



*Research article***Fuzzy boundedness of nonlinear parabolic PDE under dynamic quantization and network attacks via dynamic output feedback****Jingzhao Chen¹, Liming Ding^{2,*}, Teng-Fei Li^{2,3,*} and Jiahuan Xue¹**¹ Henan Engineering Research Center of Intelligent Manufacturing and Digital Twin, SIAS University, Zhengzhou 451150, China² School of Computer and Artificial Intelligence (School of Software), Huaihua University, Huaihua 418000, China³ College of Aeronautical Science and Engineering, Yantai Nanshan University, Yantai 265713, China*** Correspondence:** Email: dlm19830112@163.com, acelitengfei@163.com.

Abstract: The finite-time boundedness problem was investigated for nonlinear parabolic partial differential equations (PDEs) with spatially varying coefficient matrices, using a fuzzy dynamic output feedback approach. To deal with the system nonlinearities, the Takagi-Sugeno (T-S) fuzzy model was adopted, which yielded a fuzzy parabolic PDE formulation. To reduce data transmission volume while preserving system performance, dynamic quantization was applied to both control signals and measured outputs. A spatially dependent dynamic output feedback controller was then designed to cope with network attacks. Under the imposed quantization constraints, the boundedness of the closed-loop system over a finite-time interval was established. Sufficient conditions for controller synthesis and the adjustment parameters of the quantizer were derived by means of free-matrix inequalities. In addition, a generalized matrix decomposition technique was employed to decouple the cross-coupled terms that appear in the nonlinear performance criteria. The effectiveness of the proposed method was demonstrated through simulation results on a two-dimensional dynamic model.

Keywords: parabolic PDE; T-S fuzzy model; boundedness; stability**Mathematics Subject Classification:** 58J35, 03B52, 39A22, 35B35

1. Introduction

Owing to the intrinsic spatiotemporal nature of a wide range of industrial processes, conventional ordinary differential equations and finite-dimensional temporal models are generally inadequate for faithfully representing their underlying dynamics [1,2]. Parabolic partial differential equations (PDEs),

which naturally capture diffusion-dominated behaviors, offer a suitable modeling framework for such systems [3–5]. Over the past century, substantial efforts have been devoted to the theory and applications of parabolic PDEs, with numerous significant results reported in the literature [6–8]. Despite these notable advances, the control of nonlinear parabolic PDEs remains a nontrivial problem, primarily due to the complex nonlinearities, the presence of multiple uncertain control variables, and the susceptibility to both exogenous and endogenous disturbances.

Recent advances in computing, communication, and hardware technologies have spurred substantial progress in the network-based study of parabolic PDEs over the past few decades. Research conducted within the networked framework serves as a critical link between physical processes, computational resources, and communication infrastructures. As an integrated platform, such networked approaches fuse real-world dynamics with numerical algorithms, data transmission, and automated control mechanisms [9]. This integrative perspective also constitutes a foundational cornerstone for the development of future industrial systems [10].

The transmission of data over the network interconnects the physical and computing layers, offering advantages such as increased operational flexibility and reduced deployment costs. However, bandwidth limitations impose inherent constraints on communication capacity [11]. Excessive data exchange may cause network congestion, which in turn can deteriorate system stability or undermine control objectives [12, 13]. To mitigate this issue, quantizing signals prior to transmission has become a common and effective measure [14]. Nevertheless, quantization errors are inevitable and may render the desired control performance unattainable [15].

Consequently, quantized control of parabolic PDEs is a necessary and rewarding endeavor. Prior contributions addressing quantization in this context, e.g., [16, 17], are predominantly based on static quantizers, which are characterized by time-invariance and memorylessness. Such quantizers often require an infinite number of quantization levels to meet desired performance specifications, a condition seldom realizable in hardware. To circumvent this limitation, dynamic quantization offers an adaptive mechanism that adjusts the quantizer parameters in response to signal amplitude and error bounds, thus simplifying the implementation without sacrificing control quality. In this work, we adopt the dynamic quantization scheme proposed in [18].

On the other hand, the introduction of networked control technology inevitably raises security concerns pertaining to signal transmission. As noted in [19], the deployment of communication techniques in industrial control systems may expose vulnerabilities, potentially leading to system instability or even collapse of control systems. Therefore, the impact of cyber attacks cannot be ignored in the quantized control of parabolic PDEs. Recently, significant progress has been made on the control problem of parabolic PDEs under cyber attacks. For instance, a novel attack-resilient control strategy is developed to solve the fully distributed consensus problem for multi-agent systems described by parabolic PDE with spatially distributed characteristics under denial-of-service attacks [20]. Considering the appearance of multiple cyber attacks, a distributed consensus protocol with inherent robustness is designed to address the event-triggered fuzzy control problem for MASs described by nonlinear delayed parabolic PDEs [21]. However, these existing advances do not account for the influence of quantization, and, moreover, they exhibit a certain dependence on the coefficient parameters of the system. To address this issue, this paper comprehensively considers the effects of both cyber attacks and dynamic quantization, and investigates the finite-time boundedness of the system control for parabolic PDE systems with varying coefficient parameters.

Industrial systems frequently exhibit nonlinear dynamical behaviors, for which a number of effective treatment strategies have been developed, including the T-S fuzzy model [22, 23], type-2 fuzzy systems [24], and Lipschitz-based approaches [1, 25]. Among these, the T-S fuzzy framework is adopted in this study to represent the nonlinearities of parabolic PDEs, motivated by its well-established effectiveness in prior investigations [16, 17, 22]. Building on these promising results, the present work aims to design a fuzzy dynamic control law that integrates signal quantization and actuator failures for a class of parabolic PDEs.

Furthermore, control performance is a fundamental metric in control system design and has consequently attracted sustained research interest. Among the various performance indices, the extended dissipativity concept offers a notable advantage in generality, as it encompasses several conventional criteria through appropriate tuning of its weighting parameters. Meanwhile, the transient response speed, particularly the finite-time convergence property, represents another critical aspect in evaluating system behavior. Relevant contributions in this direction include observer-based finite-time control for distributed parameter systems and fuzzy delay-dependent dissipative control for Markovian jump systems with incomplete transition probabilities. However, the finite-time boundedness problem for parabolic PDEs in the presence of both quantization and cyber attacks remains unexplored, which motivates the present investigation into dissipative control strategies under finite-time constraints.

The adoption of dynamic quantizers inevitably imposes additional constraints on the controller synthesis, which often leads to conservative design conditions. This conservatism is further compounded by the intrinsic complexity of parabolic PDEs, such as nonlinear dynamics and spatiotemporal couplings, which also exacerbates the computational burden. Since the solvability of a control algorithm is directly linked to the feasibility of its associated conditions, over-conservative requirements may render the design infeasible in terms of performance robustness. Hence, reducing conservatism is essential to improving both the tractability and practical applicability of advanced control strategies. Techniques including the free-matrix inequality, slack matrix variables, and parameter-dependent Lyapunov functions have proven effective in mitigating conservatism while preserving rigorous stability guarantees. Among these, the free-matrix inequality offers a non-scaling transformation for condition conversion and is particularly well-suited to handling differential terms arising in PDE control design. Consequently, this study employs the free-matrix approach to alleviate the conservatism inherent in the proposed control conditions.

Motivated by recent advances in the field, the present study investigates the dissipative control design for a class of fuzzy parabolic PDEs. A spatially dependent dynamic output feedback strategy is developed, which integrates signal quantization and accommodates potential cyber attacks. By employing suitable decoupling techniques, sufficient conditions for the control parameters are established to guarantee the prescribed system performance index.

i) A dynamic quantizer, which is more concise than its static counterpart, is introduced to reduce data transmission while preserving system performance, thereby enhancing network resource utilization and alleviating congestion.

ii) A spatially dependent output feedback control scheme is formulated for parabolic PDEs with spatially varying coefficient matrices, using the T-S fuzzy modeling approach, under the possibility of network attacks.

iii) The design criteria for the controller parameters are derived. To mitigate the conservatism of these conditions, the free-matrix inequality and a generalized matrix decomposition technique are

incorporated.

