



*Research article***Decay rates for Caputo-type fractional evolution equations with two-parameter damping****Shi-you Lin^{1,2,*}, Ting-ting Hu¹ and Zhi-chao Lu¹**¹ School of Mathematics and Statistics, Hainan Normal University, Haikou 571158, China² Key Laboratory of Data Science and Intelligence Education, Ministry of Education, Hainan Normal University, Haikou 571158, China*** Correspondence:** Email: 050105@hainnu.edu.cn.

Abstract: This paper investigates the asymptotic decay behavior of solutions to Caputo-type fractional evolution equations with two-parameter damping of the form ${}^C D_t^\gamma u(t) + k {}^C D_t^\delta u(t) = Au(t)$, where $0 < \delta < \gamma \leq 1$ and $k > 0$. The characteristic equation $\lambda^\gamma + k\lambda^\delta = \mu$ ($\mu < 0$) has no roots in the cut plane $\mathbb{C} \setminus (-\infty, 0]$, so the resolvent family is represented entirely by a branch cut integral. Exploiting the spectral representation of the Caputo fractional (γ, δ, k) resolvent family and Watson's lemma, we establish $\|U(t)\| \leq C(1+t)^{-\delta}$ for self-adjoint A with compact resolvent. For sectorial generators with the compatibility condition $\omega < \pi(1-\gamma)$, the same decay rate $\|U(t)A^{-1}\| \leq Ct^{-\delta}$ holds. Exponential decay occurs only in the excluded case $\gamma = \delta = 1$; the lower-order fractional derivative δ governs the long-time polynomial decay.

Keywords: fractional evolution equations; decay rates; Caputo fractional resolvent; two-parameter damping; Watson's lemma

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1. Introduction

Damping plays a crucial role in both physical and mathematical formulations of fractional-order systems. Unlike traditional integer-order damping $cu'(t)$, fractional damping terms ${}^C D_t^\delta u(t)$ introduce memory-dependent dissipation effects through fractional derivatives, allowing for a more accurate description of complex systems with inherent memory characteristics. Such models naturally arise in diverse applications, including probabilistic analysis of fractional Cauchy problems [1,2], fractional boundary value problems involving nonlinear operators [3], fractional differential equations with variable coefficient operators [4], numerical schemes for fractional wave equations [5], and fractional-order oscillators with parametric and forced excitation [6,7]. The description of damped fractional

oscillations involving multiple fractional orders has been studied by Beyer and Kempfle [8], and the broader framework of distributed-order fractional models—where the order itself is integrated against a measure—was introduced by Caputo [9]; see also the comprehensive treatments in [10–12].

For the classical abstract Cauchy problem, such as the first-order evolution equation $u'(t) = Au(t)$, solutions are typically represented by the action of an operator semigroup $U(t)$ on the initial value x . In contrast, for fractional differential equations, the inherent nonlocality and memory effects associated with fractional derivatives prevent the direct application of semigroup theory. In the 1990s, research on fractional abstract Cauchy problems began to gain attention, with the primary emphasis on the existence and representation of solution operators. Bazhlekova [13] carried out a systematic investigation of fractional evolution equations, introducing the notion of a solution operator. Subsequently, Chen and Li [14] explicitly formulated the concept of a fractional resolvent operator function and rigorously established its relationship with fractional Cauchy problems. Independently, Li and Peng [15] introduced the γ -order fractional resolvent to address Riemann–Liouville (RL) fractional abstract Cauchy problems. This framework was later extended by Mei and Peng [16] to incorporate damping effects through the damped RL fractional (γ, δ, k) resolvent.

Despite the widespread recognition of fractional models in describing anomalous phenomena, most existing studies focus on well-posedness and representation theory. From a physical perspective, the transition from integer-order to fractional-order models fundamentally alters the memory structure of the system. In classical vibration theory, integer-order derivatives imply that the damping force is proportional to the instantaneous velocity, which corresponds to a Markovian process where the system has no memory of past states. However, real materials such as viscoelastic polymers or biological tissues exhibit power-law memory kernels, meaning that their current state depends strongly on the entire deformation history. Fractional derivatives introduce an additional degree of freedom—the fractional order—enabling more accurate fitting of experimental attenuation data. Therefore, providing a rigorous operator-theoretic framework for such fractionally damped systems is both mathematically novel and physically necessary for accurately characterizing the slow, non-exponential relaxation observed in complex materials.

