 - 0.35i, & z_{10}^{(0)} &= -0.45 + 0.45i, & z_{11}^{(0)} &= -0.45 - 0.5i, & z_{12}^{(0)} &= -0.45 + 0.65i, \\ z_{13}^{(0)} &= -0.45 - 0.7i, & z_{14}^{(0)} &= -0.45 + 0.9i, & z_{15}^{(0)} &= -0.45 - i, & z_{16}^{(0)} &= -0.45 + 1.4i, \\ z_{17}^{(0)} &= -0.45 - 1.6i, & z_{18}^{(0)} &= -0.35 + 2.7i, & z_{19}^{(0)} &= -0.25 - 3.5i, & z_{20}^{(0)} &= 8.5 + 20i. \end{aligned}$$

Table 6. Numerical comparison of different iterative methods for Example 4.6.

Method	$\ z^{(2)} - z^{(1)}\ $	$\ z^{(3)} - z^{(2)}\ $	$\ z^{(4)} - z^{(3)}\ $	$\ \mathcal{F}(z^{(4)})\ $	ACOC	CPU Time
P_{4s}	3.2(-1)	6.0(-7)	1.0(-45)	1.5(-357)	8.0325	9.876
PN_{4s}	3.2(-1)	6.0(-7)	1.0(-45)	1.5(-357)	8.0325	9.876
PE_{4s}	2.7(-4)	3.2(-34)	2.7(-397)	1.4(-4759)	12.0061	10.187
PEM_{4s}	6.0(-5)	2.7(-55)	2.1(-868)	1.3(-13881)	16.0028	11.167
N_4	4.4(-2)	5.0(-5)	1.7(-15)	4.4(-60)	4.0873	10.961
MMN_8	4.7(-2)	6.0(-7)	9.9(-44)	1.1(-340)	8.0216	10.844
$SEYY_4$	dvgt	dvgt	dvgt	dvgt	dvgt	dvgt
BI_2	1.2(-1)	1.0(-3)	3.4(-8)	3.5(-22)	3.1572	10.734

(dvgt: stands for divergent.)

Example 4.7. Consider the nonlinear equation [25]

$$\mathcal{F}(x) = (x - 1)^4 (x^2 + x + 5)^3 (3x - 1)^2 (x^{20} + 1) = 0.$$

The equation is a polynomial equation of degree 32 and has 32 roots, 24 of which are distinct; some roots are repeated. The approximate roots are

$$\begin{aligned}
\xi_1 &\approx -0.987688 - 0.156434i, & \xi_2 &\approx -0.987688 + 0.156434i, & \xi_3 &\approx -0.891007 - 0.453990i, \\
\xi_4 &\approx -0.891007 + 0.453990i, & \xi_5 &\approx -0.707107 - 0.707107i, & \xi_6 &\approx -0.707107 + 0.707107i, \\
\xi_7 &\approx -0.5 - 2.179450i, & \xi_8 &\approx -0.5 + 2.179450i, & \xi_9 &\approx -0.453990 - 0.891007i, \\
\xi_{10} &\approx -0.453990 + 0.891007i, & \xi_{11} &\approx -0.156434 - 0.987688i, & \xi_{12} &\approx -0.156434 + 0.987688i, \\
\xi_{13} &\approx 0.156434 - 0.987688i, & \xi_{14} &\approx 0.156434 + 0.987688i, & \xi_{15} &\approx 0.333333, \\
\xi_{16} &\approx 0.453990 - 0.891007i, & \xi_{17} &\approx 0.453990 + 0.891007i, & \xi_{18} &\approx 0.707107 - 0.707107i, \\
\xi_{19} &\approx 0.707107 + 0.707107i, & \xi_{20} &\approx 0.891007 - 0.453990i, & \xi_{21} &\approx 0.891007 + 0.453990i, \\
\xi_{22} &\approx 0.987688 - 0.156434i, & \xi_{23} &\approx 0.987688 + 0.156434i, & \xi_{24} &\approx 1.000000.
\end{aligned}$$

The multiplicities of the roots are as follows: ξ_7, ξ_8 have multiplicity 3, ξ_{15} has multiplicity 2, ξ_{24} has multiplicity 4, and all other roots are simple. The initial approximations are taken as

