



*Research article***Single-threshold maximal Baum–Katz characterizations for m -negatively associated random variables****Sen Zhang*, Yunzhi Zhu and Saisai Hou**

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Abstract: Let $S_n = \sum_{i=1}^n X_i$ and $M_n = \max_{1 \leq k \leq n} |S_k|$. We establish a single-threshold converse for identically distributed m -negatively associated (m -NA) random variables. If, for some $\varepsilon_0 > 0$,

$$\sum_{n \geq 1} n^{q/p-2} \mathbb{P}(M_n > \varepsilon_0 n^{1/p}) < \infty,$$

where $p > 0$ and $q \geq p$, then $\mathbb{E}|X_1|^q < \infty$; moreover, for $p > 1$, convergence at this single threshold also forces $\mathbb{E}X_1 = 0$. Consequently, when $1 < p < 2$ and $q \geq p$, the moment and centering conditions are equivalent both to maximal convergence at every positive threshold and to maximal convergence at some positive threshold. At the endpoint $p = 1$, we determine the threshold profile exactly: The infimum of the convergent thresholds equals $|\mathbb{E}X_1|$. The proof combines a lower maximal-tail estimate on an interlaced NA subsequence with dyadic reconstruction and a weak-law drift-recovery argument, thereby separating tail integrability from deterministic drift. We also show that the maximum cannot be replaced by the terminal sum: A symmetric NA pair-cancellation construction has a summable terminal-sum series despite an infinite q -th moment, whereas its maximal series diverges at least logarithmically. Finally, the relation with known sufficiency results is made explicit, and corresponding subcritical complete-moment and f -moment consequences are recorded. Together these results give a maximal Baum–Katz characterization under the stated dependence structure.

Keywords: Baum–Katz theorem; single-threshold converse; drift recovery; maximal partial sums; complete convergence; negative association; m -negatively associated random variables

Mathematics Subject Classification: 60F15, 60F25, 60G50

1. Introduction

Complete convergence was introduced by Hsu and Robbins [1] and refined by Baum and Katz [2]. In its classical form, for independent and identically distributed random variables with partial sums S_n , the normalization $n^{1/p}$, the summability weight $n^{q/p-2}$, and the integrability condition $\mathbb{E}|X_1|^q < \infty$ are linked: For $1 \leq p < 2$ and $q \geq p$, the moment condition together with $\mathbb{E}X_1 = 0$ is equivalent to the convergence of $\sum_{n \geq 1} n^{q/p-2} \mathbb{P}(|S_n| > \varepsilon n^{1/p})$ for every $\varepsilon > 0$.

For clarity, a sequence of events $\{A_n\}$ is said to converge completely when $\sum_{n \geq 1} \mathbb{P}(A_n) < \infty$. Weighted complete convergence inserts a deterministic weight into this series. Complete moment convergence replaces the probability by a truncated or excess moment; it is therefore stronger than the corresponding probability statement. Maximum partial sums are natural because they control the whole trajectory up to time n , not only the terminal cancellation in S_n .

Negative association was introduced by Joag-Dev and Proschan [3]. Complete convergence theory for negatively associated (NA) sequences and related dependence classes has developed in several directions; see Liang and Su [4]. NA models arise, for example, from sampling without replacement and from several occupancy and order-statistic constructions [3]. The class of m -negatively associated (m -NA) sequences was introduced by Hu, Chiang, and Taylor [5]; it allows arbitrary local dependence inside a fixed window while imposing negative association on observations sufficiently far apart. This makes it a useful abstraction when short-range interactions are unavoidable but long-range simultaneous large values remain inhibited. It is strictly weaker than requiring the whole sequence to be NA. A concrete example is obtained by concatenating independent blocks of length m , allowing arbitrary dependence within each block while keeping a common marginal law. Any indices separated by at least m then belong to distinct blocks, so the selected variables are independent and hence NA, although the full sequence need not be NA. Moment inequalities for this class were developed by Shen et al. [6], and weighted convergence results include [7–9]. The wider m -asymptotically negatively associated (m -ANA) setting is treated in [10–12]. These limit theorems also support dependence-aware statistical modeling. For example, the m -ANA theory has been used to establish consistency results for nonparametric regression and conditional value-at-risk estimators [10], while complete f -moment methods have been applied to errors-in-variables regression models [13]. We mention these applications only to place the present probability-theoretic questions in a broader context; the results below concern the underlying convergence mechanisms rather than a particular statistical model.

It is important to separate what is already known from what is proved here. Liang, Li, and Rosalsky [14, p. 424, Theorem 2.1] established a maximal complete-moment equivalence for identically distributed NA sequences. In their notation, they set their parameter $r = q/p$ and their moment-power parameter equal to $1/\alpha$. When $0 < \alpha < q$, their result gives

$$\sum_{n \geq 1} n^{q/p-2} \mathbb{E}[(M_n/n^{1/p})^\alpha - c]_+ < \infty \quad (c > 0)$$

under $\mathbb{E}|X_1|^q < \infty$ and centering. This implies both the maximal probability sufficiency and the truncated α -moment sufficiency used below. The more general weighted m -ANA theorem of Wu, Wang, and Shen [10, Theorem 2.1], specialized by taking unit coefficients and their parameter $r = q/p$, also contains this sufficiency direction. Accordingly, we do not claim these sufficiency estimates as new and do not repeat their long truncation proofs. Complete f -moment convergence for weighted sums has also been developed under extended negative dependence assumptions by Wang and Wang [13].

Two further comparisons are relevant at the probability-series level. Huang, Zhang, and Wu [15, Theorems 2.1–2.4] gave sufficient and necessary complete-convergence results for asymptotically NA weighted sums under stochastic tail comparison. Most importantly for the converse direction, Boukhari, Dzung, and Thành [16, p. 1393, Theorem 2.2] proved an inverse theorem from convergence at every positive threshold. With their moment parameter $p_B = q$, normalization exponent $\alpha_B = 1/p$, slowly varying factor $L \equiv 1$, and centering constant $c = 0$, their result overlaps the range $1 \leq p \leq q < 2$ and yields both $\mathbb{E}|X_1|^q < \infty$ and $\mathbb{E}X_1 = 0$. Their Corollary 4.1(ii) places the corresponding equivalence under the weaker m -pairwise negative-dependence condition [16, p. 1399, Corollary 4.1(ii)]. Thus the existence of an inverse theorem itself is not claimed as new. The distinction proved here is that a single fixed threshold suffices and that this implication is valid for the wider range $p > 0$, $q \geq p$, including $q \geq 2$; for $p > 1$, the same single-threshold hypothesis also identifies the drift.

Table 1 makes the logical directions explicit. In particular, specializing a theorem whose input is a moment-and-centering condition can recover the sufficiency direction used here, but cannot reverse that implication and infer the moment or drift from a convergent series.

Table 1. Comparison with the results most directly related to the present characterization.

Source	Dependence and form	Input	Output
Wu–Wang–Shen [10, Theorem 2.1]	weighted m -ANA sums	moment and centering assumptions	complete-moment, hence probability-series, sufficiency; after specialization, moment/centering $\Rightarrow$ convergence for every threshold
Wang–Wang Theorem 3.1]	[13, Sung-type weighted sums under extended negative dependence	moment, dependence, weight and growth conditions on f	broad complete f -moment sufficiency
Boukhari–Dzung– Thành Theorem 2.2 and Corollary 4.1(ii)]	maximal sums; the covers m -pairwise negative dependence	corollary convergence for every positive threshold; overlapping range $1 \leq p \leq q < 2$	recovery of the moment and drift, with an associated equivalence
Theorem (present paper)	3.1 identically distributed m -NA variables; unweighted maximal sums	convergence at one fixed threshold; $p > 0$, $q \geq p$	recovery of $\mathbb{E} X_1 ^q < \infty$, and also $\mathbb{E}X_1 = 0$ when $p > 1$

The contribution of this paper is instead the following single-threshold recovery principle and the obstruction that explains why its maximal form matters.

