
*Research article***Spectral time-fractional nonlinear Schrödinger equation: Formal Hirota framework, multi-soliton templates, and breather diagnostics****Alvaro H. Salas^{1,*}, Weaam Alhejaili² and Samir A. El-Tantawy^{3,4}**

¹ Department of Mathematics and Statistics, Universidad Nacional de Colombia, FIZMAKO Research Group, Colombia

² Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

³ Department of Physics, Faculty of Science, Al-Baha University, P.O. Box 1988, Al-Baha 65779, Saudi Arabia

⁴ Department of Physics, Faculty of Science, Port Said University, Port Said 42521, Egypt

* **Correspondence:** Email: ahsalass@unal.edu.co.

Abstract: In this work, we investigated a spectral time-fractional deformation of the focusing nonlinear Schrödinger equation in which the first-order time derivative was replaced by a translation-invariant Fourier-multiplier operator of order $0 < \alpha \leq 1$. The analysis was deliberately formulated as a formal Hirota-type construction: For $\alpha < 1$, the fractional operator did not obey the classical Leibniz and quotient rules required for an exact Hirota bilinearization, and the tau functions obtained in this work were therefore not asserted to be exact closed-form solutions of the fractional equation. Instead, the spectral action of D_t^α on exponential modes was used to preserve the classical mode-by-mode tau-function algebra while deforming the dispersion relation. This led to explicit one-, two-, and N -phase tau-function templates and to fractional Akhmediev-, Kuznetsov-Ma-, and Peregrine-type breather ansatz. The breather deformation was tied to the modulational-instability growth law through an explicitly stated reference wavenumber K , with the natural Akhmediev choice $K_a = 2\sqrt{1-2a}$ and the long-wave reference value $K_{\text{ref}} = 0.1$ used in the limiting numerical tests. The accuracy of the resulting profiles was assessed a posteriori by Fourier-spectral residuals and by scaled residual criteria that separate quantitative approximation from qualitative morphology preservation. New central time-section diagnostics showed how the fractional order modified the focusing and recurrence scales, even when two-dimensional density plots looked similar. The results indicate that the proposed profiles are useful analytical benchmarks for weak-to-moderate spectral time-fractional deformation, while the Kuznetsov-Ma case should be regarded primarily as a qualitative recurrent breather template because of its larger relative residuals.

Keywords: nonlinear Schrödinger equation; spectral fractional derivative; formal Hirota framework;

tau-function templates; modulational instability; breathers; rogue waves; residual diagnostics

Mathematics Subject Classification: 35C08, 35Q55, 35R11, 65M70

1. Introduction

Fractional calculus has become an important framework for modeling memory, hereditary effects, and anomalous transport in complex media. In particular, the introduction of noninteger-order operators enables one to incorporate nonlocal temporal response while preserving the main dispersive or diffusive structure of the underlying evolution law. Foundational contributions by Caputo [1, 2] and by Oldham and Spanier [3] helped establish the subject as a natural extension of classical differential modeling, and the theory has since developed into a broad mathematical and applied discipline, covered both in treatments addressed to physical applications [4–6] and in systematic accounts of fractional differential equations [7, 8].

Among the most important nonlinear wave equations in mathematical physics is the focusing nonlinear Schrödinger equation (NLSE),

$$i u_t + u_{xx} + 2|u|^2 u = 0, \quad (1.1)$$

or the general form with general forms for the coefficients of the dispersion and nonlinear terms

$$i u_t + \frac{P}{2} u_{xx} + Q|u|^2 u = 0, \quad (1.2)$$

where the dispersion coefficient and the nonlinearity coefficient contain the physical parameters relevant to the system under study, whether it is a plasma model or other different systems described by this equation. This equation and its family describe the balance between linear dispersion and cubic self-focusing nonlinearity in nonlinear optics, hydrodynamics, plasma physics, and Bose-Einstein condensates [9, 10]. Here, $u(x, t)$ denotes the complex wave envelope, the term u_{xx} represents the dispersive contribution, and the cubic term $2|u|^2 u$ models an intensity-dependent nonlinear response. The focusing NLSE (1.1) is also the canonical model for modulational instability (MI), through which a weakly perturbed continuous-wave background may evolve into highly localized coherent structures, including Akhmediev breathers (ABs) [10, 11], Kuznetsov-Ma breathers (KMBs), and Peregrine/rogue waves (RWs)/freak waves (FWs) [12, 13]. Among the coherent structures supported by the focusing cubic NLSE on a finite-amplitude background, three families play a central role: ABs, KMBs, and RWs. In the standard setting, one considers a continuous-wave background u_{bg} with real constant amplitude $\rho > 0$, and studies localized modulations of this carrier. ABs correspond to space-periodic, time-localized perturbations of u_{bg} ; they grow from a weakly modulated continuous wave due to MI, reach a maximal focusing, and then decay back to the background while remaining periodic in space and localized in time [10–12]. KMBs are localized in space and periodic in time, describing recurrent temporal focusing-defocusing cycles on top of a spatially localized envelope [14, 15]. The Peregrine (RWs) solution is a doubly localized rational solution of the NLSE, obtained as a limiting case of these breather families, which is localized in space and time and is commonly viewed as a canonical rogue wave (RW) prototype [12, 16]. These breathers and RW solutions have been identified in a wide variety of physical systems where weak dispersion and cubic self-focusing compete. In

nonlinear fiber optics, Akhmediev-type breathers and Peregrine solitons (RWs) have been generated in anomalous-dispersion fibers as the nonlinear stage of MI on a continuous-wave background [16, 17]. In hydrodynamics, controlled water-tank experiments have reproduced the Peregrine breather and higher-order super RWs on finite-depth and deep-water wave trains, providing quantitative laboratory analogues of oceanic FWs [18–20]. Similar structures have been reported in optical systems with Kerr nonlinearity and in other nonlinear dispersive media, illustrating the robustness of the NLSE breather hierarchy as a mechanism for extreme events [12, 16].

Plasma physics offers another natural environment for such dynamics, because many low-frequency electrostatic modes behave as weakly nonlinear, weakly dispersive waves whose envelopes are governed by NLSE-type equations. In multicomponent laboratory plasmas with negative ions, ion-acoustic waves (IAWs) at a critical negative-ion concentration satisfy an NLSE with dispersion and nonlinear coefficients determined by the plasma composition [21, 22]. Under that condition, the Lighthill criterion $PQ > 0$ is fulfilled for a broad wavenumber range, so that ion-acoustic wave (IAW) packets become modulationally unstable and evolve into strongly localized envelope structures [22, 23]. Experiments in double-plasma devices have demonstrated fundamental ion-acoustic Peregrine solitons (RWs) [24], hole-type Peregrine structures [25], second-order RWs with peak amplitudes about five times the carrier level [26], and multi-Peregrine sequences that undergo fission and spectral broadening [27, 28]. These observations are quantitatively described by the rational and breather solutions of the NLSE derived from fluid models, including Boltzmann electrons, positive ions, and negative ions [27]. In parallel, theoretical studies within the NLSE framework have addressed various types of acoustic waves in several plasma models, such as dust-acoustic and dust ion-acoustic RWs [29, 30], and magnetosonic rogons in parameter regimes relevant to space and laboratory plasmas [31, 32], emphasizing the ubiquity of breather-driven extreme events in plasmas. Several researchers have examined how plasma composition [33, 34] and (non)-Maxwellian features [35, 36] influence MI and the associated breather families in more detail. Taken together, these results reinforce the view that NLSE-type breathers provide a robust, physically meaningful description of extreme events not only in fluids and optics but also in a broad class of plasma models [12, 16].

In all the examples mentioned above, the evolution law is an integer-order NLSE, in which the first-order time derivative represents a local in time response. Many complex media, however, exhibit memory effects and anomalous transport, so that their macroscopic dynamics depend on the past history of the field rather than only on its instantaneous value [5, 6]. In disordered or structured optical materials, in soft matter systems, and in strongly coupled or viscoelastic plasmas, relaxation processes and trapping lead to broad distributions of time scales and non-exponential response kernels that cannot be represented adequately by a purely local time derivative [37, 38]. Fractional-order time derivatives provide a phenomenological way to account for such temporal nonlocality while preserving the spatial dispersion and cubic self-focusing mechanisms that underlie the NLSE [7, 8]. In a spectral formulation, the time-fractional operator is defined via its Fourier multiplier [39, 40], so that its effect can be interpreted as a deformation of the linear dispersion relation and, consequently, of the MI growth rates and breather time scales [41, 42].

From the viewpoint of MI and breather dynamics, passing from the classical NLSE to a spectral time-fractional NLSE raises several natural questions

$$i D_t^\alpha u + \frac{P}{2} u_{xx} + Q|u|^2 u = 0, \quad (1.3)$$

where D_t^α denotes the spectral fractional time derivative of order $0 < \alpha \leq 1$. One would like to understand how temporal memory modifies the MI dispersion curve, in particular, the width of the unstable band and the maximum growth rate, and hence the time scale over which ABs, KMBs, and RWs emerge from a perturbed continuous wave [43, 44]. It is also important to determine to what extent the canonical spatial morphologies of ABs, KMBs, and RWs persist when a nonlocal fractional law governs the time evolution, and how the fractionality parameter affects their amplitude, recurrence, and effective lifetime [43, 44]. Finally, for applications, it is desirable to construct analytically tractable breather-type profiles that approximate the true fractional dynamics with controlled accuracy, so that the impact of temporal memory on extreme-event statistics can be quantified in a manner explicitly connected to the classical integrable structures of the NLSE [39, 40].