The remainder of this paper is organized as follows. Section 2 describes the nonlinear parabolic PDE under consideration and states the basic assumptions. Section 3 presents the quantized dynamic output feedback controller and the associated stability conditions. Numerical simulations in Section 4 demonstrate the approach's efficacy, while Section 5 concludes the study with a summary of findings and suggestions for future work.

Notations: The symbol $\mathcal{R}^n$ denotes the n -dimensional Euclidean space, while $\mathcal{R}^{m \times n}$ represents the space of all $m \times n$ real-valued matrices. For any matrix $\mathcal{X}$, its transpose is denoted by $\mathcal{X}^T$. $\mathcal{X} > 0$ means $\mathcal{X}$ is a symmetric positive definite matrix. The matrix $\mathcal{X}^{-1}$ represents the inverse of $\mathcal{X}$. The symbol $*$ indicates the symmetric part of a matrix, i.e., $[\mathcal{X} + *] = [\mathcal{X} + \mathcal{X}^T]$. The terms $\lambda_{\min}(\mathcal{X})$ and $\lambda_{\max}(\mathcal{X})$ represent the minimum and maximum eigenvalues of $\mathcal{X}$. $\text{col}\{\mathcal{X}_1, \mathcal{X}_2\}$ signifies $= \begin{bmatrix} \mathcal{X}_1^T & \mathcal{X}_2^T \end{bmatrix}^T$. The spatial derivatives of function $\mathcal{X}(s, t)$ are denoted as follows: $\mathcal{X}_s(s, t)$ indicates its first-order partial derivative $\frac{\partial \mathcal{X}(s, t)}{\partial s}$, while $\mathcal{X}_{ss}(s, t)$ represents the second-order derivative $\frac{\partial^2 \mathcal{X}(s, t)}{\partial s^2}$.

2. Problem statements

This section introduces the studied parabolic PDE along with some essential lemmas. A nonlinear parabolic PDE with spatial varying coefficient matrices can be characterized as the parabolic PDE:

$$\begin{cases} w_t(s, t) = \frac{\partial}{\partial x}[\mathcal{A}(w(x, t))w_x(x, t)] + f(w(x, t), x) + B(x)u(x, t) + C(x)\omega(t), \\ y(x, t) = Dw(x, t). \end{cases} \quad (2.1)$$

The $x \in [0, l]$ and $t > 0$ are the variables of spatial position and temporal variable, the state vector is denoted by $w(\cdot, t)$. $\mathcal{A}(w(x, t)) \in \mathcal{R}^{n \times n}$ is a diagonal matrix, $B(x) \in \mathcal{R}^{n \times m}$ and $C(x) \in \mathcal{R}^{n \times p}$ are coefficient function matrices with respect to x , $D \in \mathcal{R}^{n \times n}$ is a constant matrix, $f(w(x, t)) \in \mathcal{R}^{n \times n}$ captures the system dynamics, $u(t)$ denotes the control input, $y(t) \in \mathcal{R}^{n \times n}$ corresponds to system outputs, and $\omega(t) \in \mathcal{R}^{n \times p}$ represents external disturbances bounded by $\int_0^\infty \|\omega(t)\|^2 dt < \bar{\omega}$.

System (2.1) is subject to the following boundary constraints:

$$w_x(0, t) = w_x(l, t), \quad t > 0. \quad (2.2)$$

Following the methodology in [15], the T-S fuzzy representation for the described system is formulated:

Plant rule i : If $\kappa_1(x, t)$ is F_{i1} , ..., and $\kappa_{\bar{k}}(x, t)$ is $F_{i\bar{k}}$, then

$$w_i(x, t) = \frac{\partial}{\partial x}[\mathcal{A}_i w_x(x, t)] + A_i w(x, t) + B_i u(t) + C_i \omega(t), \quad (2.3)$$

where $\kappa(s, t) = [\kappa_1(x, t), \kappa_2(x, t), \dots, \kappa_{\bar{k}}(x, t)]$ is the premise variables. The system contains $\bar{k}$ distinct nonlinear plants. Each fuzzy set $v \in 1, 2, \dots, \bar{k}$ corresponds to a subsystem, yielding $\kappa = 2^{\bar{k}}$ total fuzzy inference rules. The known constant matrices $A_i \in \mathcal{R}^{n \times n}$ complete the model description with $i \in \mathbb{S} = 1, 2, \dots, \kappa$. $F_{iv}(\kappa_v(x, t))$ is membership grade of $\kappa_v(x, t)$ in F_{iv} . Additionally, parameter $\kappa(x, t)$ in F_{iv} must satisfy: $\xi_i(\kappa(x, t)) = \prod_{v=1}^{\bar{k}} F_{iv}(\kappa_v(x, t))$, $h_i(\kappa(x, t)) = \xi_i(\kappa(x, t)) / \sum_{i=1}^{\kappa} \xi_i(\kappa(x, t))$,

$\sum_{i=1}^K h_i(\kappa(x, t)) = 1, h_i(\kappa(x, t)) \geq 0, i \in \mathbb{S}$. Then, $\sum_{i=1}^K h_i(\kappa(x, t))\mathcal{A}_i = \mathcal{A}(w(x, t))$, $\sum_{i=1}^K h_i(\kappa(x, t))A_i = f(w(x, t), s)$, $\sum_{i=1}^K h_i(\kappa(x, t))B_i = B(x)$, $\sum_{i=1}^K h_i(\kappa(x, t))C_i = C(x)$.

The complete fuzzy system is formulated:

$$\begin{cases} w_t(x, t) = \frac{\partial}{\partial x} \left[\sum_{i=1}^K h_i(\kappa(x, t))\mathcal{A}_i w_x(x, t) \right] + \sum_{i=1}^K h_i(\kappa(x, t))[A_i w(x, t) + B_i u(x, t) + C_i \omega(t)], \\ y(x, t) = Dw(x, t). \end{cases} \quad (2.4)$$

To simplify, $h_i(\kappa(x, t))$ is denoted by h_i in subsequent sections. The following definition is necessary to derive the main results.

Definition 1. [26] [Finite-time boundedness (FTB)] Assuming matrix $R > 0 \in \mathcal{R}^{n \times n}$ and two positive scalars c_1, c_2 , and $\mathcal{T}$, where $c_1 < c_2$, if there is a $u(x, t)$ for (2.4), it leads to the following relationship for $t \in [0, \mathcal{T}]$:

$$\sup_{t_0 \in [-\tau, 0]} \{\bar{z}(t_0), \bar{\varsigma}(t_0)\} < c_1 \Rightarrow \int_{\Omega} w^T(x, t)Rw(x, t)dx < c_2, t \in [0, \mathcal{T}], \quad (2.5)$$

where the controller state $\bar{w}(t_0) = \int_{\Omega} w^T(x, t_0)Rw(x, t_0)dx$ will be designed subsequently. Similarly, $\bar{\varsigma}(t_0) = \int_{\Omega} \varsigma^T(x, t_0)R\varsigma(x, t_0)dx$ describes the initial condition. For the system (2.1) with controller $u(x, t)$, finite-time boundedness is established relative to the parameters $(c_1, c_2, \mathcal{T})$.

In this context, we employ the quantizer defined as $q(t)$ in [27]. The error bound and range are specified as $E_q > 0$ and $R_q > 0$, and satisfy:

$$\begin{aligned} \|\varrho(q) - q\| &\leq E_q, \text{ if } \|q\| \leq R_q, \\ \|\varrho(q)\| &> R_q - E_q, \text{ if } \|q\| > R_q. \end{aligned} \quad (2.6)$$

By the dynamic quantization scheme developed in [27], $\varrho_{\mu}(q) = \mu\varrho\left(\frac{q}{\mu}\right)$, where μ is the dynamic parameter, and q is the data to be quantized. The data q can be dynamically quantized according to the adjusting parameter μ . The adaptive dynamic quantization is then designed.