- If the maximal probability series converges for one fixed threshold, then $\mathbb{E}|X_1|^q < \infty$. Theorem 3.1 proves this for identically distributed m -NA variables without centering, without an assumed moment, and for $p > 0$, $q \geq p$. When $p > 1$, the same one-threshold hypothesis also forces $\mathbb{E}X_1 = 0$; hence centering is recovered rather than imposed.
- Combining this joint moment-and-drift recovery with the cited sufficiency results gives the three-way characterization in Corollary 3.2 for $1 < p < 2$, $q \geq p$, including the endpoint $q = p$ and all $q \geq 2$. Thus one-threshold convergence automatically propagates to every positive threshold.
- Theorem 3.9 identifies the complementary phenomenon at $p = 1$: When the q -th moment is finite, the infimum of the thresholds at which the series converges is exactly $\mathbb{E}|X_1|$, and membership of the endpoint itself is not universal.

- (d) Proposition 3.7 constructs an identically distributed symmetric NA sequence for which the non-maximal series based on $|S_n|$ converges at every threshold although the q -th moment is infinite. Its maximal series is shown to diverge at least logarithmically. Thus terminal cancellation destroys the non-maximal converse, in contrast with the independent case.

The endpoint $q = p$ still requires a separation of roles. Our single-threshold proof supplies the one-threshold necessity and, for $p > 1$, recovers centering. Endpoint sufficiency is not obtained from the subcritical complete-moment theorem (whose parameter q/p must exceed one); it is instead quoted from [16, p. 1399, Corollary 4.1(ii)]. By contrast, the complete-moment transfer in Corollary 3.3 remains strictly subcritical and therefore assumes $q > p$.

Section 2 fixes the dependence convention and gives the interlacing picture. Section 3 states the converse, its Baum–Katz corollary, the critical-threshold profile, and the complete-moment transfer. The proofs of the new recovery statements are in Section 4; standard auxiliary facts are collected in Appendix A. Section 6 proves the cancellation counterexample, including its logarithmic lower rate, and records the scope of the assumptions. Finally, Section 7 explains why normalization $n^{1/r}$ with weight n^{r-2} corresponds to the moment order r^2 and records the resulting subcritical f -moment corollary.

Throughout, C denotes a positive constant, possibly depending on the parameters $p, q, \alpha, s, m, \varepsilon$ and on the distribution of X_1 but never on n or on the integration variables; its value may change from line to line. We write

$$S_0 = 0, \quad S_n = \sum_{i=1}^n X_i, \quad M_n = \max_{1 \leq k \leq n} |S_k|, \quad n \geq 1.$$

2. Definitions and roadmap

Definition 2.1 (Negative association; [3]). A finite family $Y_1, \dots, Y_N$ of random variables is NA if for every pair of nonempty disjoint index sets $A, B \subset \{1, \dots, N\}$ and every pair of coordinatewise nondecreasing functions $u: \mathbb{R}^{|A|} \rightarrow \mathbb{R}$ and $v: \mathbb{R}^{|B|} \rightarrow \mathbb{R}$,

$$\text{Cov}(u(Y_i, i \in A), v(Y_j, j \in B)) \leq 0$$

whenever the covariance exists. An infinite family is NA if every finite subfamily is NA.

We will use repeatedly the following standard closure properties (see [3]): Any subfamily of an NA family is NA; coordinatewise nondecreasing functions applied to disjoint subfamilies of an NA family form an NA family (in particular, $\varphi(Y_1), \dots, \varphi(Y_N)$ is NA for any nondecreasing $\varphi: \mathbb{R} \rightarrow \mathbb{R}$); the union of independent NA families is NA; independent random variables are NA; and if $(Y_1, \dots, Y_N)$ is NA, then so is $(-Y_1, \dots, -Y_N)$, since for coordinatewise nonincreasing u, v , one has $\text{Cov}(u, v) = \text{Cov}(-u, -v) \leq 0$.

Definition 2.2 (m -negative association; [5]). Let $m \geq 1$ be an integer. A sequence $\{X_n, n \geq 1\}$ is m -NA if for every $k \geq 2$ and all indices $1 \leq i_1 < \dots < i_k$ with $i_{j+1} - i_j \geq m$ for $1 \leq j \leq k-1$, the random variables $X_{i_1}, \dots, X_{i_k}$ are NA.

For $m = 1$, Definition 2.2 reduces to negative association of the whole sequence. Some authors use the convention $i_{j+1} - i_j > m$; this only changes the fixed separation parameter by one. The present

paper uses the convention in Definition 2.2. The single structural fact we need is that an m -NA sequence interlaces m NA subsequences: for each residue $j \in \{1, \dots, m\}$, the subsequence $\{X_{j+(t-1)m}, t \geq 1\}$ is NA because any finite set of its indices has consecutive gaps that are positive multiples of m .

The auxiliary tools have sharply separated roles. Lemmas A.1 and A.2 convert negative association into the lower-tail estimate needed for the new converse, while Lemma A.4 identifies the resulting tail series with a moment. The Rosenthal inequality in Appendix A is recorded only to explain the standard sufficiency machinery and its limitations; the revised proof cites the stronger complete-moment theorem of Liang et al. [14] instead of repeating that machinery. The remaining elementary estimates are retained there for reference in the sharpness discussion. This roadmap explains why the facts are needed while keeping the definitions and main statements together; it also makes clear that no independence is asserted between the interlaced residue classes.

3. Main results

Our standing hypotheses in this section are:

- (H) $m \geq 1$ is an integer; $\{X_n, n \geq 1\}$ is an m -NA sequence of identically distributed random variables; and $1 < p < 2, q \geq p$.

Theorem 3.1 (Single-threshold maximal converse). *Let $p > 0, q \geq p$, and let $\{X_n, n \geq 1\}$ be an identically distributed m -NA sequence. If, for some $\varepsilon_0 > 0$,*

$$\sum_{n=1}^{\infty} n^{q/p-2} \mathbb{P}(M_n > \varepsilon_0 n^{1/p}) < \infty,$$

then $\mathbb{E}|X_1|^q < \infty$. If, in addition, $p > 1$, then necessarily $\mathbb{E}X_1 = 0$.

Corollary 3.2 (Maximal Baum–Katz characterization). *Under (H), the following statements are equivalent:*

- (i) $\mathbb{E}|X_1|^q < \infty$ and $\mathbb{E}X_1 = 0$;
- (ii) for every $\varepsilon > 0$, $\sum_{n=1}^{\infty} n^{q/p-2} \mathbb{P}(M_n > \varepsilon n^{1/p}) < \infty$;
- (iii) for some $\varepsilon > 0$, $\sum_{n=1}^{\infty} n^{q/p-2} \mathbb{P}(M_n > \varepsilon n^{1/p}) < \infty$.

Corollary 3.3 (Subcritical complete-moment transfer). *Under (H), suppose further that $q > p, \mathbb{E}X_1 = 0$, and $\mathbb{E}|X_1|^q < \infty$. Then for every $0 < \alpha < q$ and every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{q/p-2} \mathbb{E}\left[\left(\frac{M_n}{n^{1/p}}\right)^{\alpha} \mathbf{1}_{\{M_n > \varepsilon n^{1/p}\}}\right] < \infty.$$

The term “subcritical” refers to the strict inequality $\alpha < q$: At $\alpha = q$, the parameter choice in [14, p. 424, Theorem 2.1] is critical and requires a logarithmic moment. Corollary 3.3 is therefore recorded as a transfer of that known NA theorem to the m -NA setting, not as an independent complete-moment theorem.