$$\begin{aligned}
z_1^{(0)} &= -1 - 0.2i, & z_2^{(0)} &= -1 + 0.2i, & z_3^{(0)} &= -0.9 - 0.5i, & z_4^{(0)} &= -0.9 + 0.5i, & z_5^{(0)} &= -0.7 - 0.7i, \\
z_6^{(0)} &= -0.7 + 0.7i, & z_7^{(0)} &= -0.5 - 2i, & z_8^{(0)} &= -0.5 + 2i, & z_9^{(0)} &= -0.5 - 0.8i, & z_{10}^{(0)} &= -0.5 + 0.8i, \\
z_{11}^{(0)} &= -0.2 - 1i, & z_{12}^{(0)} &= -0.2 + 1i, & z_{13}^{(0)} &= 0.2 - 1i, & z_{14}^{(0)} &= 0.2 + 1i, & z_{15}^{(0)} &= 0.3, \\
z_{16}^{(0)} &= 0.5 - 0.8i, & z_{17}^{(0)} &= 0.5 + 0.8i, & z_{18}^{(0)} &= 0.7 - 0.7i, & z_{19}^{(0)} &= 0.7 + 0.7i, & z_{20}^{(0)} &= 0.9 - 0.5i, \\
z_{21}^{(0)} &= 0.9 + 0.5i, & z_{22}^{(0)} &= 1 + 0.2i, & z_{23}^{(0)} &= 1 - 0.2i, & z_{24}^{(0)} &= 1.1.
\end{aligned}$$

Table 7. Numerical comparison of different iterative methods for Example 4.7.

Method	$\ z^{(2)} - z^{(1)}\ $	$\ z^{(3)} - z^{(2)}\ $	$\ z^{(4)} - z^{(3)}\ $	$\ \mathcal{F}(z^{(4)})\ $	ACOC	CPU Time
P_{4s}	4.6(-2)	7.0(-7)	4.9(-39)	1.9(-234)	4.0325	27.671
PN_{4s}	3.8(-2)	6.0(-9)	7.1(-63)	1.5(-489)	8.0413	34.938
PE_{4s}	5.6(-6)	3.1(-56)	8.4(-662)	1.8(-7927)	12.0880	29.077
PEM_{4s}	7.0(-7)	1.0(-90)	3.5(-1430)	1.5(-22860)	16.0398	72.688
N_4	1.8(-2)	6.2(-7)	2.6(-24)	6.3(-91)	4.0396	29.389
MMN_8	3.4(-2)	8.1(-9)	3.1(-61)	2.3(-480)	8.0184	36.844
$SEYY_4$	dvgt	dvgt	dvgt	dvgt	dvgt	dvgt
BI_2	1.0(-1)	4.1(-2)	2.5(-2)	5.5(7)	0.97931	5.341

(dvgt: stands for divergent.)

Table 8. Number of iterations required by different iterative methods for Examples 4.1–4.5.

Method	Number of iterations				
	Example 4.1	Example 4.2	Example 4.3	Example 4.4	Example 4.5
P_{4s}	5	4	9	5	9
PN_{4s}	4	3	5	5	5
PE_{4s}	4	3	5	3	5
PEM_{4s}	4	3	5	3	5
N_4	6	5	10	6	10
MMN_8	5	4	<i>divergent</i>	5	6
$SEYY_4$	7	5	5	<i>divergent</i>	5

Note: “divergent” indicates that the corresponding iterative method fails to satisfy the stopping criterion.

5. Analysis of numerical results

Tables 1–8 present the numerical results obtained for the seven nonlinear test problems 4.1–4.7. These examples are considered to examine the convergence behavior, accuracy, and computational efficiency of the proposed methods P_{4s} , PN_{4s} , PE_{4s} , and PEM_{4s} in comparison with the existing methods. The numerical results validate the theoretical convergence analysis established in the previous sections. The performance of the methods is analyzed in terms of absolute error, computational order of convergence (ACOC), CPU execution time, and error reduction behavior.

- (1) Table 1 presents the numerical results obtained by different iterative methods for Example 4.1. It can be observed that the proposed methods P_{4s} , PN_{4s} , PE_{4s} , and PEM_{4s} achieve very small errors and residuals. In particular, P_{4s} has a theoretical convergence order of four, while it exhibits approximately sixth-order convergence in this example, with an ACOC of 6.0915. The proposed method PN_{4s} and the existing method MMN_8 have the same theoretical order of convergence, and their error behavior is therefore compared graphically in Figure 1, which clearly demonstrates the rapid error reduction achieved by PN_{4s} .
- (2) Based on the results presented in Table 2, the proposed methods achieve very small errors and residuals with fewer iterations. The ACOC values confirm the high convergence behavior of the proposed methods. In particular, P_{4s} has a theoretical convergence order of four but exhibits approximately sixth-order convergence numerically, while PN_{4s} , PE_{4s} , and PEM_{4s} exhibit approximately eighth-, twelfth-, and sixteenth-order convergence, respectively. In particular, PN_{4s} and MMN_8 have the same theoretical order of convergence, and their error behavior is compared graphically in Figure 2, which shows the rapid error reduction achieved by the proposed method PN_{4s} .
- (3) Table 3 presents the numerical results obtained for Example 4.3. It can be observed that PN_{4s} , PE_{4s} , and PEM_{4s} achieve very small errors and residuals, with ACOC values close to four, while P_{4s} and N_4 exhibit approximately second-order convergence. The method MMN_8 diverges for the considered initial approximations, whereas the proposed methods successfully converge to the required roots. The existing method $SEYY_4$ also converges with fourth-order behavior but requires

more computational time. Figure 3 compares the error behavior of P_{4s} and N_4 , illustrating the convergence and error reduction achieved by the proposed method.