The purpose of this work is to investigate this idea for a spectral time-fractional deformation of the focusing NLSE. More precisely, we consider a model in which the standard time derivative is replaced by a translation-invariant fractional operator of order $0 < \alpha \leq 1$. Our goal is not to claim an exact classical Hirota bilinearization based on ordinary product and quotient rules, but rather to develop a formal Hirota-type framework adapted to the Fourier-symbol action of the fractional operator. Within this setting, we examine how the exponential-mode algebra and the associated dispersion relation are modified when $\alpha \neq 1$, and we use this information to construct soliton-type tau-function candidates and fractional breather ansatz. The main contribution of this work is therefore twofold. First, we show that the spectral fractional time operator admits a natural Hirota-type treatment at the level of exponential modes, leading to a modified dispersion law and a formal tau-function structure closely connected to the classical one. Second, we use this structure to derive approximate fractional Akhmediev-type breather profiles and assess their quality through residual diagnostics computed with a Fourier-spectral implementation of the fractional derivative. In this way, the paper provides a practical bridge between explicit analytical constructions and numerical validation in a nonlocal-in-time NLSE setting.

2. Preliminaries

In this section, we introduce the notation and the spectral fractional operators used throughout the paper. Since the main constructions in the sequel rely on Fourier multiplier calculus and on the action of the operators on exponential modes, we formulate the presentation in that setting from the outset. In particular, the fractional Hirota operator employed later is understood in a formal spectral sense, adapted to exponential-mode calculations, rather than as a consequence of classical product or quotient rules.

2.1. Fourier transform and basic notation

Here, we work on the real line, with variables $(x, t) \in \mathbb{R}^2$. For a sufficiently regular function f , its Fourier transform reads

$$\widehat{f}(k) = \mathcal{F}[f](k) := \int_{\mathbb{R}} e^{-ik\xi} f(\xi) d\xi, \quad k \in \mathbb{R}, \quad (2.1)$$

with inverse transform

$$f(\xi) = \mathcal{F}^{-1}[\widehat{f}](\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ik\xi} \widehat{f}(k) dk, \quad \xi \in \mathbb{R}. \quad (2.2)$$

Here, ξ denotes either x or t , depending on the context. With this convention, the classical derivative is represented in Fourier space by multiplication:

$$\mathcal{F}[\partial_\xi f](k) = ik \widehat{f}(k), \quad \mathcal{F}[\partial_\xi^n f](k) = (ik)^n \widehat{f}(k), \quad n \in \mathbb{N}. \quad (2.3)$$

This observation is the starting point for the spectral definition of fractional derivatives [39].

2.2. Spectral fractional derivative

The spectral fractional derivative of order α ($0 < \alpha \leq 1$) in the variable ξ is defined through the Fourier multiplier

$$\mathcal{F}[D_\xi^\alpha f](k) = (ik)^\alpha \widehat{f}(k), \quad k \in \mathbb{R}, \quad (2.4)$$

where $(ik)^\alpha$ is taken in the principal branch of the complex power function [39, 40].

For $k \neq 0$, we have

$$(ik)^\alpha = |k|^\alpha \exp\left(i \frac{\pi\alpha}{2} \operatorname{sgn}(k)\right). \quad (2.5)$$

When $\alpha = 1$, the multiplier reduces to ik , and therefore D_ξ^α recovers the usual first-order derivative.

Since the operator is defined through a Fourier symbol, it is linear and translation invariant. Indeed, if $(\tau_y f)(\xi) := f(\xi - y)$, then

$$D_\xi^\alpha(\tau_y f) = \tau_y(D_\xi^\alpha f), \quad y \in \mathbb{R}. \quad (2.6)$$

This property is one of the reasons why the spectral formulation is convenient for plane-wave calculations and dispersion analysis.

For the exponential modes that appear in the Hirota construction below, we use the formal identity

$$D_\xi^\alpha(e^{\lambda\xi+\delta}) = \lambda^\alpha e^{\lambda\xi+\delta}, \quad (2.7)$$

where the same branch convention is used for λ^α . This exponential-mode rule is the key ingredient in the spectral calculations that follow [39, 40].

2.3. Action in time and space

In this work, the fractional deformation is applied only in time. Thus, for a sufficiently regular function $u(x, t)$, we have

$$\mathcal{F}_t[D_t^\alpha u(x, \cdot)](\omega) = (i\omega)^\alpha \widehat{u}(x, \omega), \quad \omega \in \mathbb{R}, \quad (2.8)$$

where $\mathcal{F}_t$ denotes the Fourier transform with respect to t . Spatial derivatives remain of integer order, so that

$$\mathcal{F}_x[\partial_x^n u(\cdot, t)](k) = (ik)^n \widehat{u}(k, t), \quad n \in \mathbb{N}. \quad (2.9)$$

In particular, for the exponential wave

$$u(x, t) = e^{i(kx - \omega t) + \delta}, \quad (2.10)$$

we obtain

$$D_t^\alpha u(x, t) = (-i\omega)^\alpha u(x, t), \quad \partial_x^n u(x, t) = (ik)^n u(x, t), \quad n \in \mathbb{N}. \quad (2.11)$$

These relations will be used repeatedly in the derivation of dispersion laws and in the formal bilinear calculations.

2.4. Classical Hirota operators

For reference, we recall the classical Hirota bilinear differential operators [45]. Given functions $f(x, t)$ and $g(x, t)$, one defines

$$D_x^m D_t^n f \cdot g := (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x, t) g(x', t')|_{x'=x, t'=t}, \quad m, n \in \mathbb{N}. \quad (2.12)$$

In particular, we have

$$\begin{cases} D_x f \cdot g = f_x g - f g_x, \\ D_t f \cdot g = f_t g - f g_t, \end{cases} \quad (2.13)$$

and

$$D_x^2 f \cdot g = f_{xx} g - 2f_x g_x + f g_{xx}. \quad (2.14)$$

A basic property of the classical Hirota operators is their action on exponential functions. Then, by considering the phase functions

$$\theta_j = k_j x - \omega_j t + \delta_j, \quad j = 1, 2, \quad (2.15)$$

we have

$$D_x^m D_t^n e^{\theta_1} \cdot e^{\theta_2} = (k_1 - k_2)^m (-\omega_1 + \omega_2)^n e^{\theta_1 + \theta_2}, \quad m, n \in \mathbb{N}. \quad (2.16)$$

This exponential rule is the algebraic basis of the standard tau-function construction.

2.5. A spectral fractional Hirota operator

To adapt Hirota's exponential bookkeeping to the spectral time-fractional setting, we use the two-factor commutator

$$\mathcal{D}_t^\alpha(f, g) := (D_t^\alpha f)g - f(D_t^\alpha g), \quad (2.17)$$

where D_t^α is the spectral fractional derivative defined above. For $\alpha = 1$, this expression reduces to the classical Hirota time operator $D_t f \cdot g$. For $0 < \alpha < 1$, however, it should not be confused with an exact Hirota derivative in the classical integrable-systems sense. In particular, the Fourier-multiplier operator D_t^α does not satisfy the ordinary Leibniz rule or the quotient rule that would be needed to convert the fractional NLSE exactly into a closed bilinear tau system.

The reason the construction is still useful is more modest and purely spectral. From this point on, and throughout Sections 3 and 4, the exponential phases are written in the physical form already used in Eq (2.11), that is, with the imaginary unit included,

$$\theta_j = i(k_j x - \omega_j t) + \delta_j, \quad j = 1, 2. \quad (2.18)$$

With this convention the temporal Fourier symbol of D_t^α acting on e^{θ_j} is $(-i\omega_j)^\alpha$, and the classical spatial Hirota rule (2.16) carries the corresponding factor $i^2 = -1$, so that

$$D_x^2 e^{\theta_1} \cdot e^{\theta_2} = -(k_1 - k_2)^2 e^{\theta_1 + \theta_2}. \quad (2.19)$$

Accordingly, one has the algebraic rule

$$\mathcal{D}_t^\alpha(e^{\theta_1}, e^{\theta_2}) = [(-i\omega_1)^\alpha - (-i\omega_2)^\alpha] e^{\theta_1 + \theta_2}. \quad (2.20)$$

Thus, the interaction of two exponential modes is still represented by a simple frequency factor. This exponential-mode identity, and not any unproved fractional product rule, is the basis of the formal tau-function calculations used below.

3. Spectral time-fractional NLSE and formal bilinear framework

In this section, we introduce the spectral time-fractional NLSE studied in this work and formulate the Hirota-type framework used in the subsequent analysis. In view of the well-known fact that fractional Fourier multiplier operators do not, in general, satisfy the classical Leibniz and quotient rules in the same form as integer-order derivatives, we do not present the fractional bilinear structure below as an exact classical bilinearization. Instead, we adopt a formal spectral Hirota framework guided by the exponential-mode action of the spectral fractional derivative, the associated dispersion relation, and the requirement that the construction recover the standard Hirota system when $\alpha = 1$. This formulation is sufficient for the formal tau-function calculations and for the residual-based validation developed later.

3.1. Spectral time-fractional NLSE

Let us introduce the classical focusing cubic NLSE

$$i u_t + u_{xx} + 2|u|^2 u = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}, \quad (3.1)$$

where $u \equiv u(x, t)$ denotes the complex wave envelope [9, 46]. The linear part of focusing cubic NLSE (3.1),

$$i u_t + u_{xx} = 0,$$

admits the following plane-wave dispersion relation:

$$\omega = k^2, \quad (3.2)$$

for solutions of the form

$$u = e^{i\theta}, \quad \forall \theta = (kx - \omega t).$$

To incorporate temporal memory while keeping the spatial part local, we replace the first-order time derivative by the spectral fractional derivative of order $0 < \alpha \leq 1$:

$$i D_t^\alpha u + u_{xx} + 2|u|^2 u = 0, \quad \forall 0 < \alpha \leq 1, \quad (3.3)$$

with

$$\mathcal{F}_t[D_t^\alpha u(x, \cdot)](\omega) = (i\omega)^\alpha \widehat{u}(x, \omega). \quad (3.4)$$

When $\alpha = 1$, Eq (3.3) reduces to the classical NLSE (3.1). The linear dispersion relation of the fractional model follows by inserting the exponential mode

$$u(x, t) = e^{i\theta},$$

into the linearized equation

$$i D_t^\alpha u + u_{xx} = 0,$$

and by using the relation

$$D_t^\alpha e^{i\theta} = (-i\omega)^\alpha e^{i(kx - \omega t)},$$

we obtain

$$i(-i\omega)^\alpha - k^2 = 0, \quad (3.5)$$

which equivalently,

$$(i)^{1-\alpha} \omega^\alpha = k^2, \quad (3.6)$$

or, in the following compact form:

$$\omega^\alpha = c_\alpha k^2 \text{ \& } c_\alpha = (i)^{\alpha-1}. \quad (3.7)$$