Remark 3.4. Theorem 3.1 isolates the implication (iii) $\Rightarrow$ (i) from Corollary 3.2. It uses neither an assumed moment nor centering to begin the argument, and its moment conclusion needs neither $p < 2$ nor $q > p$; equality $q = p$ is allowed. For $p > 1$, however, the same hypothesis subsequently forces centering from the weak law because a nonzero linear drift dominates $n^{1/p}$. Thus Corollary 3.2 does not assume centering in its standing hypotheses: Condition (i) records a property recovered from (iii).

Remark 3.5 (Comparison with the literature). Under the parameter substitutions described in the Introduction, [14, p. 424, Theorem 2.1] and [10, Theorem 2.1] recover the implication (i) $\Rightarrow$ (ii) when $q > p$, and the former also gives the NA complete-moment estimate behind Corollary 3.3. At $q = p$, the sufficiency input is [16, p. 1399, Corollary 4.1(ii)]; an m -NA sequence is m -pairwise negatively dependent because every sufficiently separated pair is NA. The all-threshold converse of [16, p. 1393, Theorem 2.2] also recovers the moment and drift in the overlapping range $1 \leq p \leq q < 2$. What is proved here is the strictly weaker threshold hypothesis: The probability series at one fixed threshold already recovers the q -th moment and, for $p > 1$, the drift throughout $p > 0$, $q \geq p$. The critical profile in Theorem 3.9 and the cancellation rate in Proposition 3.7 are likewise not consequences of those cited results.

A concrete separation from the cited sufficiency theorem. Take $p = 3/2$, $q = 3$, and let $\{X_i\}$ be independent and identically distributed symmetric random variables satisfying $\mathbb{P}(|X_1| > x) = x^{-3}$ for $x \geq 1$. This sequence is m -NA for every m , $\mathbb{E}X_1 = 0$, and $\mathbb{E}|X_1|^3 = \infty$. For fixed $\varepsilon > 0$, put $u_n = \mathbb{P}(|X_1| > 2\varepsilon n^{2/3}) = (2\varepsilon)^{-3}n^{-2}$ for all sufficiently large n . Since $\max_{i \leq n} |X_i| \leq 2M_n$, independence and $1 - (1 - u)^n \geq nu/2$ whenever $0 \leq nu \leq 1$ give

$$\mathbb{P}(M_n > \varepsilon n^{2/3}) \geq 1 - (1 - u_n)^n \geq \frac{(2\varepsilon)^{-3}}{2n}$$

for all sufficiently large n . Hence $\sum_n \mathbb{P}(M_n > \varepsilon n^{2/3}) = \infty$ for every $\varepsilon > 0$. This does not contradict the sufficiency theorem of Wu, Wang, and Shen: Its moment hypothesis fails here, so that theorem is silent. Rather, the example makes the missing logical direction concrete. The divergence is a direct consequence of the contrapositive of Theorem 3.1, whereas a moment-to-convergence theorem cannot yield it by parameter substitution.

Remark 3.6. Since $|S_n| \leq M_n$, statement (i) of Corollary 3.2 also implies $\sum_{n \geq 1} n^{q/p-2} \mathbb{P}(|S_n| > \varepsilon n^{1/p}) < \infty$ for every $\varepsilon > 0$; when $q > p$, Corollary 3.3 implies the analogous complete-moment bound with $|S_n|$ in place of M_n . The converse implications, however, fail under (H); this is the content of the next proposition.

Proposition 3.7 (Failure of the non-maximal necessity). *For every $1 < p < 2$ and $q > p$, there exists an NA (hence m -NA for every $m \geq 1$) sequence $\{X_n, n \geq 1\}$ of identically distributed, symmetric random variables with $\mathbb{E}|X_1| < \infty$ and $\mathbb{E}X_1 = 0$ such that*

$$\mathbb{E}|X_1|^q = \infty \quad \text{and yet} \quad \sum_{n=1}^{\infty} n^{q/p-2} \mathbb{P}(|S_n| > \varepsilon n^{1/p}) < \infty \quad \text{for every } \varepsilon > 0.$$

For the same sequence, the corresponding maximal series has the quantitative lower rate

$$\liminf_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^{2N} n^{q/p-2} \mathbb{P}(M_n > \varepsilon n^{1/p}) \geq \frac{\varepsilon^{-q}}{4}, \quad \varepsilon > 0.$$

Consequently, the maximum in Corollary 3.2 (ii)–(iii) cannot be replaced by $|S_n|$.

Remark 3.8 (Pairwise-cancellation mechanism). The mechanism is local and occurs before any limiting argument: A heavy-tailed excursion $X_{2k-1} = Y_k$ is followed immediately by $X_{2k} = -Y_k$. Thus the terminal sum returns to zero at every even time and largely hides the heavy tail, whereas the maximum records the odd-time excursion before it is cancelled. This exact pairing is compatible with negative association but not with independent increments. Indeed, for independent identically distributed sequences, no such example exists: By the classical Baum–Katz theorem [2], convergence of the non-maximal weighted series already forces the moment condition. The example also defeats the corresponding non-maximal complete-moment and complete f -moment statements; see Remark 6.1.

Theorem 3.9 (Critical-threshold profile at $p = 1$). *Let $q > 1$, and let $\{X_n, n \geq 1\}$ be an identically distributed m -NA sequence. Define*

$$\mathcal{T}_q = \left\{ \varepsilon > 0 : \sum_{n=1}^{\infty} n^{q-2} \mathbb{P}(M_n > \varepsilon n) < \infty \right\}.$$

If $\mathbb{E}|X_1|^q = \infty$, then $\mathcal{T}_q = \emptyset$. If $\mathbb{E}|X_1|^q < \infty$ and $\mu = \mathbb{E}X_1$, then

$$(|\mu|, \infty) \subseteq \mathcal{T}_q \subseteq [|\mu|, \infty), \quad \text{and hence} \quad \inf \mathcal{T}_q = |\mu|.$$

Membership of the endpoint $|\mu|$ is not universal.

4. Proof of the converse and the NA-to- m -NA transfer

Throughout the proofs, we write $b_n = n^{1/p}$.

4.1. The negatively associated case of the sufficiency

Proposition 4.1. *Let $\{\xi_t, t \geq 1\}$ be an NA sequence of identically distributed random variables with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}|\xi_1|^q < \infty$, where $1 \leq p < 2$ and $q > p$. Then for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{q/p-2} \mathbb{P}\left(\max_{1 \leq k \leq n} \left| \sum_{t=1}^k \xi_t \right| > \varepsilon n^{1/p}\right) < \infty.$$

Proof. Write $W_n = \max_{k \leq n} |\sum_{t \leq k} \xi_t|$. Choose any $0 < \alpha < q$, and apply [14, p. 424, Theorem 2.1] with its parameters $r = q/p$ and $q_{\text{there}} = 1/\alpha$. Since $q > p$, one has $r > 1$, and $\alpha < q = pr$ places the theorem in its subcritical case. After factoring out $n^{\alpha/p}$, it gives, for every $c > 0$,

$$\sum_{n \geq 1} n^{q/p-2} \mathbb{E}[(W_n/n^{1/p})^\alpha - c]_+ < \infty.$$

Take $c = (\varepsilon/2)^\alpha$. On $\{W_n > \varepsilon n^{1/p}\}$, the integrand is at least $\varepsilon^\alpha - (\varepsilon/2)^\alpha > 0$. The asserted probability series follows. This argument records explicitly that the NA sufficiency is a specialization of an existing theorem. $\square$

4.2. Reduction from m -NA to NA

For $j \in \{1, \dots, m\}$, define the j -th interlaced subsequence and its maxima of partial sums by

$$\xi_t^{(j)} = X_{j+(t-1)m} \quad (t \geq 1), \quad V_n^{(j)} = \max_{1 \leq l \leq n} \left| \sum_{t=1}^l \xi_t^{(j)} \right|.$$

As noted after Definition 2.2, each $\{\xi_t^{(j)}, t \geq 1\}$ is an NA sequence of identically distributed random variables with the common law of X_1 .