Recently, our previous work [17] studied the following damped abstract fractional Cauchy problem:

$$\begin{cases} {}^C D_t^\gamma u(t) + k {}^C D_t^\delta u(t) = Au(t), & t > 0, \\ u(0) = x, \end{cases} \quad (\text{D-CAFCP}_x)$$

where the parameters satisfy $0 < \delta < \gamma \leq 1$, $k > 0$ is the damping coefficient, and $A : D(A) \subset X \rightarrow X$ is a densely defined closed linear operator on a Banach space $(X, \|\cdot\|)$. They introduced the concept of the Caputo fractional (γ, δ, k) resolvent family $\{U(t)\}_{t \geq 0}$ and established the existence and uniqueness of strong and mild solutions. However, the question of how fast these solutions decay as $t \rightarrow \infty$ (that is, the quantitative asymptotic behavior) remains open.

Several works have established the decay rates for specific classes of fractional evolution equations. For the undamped fractional Cauchy problem ${}^C D_t^\gamma u(t) = Au(t)$ with $0 < \gamma < 1$, it is well known that solutions exhibit polynomial decay of order $\mathcal{O}(t^{-\gamma})$ when A generates an analytic semigroup [13]. For fractional wave equations with structural damping, Affili and Valdinoci [1] established L^p – L^q decay estimates. Nevertheless, for the two-parameter damped model (D-CAFCP_x) , the interplay between the two fractional derivatives introduces additional spectral complexity that renders existing techniques insufficient. In particular, the Laplace symbol $\lambda^\gamma + k\lambda^\delta$ of the damped operator does not reduce to a

simple power, and the resolvent family involves Prabhakar-type Mittag-Leffler (ML) kernels whose asymptotic behavior is more delicate.

The present paper addresses this gap. Our main contributions are as follows:

- (i) When $-A$ is a self-adjoint, positive definite operator with compact resolvent on a Hilbert space, we prove that the Caputo fractional (γ, δ, k) resolvent family satisfies the polynomial decay estimate

$$\|U(t)\| \leq M_{\text{lin}}(1+t)^{-\delta}, \quad t \geq 0,$$

where the constant M_{lin} depends on γ , δ , k , and λ_1 , the spectral gap of $-A$. The proof relies on the asymptotic analysis of the branch cut integral via Watson's lemma, which connects the small- λ behavior of the Laplace transform with the long-time behavior of the resolvent family. A crucial structural observation is that for $0 < \delta < \gamma \leq 1$, the characteristic equation $\lambda^\gamma + k\lambda^\delta = \mu$ ($\mu < 0$) has no roots in the cut plane $\mathbb{C} \setminus (-\infty, 0]$, so the inverse Laplace transform is entirely given by the branch cut integral with no pole contribution. This result reveals that the leading-order polynomial decay rate is governed by δ rather than γ , which is a consequence of the fact that the lower-order fractional derivative λ^δ dominates the symbol $\lambda^\gamma + k\lambda^\delta$ in the low-frequency regime $\lambda \rightarrow 0$.

- (ii) For a general sectorial generator A such that $-A$ generates a bounded analytic semigroup of sectorial type ω satisfying the compatibility condition $\omega < \pi(1 - \gamma)$, which restricts the result to $0 < \delta < \gamma < 1$ and shrinks the admissible sectorial type to zero as $\gamma \rightarrow 1^-$, we establish the decay estimate

$$\|U(t)A^{-1}\| \leq Ct^{-\delta}, \quad t > 0,$$

where the constant C depends on γ , δ , k , and the sectorial data of A . This estimate has the same decay exponent δ as the self-adjoint case, showing that the polynomial decay rate is intrinsic to the two-parameter damping structure. The proof relies on the resolvent identity $R(\mu, A)A^{-1} = \mu^{-1}[R(\mu, A) + A^{-1}]$ and direct branch cut analysis via Watson's lemma.