The following lemma is necessary for the results.

Lemma 1. [28] For a real scalar $0 \leq l \in \mathcal{R}^+$ and a vector function $x : [0, l] \rightarrow \mathcal{R}^n$, if there exist matrices $\mathbb{R}, \mathbb{Z}_1$, and $\mathbb{Z}_2 \in \mathcal{R}^{n \times n}$, as well as $\mathbb{N}_1, \mathbb{N}_2$, and $\mathbb{N}_3 \in \mathcal{R}^{n \times n}$, such that:

$$\bar{\mathbb{G}} = \begin{bmatrix} \mathbb{Z}_1 & \mathbb{N}_1 & \mathbb{N}_2 \\ * & \mathbb{Z}_2 & \mathbb{N}_3 \\ * & * & \mathbb{R} \end{bmatrix} \geq 0, \quad (2.7)$$

then the following relationship stands:

$$-\int_0^l \bar{y}^T(x)\mathbb{R}\bar{y}(x)dx \leq \frac{1}{l} \int_0^l \bar{y}^T(x)\hat{\mathbb{G}}\bar{y}(x)dx, \quad (2.8)$$

$$\text{where } \bar{y}(x) = \text{col}\{y(0), y(x), y(l)\}, \hat{\mathbb{G}} = \begin{bmatrix} l\mathbb{Z}_1 - [\mathbb{N}_2 + *] & \mathbb{N}_2 - \mathbb{N}_3^T & 0 \\ * & \frac{l}{3}\mathbb{Z}_2 - [\mathbb{N}_3 + *] & 2\mathbb{N}_3 \\ * & * & 0 \end{bmatrix}.$$

Remark 1. In this article, we do not treat the boundary values and the internal distributed function as being of the same nature. Instead, we use Lemma 1 to convert the point values into an integral bound on the spatial derivative of the internal function. This inequality holds for any absolutely continuous function $w(s, t)$ on $[m_0, m]$ whose derivative is square-integrable. In this way, the boundary terms and the L^2 integral of the internal function are connected through the derivative norm, so they can be placed together in the same matrix inequality without any spatial discretization or interpolation. All derivations are kept in the continuous domain.

3. Results

This section addresses the controller with the potential attacks in practical engineering. A spatial-dependent dynamic output feedback controller for the studied system (2.4) will be designed with quantization. The controller with quantization and attacks is considered as the following form:

$$\begin{cases} \zeta_t(x, t) = D_a \zeta(x, t) + D_b \varrho_{\bar{\mu}}(y(x, t)), \\ u(x, t) = \Gamma(t)[D_c \varrho_{\mu}(\zeta(x, t)) + D_d \varrho_{\bar{\mu}}(y(x, t))], \end{cases} \quad (3.1)$$

with the undetermined gain matrices $D_a \in \mathcal{R}^{n \times n}$, $D_b \in \mathcal{R}^{n \times n}$, $D_c \in \mathcal{R}^{m \times n}$, and $D_d \in \mathcal{R}^{m \times n}$, which will be solved in the following theorem. The dynamic controller's state vector $\zeta(x, t) \in \mathcal{R}^n$ evolves from initial conditions $\zeta(x, 0) = \zeta_0(x)$, while actuator failures are characterized by the diagonal matrix $\Gamma(t) = \text{diag}\{\Gamma_{c1}(t), \dots, \Gamma_{ci}(t), \dots, \Gamma_{cm}(t)\}$ follow a Bernoulli stochastic distribution: $\text{Prob}\{\Gamma_{ci}(t) = 1\} = \bar{\Gamma}_{ci}$, $\text{Prob}\{\Gamma_{ci}(t) = 0\} = 1 - \bar{\Gamma}_{ci}$, and the binary variable $\Gamma_{ci}(t) \in \{0, 1\}$ models failure states for the i th actuator, whereas its expected value $\bar{\Gamma}_{ci} \in [0, 1]$ is a predefined parameter. Defining $\bar{\Gamma} = \text{diag}\{\bar{\Gamma}_{c1}, \dots, \bar{\Gamma}_{ci}, \dots, \bar{\Gamma}_{cm}\}$, arrives $\mathbb{E}\{\Gamma(t)\} = \bar{\Gamma}$. By applying the quantization principles specified in (2.6), the controller formulation can be alternatively expressed as:

$$u(x, t) = \Gamma(t)[D_c(\zeta(x, t) + \hat{\vartheta}(x, t)) + D_d(y(x, t) + \bar{\vartheta}(x, t))], \quad (3.2)$$

with the dynamic parameters subject to the dynamic quantizers $\hat{\vartheta}(x, t) = \hat{\mu}(x, t)\left[\varrho\left(\frac{\zeta(x, t)}{\hat{\mu}(x, t)}\right) - \frac{\zeta(x, t)}{\hat{\mu}(x, t)}\right]$, and $\bar{\vartheta}(x, t) = \bar{\mu}(x, t)\left[\varrho\left(\frac{y(x, t)}{\bar{\mu}(x, t)}\right) - \frac{y(x, t)}{\bar{\mu}(x, t)}\right]$.

By incorporating (3.2) into (2.4), the closed-loop system is obtained:

$$\begin{cases} w_t(x, t) = \frac{\partial}{\partial x} \left[\sum_{i=1}^k h_i \mathcal{A}_i w_x(x, t) \right] + \sum_{i=1}^k h_i \left[(A_i + B_i \Gamma(t) D_d D) w(x, t) + C_i \omega(t) + B_i \Gamma(t) D_c \hat{\vartheta}(t) \right. \\ \quad \left. + B_i \Gamma(t) D_c \zeta(x, t) + B_i \Gamma(t) D_d \bar{\vartheta}(t) \right], \\ y(x, t) = D w(x, t). \end{cases} \quad (3.3)$$

The subsequent outcomes can be achieved.

Theorem 1. For system (3.3), assume the existence of matrices $P_1 > 0$ and $P_2 > 0$, symmetric matrices $\mathbb{R}$, $\mathbb{Z}_1$, $\mathbb{Z}_2$, proper dimensional matrices P_{2A} , P_{2B} , U_1 , V_1 , U_2 , V_2 , $\mathbb{N}_1$, $\mathbb{N}_2$, $\mathbb{N}_3$, scalars $r_1 > 0$, $r_2 > 0$, $\rho_y > 1$, $\rho_\varsigma > 1$, $\rho_{y1} = r_1 \rho_y$, $\rho_{\varsigma 2} = r_2 \rho_\varsigma$, the prescribed parameters $\alpha > 0$, E_ϱ , R_ϱ , positive matrix R , and arbitrary matrices $\mathcal{X}$ and $\mathcal{Y}$ of appropriate dimensions, along with positive constants $\mathcal{T}$, $\bar{\varepsilon}$, δ , c_1 , c_2 ,

which fulfills:

$$\begin{bmatrix} \mathbb{Z}_1 & \mathbb{N}_1 & \mathbb{N}_2 \\ * & \mathbb{Z}_2 & \mathbb{N}_3 \\ * & * & [P_1 \mathcal{A}_i + *] - P_3 \end{bmatrix} \geq 0, \quad (3.4)$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ * & F_{22} & F_{23} \\ * & * & F_{33} \end{bmatrix} < 0, \quad (3.5)$$

$$c_2 < \frac{e^{-\delta T}(\bar{\lambda}_1 + \bar{\lambda}_2)c_1 + \bar{\omega}\alpha}{\underline{\lambda}_1}, \quad (3.6)$$