The decomposition can be visualized as

$$\begin{array}{c|cccc} j=1 & X_1 & X_{1+m} & X_{1+2m} & \cdots \\ j=2 & X_2 & X_{2+m} & X_{2+2m} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ j=m & X_m & X_{2m} & X_{3m} & \cdots \end{array},$$

and every initial segment $\{1, \dots, k\}$ is the union of initial segments of these rows. This deterministic observation, rather than any independence between rows, is what drives the next lemma.

Lemma 4.2 (Reduction). *Let $\{X_n\}$ be m -NA. For all $n \geq 1$, $x > 0$, and $\alpha > 0$,*

$$\mathbb{P}(M_n > mx) \leq \sum_{j=1}^m \mathbb{P}(V_n^{(j)} > x), \quad (4.1)$$

$$\mathbb{E}[M_n^\alpha \mathbf{1}_{\{M_n > mx\}}] \leq m^\alpha \sum_{j=1}^m \mathbb{E}[(V_n^{(j)})^\alpha \mathbf{1}_{\{V_n^{(j)} > x\}}]. \quad (4.2)$$

Proof. For $1 \leq k \leq n$, splitting the index set $\{1, \dots, k\}$ into residue classes modulo m gives $S_k = \sum_{j=1}^m \sum_{t \leq l_j(k)} \xi_t^{(j)}$ with $l_j(k) \leq \lceil k/m \rceil \leq n$, whence $M_n \leq \sum_{j=1}^m V_n^{(j)}$. If $V_n^{(j)} \leq x$ for every j , then $M_n \leq mx$; this proves (4.1). For (4.2), let j^* be an index attaining $\max_j V_n^{(j)}$. On $\{M_n > mx\}$, we have $m V_n^{(j^*)} \geq \sum_j V_n^{(j)} \geq M_n > mx$, so $V_n^{(j^*)} > x$ and

$$M_n^\alpha \mathbf{1}_{\{M_n > mx\}} \leq m^\alpha (V_n^{(j^*)})^\alpha \mathbf{1}_{\{V_n^{(j^*)} > x\}} \leq m^\alpha \sum_{j=1}^m (V_n^{(j)})^\alpha \mathbf{1}_{\{V_n^{(j)} > x\}}. \quad \square$$

4.3. The single-threshold converse and the characterization

Proof of Theorem 3.1. Set $b_n = n^{1/p}$, suppose the series converges at $\varepsilon_0 > 0$, and put $P_n = \mathbb{P}(|X_1| > 2\varepsilon_0 n^{1/p})$. Since $X_i = S_i - S_{i-1}$, $\max_{i \leq n} |X_i| \leq 2M_n$. Hence

$$\mathbb{P}(M_n > \varepsilon_0 b_n) \geq \mathbb{P}\left(\max_{i \in J_n} |X_i| > 2\varepsilon_0 b_n\right), \quad J_n = \{1, 1+m, 1+2m, \dots\} \cap \{1, \dots, n\}.$$

The family $(X_i)_{i \in J_n}$ is NA and identically distributed, and $|J_n| = \lceil n/m \rceil \geq n/m$. Lemma A.2 yields

$$\mathbb{P}(M_n > \varepsilon_0 b_n) \geq \frac{c_0}{m} \min(1, nP_n).$$

Consequently,

$$\sum_{n \geq 1} n^{q/p-2} \min(1, nP_n) < \infty. \quad (4.3)$$

For $j \geq 0$ and $2^j < n \leq 2^{j+1}$, monotonicity of P_n gives

$$n^{q/p-2} \geq \kappa 2^{j(q/p-2)}, \quad \min(1, nP_n) \geq \min(1, 2^j P_{2^{j+1}}),$$

where $\kappa = \min(1, 2^{q/p-2})$. Summing within dyadic blocks in (4.3),

$$\sum_{j \geq 0} 2^{j(q/p-1)} \min(1, 2^j P_{2^{j+1}}) < \infty. \quad (4.4)$$

Because $q \geq p$, the factor $2^{j(q/p-1)}$ is at least one. Thus the minimum in (4.4) tends to zero and equals $2^j P_{2^{j+1}}$ for all sufficiently large j . It follows that

$$\sum_{j \geq j_0} 2^{jq/p} P_{2^{j+1}} < \infty.$$

For $2^{j+1} < n \leq 2^{j+2}$, $n^{q/p-1} \leq 2^{(j+2)(q/p-1)}$ and $P_n \leq P_{2^{j+1}}$, whence

$$\sum_{n > 2^{j_0+1}} n^{q/p-1} P_n \leq C \sum_{j \geq j_0} 2^{jq/p} P_{2^{j+1}} < \infty.$$

Lemma A.4, with $a = p$, $s = q$, and $c = 2\varepsilon_0$, now gives $\mathbb{E}|X_1|^q < \infty$. No centering or upper restriction on p has been used for this conclusion.

Suppose now that $p > 1$, and put $\mu = \mathbb{E}X_1$, which is well defined because $q \geq p > 1$. We give the weak-law details needed in the present m -NA setting. For any one of the NA residue subsequences $\{\xi_t\}$, let $T_N(x) = (-N) \vee (x \wedge N)$ and $\eta_{t,N} = T_N(\xi_t)$. Monotone transforms preserve NA, and therefore

$$\text{Var}\left(\sum_{t=1}^N \eta_{t,N}\right) \leq N \text{Var}(\eta_{1,N}).$$

Moreover,

$$\mathbb{P}(\exists t \leq N : \xi_t \neq \eta_{t,N}) \leq N\mathbb{P}(|\xi_1| > N) \rightarrow 0$$

because $\mathbb{E}|\xi_1| < \infty$. Also

$$\frac{\text{Var}(\eta_{1,N})}{N} \leq \frac{\mathbb{E}\eta_{1,N}^2}{N} \rightarrow 0,$$

since $\eta_{1,N}^2/N \leq |\xi_1|$ and the left integrand tends pointwise to zero. Finally, $\mathbb{E}\eta_{1,N} \rightarrow \mu$. Chebyshev's inequality and the last three estimates show that $N^{-1} \sum_{t=1}^N \xi_t \rightarrow \mu$ in probability. If $N_j(n)$ is the number of indices not exceeding n in residue class j , then

$$\frac{S_n}{n} = \sum_{j=1}^m \frac{N_j(n)}{n} \left(\frac{1}{N_j(n)} \sum_{t=1}^{N_j(n)} \xi_t^{(j)} \right), \quad \frac{N_j(n)}{n} \rightarrow \frac{1}{m}.$$

Recombining the finitely many residue classes therefore yields

$$\frac{S_n}{n} \rightarrow \mu \quad \text{in probability.}$$

If $\mu \neq 0$, then $|S_n|/n^{1/p} \rightarrow \infty$ in probability, so $\mathbb{P}(M_n > \varepsilon_0 n^{1/p}) \rightarrow 1$. Since $q/p - 2 \geq -1$, the defining series must then diverge, a contradiction. Hence $\mu = 0$. $\square$

Proof of Corollary 3.2. (i) $\Rightarrow$ (ii). First suppose $q > p$. Fix $\varepsilon > 0$. By (4.1) with $x = (\varepsilon/m)b_n$,

$$\sum_{n \geq 1} n^{q/p-2} \mathbb{P}(M_n > \varepsilon b_n) \leq \sum_{j=1}^m \sum_{n \geq 1} n^{q/p-2} \mathbb{P}\left(V_n^{(j)} > \frac{\varepsilon}{m} b_n\right),$$

and each inner series is finite by Proposition 4.1 applied to the NA sequence $\{\xi_t^{(j)}\}$.