- (iii) We characterize the influence of the damping parameters δ , γ , and k on the decay rate. A central finding is that δ is the parameter that determines the long-time polynomial decay rate of the resolvent family in the self-adjoint setting. In particular, since $0 < \delta < \gamma \leq 1$, the fractionally damped system decays more slowly than its undamped counterpart ${}^C D_t^\gamma u = Au$, which decays at rate $O(t^{-\gamma})$. This apparent slowdown has a clear physical and Tauberian interpretation: The power-law memory kernel associated with ${}^C D_t^\delta u$ prolongs the relaxation process by retaining a longer history of past states, thereby slowing the return to equilibrium. The damping coefficient k modulates the prefactor but does not alter the leading-order polynomial exponent.
- (iv) As an application, we derive explicit decay estimates for the fractionally damped beam equation considered in [17], demonstrating that the energy of the beam decays polynomially with rate $O(t^{-\delta})$.

The remainder of this paper is organized as follows. Section 2 collects the necessary preliminaries on fractional calculus, ML functions, and the Caputo fractional (γ, δ, k) resolvent family. Section 3 performs a detailed Laplace transform analysis of the resolvent family, including the characteristic equation, spectral decomposition, and branch cut contribution. Section 4 establishes the main decay

results for the linear equation, treating both the self-adjoint case ($0 < \delta < \gamma \leq 1$) and the general sectorial case ($0 < \delta < \gamma < 1$ with $\omega < \pi(1 - \gamma)$). Section 5 extends the decay analysis to the semilinear equation. Section 6 presents the application to the fractionally damped beam equation together with concluding remarks.

2. Preliminaries

This section introduces key concepts of fractional integrals, fractional derivatives, and the ML function. These tools form the analytical foundation for the decay analysis in subsequent sections.

Throughout, X denotes a complex Banach space equipped with the norm $\|\cdot\|$, and $A : D(A) \subset X \rightarrow X$ is a closed linear operator. The resolvent set of A is denoted by $\rho(A)$, and the resolvent operator is $R(\lambda, A) := (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. For two functions $f_1, f_2 : [0, \infty) \rightarrow X$ that are integrable on every compact interval, their convolution is defined by

$$(f_1 * f_2)(t) := \int_0^t f_1(s)f_2(t-s) \, ds, \quad t \geq 0.$$

Given a locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$, its Laplace transform (LT) is given by

$$\mathcal{L}\{f\}(\mu) \equiv \hat{f}(\mu) := \int_0^\infty e^{-\mu t} f(t) \, dt,$$

where the integral converges for the given complex parameter μ . Hereafter, $\mathcal{L}$ denotes the LT with respect to the variable t .

Definition 2.1 (RL fractional integral). *Let $\nu > 0$. The RL fractional integral of order ν is*

$$(J_t^\nu f)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\theta)^{\nu-1} f(\theta) \, d\theta, \quad \nu \in (0, 1],$$

and $(J_t^0 f)(t) = f(t)$.

Definition 2.2 (Caputo fractional derivative). *Let X be a Banach space and $u \in AC_{\text{loc}}([0, \infty); X)$, so that $u' \in L^1_{\text{loc}}(\mathbb{R}_+; X)$. For $0 < \gamma \leq 1$ and $t > 0$, the Caputo derivative is defined by*

$${}^C D_t^\gamma u(t) = J_t^{1-\gamma} u'(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} u'(\tau) \, d\tau.$$

Definition 2.3 (RL fractional derivative). *For a function $u(t)$, the RL derivative is defined by*

$${}^{RL} D_t^\gamma u(t) = \frac{d}{dt} J_t^{1-\gamma} u(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t (t-s)^{-\gamma} u(s) \, ds,$$

where $\Gamma(\cdot)$ denotes the Gamma function and $J_t^{1-\gamma}$ represents the RL fractional integral operator of order $1 - \gamma$.