with

$$\begin{aligned} F_{11} &= \begin{bmatrix} F_{11}^{11} & \frac{1}{l}[\mathbb{N}_2 - \mathbb{N}_3^T] & \frac{\pi^2}{4l^2}P_3 \\ * & \frac{1}{3}\mathbb{Z}_2 - \frac{1}{l}[\mathbb{N}_3 + *] & \frac{2}{l}\mathbb{N}_3 \\ * & * & F_{11}^{33} \end{bmatrix}, \quad F_{11}^{11} = \mathbb{Z}_1 - \frac{1}{l}[\mathbb{N}_2 + *] - \frac{\pi^2}{4l^2}P_3, \\ F_{11}^{33} &= [P_1 A_i + b\mathcal{Y}V_2 D + *] + \delta P_1 - \frac{\pi^2}{4l^2}P_3, \\ F_{12} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{X}V_1 + bD^T P_{2B}^T & \mathcal{Y}V_2 & \mathcal{X}V_1 & P_1 C_i \end{bmatrix}, \quad F_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\varepsilon}[P_1 B_i \bar{\Gamma} - \mathcal{X}U_1] & F_{13}^2 & \frac{2E_\theta \rho_{y1}}{R_\theta} D^T & 0 \end{bmatrix}, \\ F_{13}^2 &= \bar{\varepsilon}[P_1 B_u \bar{\Gamma} - \mathcal{Y}U_2] + [DV_2]^T F_{22} = \begin{bmatrix} \Pi_1 & P_{2B} & 0 & 0 \\ * & -r_1 I & 0 & 0 \\ * & * & -r_2 I & 0 \\ * & * & * & -\frac{\alpha}{l} I \end{bmatrix}, \\ F_{23} &= \begin{bmatrix} V_1^T & 0 & 0 & \frac{2E_\theta \rho_{s2}}{R_\theta} I \\ 0 & bV_2^T & 0 & 0 \\ V_1^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} F_{33} &= -\text{diag}\{\bar{\varepsilon}[U_1 + *], \bar{\varepsilon}[U_2 + *], r_1 I, r_2 I\}, \quad \Pi_1 = [P_{2A} + *] + \delta P_2, \\ \underline{\lambda}_1 &= \lambda_{\min}(R^{-\frac{1}{2}}P_1 R^{-\frac{1}{2}}), \quad \bar{\lambda}_1 = \lambda_{\max}(R^{-\frac{1}{2}}P_1 R^{-\frac{1}{2}}), \quad \underline{\lambda}_2 = \lambda_{\min}(R^{-\frac{1}{2}}P_2 R^{-\frac{1}{2}}), \quad \bar{\lambda}_2 = \lambda_{\max}(R^{-\frac{1}{2}}P_2 R^{-\frac{1}{2}}). \end{aligned}$$

The adjusting parameters $\bar{\mu}(s, t)$ and $\hat{\mu}(s, t)$ undergo continuous online updating according to the following scheme:

$$\begin{aligned} \frac{\rho_y}{R_\theta} \|y(x, t)\| &\leq \bar{\mu}(x, t) \leq \frac{2\rho_y}{R_\theta} \|y(x, t)\|, \\ \frac{\rho_s}{R_\theta} \|s(x, t)\| &\leq \hat{\mu}(x, t) \leq \frac{2\rho_s}{R_\theta} \|s(x, t)\|. \end{aligned} \quad (3.7)$$

Then, the FTB of system (3.3) is guaranteed under the designed controller (3.1). The quantized dynamic output feedback control scheme described by (3.1), incorporating the adaptation law specified

in (3.7), guarantees asymptotic stabilization of system (2.4). The corresponding control gains are determined as follows: $D_a = P_2^{-1}P_{2A}$, $D_b = P_2^{-1}P_{2B}$, $D_c = U_1^{-1}V_1$, $D_d = U_2^{-1}V_2$.

Proof: Consider the Lyapunov function candidate for (3.3) :

$$\mathcal{G}(t) = \mathcal{G}_1(t) + \mathcal{G}_2(t), \quad (3.8)$$

where $\mathcal{G}_1(t) = \int_0^l w^T(x, t)P_1 w(x, t)dx$, $\mathcal{G}_2(t) = \int_0^l \varsigma^T(x, t)P_2 \varsigma(x, t)dx$, $P_1 > 0$ and $P_2 > 0$ represent unspecified constant matrices. The infinitesimal operator $\mathcal{L}$ is applied to the function $\mathcal{G}_1(t)$ for analysis,

$$\begin{aligned} \mathbb{E}\{\mathcal{L}\mathcal{G}_1(t)\} &= \left[\int_0^l w^T(x, t)P_1 w_t(x, t)dx + * \right] \\ &= \left[\int_0^l w^T(x, t)P_1 \frac{\partial}{\partial x} \left(\sum_{i=1}^K h_i \mathcal{A}_i w_x(x, t) \right) dx + * \right] + \int_0^l w^T(x, t) \left[\sum_{i=1}^K h_i P_1 A_i + * \right] w(x, t) \quad (3.9) \\ &\quad + \left[\int_0^l w^T(x, t) \sum_{i=1}^K h_i P_1 B_i \mathbb{E}\{u(t)\} dx + * \right] + \left[\int_0^l w^T(x, t) \sum_{i=1}^K h_i P_1 C_i \omega(t) dx + * \right]. \end{aligned}$$

By considering the boundary constraints specified in (2.2) and employing integration by parts,

$$\begin{aligned} &\left[\int_0^l w^T(x, t)P_1 \frac{\partial}{\partial x} \left(\sum_{i=1}^K h_i \mathcal{A}_i w_x(x, t) \right) dx + * \right] \\ &= \left[z^T(s, t) \sum_{i=1}^K h_i P_1 \mathcal{A}_i z_s(s, t) \Big|_{s=0}^{s=l} + * \right] - \int_0^l z_s^T(s, t) \left[\sum_{i=1}^K h_i P_1 \mathcal{A}_i + * \right] z_s(s, t) ds \quad (3.10) \\ &= - \int_0^l w_x^T(x, t) \left[\sum_{i=1}^K h_i P_1 \mathcal{A}_i + * \right] w_x(x, t) dx. \end{aligned}$$

The following relationship can be derived with Wirtinger's inequality:

$$\begin{aligned} \int_0^l w_x^T(x, t)P_3 w_x(x, t)dx &\leq \frac{\pi^2}{4l^2} \int_{\Omega} w^T(x, t)P_3 w(x, t)dx + \frac{\pi^2}{4l^2} \int_{\Omega} w^T(0, t)P_3 w(0, t)dx \quad (3.11) \\ &\quad - \left[\frac{\pi^2}{4l^2} \int_{\Omega} w^T(x, t)P_3 w(0, t)dx + * \right]. \end{aligned}$$

By Lemma 1, if

$$\begin{bmatrix} \mathbb{Z}_1 & \mathbb{N}_1 & \mathbb{N}_2 \\ * & \mathbb{Z}_2 & \mathbb{N}_3 \\ * & * & [P_1 \mathcal{A}_i + *] - P_3 \end{bmatrix} \geq 0, \quad (3.12)$$

derives:

$$- \int_0^l w_x^T(x, t) \left[\sum_{i=1}^K h_i P_1 \mathcal{A}_i + * \right] w_x(x, t) dx \leq - \int_0^l \sum_{i=1}^K h_i [\bar{w}^T(x, t) \hat{\mathbb{M}} \bar{w}(x, t) dx + w_x^T(x, t)P_3 w_x(x, t)] dx \quad (3.13)$$

with $\bar{w}(x, t) = \text{col}\{w(0, t), w(l, t), w(x, t)\}$, $\hat{\mathbb{M}} = \begin{bmatrix} \mathbb{Z}_1 - \frac{1}{l}[\mathbb{N}_2 + *] & \frac{1}{l}[\mathbb{N}_2 - \mathbb{N}_3^T] & 0 \\ * & \frac{1}{3}\mathbb{Z}_2 - \frac{1}{l}[\mathbb{N}_3 + *] & \frac{2}{l}\mathbb{N}_3 \\ * & * & [P_1\mathcal{A}_i + *] - P_3 \end{bmatrix}$.