If $q = p$, condition (i) and [16, p. 1399, Corollary 4.1(ii)], with its moment parameter equal to p , normalization exponent $1/p$, and slowly varying function identically one, give the same conclusion. Indeed, m -NA implies the m -pairwise negative dependence assumed there.

(ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (i) follows from both conclusions of Theorem 3.1, since $p > 1$. $\square$

Proof of Theorem 3.9. If $\mathbb{E}|X_1|^q = \infty$, Theorem 3.1, with $p = 1$, shows that the defining series cannot converge at any threshold; hence $\mathcal{T}_q = \emptyset$.

Suppose that $\mathbb{E}|X_1|^q < \infty$, set $\mu = \mathbb{E}X_1$, and write $Z_i = X_i - \mu$, $T_k = \sum_{i=1}^k Z_i$, and $M_n^Z = \max_{k \leq n} |T_k|$. Translation preserves m -NA. If $\varepsilon > |\mu|$, then

$$M_n \leq |\mu|n + M_n^Z,$$

and the interlacing reduction (4.1), followed by Proposition 4.1 with $p = 1$, gives

$$\sum_{n=1}^{\infty} n^{q-2} \mathbb{P}(M_n^Z > (\varepsilon - |\mu|)n) < \infty.$$

Thus $(|\mu|, \infty) \subseteq \mathcal{T}_q$.

The weak-law argument in the proof of Theorem 3.1 gives $S_n/n \rightarrow \mu$ in probability. For $0 < \varepsilon < |\mu|$,

$$\mathbb{P}(M_n > \varepsilon n) \geq \mathbb{P}(|S_n| > \varepsilon n) \rightarrow 1.$$

Since $\sum_{n \geq 1} n^{q-2} = \infty$ for $q > 1$, such an ε does not belong to $\mathcal{T}_q$. This proves the two inclusions and the formula for the infimum.

Finally, fix $\mu > 0$. If $X_n \equiv \mu$, then $M_n = \mu n$, so the strict-threshold series at $\varepsilon = \mu$ is zero. In contrast, let $X_n = \mu + R_n$, where R_n are independent Rademacher variables. Along the even integers,

$$\mathbb{P}(M_n > \mu n) \geq \mathbb{P}\left(\sum_{i=1}^n R_i > 0\right) \rightarrow \frac{1}{2},$$

and the endpoint series diverges. Both examples are identically distributed NA sequences with moments of every order, proving that endpoint membership is not universal. $\square$

5. Subcritical complete-moment transfer

Again we first treat the NA case and then reduce.

Proposition 5.1. *Let $\{\xi_t, t \geq 1\}$ be an NA sequence of identically distributed random variables with $\mathbb{E}\xi_1 = 0$ and $\mathbb{E}|\xi_1|^q < \infty$, where $1 < p < 2$, $q > p$, and let $0 < \alpha < q$. Then, with $W_n = \max_{k \leq n} |\sum_{t \leq k} \xi_t|$, for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{q/p-2} \mathbb{E}\left[\left(\frac{W_n}{n^{1/p}}\right)^\alpha \mathbf{1}_{\{W_n > \varepsilon n^{1/p}\}}\right] < \infty.$$

Proof. Use [14, p. 424, Theorem 2.1] with $r = q/p$ and $q_{\text{there}} = 1/\alpha$, exactly as in the proof of Proposition 4.1. For $V_n = W_n/n^{1/p}$, it gives

$$\sum_{n \geq 1} n^{q/p-2} \mathbb{E}[(V_n^\alpha - (\varepsilon/2)^\alpha)_+] < \infty.$$

On $\{V_n > \varepsilon\}$,

$$V_n^\alpha \leq (1 - 2^{-\alpha})^{-1} (V_n^\alpha - (\varepsilon/2)^\alpha)_+,$$

which proves the displayed assertion. This is precisely the known NA complete-moment result needed for the interlacing transfer. $\square$

Proof of Corollary 3.3. Fix $\varepsilon > 0$ and $0 < \alpha < q$. By (4.2) with $x = (\varepsilon/m)b_n$,

$$\mathbb{E}[M_n^\alpha \mathbf{1}_{\{M_n > \varepsilon b_n\}}] \leq m^\alpha \sum_{j=1}^m \mathbb{E}[(V_n^{(j)})^\alpha \mathbf{1}_{\{V_n^{(j)} > (\varepsilon/m)b_n\}}].$$

Dividing by b_n^α , multiplying by $n^{q/p-2}$, summing over n , and applying Proposition 5.1 to each NA subsequence $\{\xi_t^{(j)}\}$ (with threshold ε/m) gives the claim. $\square$

6. Sharpness

6.1. Failure of the non-maximal necessity

Proof of Proposition 3.7. Let $q > p > 1$. Let $\{Y_k, k \geq 1\}$ be independent, identically distributed, and symmetric, with

$$\mathbb{P}(|Y_1| > x) = x^{-q}, \quad x \geq 1,$$

so that $|Y_1| \geq 1$ almost surely. Then $\mathbb{E}|Y_1| = 1 + \int_1^\infty x^{-q} dx < \infty$ (since $q > 1$) and $\mathbb{E}|Y_1|^q \geq q \int_1^\infty x^{q-1} \cdot x^{-q} dx = \infty$. Define

$$X_{2k-1} = Y_k, \quad X_{2k} = -Y_k, \quad k \geq 1.$$

The sequence satisfies the hypotheses. By symmetry of Y_1 , every X_n has the law of Y_1 ; the sequence is identically distributed, symmetric, integrable, and centered, with $\mathbb{E}|X_1|^q = \infty$. It is NA. First, each pair $(Y_k, -Y_k)$ is NA because for nondecreasing f, g with finite covariance, the function $h(y) = g(-y)$ is nonincreasing, and for an independent copy Y' of $Y = Y_k$,

$$2 \operatorname{Cov}(f(Y), h(Y)) = \mathbb{E}[(f(Y) - f(Y'))(h(Y) - h(Y'))] \leq 0,$$

since the two factors always have opposite signs. Second, the pairs $\{(Y_k, -Y_k)\}_{k \geq 1}$ are independent of one another, and the union of independent NA families is NA [3]. Being NA, the sequence is m -NA for every $m \geq 1$.

The non-maximal series converges. The partial sums telescope: $S_{2n} = 0$ and $S_{2n-1} = Y_n$ for all $n \geq 1$. Hence, for any $\varepsilon > 0$, only odd indices contribute, and for $n \geq \varepsilon^{-p}$ (so that $\varepsilon n^{1/p} \geq 1$),

$$n^{q/p-2} \mathbb{P}(|S_n| > \varepsilon n^{1/p}) \leq n^{q/p-2} \mathbb{P}(|Y_1| > \varepsilon n^{1/p}) = \varepsilon^{-q} n^{q/p-2} n^{-q/p} = \varepsilon^{-q} n^{-2}.$$

The finitely many remaining terms are bounded by $n^{q/p-2}$, so the series converges for every $\varepsilon > 0$.