Thus, the spectral fractional deformation preserves a power-law dispersion structure, with the fractional order entering through the temporal frequency relation. In the formal calculations below, one may absorb the constant factor c_α into a rescaling of the time variable, so that the working dispersion relation is written in the normalized form

$$\omega^\alpha = k^2. \quad (3.8)$$

3.2. Dependent-variable transformation and formal bilinear ansatz

In the classical Hirota method, one introduces tau functions $f \equiv f(x, t)$ and $g \equiv g(x, t)$ through

$$u(x, t) = \frac{g}{f}, \quad f \neq 0. \quad (3.9)$$

For $\alpha = 1$, the standard focusing NLSE can be represented by a genuine bilinear system of the schematic form

$$\begin{cases} (iD_t + D_x^2)g \cdot f = 0, \\ D_x^2 f \cdot f - 2|g|^2 = 0. \end{cases} \quad (3.10)$$

For the spectral time-fractional equation, we do not claim such an exact classical bilinearization. Instead, using the exponential identity $D_t^\alpha e^{i(kx - \omega t) + \delta} = (-i\omega)^\alpha e^{i(kx - \omega t) + \delta}$ and the commutator rule (2.20), we introduce the formal spectral Hirota system

$$\begin{cases} (i\mathcal{D}_t^\alpha + D_x^2)g \cdot f = 0, \\ D_x^2 f \cdot f - 2|g|^2 = 0. \end{cases} \quad (3.11)$$

Equation (3.11) is exact only in the integer-order limit $\alpha = 1$. For $0 < \alpha < 1$, it is a generating ansatz for explicit exponential templates whose validity must be checked by direct substitution or by residual diagnostics in the original fractional equation. This convention is used throughout the rest of the paper.

3.3. Consistency with the linear dispersion relation

To verify that the formal bilinear system is compatible with the linear dispersion relation of Eq (3.3), consider the simple tau-function choice

$$\begin{cases} f(x, t) = 1 + e^{\eta(x, t)}, \\ g(x, t) = e^{\theta(x, t)}, \end{cases} \quad (3.12)$$

with

$$\eta(x, t) = i(\kappa x - \Omega t) + \eta_0 \text{ \& } \theta(x, t) = i(kx - \omega t) + \theta_0, \quad (3.13)$$

where $\kappa, \Omega, k, \omega, \eta_0, \theta_0 \in \mathbb{C}$.

Using the classical spatial Hirota rule and the spectral fractional time rule, we have

$$\begin{cases} D_x^2 e^{\theta_1} \cdot e^{\theta_2} = -(k_1 - k_2)^2 e^{\theta_1 + \theta_2}, \\ \mathcal{D}_t^\alpha(e^{\theta_1}, e^{\theta_2}) = [(-i\omega_1)^\alpha - (-i\omega_2)^\alpha] e^{\theta_1 + \theta_2}. \end{cases} \quad (3.14)$$

Substituting these relations into the first equation of system (3.11) shows that the leading exponential contribution enforces

$$i(-i\omega)^\alpha - k^2 = 0, \quad (3.15)$$

which is precisely the linear spectral dispersion relation (3.5) obtained in Section 3.1, and therefore, after the constant c_α of (3.7) has been absorbed into the rescaling of time, the normalized relation Eq (3.8). In this sense, the formal bilinear ansatz is consistent with the linear spectral content of the time-fractional NLSE.

3.4. Breather family for general cubic NLSE

For later comparison with the fractional setting, it is convenient to consider the general cubic NLSE [47–49]

$$i u_t + \frac{P}{2} u_{xx} + Q|u|^2 u = 0, \quad (3.16)$$

where P indicates the coefficient of linear dispersion and Q is the coefficient of cubic nonlinearity. In the classical integer-order model, these parameters govern the spatial dispersive scale and the nonlinear self-focusing strength, respectively. In the fractional setting studied later, the temporal deformation acts through operator D_t^α ; therefore, the roles of P and Q as dispersion and nonlinearity coefficients remain unchanged, and only the temporal evolution law is modified.

A convenient background solution of Eq (3.16) reads

$$u_{\text{bg}}(t) = \sqrt{\frac{P}{Q}} e^{iPt}. \quad (3.17)$$

The rational-type solutions, ABs, KMBs, and RWs, are localized modulations of this background. A unified breather expression on this background reads [10, 11, 50]

$$u_a(x, t) = \sqrt{\frac{P}{Q}} e^{iPt} \left[1 + \frac{2(1 - 2a) \cosh(l_1 Pt) + i l_1 \sinh(l_1 Pt)}{\sqrt{2a} \cos(l_2 x) - \cosh(l_1 Pt)} \right], \quad (3.18)$$

with $l_1 = \sqrt{8a(1 - 2a)}$ and $l_2 = \sqrt{4(1 - 2a)}$. Control parameter a determines the qualitative regime for: (i) $0 < a < 0.5$: ABs (periodic in x , localized in t) can exist and propagate as shown in Figure 1, (ii) $a > 0.5$: KMB (localized in x , periodic in t) can arise and propagate as illustrated in Figure 2, and (iii) $a \rightarrow 0.5$: Peregrine (RW) (localized in both x and t) can be generated as evident in Figure 3.

The solution (3.18) is used to illustrate the canonical morphology of the classical breather solutions before introducing the time-fractional deformation. In particular, it provides a convenient reference configuration against which the effect of memory parameter α can be assessed in the later numerical sections. For $0 < a < 0.5$, one has $l_1, l_2 \in \mathbb{R}$, so the factor $\cos(l_2 x)$ produces spatial periodicity while the hyperbolic dependence in time yields temporal localization. This is the characteristic Akhmediev regime associated with MI on a plane-wave background. For $a > 0.5$, the relevant parameter becomes

purely imaginary, and the solution changes into a KMB, localized in space and recurrent in time. In the limit $a \rightarrow 0.5$, the characteristic spatial and temporal periods diverge and the breather degenerates into a single isolated extreme event, namely the RW.

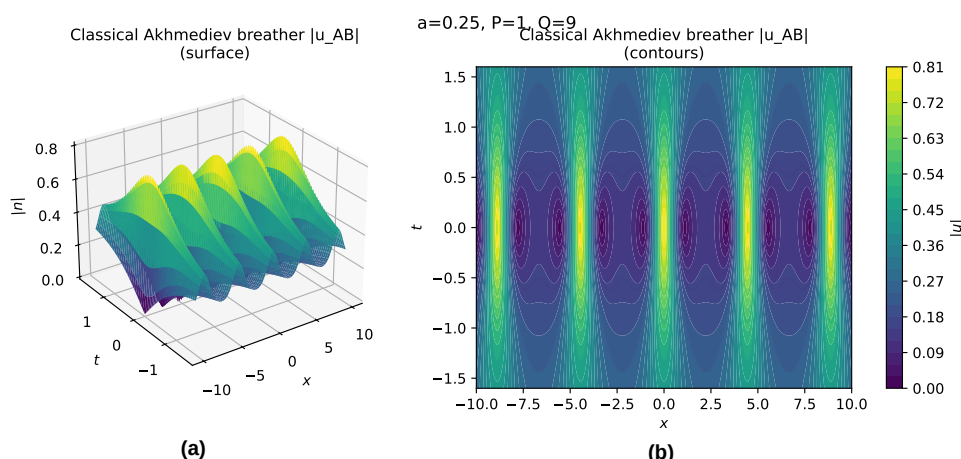


Figure 1. Classical Akhmediev breather regime corresponding to Eq (3.18) with $a = 0.25$ and $(P, Q) = (1, 9)$. (a) Surface plot and (b) contour plot of the modulus $|u_{AB}(x, t)|$ in the dimensionless (x, t) -plane. The horizontal axes represent the spatial variable x and the temporal variable t , while the colors and contour lines indicate the magnitude levels of $|u_{AB}(x, t)|$. The figure shows spatially periodic modulation and temporal localization, which are the defining features of the Akhmediev breather.

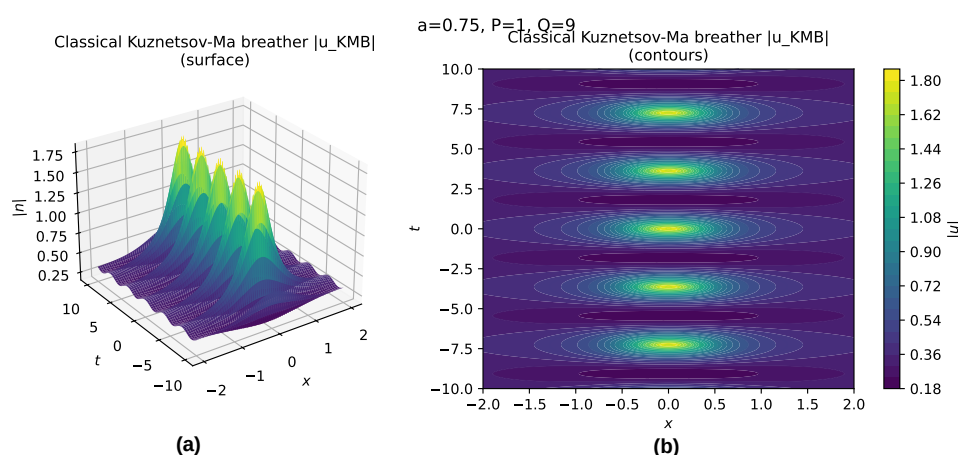


Figure 2. Classical Kuznetsov-Ma breather regime corresponding to Eq (3.18) with $a = 0.75$ and $(P, Q) = (1, 9)$. (a) Surface plot and (b) contour plot of the modulus $|u_{KMB}(x, t)|$ in the dimensionless (x, t) -plane. The horizontal axes represent x and t , and the colors and contour lines represent the magnitude levels of $|u_{KMB}(x, t)|$. The solution is localized in space and periodic in time, as expected for the Kuznetsov-Ma breather.