Then, the following relationship can be found if (3.12) stands:

$$\begin{aligned} & \left[\int_0^l w^T(x, t) P_1 \frac{\partial}{\partial x} \left(\sum_{i=1}^k h_i \mathcal{A}_i w_x(x, t) \right) dx + * \right] \\ & \leq \int_0^l w^T(0, t) \left[\mathbb{Z}_1 - \frac{1}{l}[\mathbb{N}_2 + *] - \frac{\pi^2}{4l^2} \right] w(0, t) dx + \left[\int_0^l w^T(0, t) \frac{\mathbb{N}_2 - \mathbb{N}_3^T}{l} w(l, t) dx + * \right] \\ & \quad + \left[\frac{\pi^2}{4l^2} \int_0^l w^T(0, t) P_3 w(x, t) dx + * \right] + \int_0^l w^T(l, t) \left[\frac{1}{3}\mathbb{Z}_2 - \frac{1}{l}[\mathbb{N}_3 + *] \right] w(l, t) dx \\ & \quad + \left[\int_0^l w^T(l, t) \frac{\mathbb{N}_3}{l} z(s, t) dx + * \right] - \frac{\pi^2}{4l^2} \int_0^l w^T(x, t) P_3 w(x, t) dx. \end{aligned} \quad (3.14)$$

Moreover,

$$\begin{aligned} \left[\int_0^l w^T(x, t) \sum_{i=1}^k h_i P_1 B_i \mathbb{E}\{u(t)\} dx + * \right] &= \left[\int_0^l w^T(x, t) \sum_{i=1}^k h_i P_1 B_i \bar{\Gamma} D_c \varsigma(x, t) dx + * \right] \\ & \quad + \left[\int_0^l w^T(x, t) \sum_{i=1}^k h_i P_1 B_i \bar{\Gamma} D_c \hat{\vartheta}(x, t) dx + * \right] \\ & \quad + \int_0^l w^T(x, t) \sum_{i=1}^k h_i [P_1 B_i \bar{\Gamma} D_d D + *] w(x, t) dx \\ & \quad + \left[\int_0^l w^T(x, t) \sum_{i=1}^k h_i P_1 B_i \bar{\Gamma} D_d \bar{\vartheta}(x, t) dx + * \right]. \end{aligned} \quad (3.15)$$

By computing the time derivative of $\mathcal{G}_2(t)$,

$$\begin{aligned} \mathbb{E}\{\mathcal{L}\mathcal{G}_2(t)\} &= \int_0^l \varsigma^T(x, t) [P_2 D_a + *] \varsigma_t(x, t) dx \\ & \quad + \left[\int_0^l \varsigma^T(x, t) P_2 D_b D w(x, t) dx + * \right] \\ & \quad + \left[\int_0^l \varsigma^T(x, t) P_2 D_b \bar{\vartheta}(x, t) dx + * \right]. \end{aligned} \quad (3.16)$$

The adaptive laws defined in (3.7) and (2.6) yield the following mathematical relationship:

$$\begin{aligned} \left\| \frac{y(x, t)}{\hat{\mu}(x, t)} \right\| \leq R_\varrho &\Rightarrow \left\| \varrho \left(\frac{y(x, t)}{\mu(x, t)} \right) - \frac{y(x, t)}{\mu(x, t)} \right\| < E_\varrho, \\ \left\| \frac{\varsigma(x, t)}{\hat{\mu}(x, t)} \right\| \leq R_\varrho &\Rightarrow \left\| \varrho \left(\frac{\varsigma(x, t)}{\hat{\mu}(x, t)} \right) - \frac{\varsigma(x, t)}{\hat{\mu}(x, t)} \right\| < E_\varrho. \end{aligned} \quad (3.17)$$

With Lemma 1, computing the Euclidean norm with respect to $\bar{\vartheta}(x, t)$ and $\hat{\vartheta}(t)$, arrives:

$$\|\bar{\vartheta}(x, t)\| \leq \frac{2E_\varrho \rho_y}{R_\varrho} \|y(x, t)\|, \quad \|\hat{\vartheta}(x, t)\| \leq \frac{2E_\varrho \rho_\varsigma}{R_\varrho} \|\varsigma(x, t)\|. \quad (3.18)$$

The following inequalities stand for arbitrary positive scalars r_1 and r_2 : For any positive scalars r_1 and r_2 , the following relationship can be established:

For scalars $r_1 \geq 0$ and $r_2 \geq 0$, the subsequent relationship can be established:

$$\begin{aligned} \int_0^l r_1 \bar{\vartheta}^T(x, t) \bar{\vartheta}(x, t) dx &\leq \int_0^l \bar{H}_1 w^T(x, t) D^T D w(x, t) dx, \\ \int_0^l r_2 \hat{\vartheta}^T(x, t) \hat{\vartheta}(x, t) dx &\leq \int_0^l \bar{H}_2 \varsigma^T(x, t) \varsigma(x, t) dx, \end{aligned} \quad (3.19)$$

with $\bar{H}_1 = \frac{4E_\vartheta^2 \rho_x^2 r_1}{R_\vartheta^2}$ and $\bar{H}_2 = \frac{4E_\varsigma^2 \rho_\varsigma^2 r_2}{R_\varsigma^2}$. By adding and subtracting $\int_0^l r_1 \bar{\vartheta}^T(x, t) \bar{\vartheta}(x, t) dx$, $\int_0^l r_2 \hat{\vartheta}^T(x, t) \hat{\vartheta}(x, t) dx$, $\int_0^l w_x^T(x, t) P_3 w_x(x, t) dx$ to the derivative of $\mathcal{G}(t)$, one can obtain:

$$\mathbb{E}\{\mathcal{L}\mathcal{G}(t) + \delta\mathcal{G}(t) - \alpha\|\omega(t)\|^2\} \leq \int_0^l \sum_{i=1}^K h_i \hat{w}^T(x, t) F_1 \hat{w}(x, t) dx, \quad (3.20)$$

where

$$\begin{aligned} F_1 &= \begin{bmatrix} \bar{F}_{11} & \bar{F}_{12} \\ * & \bar{F}_{22} \end{bmatrix}, \quad \bar{F}_{11} = \begin{bmatrix} F_{11}^{11} & \frac{1}{l}[\mathbb{N}_2 - \mathbb{N}_3^T] & \frac{\pi^2}{4l^2}P_3 \\ * & \frac{1}{3}\mathbb{Z}_2 - \frac{1}{l}[\mathbb{N}_3 + *] & \frac{2}{l}\mathbb{N}_3 \\ * & * & \bar{\varsigma}_1 \end{bmatrix}, \\ \bar{F}_{12} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{F}_{12}^1 & P_1 B_i \bar{\Gamma} D_d & P_1 B_i \bar{\Gamma} D_c & P_1 C_i \end{bmatrix}, \quad \bar{F}_{12}^1 = P_1 B_i \bar{\Gamma} D_c + [P_2 D_b]^T, \\ F_{22} &= \begin{bmatrix} \bar{\Pi}_1 & P_2 D_b & 0 & 0 \\ * & -r_1 I & 0 & 0 \\ * & * & -r_2 I & 0 \\ * & * & * & -\frac{\alpha}{l} I \end{bmatrix}, \quad \bar{\varsigma}_1 = [P_1 A_i + P_1 B_i \bar{\Gamma} D_d D + *] + \frac{4E_\vartheta^2 \rho_x^2 r_1}{R_\vartheta^2} D^T D + \delta P_1 - \frac{\pi^2}{4l^2} P_3, \\ \bar{\Pi}_1 &= [P_2 D_a + *] + \delta P_2 + \frac{4E_\varsigma^2 \rho_\varsigma^2 r_2}{R_\varsigma^2} I, \quad \hat{w}(x, t) = \text{col}\{w(0, t), w(l, t), w(x, t), \varsigma(x, t), \bar{\vartheta}(x, t) \hat{\vartheta}(x, t), \omega(t)\}. \end{aligned}$$