Lemma 2.4 (Caputo–RL relation [18]). *For $0 < \gamma < 1$, the following relation holds:*

$${}^C D_t^\gamma u(t) = {}^{RL} D_t^\gamma u(t) - \frac{u(0) t^{-\gamma}}{\Gamma(1-\gamma)}. \quad (2.1)$$

Lemma 2.5 (Laplace transform identities [12]). *When the transforms exist, the Laplace identities read*

$$\mathcal{L}\{J_t^\mu f\}(\lambda) = \lambda^{-\mu} \hat{f}(\lambda), \quad \mathcal{L}\{^C D_t^\nu u\}(\lambda) = \lambda^\nu \hat{u}(\lambda) - \lambda^{\nu-1} u(0^+).$$

Definition 2.6 (Generalized ML function [19]). *Let $z, \rho, \zeta \in \mathbb{C}$ with $\operatorname{Re}(\rho) > 0$. The generalized ML function is*

$$E_{\rho, \sigma}^\zeta(z) = \sum_{m=0}^{\infty} \frac{(\zeta)_m}{\Gamma(\rho m + \sigma)} \frac{z^m}{m!},$$

where $(\zeta)_m = \Gamma(\zeta + m)/\Gamma(\zeta)$ is the Pochhammer symbol.

Remark 2.7. *By specializing the parameter to $\zeta = 1$, the generalized ML function reduces to the classical two-parameter ML function, that is, $E_{\rho, \sigma}^1(z) = E_{\rho, \sigma}(z)$ for all $z \in \mathbb{C}$. Furthermore, in the special case $\sigma = 1$, this simplifies to the one-parameter ML function E_ρ , namely, $E_{\rho, 1}^1(z) = E_\rho(z)$.*

Definition 2.8 (Prabhakar integral kernel). *Let $\gamma, \delta, \zeta \in \mathbb{C}$ with $\operatorname{Re}(\gamma) > 0$ and $\operatorname{Re}(\delta) > 0$, and let $a \in \mathbb{C}$. The Prabhakar integral kernel $e_{\gamma, \delta, a}^\zeta : \mathbb{R} \rightarrow \mathbb{R}$ is defined as*

$$e_{\gamma, \delta, a}^\zeta(t) = t^{\delta-1} E_{\gamma, \delta}^\zeta(at^\gamma) \quad \text{for } t > 0,$$

and $e_{\gamma, \delta, a}^\zeta(0) = 0$. For the special case $\zeta = 1$, the kernel simplifies to $e_{\gamma, \delta, a} : \mathbb{R} \rightarrow \mathbb{R}$, given by

$$e_{\gamma, \delta, a}(t) = t^{\delta-1} E_{\gamma, \delta}(at^\gamma) \quad \text{for } t > 0, \quad \text{and} \quad e_{\gamma, \delta, a}(0) = 0.$$

Definition 2.9 (ML integral operator [16]). *Let $\gamma, \delta \in \mathbb{C}$ with $\operatorname{Re}(\gamma) > 0$ and $\operatorname{Re}(\delta) > 0$, and let $a \in \mathbb{C}$. The ML integral operator $I_t^{\gamma, \delta, a}$ acts on locally integrable functions $f \in L_{\text{loc}}^1(\mathbb{R}_+)$ as follows:*

$$I_t^{\gamma, \delta, a} f(t) = (e_{\gamma, \delta, a} * f)(t) \quad \text{for } t > 0,$$

and $I_t^{\gamma, \delta, a} f(0) = 0$, where $e_{\gamma, \delta, a}$ is the kernel defined in Definition 2.8.

We repeatedly utilize the LT formula for the generalized ML kernel $e_{\gamma, \delta, a}^\zeta(t)$ [20, 21]:

$$\mathcal{L}\{e_{\gamma, \delta, a}^\zeta\}(\lambda) = \frac{\lambda^{\zeta\gamma-\delta}}{(\lambda^\gamma - a)^\zeta}, \quad \operatorname{Re} \lambda > |a|^{1/\gamma}. \quad (2.2)$$

The following identity and convolution properties of the ML kernels are essential for our analysis.