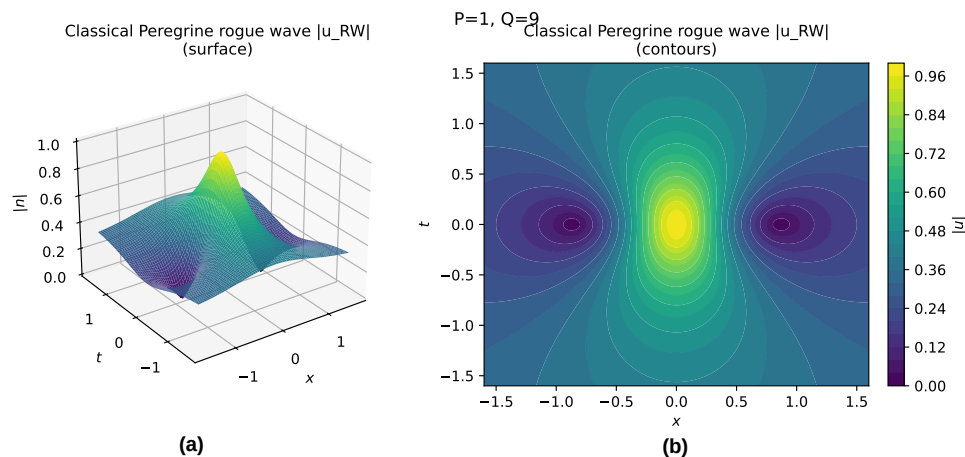


Figure 3. Classical Peregrine rogue-wave limit of Eq (3.18), obtained as $a \rightarrow 0.5$, for $(P, Q) = (1, 9)$. (a) Surface plot and (b) contour plot of the modulus $|u_{RW}(x, t)|$ in the dimensionless (x, t) -plane. The horizontal axes correspond to x and t , while the colors and contour lines indicate the levels of $|u_{RW}(x, t)|$. The profile is localized in space and time and represents the limiting rogue wave generated by maximal nonlinear focusing on the background.

4. Formal fractional N -soliton tau-functions

In this section, we construct formal one- and two-soliton tau-function candidates for the spectral time-fractional NLSE and explain how the same algebraic pattern extends to the general N -phase case. In agreement with the discussion of the previous section, we do not interpret these expressions as exact classical Hirota solutions for $0 < \alpha < 1$. Rather, they are introduced as formal spectral Hirota ansatz motivated by the exponential action of the operator D_t^α , by the associated fractional dispersion relation, and by the requirement that the construction recover the standard integrable NLSE formulas when $\alpha = 1$. The key structural observation is that the spectral fractional Hirota operator acts algebraically on exponential modes in exactly the way needed for the usual mode-by-mode Hirota bookkeeping. Consequently, the combinatorial tau-function pattern may be retained in formal form, while the fractional order α enters through the modified dispersion relations associated with each phase. We begin with the formal bilinear system

$$\begin{cases} (iD_t^\alpha + D_x^2)g \cdot f = 0, \\ D_x^2 f \cdot f - 2|g|^2 = 0, \end{cases} \quad (4.1)$$

together with the dependent-variable transformation

$$u(x, t) = \frac{g(x, t)}{f(x, t)}, \quad (4.2)$$

where for a single phase, we adopt the standard one-mode Hirota ansatz

$$f(x, t) = 1 + e^{\eta(x, t)} \text{ \& } g(x, t) = e^{\eta(x, t)}, \quad (4.3)$$

with

$$\eta(x, t) \equiv \eta = i(kx - \omega t) + \delta, \quad \forall k, \omega, \delta \in \mathbb{C}. \quad (4.4)$$

At the level of the first bilinear equation, the required exponential rules read

$$\begin{cases} D_x^2 e^{\eta_1} \cdot e^{\eta_2} = -(k_1 - k_2)^2 e^{\eta_1 + \eta_2}, \\ \mathcal{D}_t^\alpha(e^{\eta_1}, e^{\eta_2}) = [(-i\omega_1)^\alpha - (-i\omega_2)^\alpha] e^{\eta_1 + \eta_2}, \end{cases} \quad (4.5)$$

with $\eta_j = i(k_j x - \omega_j t) + \delta_j$.

Substituting (4.3) into the first relation in Eq (4.1) yields

$$(i\mathcal{D}_t^\alpha + D_x^2) g \cdot f = (i\mathcal{D}_t^\alpha + D_x^2) e^\eta \cdot (1 + e^\eta) = (i\mathcal{D}_t^\alpha + D_x^2) e^\eta \cdot 1 + (i\mathcal{D}_t^\alpha + D_x^2) e^\eta \cdot e^\eta. \quad (4.6)$$

The self-interaction term vanishes, and therefore, we get

$$(i\mathcal{D}_t^\alpha + D_x^2) g \cdot f = [i(-i\omega)^\alpha - k^2] e^\eta. \quad (4.7)$$

Hence, the one-phase mode must satisfy the fractional dispersion relation

$$i(-i\omega)^\alpha - k^2 = 0. \quad (4.8)$$

When $\alpha = 1$, relation (4.8) reduces to the classical relation $\omega = k^2$. It should be stressed that the templates constructed in this section are fixed by the first equation of system (4.1) alone; the second equation is not imposed on them, and is left to the residual diagnostics of Section 6. This is consistent with the formal status of the construction stated in Section 3.2.

Accordingly, the associated one-phase field is written formally as follows:

$$u_1(x, t) = \frac{e^\eta}{1 + e^\eta}, \quad (4.9)$$

with η constrained by Eq (4.8). This expression should be understood as a formal one-soliton tau-function candidate: Its algebraic structure is the same as in the classical Hirota construction, whereas the time dependence is deformed through the fractional dispersion law. In the integrable limit $\alpha = 1$, the usual integer-order one-soliton behavior is recovered after the standard reparameterization of phase, shift, and overall complex amplitude.

For two-solitons, we generalize the tau-functions by superposing two exponential modes and their interaction term. Accordingly, the standard Hirota ansatz reads

$$\begin{cases} f(x, t) = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}, \\ g(x, t) = e^{\eta_1} + e^{\eta_2}, \end{cases} \quad (4.10)$$

with $\eta_j(x, t) = i(k_j x - \omega_j t) + \delta_j, \quad \forall k_j, \omega_j, \delta_j \in \mathbb{C}, j = 1, 2$.

As in the one-phase case, the terms proportional to e^{η_1} and e^{η_2} in the first bilinear equation must vanish separately. This yields the two fractional dispersion relations

$$i(-i\omega_j)^\alpha - k_j^2 = 0, \quad \forall j = 1, 2. \quad (4.11)$$

Thus, each phase obeys its own fractional spectral law, and the fractional operator modifies the temporal dispersion of each component without altering the basic mode-superposition principle.

The remaining nontrivial information is encoded in the mixed exponential contributions, especially those proportional to $e^{\eta_1+\eta_2}$. Guided by the classical Hirota algebra and by the spectral fractional Korteweg–de Vries (KdV) analogue, we retain the standard pairwise interaction factor

$$A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2. \quad (4.12)$$

Expression (4.12) is the pairwise interaction factor of the classical KdV-type tau-function algebra, and it is retained here in that role: it is the factor with respect to which the present construction is a formal continuation for $0 < \alpha < 1$. It is not claimed to reproduce the two-soliton interaction coefficient of the focusing NLSE, whose Hirota factors are built from the conjugate combinations $k_i + \bar{k}_j$ and are therefore not of the form (4.12). In other words, the fractional order enters through the dispersion relations (4.11), while the pairwise interaction factor is preserved, as a template, at the level of the formal tau structure.

The resulting two-phase field is therefore written as

$$u_2(x, t) = \frac{g(x, t)}{f(x, t)} = \frac{e^{\eta_1} + e^{\eta_2}}{1 + e^{\eta_1} + e^{\eta_2} + \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 e^{\eta_1 + \eta_2}}. \quad (4.13)$$

As before, this should be regarded as a formal two-soliton tau-function candidate rather than as an exact-solution claim for the fractional equation. Its role is to preserve the classical interaction pattern while enabling the time dependence of each phase to be modified by the fractional dispersion law.

The same idea extends directly to the general N -phase case. The following phase function is considered:

$$\eta_j(x, t) = i(k_j x - \omega_j t) + \delta_j, \quad \forall j = 1, \dots, N, \quad (4.14)$$

where each phase satisfies the fractional dispersion relation

$$i(-i\omega_j)^\alpha - k_j^2 = 0, \quad \forall j = 1, \dots, N. \quad (4.15)$$

The corresponding formal tau-function pattern is built from all subsets of the phase set $\{\eta_1, \dots, \eta_N\}$, in the same combinatorial manner as in the classical Hirota theory:

$$\begin{cases} f(x, t) = 1 + \sum_{j=1}^N e^{\eta_j} + \sum_{1 \leq i < j \leq N} A_{ij} e^{\eta_i + \eta_j} + \sum_{1 \leq i < j < \ell \leq N} A_{ij} A_{i\ell} A_{j\ell} e^{\eta_i + \eta_j + \eta_\ell} + \dots, \\ g(x, t) = \sum_{j=1}^N C_j e^{\eta_j} + \sum_{1 \leq i < j \leq N} C_{ij} e^{\eta_i + \eta_j} + \dots, \end{cases} \quad (4.16)$$

with pairwise interaction coefficients

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2, \quad \forall 1 \leq i < j \leq N. \quad (4.17)$$

The constants C_j , C_{ij} , and higher-order coefficients determine the relative amplitudes and phases in the numerator and are chosen according to the desired scalar reduction or normalization. Pattern (4.16) is written as a generic combinatorial template; in the standard integrable reduction of the focusing NLSE the numerator and the denominator carry, respectively, the odd- and the even-order combinations of phases, which is obtained by retaining only those subsets of phases in each sum.