Let $D_c = U_1^{-1} V_1$, $D_d = U_2^{-1} V_2$, which means $P_1 B_i \bar{\Gamma} D_c = [P_1 B_i \bar{\Gamma} - \mathcal{X} U_1] U_1^{-1} V_1 + \mathcal{X} V_1$, $P_1 B_i \bar{\Gamma} D_d = [P_1 B_i \bar{\Gamma} - \mathcal{Y} U_2] U_2^{-1} V_2 + \mathcal{Y} V_2$, for any given $\mathcal{X}$ and $\mathcal{Y}$. Applying the Schur complement lemma, F_1 admits the following representation:

$$\bar{F}_1 + [\Lambda_{11} \text{diag}\{U_1^{-1} V_1, U_2^{-1} V_2\} \Lambda_{12} + *], \quad (3.21)$$

where

$$\begin{aligned} \bar{F}_1 &= \begin{bmatrix} \hat{F}_{11} & F_{12} \\ * & F_{22} \end{bmatrix}, \quad \hat{F}_{11} = \begin{bmatrix} F_{11}^{11} & \frac{1}{l}[\mathbb{N}_2 - \mathbb{N}_3^T] & \frac{\pi^2}{4l^2}P_3 \\ * & \frac{1}{3}\mathbb{Z}_2 - \frac{1}{l}[\mathbb{N}_3 + *] & \frac{2}{l}\mathbb{N}_3 \\ * & * & \hat{\varsigma}_1 \end{bmatrix}, \\ \hat{\varsigma}_1 &= [P_1 A_i + b \mathcal{Y} V_2 + *] + \frac{4E_\vartheta^2 \rho_x^2 r_1}{R_\vartheta^2} I + \delta P_1 - \frac{\pi^2}{4l^2} P_3, \\ \Lambda_{11}^T &= \begin{bmatrix} 0 & 0 & [P_1 B_i \bar{\Gamma} - \mathcal{X} U_1]^T & 0 & 0 & 0 & 0 \\ 0 & 0 & [P_1 B_i \bar{\Gamma} - \mathcal{Y} U_2]^T & 0 & 0 & 0 & 0 \end{bmatrix}^T, \quad \Lambda_{12} = \begin{bmatrix} 0 & 0 & 0 & I & 0 & I & 0 \\ 0 & 0 & D & 0 & I & 0 & 0 \end{bmatrix}. \end{aligned}$$

If $F_1 < 0$, the condition described in Eq (3.5) is deduced using the Schur complement, signifying:

$$\mathbb{E}\{\mathcal{L}\mathcal{G}(t) + \delta\mathcal{G}(t) - \alpha\|\omega(t)\|^2\} < 0. \quad (3.22)$$

Integrating Eq (3.22) over the interval $[0, \mathcal{T}]$ and applying the initial condition leads to:

$$e^{\delta\mathcal{T}} \mathbb{E}\mathcal{G}(\mathcal{T}) - \mathcal{G}(0) < \int_0^{\mathcal{T}} e^{\delta t} \alpha \|\omega(t)\|^2 dt. \quad (3.23)$$

Furthermore, inequality (3.23) provides:

$$\begin{aligned} \mathbb{E}\{\mathcal{G}(\mathcal{T})\} &< e^{-\delta\mathcal{T}} [\mathcal{G}(0) + \int_0^{\mathcal{T}} e^{\delta t} \alpha \|\omega(t)\|^2 dt] \\ &< e^{-\delta\mathcal{T}} \left[\int_{\Omega} w^T(x, 0) P_1 w(x, 0) dx + \int_0^{\mathcal{T}} \varsigma^T(x, 0) P_2 \varsigma(x, 0) dx + e^{-\delta\mathcal{T}} \bar{\omega} \alpha \right] \\ &< e^{-\delta\mathcal{T}} (\bar{\lambda}_1 + \bar{\lambda}_2) c_1 + \bar{\omega} \alpha. \end{aligned} \quad (3.24)$$

Conversely,

$$\begin{aligned} \mathbb{E}\{\mathcal{G}(\mathcal{T})\} &= \int_{\Omega} w^T(x, \mathcal{T}) R^{\frac{1}{2}} (R^{-\frac{1}{2}} P_1 R^{-\frac{1}{2}}) R^{\frac{1}{2}} w(x, \mathcal{T}) dx \\ &> \underline{\lambda}_1 \int_{\Omega} w^T(x, \mathcal{T}) R w(x, \mathcal{T}) dx. \end{aligned} \quad (3.25)$$

Thus, $\forall t \in [0, \mathcal{T}]$,

$$\int_{\Omega} w^T(x, t) R w(x, t) dx < \frac{e^{-\delta\mathcal{T}} (\bar{\lambda}_1 + \bar{\lambda}_2) c_1 + \bar{\omega} \alpha}{\underline{\lambda}_1}. \quad (3.26)$$

Using Eq (3.6), it can be deduced that $c_2 < \int_{\Omega} w^T(x, t) R w(x, t) dx$.

Consequently, the FTB is assured through the utilization of the dynamic output feedback controller (3.1). This completes the proof. $\square$

Remark 2. The parameters $\bar{\mu}(x, t)$ and $\hat{\mu}(x, t)$ are dynamically updated following the adaptation law proposed in [18] as: Let $\mathbb{K}_1 = \bar{\mu}(x, t)$, $\mathbb{K}_2 = \hat{\mu}(x, t)$, $\mathbb{L}_1 = y(x, t)$, $\mathbb{L}_2 = \varsigma(x, t)$,

$$\mathbb{K}_p = \begin{cases} \text{floor}\left(\frac{2\rho_y}{R_q} |\mathbb{L}_p(x, t)| \times 10^\ell\right) \times 10^{-\ell}, & 0 \leq \frac{\rho_y}{R_q} |\mathbb{L}_p(x, t)| < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq \frac{\rho_y}{R_q} |\mathbb{L}_p(x, t)| < 1, \\ \text{floor}\left(\frac{2\rho_y}{R_q} |\mathbb{L}_p(x, t)|\right), & \frac{\rho_y}{R_q} |\mathbb{L}_p(x, t)| > 1, \end{cases}$$

where the $\text{floor}(\zeta)$ means the largest integer of ζ and satisfies $\text{floor}(\zeta) \leq \zeta$. $\ell = \min\{\ell \in \mathcal{N}^+ | \frac{2\rho_y}{R_q} |\mathbb{L}_p(x, t)| \times 10^\ell > 1\}$. When the adjusting parameter $\bar{\mu}(x, t)$ and $\hat{\mu}(x, t)$ are small enough, in practice we consider them as 0 and do not process them through the quantizer.

4. Simulations study

In this section, a numerical example is presented to illustrate the effectiveness and superiority of the proposed control design method. For system (2.1), considering its two-dimensional dynamic model, the system parameters are selected as follows:

$$\mathcal{A}(w(x, t)) = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.1 - 0.1 \sin(w_1(x, t)) \end{bmatrix}, \quad B(x) = \begin{bmatrix} -0.33 & 0 \\ 0 & -0.78 \end{bmatrix}, \quad C(x) = \begin{bmatrix} 0.92 & 1.3 \\ 0.77 & 0.52 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.078 & 0.16 \\ -0.22 & -0.12 \end{bmatrix}, \quad f(w(x, t), x) = \begin{bmatrix} -0.34w_1(x, t) + 0.78w_2(x, t) \\ -1.82w_1(x, t) - w_2^3(x, t) + 0.33w_2(x, t) \end{bmatrix}, \quad l = 1.$$

with:

$$w_x(0, t) = w_x(l, t) = 0, \quad w_0(x) = \begin{bmatrix} 0.14 - 0.24 \cos(2\pi x) \\ 0.22 + 0.12 \cos(2\pi x) \end{bmatrix}.$$