Lemma 2.10 (ML function identity [20]). *For every positive real number t , the following identity involving ML functions holds:*

$$E_{\gamma-\delta}(-kt^{\gamma-\delta}) + k e_{\gamma-\delta, 1+\gamma-\delta, -k}(t) = 1, \quad t > 0. \quad (2.3)$$

Lemma 2.11 (Convolution properties of ML kernels [21]). *Let $\gamma, \delta, \xi, \vartheta > 0$ and $\delta > \nu > 0$. Then:*

$$\frac{t^{\xi-1}}{\Gamma(\xi)} * e_{\gamma, \delta, a}^\vartheta(t) = e_{\gamma, \delta+\xi, a}^\vartheta(t), \quad (2.4)$$

$$\frac{d}{dt} \left[\frac{t^{-\nu}}{\Gamma(1-\nu)} \right] * e_{\gamma, \delta, a}^\vartheta(t) = e_{\gamma, \delta-\nu, a}^\vartheta(t). \quad (2.5)$$

Remark 2.12. Specifically for Eq (2.4), when $\xi = 1$,

$$\int_0^t e_{\gamma,\delta,a}^\theta(s) ds = e_{\gamma,\delta+1,a}^\theta(t).$$

We now recall the definition and fundamental properties of the Caputo fractional (γ, δ, k) resolvent family, which was introduced in [17] and forms the cornerstone of the present work.

Definition 2.13 (Caputo fractional (γ, δ, k) resolvent [17]). *A family $\{U(t)\}_{t \geq 0}$ of bounded linear operators on a Banach space X is called a Caputo fractional (γ, δ, k) resolvent if the following conditions are satisfied:*

(a) For each $x \in X$, $U(\cdot)x \in C([0, \infty), X)$, and

$$U(0) = I. \quad (2.6)$$

(b) The operators commute for all nonnegative arguments, i.e.,

$$U(v)U(\tau) = U(\tau)U(v), \quad \forall \tau, v \geq 0.$$

(c) For all $\tau, v > 0$, the identity holds:

$$U(v)I_\tau^{\gamma-\delta,\gamma,-k}U(\tau) - I_v^{\gamma-\delta,\gamma,-k}U(v)U(\tau) = I_\tau^{\gamma-\delta,\gamma,-k}U(\tau) - I_v^{\gamma-\delta,\gamma,-k}U(v). \quad (2.7)$$

Remark 2.14. By the strong continuity of the operator family, $\sup_{0 \leq t \leq b} \|U(t)\| < \infty$ for every $b > 0$, the subsequent integrals and convolution terms, such as those involving $I_t^{\gamma-\delta,\gamma,-k}U(t)$ and $\int_0^t U(t-\sigma)f(\sigma) d\sigma$ with $f \in L_{\text{loc}}^1([0, \infty); X)$, are well defined and are used throughout this paper.

Definition 2.15 (Generator of the resolvent family [17]). *The generator A of a Caputo fractional (γ, δ, k) resolvent family $\{U(r)\}_{r \geq 0}$ is defined through its domain*

$$D(A) = \left\{ x \in X : \lim_{r \rightarrow 0^+} \frac{U(r)x - x}{r^\gamma} \text{ exists} \right\},$$

and for each $x \in D(A)$,

$$Ax = \frac{\Gamma(2\gamma)}{\Gamma(\gamma)} \lim_{r \rightarrow 0^+} \frac{U(r)x - x}{r^\gamma}.$$

The following proposition collects the fundamental properties of the generator, which serve as the logical hub for all subsequent analysis.