The formal N -phase field is then

$$u_N(x, t) = \frac{g(x, t)}{f(x, t)}. \quad (4.18)$$

The fractional order α deforms the time dependence of the phases but does not change the formal pairwise interaction factors (4.17).

When $\alpha = 1$, the above scheme reduces to the standard integer-order Hirota construction for the focusing NLSE. For $0 < \alpha < 1$, it provides a consistent formal family of fractional N -soliton tau-function candidates that will be used as analytical templates in the later residual-based discussion.

5. Fractional MI and associated fractional breathers

In this section, we construct approximate fractional versions of the three standard breather families of the focusing NLSE, namely the ABs, the KMBs, and the RWs. The purpose of this construction is not to claim new exact closed-form solutions of the spectral time-fractional NLSE but rather to build explicit analytical ansatz that remain consistent with the classical breather morphology and with the modified time scale suggested by fractional MI. This point is important: The formulas below are introduced as residual-validated approximate profiles, and their quality is assessed afterward through the residual of the spectral time-fractional equation. The guiding idea is simple. We begin from the classical breather expressions at $\alpha = 1$, identify the factors responsible for MI-driven time oscillation or time localization, and then deform only those temporal factors according to the fractional spectral time scale, while preserving the spatial shape of the classical breather. This yields explicit formulas that reduce exactly to the integer-order breathers when $\alpha = 1$ and provide a coherent family of approximate fractional breather profiles for $0 < \alpha \leq 1$ [43, 44].

5.1. Linear stability analysis, MI wavenumber, and fractional clock

We consider the normalized spectral time-fractional NLSE

$$iD_t^\alpha u + u_{xx} + 2|u|^2 u = 0, \quad 0 < \alpha \leq 1. \quad (5.1)$$

The linear stability calculation below is formal in the same spectral sense as the previous sections: Exponential perturbations are used to read the Fourier symbol of D_t^α . Let

$$u_0(x, t) = \rho e^{2ip^2 t}, \quad \rho > 0, \quad (5.2)$$

and perturb the background as

$$u(x, t) = [\rho + f(x, t) + ig(x, t)] e^{2ip^2 t}, \quad |f| + |g| \ll \rho. \quad (5.3)$$

Keeping only first-order terms gives

$$\begin{cases} D_t^\alpha f + g_{xx} = 0, \\ -D_t^\alpha g + f_{xx} + 4\rho^2 f = 0. \end{cases} \quad (5.4)$$

For exponential perturbations $f = Fe^{\lambda t + iKx}$ and $g = Ge^{\lambda t + iKx}$, with the spectral rule $D_t^\alpha e^{\lambda t} = \lambda^\alpha e^{\lambda t}$, Eq (5.4) yields

$$\lambda^{2\alpha} = K^2(4\rho^2 - K^2). \quad (5.5)$$

Consequently, the unstable band is

$$0 < K < 2\rho, \quad (5.6)$$

and the fractional growth rate is

$$\lambda_\alpha(K; \rho) = \left[K^2(4\rho^2 - K^2) \right]^{1/(2\alpha)}. \quad (5.7)$$

The corresponding classical growth rate is

$$\lambda_{\text{cl}}(K; \rho) = \left[K^2(4\rho^2 - K^2) \right]^{1/2}, \quad (5.8)$$

so that the fractional internal time used in the breather ansatz is

$$\tau_\alpha(t; K, \rho) = \Lambda_\alpha(K; \rho)t, \quad \Lambda_\alpha(K; \rho) = \frac{\lambda_\alpha(K; \rho)}{\lambda_{\text{cl}}(K; \rho)} = \lambda_{\text{cl}}(K; \rho)^{1/\alpha-1}. \quad (5.9)$$

This expression is now stated with all parameters visible. In the Akhmediev regime $0 < \alpha < 1/2$, the classical breather modulation wavenumber is

$$K_a = \ell_2 = 2\sqrt{1-2a}, \quad (5.10)$$

after the usual unit-background normalization. Therefore, a breather-tied clock can be obtained by taking $K = K_a$ in (5.9). In the limiting Peregrine case $a \rightarrow 1/2$, one has $K_a \rightarrow 0$; since $1/\alpha - 1 > 0$ for $0 < \alpha < 1$, the ratio in (5.9) then degenerates, $\Lambda_\alpha \rightarrow 0$, so that the fractional internal time collapses and the breather-tied clock ceases to be usable. For this reason, and for the Kuznetsov-Ma case where the spatial modulation is obtained by analytic continuation, the numerical residual tests below use the explicitly declared long-wave reference value

$$K = K_{\text{ref}} = 0.1, \quad \rho = 1. \quad (5.11)$$

This choice is inside the MI band and keeps the Peregrine limit comparable with the nearby Akhmediev regime. No exactness claim depends on this choice; it sets only the fractional time scale of the approximate profiles before residual validation.

5.2. Unified deformation rule

The classical breather formulas in Eq (3.18) contain the temporal factors responsible for growth, recurrence, and decay. The deformation rule used here is intentionally explicit: In those MI-controlled factors, we replace t by $\tau_\alpha(t; K, \rho)$ from (5.9). For Akhmediev breathers, one may use the breather-tied value $K = K_a$ in (5.10); in the residual experiments reported below, the value actually used is always stated, and the limiting tests use $K_{\text{ref}} = 0.1$. The carrier factor and the spatial profile are kept in their classical form. This gives closed analytical ansatz that recover the integer-order breathers when $\alpha = 1$, but for $0 < \alpha < 1$, they remain approximate and must pass the residual checks of Section 6.

In the generalized cubic NLSE

$$i u_t + \frac{P}{2} u_{xx} + Q|u|^2 u = 0, \quad (5.12)$$

parameter P is the coefficient of linear dispersion and Q is the coefficient of cubic nonlinearity. These interpretations remain unchanged in the fractional setting considered here; the time-fractional deformation acts through the operator D_t^α , thereby modifying the temporal evolution law rather than the physical roles of P and Q . Fractional breathers to the fractional NLSE

$$iD_t^\alpha u + \frac{P}{2}u_{xx} + Q|u|^2u = 0, \quad (5.13)$$

are discussed as follows:

- (i) Fractional AB (FAB): For the Akhmediev regime $0 < a < 0.5$, the classical breather is periodic in space and localized in time. The corresponding FAB ansatz reads

$$u_{\text{AB}}^{(\alpha)}(x, t) \equiv u_{\text{AB}}^{(\alpha)} = \sqrt{\frac{P}{Q}} e^{iPt} \left[1 + \frac{2(1-2a) \cosh[l_1 P \tau_\alpha(t)] + i l_1 \sinh[l_1 P \tau_\alpha(t)]}{\sqrt{2a} \cos(l_2 x) - \cosh[l_1 P \tau_\alpha(t)]} \right], \quad (5.14)$$

where $l_1 = \sqrt{8a(1-2a)}$, $l_2 = \sqrt{4(1-2a)}$, and $0 < a < 0.5$.

For $\alpha = 1$, the identity $\tau_\alpha(t) = t$ recovers the classical integer-order AB. For $0 < \alpha < 1$, the spatial modulation pattern remains periodic in x , while the growth and decay in time occur on the modified fractional MI scale.

- (ii) Fractional KMB (FKMB): For $a > 0.5$, the classical KMB is localized in space and periodic in time. It is obtained from (3.18) by the analytic continuation $l_1 = i\tilde{l}_1$, $l_2 = i\tilde{l}_2$, which leaves the denominator unchanged but reverses the sign of both terms of the numerator. The corresponding FKMB ansatz therefore reads

$$u_{\text{KMB}}^{(\alpha)}(x, t) \equiv u_{\text{KMB}}^{(\alpha)} = \sqrt{\frac{P}{Q}} e^{iPt} \left[1 - \frac{2(2a-1) \cos[\tilde{l}_1 P \tau_\alpha(t)] + i \tilde{l}_1 \sin[\tilde{l}_1 P \tau_\alpha(t)]}{\sqrt{2a} \cosh(\tilde{l}_2 x) - \cos[\tilde{l}_1 P \tau_\alpha(t)]} \right], \quad (5.15)$$

where $\tilde{l}_1 = \sqrt{8a(2a-1)}$, $\tilde{l}_2 = \sqrt{4(2a-1)}$, and $a > 0.5$.

Again, the limit $\alpha = 1$ gives the classical KMB, while for $0 < \alpha < 1$, the oscillation period in time is altered through the fractional effective clock $\tau_\alpha(t)$. In particular, the whole family (5.14)-(5.15) shares at $\alpha = 1$ the peak value $\sqrt{P/Q}(1 + 2\sqrt{2a})$, which gives $\sqrt{P/Q}(1 + \sqrt{2})$ in the Akhmediev case $a = 1/4$, $3\sqrt{P/Q}$ in the Peregrine limit $a \rightarrow 1/2$, and $\sqrt{P/Q}(1 + \sqrt{6})$ for the Kuznetsov-Ma value $a = 3/4$ used in Section 7.3.

- (iii) Fractional rogue waves (FRWs): The RWs arise as the limiting case $a \rightarrow 0.5$ of the classical breather family, when both the spatial and temporal periods diverge. Applying the same time-deformation rule yields the FRW ansatz

$$u_{\text{RW}}^{(\alpha)}(x, t) \equiv u_{\text{RW}}^{(\alpha)} = \sqrt{\frac{P}{Q}} e^{iPt} \left[1 - \frac{4(1 + 2iP\tau_\alpha(t))}{1 + 4x^2 + 4P^2\tau_\alpha(t)^2} \right]. \quad (5.16)$$

This profile preserves the rational spatial structure of the classical Peregrine solution and transfers the temporal localization to the fractional time scale. In the integer-order limit $\alpha = 1$, it reduces to the standard rogue-wave formula.