The states of the open-loop system are subsequently determined and illustrated in Figures 1 and 2 with $\omega(t) = \begin{bmatrix} 0.03e^{-0.4t} + 0.02e^{-0.4t} \cos(2\pi t) \\ 0.02e^{-0.4t} + 0.01e^{-0.4t} \sin(2\pi t) \end{bmatrix}$ and $\bar{\omega} = 1$. The dynamic instability of the parabolic PDE, as characterized by Eqs (2.1) to (2.2), is clearly demonstrated. Based on Figures 1 and 2, a range of the system state can be characterized: $\mathcal{D}^1 = [\mathfrak{I}_{min}^1, \mathfrak{I}_{max}^1]$ with $\mathfrak{I}_{min}^1 \leq 0$ and $\mathfrak{I}_{max}^1 \geq 2.5$. Select the premise variable as $\varkappa^1(x, t) = w_1^2(x, t)$, $\varkappa^2(x, t) = \sin(w_1(x, t))$, assume

$$\begin{aligned} \varkappa^1(x, t) &= h_1^1(\varkappa(x, t))\mathfrak{I}_{max}^1 + h_2^1(\varkappa(x, t))\mathfrak{I}_{min}^1, \\ \varkappa^2(x, t) &= h_1^2(\varkappa(x, t))\mathfrak{I}_{max}^2 + h_2^2(\varkappa(x, t))\mathfrak{I}_{min}^2, \end{aligned}$$

then we have:

$$h_1^1(\varkappa^1(x, t)) = \frac{\varkappa^1(x, t) - \mathfrak{I}_{min}^1}{\mathfrak{I}_{max}^1 - \mathfrak{I}_{min}^1}, \quad h_2^1(\varkappa^1(x, t)) = \frac{\mathfrak{I}_{max}^1 - \varkappa^1(x, t)}{\mathfrak{I}_{max}^1 - \mathfrak{I}_{min}^1}, \quad h_1^2(\varkappa^2(x, t)) = \frac{\varkappa^2(x, t) - \mathfrak{I}_{min}^2}{\mathfrak{I}_{max}^2 - \mathfrak{I}_{min}^2}, \quad h_2^2(\varkappa^2(x, t)) = \frac{\mathfrak{I}_{max}^2 - \varkappa^2(x, t)}{\mathfrak{I}_{max}^2 - \mathfrak{I}_{min}^2}.$$

The nonlinear expression $\mathcal{A}$ and f are characterized as:

$$\mathcal{A}(w(x, t)) = \sum_{i=1}^4 h_i \mathcal{A}_i, \quad f(w(x, t)) = \sum_{i=1}^4 h_i A_i w(x, t),$$

where

$$\begin{aligned} \mathcal{A}_1 &= \mathcal{A}_3 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.1 - 0.1 \mathfrak{I}_{max}^2 \end{bmatrix}, \quad \mathcal{A}_2 = \mathcal{A}_4 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.1 - 0.1 \mathfrak{I}_{min}^2 \end{bmatrix}, \\ A_1 &= A_2 = \begin{bmatrix} -0.34 & 0.78 \\ -1.82 & 0.33 - \mathfrak{I}_{max}^1 \end{bmatrix}, \quad A_3 = A_4 = \begin{bmatrix} -0.34 & 0.78 \\ -1.82 & 0.33 - \mathfrak{I}_{min}^1 \end{bmatrix}, \end{aligned}$$

$$h_i(\varkappa(x, t)) = h_i^1(\varkappa^1(x, t))h_i^2(\varkappa^2(x, t)), \quad \text{with } i = \{1, \dots, 4\}, \quad \text{and } \bar{i}, \hat{i} = 1, 2.$$

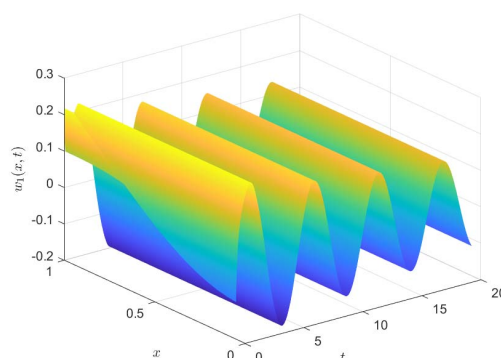


Figure 1. Open-loop state distribution $w_1(x, t)$.

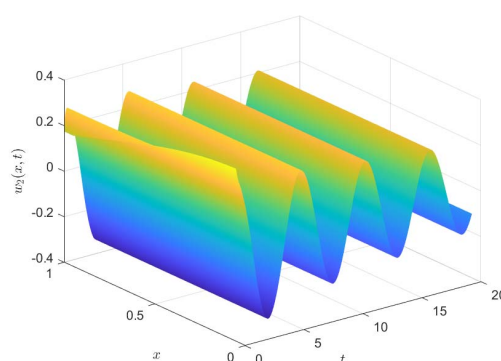


Figure 2. Open-loop state distribution $w_2(x, t)$.

The quantization parameters are selected according to practical implementation requirements. The error bound E_ϱ depends on bandwidth limitations of the communication channel, whereas the range R_ϱ is governed by the input signal magnitude. For this study, we employ $E_\varrho = 0.05$ and $R_\varrho = 120$, with complete specifications detailed below: $\bar{\varepsilon} = 4.7$, $R = 2I$, $c_1 = 0.59$, $c_2 = 190$, $\mathcal{X} = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.6 \end{bmatrix}$, $\mathcal{Y} = 1.2I$, $\mathcal{T} = 5$, and $\delta = 0.004$. The prescript performance is chosen as $\alpha = 0.2$. Figure 3 partially depicts the Bernoulli stochastic process, which exhibits a mean value of $\bar{\Gamma} = 0.5$. By solving conditions in Theorem 1, we derive the following design parameters:

$$\begin{aligned}
 D_a &= \begin{bmatrix} -1.0791 & 0.0061 \\ 0.0061 & -1.1121 \end{bmatrix}, D_b = \begin{bmatrix} 0.0362 & 0.0916 \\ -0.0091 & 0.0106 \end{bmatrix}, D_c = \begin{bmatrix} 27.3180 & -30.6727 \\ -31.6077 & 193.0735 \end{bmatrix} \times 10^4, \\
 D_d &= \begin{bmatrix} -87.4969 & -5.8937 \\ 20.1672 & -49.2785 \end{bmatrix}, P_1 = \begin{bmatrix} 189.8520 & 0.8850 \\ 0.8850 & 112.0809 \end{bmatrix}, P_2 = \begin{bmatrix} 1.0635 & 0.0000 \\ 0.0000 & 1.0635 \end{bmatrix} \times 10^6, \\
 \mathbb{Z}_1 &= \begin{bmatrix} 518.4826 & 150.6029 \\ 150.6029 & 119.5983 \end{bmatrix}, \mathbb{Z}_2 = \begin{bmatrix} 1.0554 & 0.2931 \\ 0.2931 & 0.2641 \end{bmatrix} \times 10^3, \mathbb{N}_1 = \begin{bmatrix} 373.7564 & 126.5726 \\ 128.3374 & 61.5812 \end{bmatrix}, \\
 \mathbb{N}_2 &= \begin{bmatrix} 243.4343 & 75.2094 \\ 76.1758 & 49.3257 \end{bmatrix}, \mathbb{N}_3 = \begin{bmatrix} 339.9622 & 77.8044 \\ 76.9094 & 107.4104 \end{bmatrix}, \rho_y = 1.4969, \rho_\varsigma = 1.9317.
 \end{aligned}$$

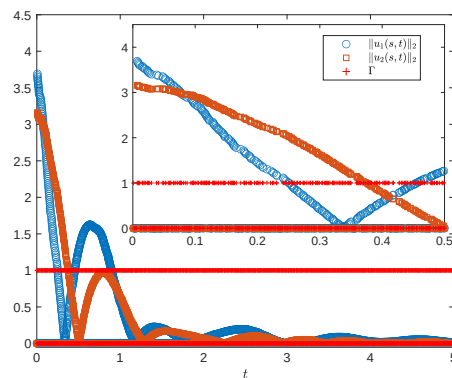


Figure 3. Control input $\|u_i(s, t)\|_2$ ($i = 1, 2$) and stochastic process Γ .