Proposition 2.16 (Properties of the generator [17]). *Let $\{U(t)\}_{t \geq 0}$ be a Caputo fractional (γ, δ, k) resolvent on a Banach space X with generator A . Then the following properties hold:*

(1a) For all $s \geq 0$, if $x \in D(A)$, then $U(s)x$ remains in $D(A)$ and

$$AU(s)x = U(s)Ax.$$

(1b) For any $x \in X$, the action of the operator $U(t)$ on x satisfies the following relation:

$$U(t)x = x + A I_t^{\gamma-\delta,\gamma,-k}U(t)x, \quad t \geq 0. \quad (2.8)$$

(1c) If $x \in D(A)$ and $t > 0$, then

$$U(t)x = x + I_t^{\gamma-\delta, \gamma, -k} U(t)Ax. \quad (2.9)$$

(1d) The operator A is closed and densely defined in X .

Remark 2.17. The logical chain of Proposition 2.16 proceeds as follows: (1a) follows from the commutativity of the resolvent family and the definition of the generator; (1b) is established using the limit relation

$$\lim_{r \rightarrow 0^+} \frac{\Gamma(2\gamma)}{\Gamma(\gamma)} \frac{I_r^{\gamma-\delta, \gamma, -k} U(r)x}{r^\gamma} = x, \quad (2.10)$$

together with the uniform continuity of $E_{\gamma-\delta, \gamma}(-kr^{\gamma-\delta})$ on $[0, 1]$; (1c) is obtained from (1a) and (1b) by extracting A from the integral using the commutativity and the dominated convergence theorem; and (1d) follows from (1c) and the limit relation (2.10) via an approximation argument.

For exponentially bounded resolvent families, the following equivalent characterization provides the crucial link between the resolvent family and the spectral properties of the generator.

Theorem 2.18 (Equivalent characterization [17]). *For a closed operator A , the following statements are equivalent:*

- (i) A generates an exponentially bounded Caputo fractional (γ, δ, k) resolvent family $\{U(t)\}_{t \geq 0}$ with $\|U(t)\| \leq M_{\text{gen}} e^{\psi t}$.
- (ii) The spectral interval $(\psi^\gamma + k\psi^\delta, \infty) \subset \rho(A)$, and a bounded operator family $\{U(t)\}_{t \geq 0}$ exists such that:
 - (1) The map $t \mapsto U(t)x$ is strongly continuous and $U(0) = I$;
 - (2) $U(\nu)U(\tau) = U(\tau)U(\nu)$ for all $\tau, \nu \geq 0$;
 - (3) $\|U(t)\| \leq M_{\text{gen}} e^{\psi t}$ for $t > 0$;
 - (4) Its LT satisfies:

$$\mathcal{L}\{U(\cdot)x\}(\lambda) = R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})x, \quad \lambda > \psi, \quad x \in X. \quad (2.11)$$

The LT formula (2.11) is the key tool for our decay analysis, as it connects the long-time behavior of $U(t)$ to the spectral properties of A through the symbol $\lambda^\gamma + k\lambda^\delta$.

Remark 2.19 (Dependency on the companion paper [17]). *For the reader's convenience, we explicitly map each object imported from the companion paper [17] to the precise result justifying it:*

- Definition 2.13 (the Caputo fractional (γ, δ, k) resolvent family) corresponds to [17, Definition 2.1];
- Definition 2.15 (the generator) corresponds to [17, Definition 2.4];
- Proposition 2.16 (properties of the generator) corresponds to [17, Theorem 2.6];
- Theorem 2.18 (the equivalent characterization via the Laplace transform formula (2.11)) corresponds to [17, Theorem 3.2], which also yields the uniform bound $\|U(t)\| \leq M_{\text{gen}}$ used throughout the present paper (to avoid overloading, we consistently use M_{gen} to denote the generation bound of Theorem 2.18, M_{lin} to denote the linear decay constants of Theorem 4.1 and (5.6) below, and M_V to denote the kernel constant of (5.7) below);

- The integrated convolution relation (5.4) between $V(t)$ and $U(t)$ follows from the Laplace transform identity $(\lambda^{\gamma-1} + k\lambda^{\delta-1})/(\lambda^{\gamma} + k\lambda^{\delta}) = 1/\lambda$ together with [17, Theorem 3.2];
- The formal Prabhakar-type expansion of $V(t)$ sketched in Appendix A is obtained by applying the Prabhakar convolution structure of [17, Section 2] eigenfunction-by-eigenfunction; its convergence is not used in any proof.