The maximal series diverges logarithmically. For every $n \geq 1$,

$$M_{2n} = \max_{1 \leq k \leq n} |Y_k|.$$

For all sufficiently large n , put $a_n = \varepsilon^{-q}(2n)^{-q/p}$. Since $q > p$, one has $na_n \rightarrow 0$, and independence gives

$$\mathbb{P}(M_{2n} > \varepsilon(2n)^{1/p}) = 1 - (1 - a_n)^n \sim na_n.$$

Consequently,

$$(2n)^{q/p-2} \mathbb{P}(M_{2n} > \varepsilon(2n)^{1/p}) \sim \frac{\varepsilon^{-q}}{4n}.$$

Summing only these even-indexed terms yields

$$\liminf_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^{2N} n^{q/p-2} \mathbb{P}(M_n > \varepsilon n^{1/p}) \geq \frac{\varepsilon^{-q}}{4},$$

which proves the quantitative assertion. $\square$

Remark 6.1. The same example defeats the non-maximal complete moment statements. For $f(x) = x^\alpha$ with $0 < \alpha < q$ and $v \geq 1$, the tail $\mathbb{P}(|Y_1| > x) = x^{-q}$ gives $\mathbb{E}[|Y_1|^\alpha \mathbf{1}_{\{|Y_1| > v\}}] = v^{\alpha-q} + \frac{\alpha}{q-\alpha} v^{\alpha-q} = C v^{\alpha-q}$, so for large odd n ,

$$n^{q/p-2} \mathbb{E}\left[\left(\frac{|S_n|}{n^{1/p}}\right)^\alpha \mathbf{1}_{\{|S_n| > \varepsilon n^{1/p}\}}\right] = C n^{q/p-2} n^{-\alpha/p} (\varepsilon n^{1/p})^{\alpha-q} = C_\varepsilon n^{-2},$$

which is summable, while $\mathbb{E}|X_1|^q = \infty$. By Proposition 3.7, the corresponding maximal series diverge at least logarithmically.

6.2. Two diagnostics for the scaling pair $(n^{1/r}, n^{r-2})$

The next two statements explain, ahead of Section 7, why the moment order attached to the normalization $n^{1/r}$ and the weight n^{r-2} is r^2 and why the range $1 < r < 2$ cannot be enlarged.

Proposition 6.2 (The condition $\mathbb{E}|X_1|^r < \infty$ is not sufficient). *Let $1 < r < 2$. There is an independent (hence NA and m -NA for every m) sequence of identically distributed, symmetric random variables such that $\mathbb{E}|X_1|^r < \infty$ and, for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{r-2} \mathbb{P}(M_n > \varepsilon n^{1/r}) = \infty.$$

Proof. Let X_1 be symmetric with $\mathbb{P}(|X_1| > x) = x^{-r^2}$ for $x \geq 1$. Then $\mathbb{E}|X_1|^u < \infty$ if and only if $u < r^2$; in particular, $\mathbb{E}|X_1|^r < \infty$ (as $r < r^2$) and $\mathbb{E}|X_1|^{r^2} = \infty$. Since $n^{r-2} = n^{q/p-2}$ for $p = r$, $q = r^2$, the divergence for every ε is exactly the contrapositive of the implication (iii) $\Rightarrow$ (i) of Corollary 3.2. $\square$

Proposition 6.3 (The range $1 < r < 2$ is essential). *Let $r \geq 2$, and let $X_1, X_2, \dots$ be independent Rademacher variables. Then all moments of X_1 are finite, yet for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{r-2} \mathbb{P}(|S_n| > \varepsilon n^{1/r}) = \infty,$$

and a fortiori the corresponding maximal series diverges.

Proof. By the central limit theorem, $S_n/\sqrt{n} \Rightarrow Z$ with Z standard normal. If $r > 2$, then for every fixed $\tau > 0$ and all large n , $\varepsilon n^{1/r} \leq \tau \sqrt{n}$, so $\liminf_n \mathbb{P}(|S_n| > \varepsilon n^{1/r}) \geq \mathbb{P}(|Z| > \tau)$; letting $\tau \downarrow 0$ gives $\mathbb{P}(|S_n| > \varepsilon n^{1/r}) \rightarrow 1$, while $n^{r-2} \geq 1$. If $r = 2$, the weight is 1 and $\mathbb{P}(|S_n| > \varepsilon \sqrt{n}) \rightarrow \mathbb{P}(|Z| > \varepsilon) > 0$. In either case, the terms do not tend to zero. $\square$

7. The scaled form and complete f -moment convergence

If the normalization is fixed as $n^{1/r}$ and the weight as n^{r-2} , the Baum–Katz relation $q/p - 2 = r - 2$ with $p = r$ forces $q = r^2$. Substituting $p = r$, $q = r^2$ in Corollaries 3.2 and 3.3 (note $r^2 > r$ for $r > 1$) gives:

Corollary 7.1 (Scaled maximal Baum–Katz form). *Let $1 < r < 2$, let $m \geq 1$, and let $\{X_n\}$ be an m -NA sequence of identically distributed random variables. Then the following are equivalent: (i) $\mathbb{E}|X_1|^{r^2} < \infty$ and $\mathbb{E}X_1 = 0$; (ii) $\sum_{n \geq 1} n^{r-2} \mathbb{P}(M_n > \varepsilon n^{1/r}) < \infty$ for every $\varepsilon > 0$; and (iii) the series in (ii) converges for some $\varepsilon > 0$. Moreover, under (i), for every $0 < \alpha < r^2$ and $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{r-2} \mathbb{E} \left[\left(\frac{M_n}{n^{1/r}} \right)^{\alpha} \mathbf{1}_{\{M_n > \varepsilon n^{1/r}\}} \right] < \infty.$$

Proposition 6.2 shows that the moment order r^2 cannot be lowered to r , and Proposition 6.3 shows that the restriction $r < 2$ cannot be dropped.

The next statement is a short consequence of the maximal characterization and the known complete-moment estimate. It is included to make the scaling and the distinction between raw truncated moments and excess moments explicit. Wang and Wang [13] proved broad complete f -moment results for Sung-type weighted sums under extended negative dependence conditions. Their theorem and the present corollary use different dependence hypotheses and functional formulations; nevertheless, we do not present the sufficiency below as a separate foundational advance.

Corollary 7.2 (Subcritical complete f -moment consequence). *Let $1 < r < 2$, $q = r^2$, let $m \geq 1$, and let $\{X_n\}$ be an m -NA sequence of identically distributed random variables. Let $f: [0, \infty) \rightarrow [0, \infty)$ be nondecreasing with $f(x) > 0$ for $x > 0$, and suppose there are constants $C > 0$ and $0 < \alpha < q$ such that*

$$f(x) \leq C(1 + x^{\alpha}), \quad x \geq 0.$$

Then the following are equivalent:

- (i) $\mathbb{E}|X_1|^q < \infty$ and $\mathbb{E}X_1 = 0$;
- (ii) for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} \mathbb{E} \left[f \left(\frac{M_n}{n^{1/r}} \right) \mathbf{1}_{\{M_n > \varepsilon n^{1/r}\}} \right] < \infty.$$

Moreover, in (ii), convergence for a single $\varepsilon > 0$ already implies (i).