The dynamic behavior of system states under controller (3.1) is presented in Figures 4 and 5. Figure 6 shows the evolution of the norms $\|w_i(x, t)\|_2$ ($i = 1, 2$), while corresponding control input norms $\|u_i(x, t)\|_2$ ($i = 1, 2$) are demonstrated in Figure 3.

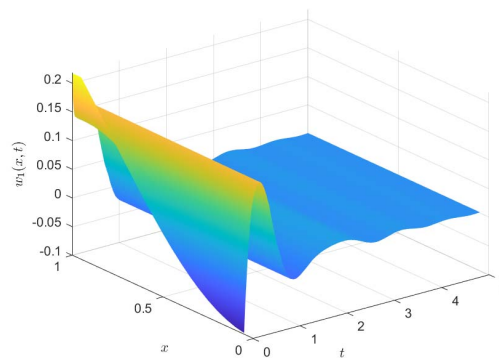


Figure 4. Close-loop state distribution $w_1(x, t)$.

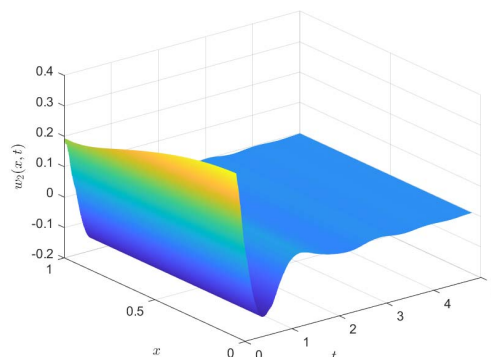


Figure 5. Close-loop state distribution $w_2(x, t)$.

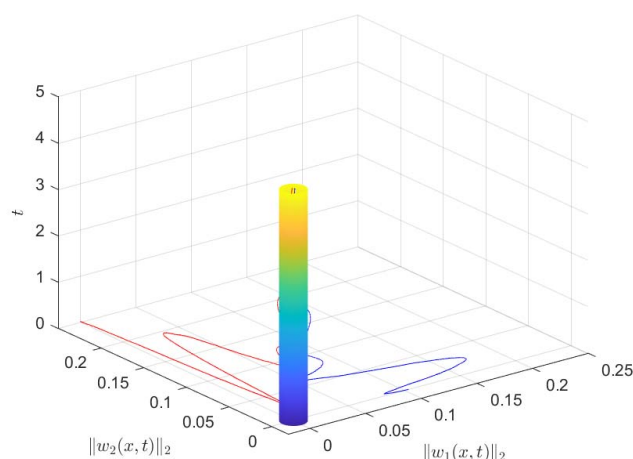


Figure 6. Close-loop state norm distribution $\|w_i(x, t)\|_2$ ($i = 1, 2$).

Through numerical simulations, the effectiveness of the developed control approach in achieving extended FTB for parabolic PDE systems has been confirmed. These experimental results additionally demonstrate the controller's strong robustness.

Comparative explanations: Fuzzy finite-time boundedness for a class of nonlinear parabolic PDEs under dynamic quantization in the presence of network attacks. Sufficient conditions for the controller synthesis are derived via several key inequalities. The comparison with some existing results is summarized as follows:

i) Compared with [3, 25], communication channels are used here for output transmission, so that the control method can be implemented more simply, directly, and cheaply. Meanwhile, a dynamic quantizer is developed to reduce the transmitted data without sacrificing performance, which gives it an advantage over the static quantizer adopted in [16, 17]. The superiority of the dynamic quantized control over the static one is further confirmed by the simulation results shown in Figures 7 and 8.

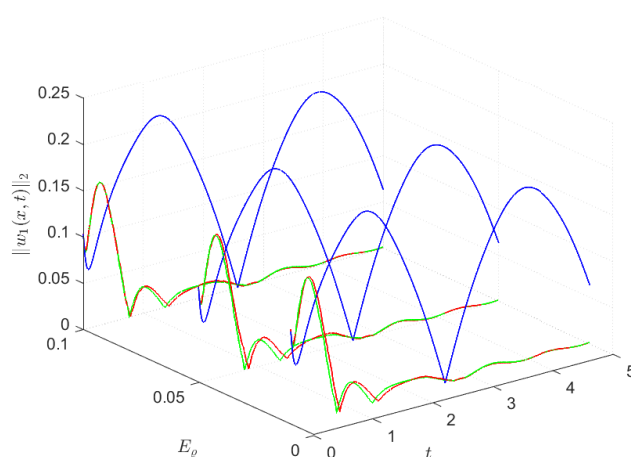


Figure 7. $\|w_1(x, t)\|_2$ with different E_q .

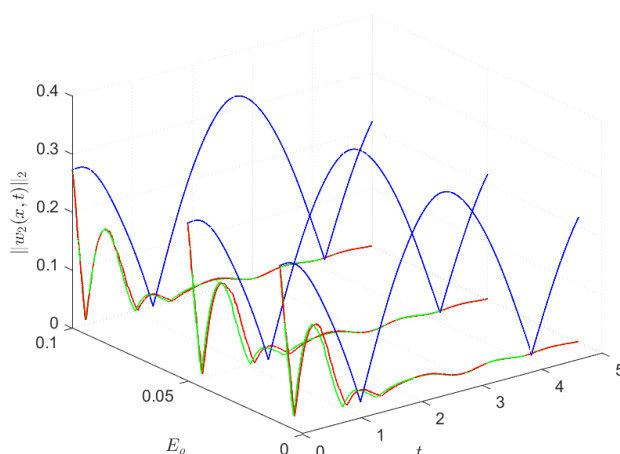


Figure 8. $\|w_2(x, t)\|_2$ with different E_q .

ii) In contrast to the dissipative control approaches outlined in [4, 26] for nonlinear parabolic PDEs, frequent network attacks are accounted for in the present work, and a quantized dynamic output feedback law is constructed in response. Sufficient conditions for finite-time boundedness are subsequently derived.

iii) To alleviate the conservatism of the established criteria, arbitrary matrices $\mathcal{X}$ and $\mathcal{Y}$, together with free matrices $\mathbb{N}_1, \mathbb{N}_2$, and $\mathbb{N}_3$, are introduced in this study. These are employed to ensure finite-time boundedness of the nonlinear parabolic PDEs. The resulting design can be reduced to the decoupling approach in [15] by setting $\mathcal{X} = \mathcal{Y} = I$.

5. Conclusions

The research has focused on the fuzzy boundedness analysis of nonlinear parabolic PDEs via the T-S fuzzy model. The output signals and state signals of the dynamic controller have been quantized using the dynamic quantizer, which not only achieves the minimal data transmission volume but also ensures system performance. Considering potential network attacks on the actuator side, dynamic output feedback controllers with quantization have been specifically designed to address this concern. Subsequently, sufficient criteria for the finite-time boundedness under the designed controllers have been established based on the free matrix inequality. The efficacy of the developed control approach has been verified by numerical simulations. By the definition of FTB, the smaller the value of c_2 is, the better the performance of the system at the terminal time $\mathcal{T}$ will be. However, according to the inequality in Theorem 1, when the value of c_2 is chosen too small, the inequality (3.6) becomes increasingly difficult to satisfy. How to determine the optimal value of c_2 to pursue a better bound of the system state and how to simplify the adjusting parameter μ of the dynamic quantizers are the remaining problems faced in practical applications. Future research will contain these problems.

Author contributions

T. Li and L. Ding: Conceptualization, writing-review and editing; J. Chen and T. Li: Methodology,

writing-original draft; J. Chen, J. Xue, and L. Ding: Software. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that no Generative AI tools were used in the preparation of this manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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