For numerical illustration, it is convenient to collect the three regimes into a single parameter-dependent expression. Accordingly, the general formula for fractional breathers can be written in the following compact form:

$$u_a^{(\alpha)}(x, t) = \sqrt{\frac{P}{Q}} e^{iPt} \left[1 + \frac{2(1-2a) \cosh[l_1 P \tau_a(t)] + i l_1 \sinh[l_1 P \tau_a(t)]}{\sqrt{2a} \cos(l_2 x) - \cosh[l_1 P \tau_a(t)]} \right], \quad (5.17)$$

with

$$l_1 = \sqrt{8a(1-2a)} \text{ and } l_2 = \sqrt{4(1-2a)}, \quad (5.18)$$

and interpret the cases $a < 0.5$, $a > 0.5$, and $a \rightarrow 0.5$ through the usual analytic continuation and limiting procedure.

6. Residual analysis and numerical experiments

In this section, we describe how the spectral fractional time derivative D_t^α is implemented numerically and how this implementation is used to evaluate the residual of the approximate fractional breather profiles introduced in the previous section. The numerical procedure is based on a Fourier-interpolation approach in time: The solution is sampled over a finite time window, extended periodically via its discrete Fourier interpolant, differentiated in the Fourier domain by multiplying by the symbol $(i\omega)^\alpha$, and then transformed back to physical time. This construction is natural for the spectral fractional derivative and provides an efficient framework for assessing the quality of the breather ansatz [39, 41, 42]. Since approximate breathers are localized or effectively confined to a finite space-time region, the computational parameters are chosen based on their characteristic scales. More precisely, the spatial interval $[-L, L]$ is selected large enough to contain one or more full spatial periods (for periodic profiles) or the effective spatial support of the localized structure, while the time window $[0, T]$ is taken sufficiently large to capture the entire focusing-defocusing cycle or the full time localization of the event. For notational simplicity the time window is written below as $[0, T]$; the symmetric windows used in Section 7 are obtained by an obvious translation, which does not affect the spectral construction. The number of grid points, denoted by M in space and N in time (written N_x and N_t , respectively, in Section 7), is then increased until the residual norms remain stable under further refinement. This choice of L , T , M , and N improves the reproducibility of the numerical procedure and clarifies the relation between the grid and the intrinsic scales of the breather profiles.

6.1. Discrete Fourier grid and periodic interpolation in time

Let $f(t)$ be a sufficiently smooth time-dependent function defined on a finite interval $[0, T]$. We then sample the function $f(t)$ at N equally spaced grid points

$$t_n = n\Delta t, \quad n = 0, 1, \dots, N-1, \quad \Delta t = \frac{T}{N}. \quad (6.1)$$

The samples $f_n = f(t_n)$ are stored in a vector $\{f_0, \dots, f_{N-1}\}$. For the spectral method, we declare that $f(t)$ is T -periodic and that these N samples represent one full period. Thus, f is approximated by the discrete Fourier series

$$f(t) \approx \sum_{m=-N/2}^{N/2-1} \widehat{f}_m e^{i\omega_m t}, \quad \forall \omega_m = \frac{2\pi m}{T}, \quad (6.2)$$

where the coefficients $\widehat{f}_m$ are computed through the discrete Fourier transform of the sampled data [41]. In practice, this is carried out using the fast Fourier transform (FFT). Using a standard FFT indexing convention, the discrete frequencies are assigned by

$$\omega_k = \begin{cases} \frac{2\pi k}{T}, & 0 \leq k \leq N/2, \\ \frac{2\pi(k-N)}{T}, & N/2 < k \leq N-1, \end{cases} \quad (6.3)$$

where $k = 0, 1, \dots, N-1$. This representation turns the sampled time series into a smooth periodic interpolant, which is the natural setting for the spectral realization of fractional differentiation.

6.2. Fractional derivative as a Fourier multiplier

The defining property of the spectral fractional derivative reads

$$\mathcal{F}_t[D_t^\alpha f](\omega) = (i\omega)^\alpha \widehat{f}(\omega), \quad \forall 0 < \alpha \leq 1. \quad (6.4)$$

On the discrete Fourier grid, this becomes a pointwise multiplication of the Fourier coefficients by the multiplier $(i\omega_k)^\alpha$. The numerical algorithm can be summarized as follows:

- (i) Compute the FFT of the samples f_n :

$$F_k = \text{FFT}(f_n), \quad \forall k = 0, 1, \dots, N-1. \quad (6.5)$$

- (ii) Define the fractional multiplier

$$M_k^{(\alpha)} = (i\omega_k)^\alpha = |\omega_k|^\alpha \exp\left(i\frac{\pi\alpha}{2} \text{sgn}(\omega_k)\right), \quad (6.6)$$

using the principal branch of the complex power. For the zero mode, we set

$$M_0^{(\alpha)} = 0, \quad (6.7)$$

so that the derivative of a constant vanishes [39, 40].

- (iii) Multiply in the Fourier domain:

$$G_k = M_k^{(\alpha)} F_k, \quad k = 0, 1, \dots, N-1. \quad (6.8)$$

- (iv) Apply the inverse FFT to recover the numerical values of the fractional derivative:

$$(D_t^\alpha f)(t_n) \approx g_n = \text{IFFT}(G_k), \quad n = 0, 1, \dots, N-1. \quad (6.9)$$

This is the exact analogue of the usual Fourier spectral differentiation for integer-order derivatives, with the only difference being that the exponent of the multiplier is the noninteger α .

6.3. Interpolation viewpoint and spectral accuracy

The previous procedure admits a simple interpretation as interpolation. The FFT first constructs the trigonometric interpolant $p(t)$ that matches the data at the grid points. The multiplication by $(i\omega)^\alpha$ then computes the spectral fractional derivative of this interpolant exactly in Fourier space, and the inverse FFT evaluates that derivative back at the sampling nodes. In this sense, the numerical error comes from approximating the original function by its periodic trigonometric interpolant on the chosen time window. When the function is sufficiently smooth, and the chosen interval is large enough so that the profile is negligible near the endpoints, the periodic extension introduces only a small error, and the method exhibits spectral accuracy as N increases [41, 42]. This is precisely why the time window T is tied to the breather's characteristic lifetime.

6.4. Residual computation for the fractional breathers

We now outline how to use this numerical fractional derivative to compute the residual of our approximate fractional breathers $u^{(\alpha)}(x, t)$ and to measure their accuracy.

- (i) **Spatial discretization:** We discretize space on a periodic interval $[-L, L]$ with M grid points as follows:

$$x_j = -L + j\Delta x, \quad j = 0, \dots, M-1, \quad \Delta x = \frac{2L}{M}. \quad (6.10)$$

Here, we represent $u^{(\alpha)}(x, t)$ at the grid points x_j and use a standard Fourier spectral derivative in x to compute $u_{xx}^{(\alpha)}(x_j, t_n)$, again by multiplying the spatial FFT by $(ik)^2 = -k^2$ in the wavenumber domain [42].

- (ii) **Time discretization and spectral fractional operator D_t^α :** For each fixed spatial point x_j , we view $u^{(\alpha)}(x_j, t)$ as a function of t and store its values at the time grid $\{t_n\}_{n=0}^{N-1}$. Applying the Fourier-interpolation procedure above to this time series yields approximate values of $D_t^\alpha u^{(\alpha)}(x_j, t_n)$ for all j and n .
- (iii) **Residual:** At each grid point (x_j, t_n) , we then evaluate

$$R^{(\alpha)}(x_j, t_n) = i D_t^\alpha u^{(\alpha)}(x_j, t_n) + u_{xx}^{(\alpha)}(x_j, t_n) + 2 |u^{(\alpha)}(x_j, t_n)|^2 u^{(\alpha)}(x_j, t_n). \quad (6.11)$$

If $u^{(\alpha)}$ is an exact solution, $R^{(\alpha)}$ would be identically zero. For the fractional breather ansatz used here, $R^{(\alpha)}$ is generally nonzero and must be interpreted with a stated accuracy criterion.

For interpretation we use the following quantitative accuracy convention. Let

$$\eta_{\text{rms}} = \left(\frac{1}{MN} \sum_{j,n} \mathcal{E}_\alpha(x_j, t_n)^2 \right)^{1/2}, \quad \eta_{\text{max}} = \max_{j,n} \mathcal{E}_\alpha(x_j, t_n), \quad (6.12)$$

where $\mathcal{E}_\alpha$ is the scaled residual defined in (7.2) below. We call an ansatz quantitatively accurate on the selected window when $\eta_{\text{rms}} \leq 0.10$ and $\eta_{\text{max}} \leq 0.50$. We call it a moderate residual approximation when $0.10 < \eta_{\text{rms}} \leq 0.35$ but the large values of η_{max} are confined near focusing or boundary layers. If $\eta_{\text{rms}} > 0.35$, the ansatz is reported only as a qualitative morphological template. This criterion addresses the fact that a maximum scaled residual close to one does not represent a uniformly small relative error, even when the space-time morphology remains visually meaningful.

7. Numerical examples

The numerical tests in this section have three purposes. First, they verify that the approximate profiles reduce to the classical breathers when $\alpha = 1$. Second, they quantify the residual of the fractional ansatz on finite space-time windows. Third, they add central time-section plots at $x = 0$, because two-dimensional density plots may hide the effect of α on the focusing or recurrence time scale.

We use the residual

$$R_\alpha(x, t) = iD_t^\alpha u^{(\alpha)}(x, t) + \frac{P}{2} \partial_{xx} u^{(\alpha)}(x, t) + Q|u^{(\alpha)}(x, t)|^2 u^{(\alpha)}(x, t), \quad (7.1)$$

and the scaled residual

$$\mathcal{E}_\alpha(x, t) = \frac{|R_\alpha(x, t)|}{|iD_t^\alpha u^{(\alpha)}| + \left| \frac{P}{2} \partial_{xx} u^{(\alpha)} \right| + |Q|u^{(\alpha)}|^2 u^{(\alpha)}| + 10^{-30}}. \quad (7.2)$$

For all three fractional examples below we set $P = Q = 2$, $\rho = 1$, and $K = K_{\text{ref}} = 0.1$ in the fractional clock. This value is not hidden in the computation; it is the reference long-wave MI scale used to compare the three families on the same footing. The natural Akhmediev relation $K_a = 2\sqrt{1 - 2a}$ is given in (5.10) for applications where the clock is tied directly to the breather modulation parameter.