Proof. Put $V_n = M_n/n^{1/r}$. Assume (i). The growth assumption gives

$$f(V_n) \mathbf{1}_{\{V_n > \varepsilon\}} \leq C \mathbf{1}_{\{V_n > \varepsilon\}} + C V_n^{\alpha} \mathbf{1}_{\{V_n > \varepsilon\}},$$

and after multiplying by n^{r-2} and summing, the first term is summable by Corollary 7.1 (ii) and the second by the complete moment bound of Corollary 7.1; this proves (ii) for every $\varepsilon > 0$.

Conversely, suppose the series in (ii) converges for some $\varepsilon > 0$. Since f is nondecreasing and $f(\varepsilon) > 0$,

$$f(V_n)\mathbf{1}_{\{V_n > \varepsilon\}} \geq f(\varepsilon)\mathbf{1}_{\{V_n > \varepsilon\}},$$

so $\sum_{n \geq 1} n^{r-2} \mathbb{P}(M_n > \varepsilon n^{1/r}) < \infty$, and Corollary 7.1 (iii) yields both $\mathbb{E}|X_1|^q < \infty$ and $\mathbb{E}X_1 = 0$. $\square$

Remark 7.3. Since $x \mapsto f(x)\mathbf{1}_{\{x > \varepsilon\}}$ is nondecreasing and $|S_n| \leq M_n$, statement (i) of Corollary 7.2 also implies the non-maximal bound $\sum_{n \geq 1} n^{r-2} \mathbb{E}[f(|S_n|/n^{1/r})\mathbf{1}_{\{|S_n| > \varepsilon n^{1/r}\}}] < \infty$ for every $\varepsilon > 0$. The converse of this non-maximal form is false under the present hypotheses, by Proposition 3.7 and Remark 6.1.

Remark 7.4. The positivity assumption $f(x) > 0$ for $x > 0$ is used only to pass from the f -moment series to the probability series; in an individual application, it can be weakened to $f(\varepsilon) > 0$ at the particular threshold where convergence is assumed. Note also that Corollary 7.2 is an equivalence with the Baum–Katz moment and drift condition in (i), not with $\mathbb{E}f(|X_1|) < \infty$; for subcritical f , the latter condition is in general strictly weaker (Example 7.7).

7.1. Examples

The first two examples exhibit admissible functions f ; the last two delimit the theorem.

Example 7.5 (Powers and logarithmic perturbations). Let $1 < r < 2$, $q = r^2$. The function $f(x) = x^\alpha$ with $0 < \alpha < q$ satisfies the hypotheses of Corollary 7.2 directly. The function $f(x) = x^\alpha \log(1 + x)$ with $0 < \alpha < q$ satisfies them as well: Choosing $\delta > 0$ with $\alpha + \delta < q$, one has $f(x) \leq C_\delta(1 + x^{\alpha+\delta})$. The same device applies to $f(x) = x^\alpha L(x)$ for any nondecreasing slowly varying L , since $L(x) = O(x^\delta)$ for every $\delta > 0$; examples are $L(x) = \log^\gamma(e + x)$, $\gamma > 0$, and $L(x) = \log \log(e^e + x)$.

Example 7.6 (A concrete parameter choice). Take $r = 3/2$, so the normalization is $n^{2/3}$, the weight is $n^{-1/2}$, and the moment order is $q = r^2 = 9/4$. For an m -NA, identically distributed, centered sequence, $\mathbb{E}|X_1|^{9/4} < \infty$ is equivalent to $\sum_{n \geq 1} n^{-1/2} \mathbb{P}(M_n > \varepsilon n^{2/3}) < \infty$, and it implies the complete f -moment bounds of Corollary 7.2 for every nondecreasing f with $f(x) = O(x^\alpha)$, $\alpha < 9/4$.

Example 7.7 (The condition $\mathbb{E}f(|X_1|) < \infty$ is not sufficient). Fix $0 < \alpha < q = r^2$ and $f(x) = x^\alpha$. Let $\{X_n\}$ be independent, identically distributed, and symmetric, with $\mathbb{P}(|X_1| > x) = x^{-q}$ for $x \geq 1$. Then $\mathbb{E}f(|X_1|) = \mathbb{E}|X_1|^\alpha < \infty$ but $\mathbb{E}|X_1|^q = \infty$, so, by Corollary 3.2, the weighted probability series for M_n diverges for every $\varepsilon > 0$, and hence so does the f -moment series in Corollary 7.2 (ii), which dominates $f(\varepsilon) \sum_{n \geq 1} n^{r-2} \mathbb{P}(M_n > \varepsilon n^{1/r})$.

Example 7.8 (Endpoint caution). Corollary 7.2 covers $f(x) = O(x^\alpha)$ only for $\alpha < q = r^2$; it makes no assertion about the endpoint $f(x) = x^q$, nor about $f(x) = x^q \log(1 + x)$. The need for separate endpoint hypotheses is suggested by Lemma A.8: There are distributions with $\mathbb{E}|X_1|^q < \infty$ but $\mathbb{E}\{|X_1|^q \log(1 + |X_1|)\} = \infty$, so no logarithmically perturbed endpoint condition can be deduced from the finite q -th moment alone. Endpoint complete moment results must be proved, or quoted, separately.

8. Conclusions

The new point established here is the joint single-threshold recovery principle. For identically distributed m -NA variables, convergence at one fixed threshold forces the q -th moment for $p > 0$,

$q \geq p$ and also forces zero drift when $p > 1$. Consequently, for $1 < p < 2$, $q \geq p$, one-threshold convergence, every-threshold convergence, and the moment-plus-centering condition are equivalent. The closest previous converse, [16, p. 1393, Theorem 2.2], assumes convergence at every positive threshold and overlaps the range $1 \leq p \leq q < 2$; the single-threshold quantifier and the wider range above are therefore the precise distinctions of Theorem 3.1. The sufficiency directions, including the endpoint input at $q = p$, are deliberately attributed to [10, 14, 16]; the subcritical complete-moment and f -moment statements remain consequences rather than separate claims of priority. At $p = 1$, the threshold profile changes: Its infimum equals $|\mathbb{E}X_1|$, and the critical endpoint can either belong or fail to belong to the convergence set.

The cancellation example shows why the converse is genuinely maximal: Under NA, the terminal sums may repeatedly cancel even when individual increments have an infinite q -th moment, while the maximal series retains an explicit logarithmic divergence rate. Such a construction is incompatible with independence, where the classical Baum–Katz converse applies. Other dependence classes may or may not permit the same mechanism; the decisive issue is whether their structural constraints exclude local near-perfect cancellation. Further questions include conditions restoring a non-maximal converse, the exact endpoint membership in broader threshold profiles, and analogues of single-threshold drift recovery under asymptotic negative dependence. The scaled application also suggests studying rates when $n^{1/r}$ is replaced by a regularly varying normalization.

Author contributions

Sen Zhang: Conceptualization, methodology, formal analysis, writing—original draft, writing—review and editing. Yunzhi Zhu: Conceptualization, methodology, formal analysis, validation, writing—original draft, writing—review and editing. Saisai Hou: Validation, investigation, writing—review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

During the preparation of this manuscript, the authors used Anthropic’s Claude to polish the English language and to assist with the LaTeX typesetting. All mathematical content is the authors’ own, and the authors take full responsibility for the final content of the manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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A. Auxiliary facts

The following short facts are collected here so that the proof of the new single-threshold converse remains easy to follow.

Lemma A.1 (Negative orthant bounds). *Let $Z_1, \dots, Z_N$ be NA. Then for all real numbers $x_1, \dots, x_N$,*

$$\mathbb{P}\left(\bigcap_{i=1}^N \{Z_i \leq x_i\}\right) \leq \prod_{i=1}^N \mathbb{P}(Z_i \leq x_i) \quad \text{and} \quad \mathbb{P}\left(\bigcap_{i=1}^N \{Z_i \geq x_i\}\right) \leq \prod_{i=1}^N \mathbb{P}(Z_i \geq x_i).$$

This standard orthant consequence of negative association is recalled without proof; see [3].