- (4) Table 4 presents the results for Example 4.4. The proposed methods show better performance than the existing methods in terms of absolute error and convergence rate. In particular, P_{4s} , PN_{4s} , PE_{4s} , and PEM_{4s} achieve high convergence orders, while $SEYY_4$ diverges for the considered initial approximations. Figure 4 presents a comparative error analysis of PN_{4s} and MMN_8 for all eleven roots. The graphical results demonstrate that the proposed method PN_{4s} achieves a rapid decrease in error and exhibits competitive convergence behavior compared with MMN_8 .
- (5) The numerical results for Example 4.5 are presented in Table 5. It is evident from the table that the proposed methods yield good results in terms of error, convergence rate, and CPU time. In particular, P_{4s} exhibits approximately second-order convergence, while PN_{4s} , PE_{4s} , and PEM_{4s} exhibit fourth-order convergence numerically. Figure 5 depicts the error analysis of P_{4s} and N_4 for the seven roots. The graphical comparison confirms that P_{4s} provides an effective error reduction process and exhibits improved numerical performance over N_4 .
- (6) The numerical results for Example 4.6 are presented in Table 6. The proposed methods provide highly accurate approximations with very small errors and residuals. In particular, P_{4s} and PN_{4s} exhibit approximately eighth-order convergence, while PE_{4s} and PEM_{4s} demonstrate approximately twelfth- and sixteenth-order convergence, respectively. Moreover, the proposed methods exhibit superior numerical performance in terms of accuracy and convergence when computing the 20 distinct complex roots of the given polynomial equation.
- (7) The numerical results for Example 4.7 are presented in Table 7. The proposed methods demonstrate excellent numerical performance in terms of accuracy and convergence for simultaneously computing the roots with different multiplicities. In particular, P_{4s} , PN_{4s} , PE_{4s} , and PEM_{4s} attain approximately fourth-, eighth-, twelfth-, and sixteenth-order convergence, respectively. The proposed methods also produce extremely small residuals, confirming their high accuracy. In contrast, the third-order BI_2 method exhibits poor numerical behavior for this example, with an observed convergence order of only 0.97931 and a relatively large residual, indicating that its expected third-order convergence is not achieved for this problem.

Overall, the numerical experiments demonstrate the effectiveness and reliability of the proposed methods for the simultaneous approximation of multiple roots of nonlinear equations. The observed higher convergence orders lead to a substantial reduction in the number of iterations required to achieve a prescribed level of accuracy, while the use of only first-derivative evaluations helps reduce the computational cost associated with higher-order schemes. These features are particularly advantageous for problems involving multiple roots, where rapid error reduction and accurate approximations are essential. The proposed methods are further validated through seven numerical examples covering a diverse range of problems, including the Van der Waals equation, a beam problem, a nonlinear equation, a polynomial equation of degree 12 having complex roots, a nonlinear equation having infinitely many roots, a polynomial equation of degree 20 with distinct complex roots, and a polynomial equation of degree 32 having 32 roots, including 24 distinct roots with some prescribed multiplicities. In comparison with closely related existing methods, the main novelty of the proposed framework lies in its ability to systematically enhance the convergence order while retaining a relatively simple

computational structure based on first-derivative information. More specifically, when combined with an iterative method of order p , the proposed framework achieves convergence of order $4p$ for polynomial equations and $2p$ for general non-polynomial equations. Thus, the proposed approach provides a practical balance between convergence speed, computational effort, and accuracy, making it a competitive alternative to existing simultaneous root-finding methods.

6. Conclusions

This manuscript presents a novel iterative approach for the simultaneous approximation of multiple roots of nonlinear equations. The proposed algorithm achieves a convergence order of $4p$ for polynomial equations and $2p$ for non-polynomial equations, where p denotes the convergence order of the underlying multiple-root iterative method used in the first step. The merit of the proposed method is supported by the theoretical convergence analysis and numerical experiments on seven test problems, which confirm the theoretical results. Furthermore, comparisons with the existing methods N_4 and MMN_8 demonstrate the computational efficiency and accuracy of the proposed scheme. These results indicate that the proposed approach provides an effective framework for the simultaneous approximation of multiple roots and offers promising prospects for the development of efficient simultaneous root-finding methods.

Author contributions

Monika Panwar: Conceptualization, methodology, writing – review and editing, writing – original draft. Sonia Bhalla: Conceptualization, methodology, formal analysis, supervision. Higinio Ramos: Investigation, validation, formal analysis, resources, funding acquisition. Ramandeep Behl: Conceptualization, validation, supervision, writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Higinio Ramos is an editorial board member for AIMS Mathematics and was not involved in the editorial review or the decision to publish this article.

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