7.1. Fractional Peregrine-type profile

We first consider the approximate fractional rogue-wave profile (5.16) on $x \in [-4, 4]$ and $t \in [-2, 2]$, with $N_x = 241$ and $N_t = 1024$. The tested fractional orders are $\alpha \in \{1, 0.7, 0.5, 0.3\}$, and the corresponding profiles are displayed in Figures 4 and 5. The exact integer-order formula gives a direct analytic residual of order 10^{-14} when $\alpha = 1$; the finite-window Fourier residual reported in Table 1 also contains periodic-extension error from the spectral differentiation on a finite interval.

Table 1 shows that the Peregrine-type ansatz is quantitatively accurate in the integer-order check and remains a moderate residual approximation for the fractional values tested. The maximum scaled residual approaches 1 for small α , so the corresponding profiles should not be described as exact or uniformly small-error solutions; they are residual-tested analytical approximations.

Table 1. Global residual diagnostics for the approximate fractional Peregrine-type ansatz. The final column classifies the result according to (6.12).

α	Λ_α	$\max u_{\text{RW}} $	$\max R $	RMS $ R $	η_{max}	η_{rms}	Regime
1	1	3	0.790528	0.445625	0.270016	0.0418048	quantitative
0.7	0.501428	3	14.8111	2.88898	0.849774	0.245062	moderate
0.5	0.19975	3	18.7756	5.04954	0.988809	0.323262	moderate
0.3	0.0233239	3	21.0590	6.67133	0.999385	0.349253	moderate/limit

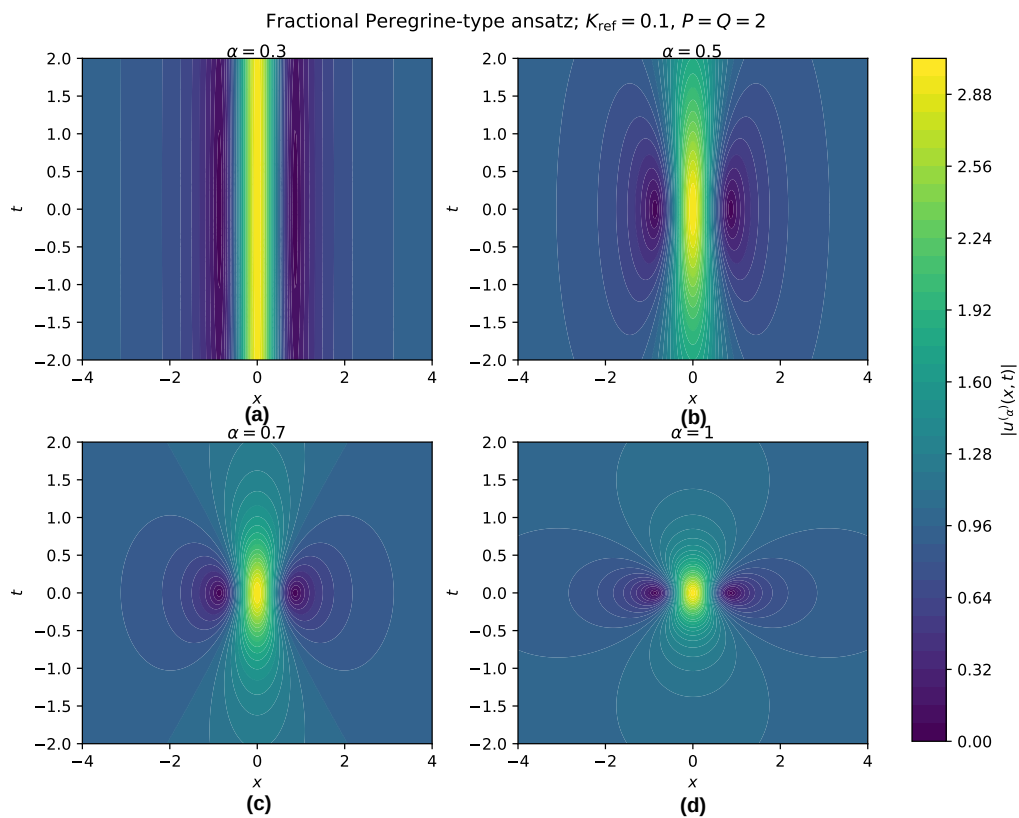


Figure 4. Density/contour plots of the fractional Peregrine-type profile $|u_{\text{RW}}^{(\alpha)}(x, t)|$ for (a) $\alpha = 0.3$, (b) $\alpha = 0.5$, (c) $\alpha = 0.7$, and (d) $\alpha = 1$, with $P = Q = 2$ and $K_{\text{ref}} = 0.1$. The spatial localization is preserved, while the fractional clock changes the temporal width of the focusing event.

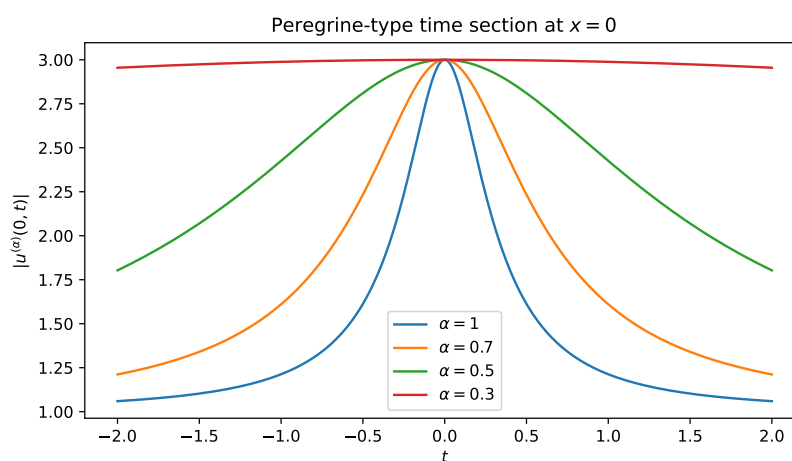


Figure 5. Central time section $|u_{\text{RW}}^{(\alpha)}(0, t)|$. This plot makes the effect of α visible: decreasing α broadens the event on the chosen reference clock, although all curves retain the same peak value at $t = 0$.

7.2. Fractional Akhmediev-type profile

We next use the approximate fractional Akhmediev profile (5.14) with $a = 0.25$. The spatial period is $L_x = 2\pi/\ell_2 \approx 4.44288$, and the computational window is $x \in [-8.8857, 8.8857]$, $t \in [-8, 8]$, with $N_x = 128$ and $N_t = 1024$. The tested fractional orders are $\alpha \in \{1, 0.75, 0.55, 0.4\}$, and the corresponding profiles are displayed in Figures 6 and 7.

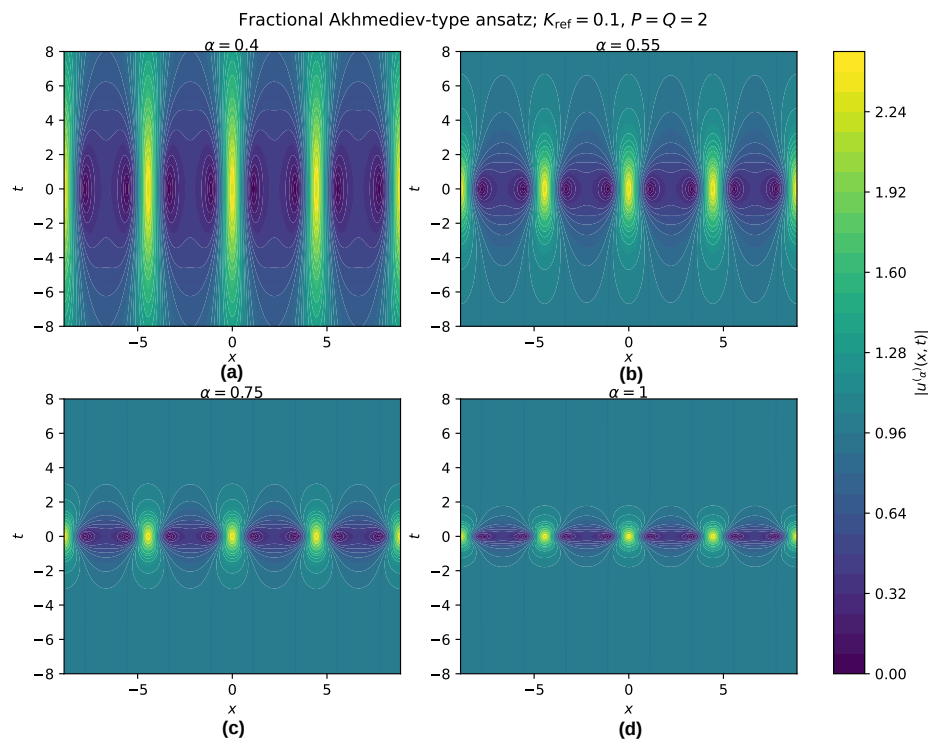


Figure 6. Density/contour plots of the fractional Akhmediev-type profile $|u_{AB}^{(\alpha)}(x, t)|$ for $a = 0.25$ and (a) $\alpha = 0.4$, (b) $\alpha = 0.55$, (c) $\alpha = 0.75$, and (d) $\alpha = 1$. The spatial periodicity is retained, while the fractional order changes the temporal growth and decay scale.

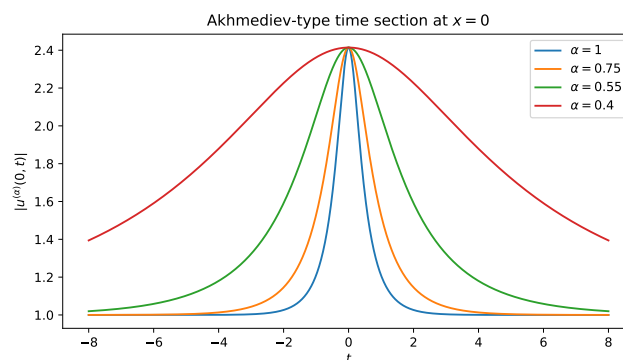


Figure 7. Central time section $|u_{AB}^{(\alpha)}(0, t)|$ for the Akhmediev-type profile. The curves make explicit the clock deformation that is hard to see in a surface plot alone.