Lemma A.2 (Lower bound for the maximal tail). *Let $Z_1, \dots, Z_N$ be NA and identically distributed. Then for every $x > 0$,*

$$\mathbb{P}\left(\max_{1 \leq i \leq N} |Z_i| > x\right) \geq c_0 \min(1, N \mathbb{P}(|Z_1| > x)), \quad c_0 = \frac{1}{2}(1 - e^{-1}).$$

Proof. Write $\bar{F}(x) = \mathbb{P}(Z_1 > x)$ and $\bar{G}(x) = \mathbb{P}(Z_1 < -x)$, so that $\mathbb{P}(|Z_1| > x) = \bar{F}(x) + \bar{G}(x)$. By Lemma A.1,

$$\mathbb{P}\left(\max_i Z_i > x\right) \geq 1 - \prod_{i=1}^N \mathbb{P}(Z_i \leq x) = 1 - (1 - \bar{F}(x))^N \geq 1 - e^{-N\bar{F}(x)},$$

and likewise $\mathbb{P}(\min_i Z_i < -x) \geq 1 - e^{-N\bar{G}(x)}$. Since $1 - e^{-u} \geq (1 - e^{-1}) \min(1, u)$ for $u \geq 0$ (by concavity on $[0, 1]$ and monotonicity beyond), and since $\{\max_i |Z_i| > x\}$ contains both $\{\max_i Z_i > x\}$ and $\{\min_i Z_i < -x\}$,

$$\mathbb{P}\left(\max_i |Z_i| > x\right) \geq (1 - e^{-1}) \min(1, N \max(\bar{F}(x), \bar{G}(x))) \geq \frac{1-e^{-1}}{2} \min(1, N(\bar{F}(x) + \bar{G}(x))),$$

where the last step uses $\max(\bar{F}, \bar{G}) \geq (\bar{F} + \bar{G})/2$ and $\min(1, u/2) \geq \frac{1}{2} \min(1, u)$. □

Lemma A.3 (Rosenthal-type maximal inequality; [17]). *Let $s \geq 2$, and let $Z_1, \dots, Z_N$ be NA with $\mathbb{E}Z_i = 0$ and $\mathbb{E}|Z_i|^s < \infty$ for all i . Then there is a constant C_s , depending only on s , such that*

$$\mathbb{E} \max_{1 \leq k \leq N} \left| \sum_{i=1}^k Z_i \right|^s \leq C_s \left\{ \sum_{i=1}^N \mathbb{E}|Z_i|^s + \left(\sum_{i=1}^N \mathbb{E}Z_i^2 \right)^{s/2} \right\}.$$

Lemma A.3 follows from Shao's comparison theorem [17, Theorem 1], which transfers moment bounds for sums of independent variables to negatively associated summands, together with the classical Doob–Rosenthal maximal inequality. Its assumptions—centering, a finite moment of order at least two, and negative association of the whole row—make it a sufficiency tool only. It cannot prove Theorem 3.1, which starts from a probability series and assumes no moment or centering; for $q < 2$, any Rosenthal-based sufficiency proof must additionally truncate the variables.

The next four lemmas are elementary moment computations. The tail-sum, truncated-moment, and tail-moment identities are standard; see, for example, Wang [18]. To avoid repeating standard material, the routine proofs of Lemmas A.4–A.6 are omitted; the weighted integral calculation is kept because its exact exponents are useful for comparison.

Lemma A.4 (Tail-sum equivalence). *Let X be a random variable, and let $a > 0$, $s > 0$, $c > 0$. Then*

$$\mathbb{E}|X|^s < \infty \iff \sum_{n=1}^{\infty} n^{s/a-1} \mathbb{P}(|X| > c n^{1/a}) < \infty.$$

This standard tail-sum criterion follows by comparing the layer-cake integral with unit intervals; we omit the routine proof.

Lemma A.5 (Truncated-moment series). *Let $0 < q < s$, $a > 0$, $c > 0$, and suppose $\mathbb{E}|X|^q < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{(q-s)/a-1} \mathbb{E}[|X|^s \mathbf{1}_{\{|X| \leq c n^{1/a}\}}] < \infty.$$

This estimate is a standard summation-by-parts consequence of $\mathbb{E}|X|^q < \infty$; the proof is omitted.

Lemma A.6 (Tail-moment series). *Let $0 \leq \alpha < q$, $a > 0$, $c > 0$, and suppose $\mathbb{E}|X|^q < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{(q-\alpha)/a-1} \mathbb{E}[|X|^\alpha \mathbf{1}_{\{|X| > c n^{1/a}\}}] < \infty.$$

By Tonelli's theorem, the left-hand side is bounded by a constant multiple of $\mathbb{E}|X|^q$; we omit the standard calculation.

Lemma A.7 (Weighted tail-integral series). *Let $0 < \alpha < q$, $a > 0$, $c > 0$, and suppose $\mathbb{E}|X|^q < \infty$. Put $R(v) = \int_v^\infty u^{\alpha-1} \mathbb{P}(|X| > u) du$ for $v > 0$. Then*

$$\sum_{n=1}^{\infty} n^{(q-\alpha)/a-1} R(c n^{1/a}) < \infty.$$

Proof. First, $R(c) \leq \mathbb{E}|X|^q \int_c^\infty u^{\alpha-1-q} du < \infty$ by Markov's inequality, so every term is finite. Let $\gamma = (q - \alpha)/a > 0$. Since R is nonincreasing and $x^{\gamma-1}$ is comparable to $n^{\gamma-1}$ on $[n-1, n]$ for $n \geq 2$,

$$\sum_{n \geq 2} n^{\gamma-1} R(c n^{1/a}) \leq C \int_1^\infty x^{\gamma-1} R(c x^{1/a}) dx = C a c^{-a\gamma} \int_c^\infty v^{a\gamma-1} R(v) dv$$

by the substitution $v = c x^{1/a}$. Since $a\gamma = q - \alpha$, Tonelli's theorem gives

$$\int_c^\infty v^{q-\alpha-1} \int_v^\infty u^{\alpha-1} \mathbb{P}(|X| > u) du dv = \int_c^\infty u^{\alpha-1} \mathbb{P}(|X| > u) \int_c^u v^{q-\alpha-1} dv du \leq \frac{1}{q-\alpha} \cdot \frac{\mathbb{E}|X|^q}{q}$$

because $\int_c^u v^{q-\alpha-1} dv \leq u^{q-\alpha}/(q-\alpha)$ and $\int_0^\infty u^{q-1} \mathbb{P}(|X| > u) du = \mathbb{E}|X|^q/q$. $\square$

Finally we record an elementary fact used in Example 7.8.

Lemma A.8 (No automatic logarithmic upgrade). *For every $s > 0$, there is a symmetric random variable X with $\mathbb{E}|X|^s < \infty$ and $\mathbb{E}\{|X|^s \log(1 + |X|)\} = \infty$.*

Proof. Let X be symmetric with $\mathbb{P}(|X| = 2^k) = c k^{-2} 2^{-ks}$ for $k \geq 1$, where $c > 0$ is chosen so small that the remaining mass can be placed at 0. Then $\mathbb{E}|X|^s = c \sum_k k^{-2} < \infty$, while $\log(1 + 2^k) \geq k \log 2$ yields $\mathbb{E}\{|X|^s \log(1 + |X|)\} \geq c \log 2 \sum_k k^{-1} = \infty$. $\square$



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