We emphasize that, under the hypotheses of the present paper (self-adjoint negative-definite A with compact resolvent on a Hilbert space, or more generally $-A$ generating a bounded analytic semigroup), the existence of an exponentially bounded (γ, δ, k) resolvent family is guaranteed by [17, Theorem 3.2], so all subsequent analysis is well-posed. The “DOI: 10.3390/axioms15030179” is publicly accessible.

3. Laplace transform analysis of the resolvent family

In this section, we perform a detailed Laplace transform analysis of the Caputo fractional (γ, δ, k) -resolvent family $\{U(t)\}_{t \geq 0}$ established in the preceding sections. The key observation is that the Laplace transform formula derived in Theorem 2.18 provides a direct link between the spectral properties of the generator A and the asymptotic behavior of $U(t)$. By decomposing the inverse Laplace transform into contributions from poles and branch cuts, we obtain a spectral representation of $U(t)$ that serves as the foundation for the decay estimates in Section 4.

3.1. Characteristic equation and root structure

Throughout this section, we assume that $-A$ generates a bounded analytic semigroup on X . Then $-A$ is sectorial of some type $\omega \in (0, \pi/2)$: the spectrum satisfies

$$\sigma(A) \subset \{\mu \in \mathbb{C} : |\arg(-\mu)| \leq \omega\} \cup \{0\},$$

which is in general a proper superset of $(-\infty, 0]$, and the resolvent bound $\|R(\mu, A)\| \leq C_{\omega}/|\mu|$ holds in the sector $\{\mu \in \mathbb{C} : |\arg \mu| < \pi - \omega\}$; in particular, $(0, \infty) \subset \rho(A)$. We distinguish two regimes used throughout the paper: (a) the real-spectrum regime $\sigma(A) \subset (-\infty, 0]$ (in particular, the self-adjoint setting of Section 4), in which the pole analysis of Proposition 3.3 applies to every spectral point for all $0 < \delta < \gamma \leq 1$; and (b) the general sectorial regime, in which the absence of poles and the resolvent bounds along the branch cut are guaranteed only under the restriction $\gamma < 1$ together with the compatibility condition $\omega < \pi(1 - \gamma)$ (see Proposition 3.6 and Theorem 4.5). For each real $\mu < 0$ (in particular, for each $\mu \in \sigma(A)$ in regime (a)), the characteristic equation associated with the Laplace transform of $U(t)$ is

$$\lambda^{\gamma} + k\lambda^{\delta} = \mu, \quad 0 < \delta < \gamma \leq 1, \quad k > 0. \quad (3.1)$$

We study the roots of (3.1) in the complex plane $\mathbb{C}$, where λ^{γ} and λ^{δ} are understood via the principal branch of the complex power function with branch cut along the negative real axis.

Lemma 3.1. *The characteristic equation (3.1) has no roots in the closed right half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\}$ for any $\mu < 0$.*

Proof. Suppose λ is a root of (3.1) with $\operatorname{Re} \lambda \geq 0$. Write $\lambda = re^{i\theta}$ with $r \geq 0$ and $|\theta| \leq \pi/2$. Then

$$\lambda^{\gamma} + k\lambda^{\delta} = r^{\gamma}e^{i\gamma\theta} + kr^{\delta}e^{i\delta\theta}.$$

Taking the real part and using $\cos(\gamma\theta) \geq 0$, $\cos(\delta\theta) \geq 0$ for $|\theta| \leq \pi/2$ (since $0 < \delta < \gamma \leq 1$), we obtain

$$\operatorname{Re}(\lambda^\gamma + k\lambda^\delta) = r^\gamma \cos(\gamma\theta) + kr^\delta \cos(\delta\theta) \geq 0.$$

However, $\mu < 0$ implies $\operatorname{Re} \mu < 0$, which contradicts $\lambda^\gamma + k\lambda^\delta = \mu$. Hence no such root exists. $\square$