As reported in Table 2, the Akhmediev ansatz satisfies the accuracy criterion in a near-quantitative sense for $\alpha = 1$ and remains in the moderate residual class for the tested fractional values. As in the Peregrine case, the largest scaled residuals occur near the focusing region, where the three terms of the equation are largest and most sensitive to the nonlocal time derivative.

Table 2. Global residual diagnostics for the approximate fractional Akhmediev-type ansatz.

α	Λ_α	$\max u_{AB} $	$\max R $	RMS $ R $	$\eta_{\max}$	η_{rms}	Regime
1	1	2.41421	0.920173	0.618592	0.567524	0.0705177	near-quantitative
0.75	0.584560	2.41421	6.43783	1.81473	0.826233	0.194139	moderate
0.55	0.267714	2.41421	9.06813	3.23297	0.969746	0.288901	moderate
0.40	0.089275	2.41421	10.4190	4.08055	0.993159	0.325877	moderate

7.3. Fractional Kuznetsov-Ma-type profile

Finally, we consider the fractional Kuznetsov-Ma-type profile (5.15) with $a = 0.75$. This value gives $\sqrt{P/Q} = 1$, $\tilde{\ell}_1 = \sqrt{3} \approx 1.73205$, and $\tilde{\ell}_2 = \sqrt{2} \approx 1.41421$. The classical recurrence period is $T_{\text{KMB}} = 2\pi/(P\tilde{\ell}_1) \approx 1.8138$. The computational window is $x \in [-5.65685, 5.65685]$ and $t \in [-8, 8]$, with fractional orders $\alpha \in \{1, 0.75, 0.65, 0.55\}$, and the corresponding profiles are displayed in Figures 8 and 9.

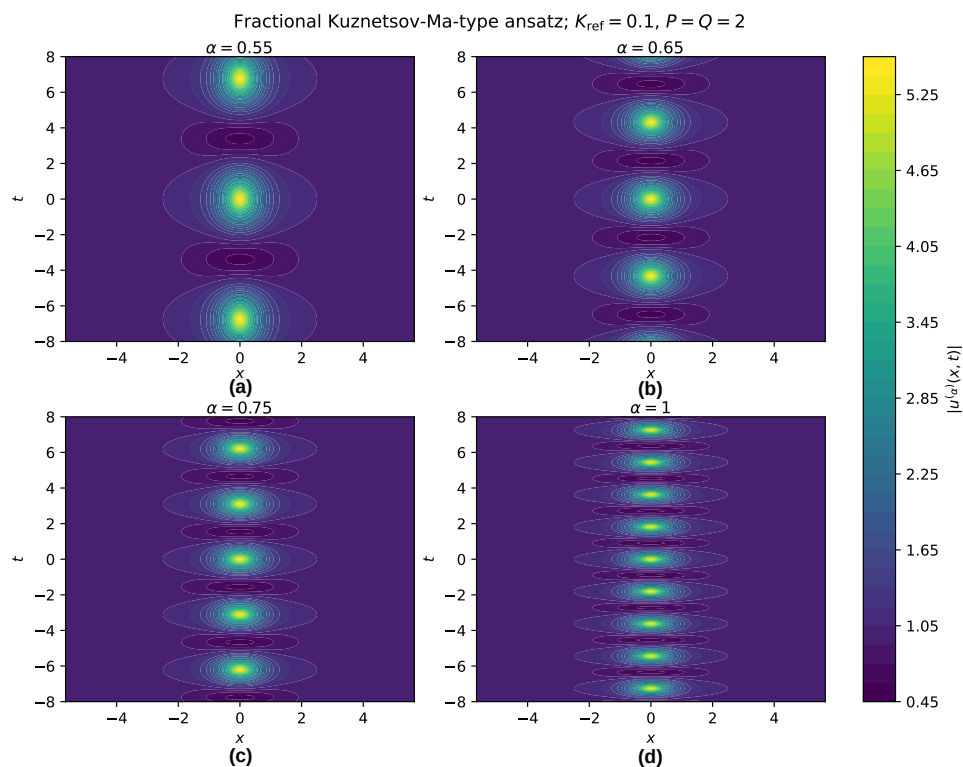


Figure 8. Density/contour plots of the fractional Kuznetsov-Ma-type profile $|u_{\text{KMB}}^{(\alpha)}(x, t)|$ for $a = 0.75$ and (a) $\alpha = 0.55$, (b) $\alpha = 0.65$, (c) $\alpha = 0.75$, and (d) $\alpha = 1$. The spatial localization is preserved, while the fractional clock changes the recurrence period in time.

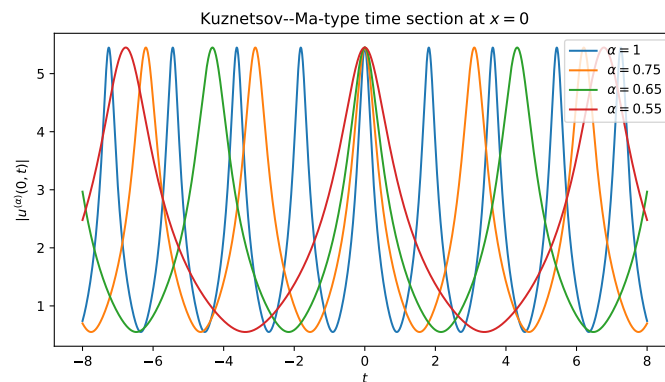


Figure 9. Central time section $|u_{\text{KMB}}^{(\alpha)}(0, t)|$. The recurrence period is visibly modified by the fractional clock, but the residual levels in Table 3 show that this family is best interpreted as a qualitative recurrent-breather template rather than a quantitative fractional solution on the full window.

As listed in Table 3, the Kuznetsov-Ma residuals are much larger than those for the Akhmediev and Peregrine profiles. This is consistent with the recurrent temporal structure of the KMB and with the finite-window periodic extension used by the Fourier fractional derivative. According to the accuracy criterion (6.12), the KMB profiles should therefore be presented as qualitative morphological ansatz only. This clarification prevents the scaled residuals close to 1 from being misread as small relative errors.

Table 3. Global residual diagnostics for the approximate fractional Kuznetsov-Ma-type ansatz.

α	Λ_α	$\max u_{\text{KMB}} $	$\max R $	$\text{RMS } R $	$\eta_{\max}$	η_{rms}	Regime
1	1	5.44949	23.5750	17.1082	0.999997	0.580782	qualitative
0.75	0.584560	5.44949	259.315	19.2318	0.997953	0.692424	qualitative
0.65	0.420087	5.44949	264.090	23.0258	0.885927	0.725169	qualitative
0.55	0.267714	5.44949	264.060	17.1492	0.912708	0.741353	qualitative

Overall, the numerical evidence supports the following interpretation. The fractional breather formulas are reliable explicit benchmarks near the integer-order regime and useful moderate approximations for the Peregrine and Akhmediev tests considered here. Away from this regime, and especially for the Kuznetsov-Ma recurrence, they should be used as morphology-preserving templates whose acceptance must be based on residual diagnostics rather than on formal Hirota algebra alone.

8. Conclusions

We studied a spectral time-fractional deformation of the focusing nonlinear Schrödinger equation from an explicitly formal Hirota perspective. The main point is not that the fractional equation remains integrable or that it admits an exact classical bilinearization. For $0 < \alpha < 1$, the spectral fractional derivative lacks the classical product and quotient rules needed for such a claim. The contribution is instead a controlled exponential-mode framework: The Fourier-multiplier action of D_t^α gives a simple

algebraic rule on exponentials, and this rule generates tau-function templates with fractional dispersion relations.

The resulting one-, two-, and N -phase tau structures are therefore formal analytical templates. They reduce to the standard Hirota structures when $\alpha = 1$, but for $0 < \alpha < 1$, they must be checked against the original spectral time-fractional NLSE. The same philosophy was applied to breathers on a finite background. Fractional Akhmediev-, Kuznetsov-Ma-, and Peregrine-type profiles were constructed by replacing the MI-controlled time factors by the fractional clock $\tau_\alpha(t; K, \rho)$. The dependence on the perturbation wavenumber was made explicit, including the natural Akhmediev relation $K_a = 2\sqrt{1 - 2a}$ and the long-wave reference value $K_{\text{ref}} = 0.1$ used in the limiting residual tests.

The residual analysis provides the practical acceptance mechanism for these profiles. By introducing unscaled and scaled residual diagnostics, together with a stated accuracy criterion, the numerical section distinguishes quantitative approximation from qualitative morphology preservation. The Peregrine- and Akhmediev-type profiles remain useful residual-tested approximations in the weak-to-moderate fractional regimes examined here, while the Kuznetsov-Ma-type profile has larger relative residuals and should be used mainly as a recurrent-breather template. The added central time-section plots show the effect of the fractional order on focusing and recurrence scales more clearly than surface plots alone.

In summary, the paper provides a practical research framework rather than an unqualified exact-solution theory: formal spectral Hirota templates, explicit fractional breather ansatz, a transparent MI-clock calibration, and Fourier-spectral residual validation. This combination gives useful benchmark profiles for numerical studies of NLSE-type models with temporal memory and suggests natural extensions to coupled and vector systems, where exactness would again have to be assessed through the original fractional equations and not only through formal tau algebra.

Author contributions

The authors conceived the study, developed the analytical methodology, performed the numerical simulations, analyzed the results, and wrote the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no Generative-AI tools were used to produce scientific results, data, or conclusions in this manuscript. Computational assistance was limited to standard numerical and symbolic software used for algebraic manipulation, simulation, and figure generation.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

Data and code availability

All numerical simulations reported in this study were carried out using Mathematica 14.2. The data supporting the findings of this work are generated directly from the governing equations and computational procedures described in the manuscript. No external datasets were used. The code used to produce the numerical results is available from the corresponding author upon reasonable request.

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