Remark 3.2. Lemma 3.1 shows that the characteristic equation has no roots in the closed right half-plane. Combined with Proposition 3.3, this yields a complete picture: For $0 < \delta < \gamma \leq 1$, there are no roots anywhere in the cut plane $\mathbb{C} \setminus (-\infty, 0]$; only in the integer-order case $\gamma = \delta = 1$ does a (real negative) root exist. (Here the characteristic equation is taken with a real $\mu < 0$, so this statement covers the real-spectrum regime (a); for a general sectorial generator with non-real spectrum, the pole-free property relevant to the decay analysis is guaranteed only for $0 < \delta < \gamma < 1$ under the compatibility condition $\omega < \pi(1 - \gamma)$; see Proposition 3.6 and Theorem 4.5.)

Proposition 3.3. Let $\mu < 0$ and $f(\lambda) := \lambda^\gamma + k\lambda^\delta$.

- (i) For $0 < \delta < \gamma < 1$, the characteristic equation $\lambda^\gamma + k\lambda^\delta = \mu$ has no root in the cut plane $\mathbb{C} \setminus (-\infty, 0]$.
- (ii) For $\gamma = 1$ and $0 < \delta < 1$, the characteristic equation $\lambda + k\lambda^\delta = \mu$ has no root in the cut plane $\mathbb{C} \setminus (-\infty, 0]$.
- (iii) For $\gamma = \delta = 1$ (integer-order damping), the equation reduces to $(1 + k)\lambda = \mu$, which has the unique root $\lambda = \mu/(1 + k)$.

Proof. (i) Let $0 < \delta < \gamma < 1$ (the case $\gamma = 1$ is treated separately in (ii)). We show that $f(\lambda) \neq \mu$ for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ by examining three regions.

Region 1: $\operatorname{Re} \lambda \geq 0$. Write $\lambda = re^{i\theta}$ with $r \geq 0$ and $|\theta| \leq \pi/2$. Since $0 < \delta < \gamma \leq 1$, we have $\cos(\gamma\theta) \geq 0$ and $\cos(\delta\theta) \geq 0$ for $|\theta| \leq \pi/2$, hence

$$\operatorname{Re} f(\lambda) = r^\gamma \cos(\gamma\theta) + kr^\delta \cos(\delta\theta) \geq 0.$$

But $\mu < 0$ implies $\operatorname{Re} \mu < 0$, so $f(\lambda) \neq \mu$ in this region.

Region 2: $\operatorname{Re} \lambda < 0$ and $\operatorname{Im} \lambda > 0$. Write $\lambda = re^{i\theta}$ with $\pi/2 < \theta < \pi$. Then $\gamma\theta \in (\gamma\pi/2, \gamma\pi) \subset (0, \gamma\pi] \subset (0, \pi)$ and $\delta\theta \in (\delta\pi/2, \delta\pi) \subset (0, \delta\pi] \subset (0, \pi)$, so $\sin(\gamma\theta) > 0$ and $\sin(\delta\theta) > 0$. Hence

$$\operatorname{Im} f(\lambda) = r^\gamma \sin(\gamma\theta) + kr^\delta \sin(\delta\theta) > 0.$$

Since $\operatorname{Im} \mu = 0$, we have $f(\lambda) \neq \mu$ in this region.

Region 3: $\operatorname{Re} \lambda < 0$ and $\operatorname{Im} \lambda < 0$. By conjugation symmetry, $f(\bar{\lambda}) = \overline{f(\lambda)}$, so $\operatorname{Im} f(\lambda) < 0$ whenever $\operatorname{Im} \lambda < 0$. Since $\operatorname{Im} \mu = 0$, again $f(\lambda) \neq \mu$.

Combining the three regions, $f(\lambda) \neq \mu$ for all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$.

(ii) Let $\gamma = 1$ and $0 < \delta < 1$. We verify that $f(\lambda) = \lambda + k\lambda^\delta \neq \mu$ for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ by examining three regions.

Region 1: $\operatorname{Re} \lambda \geq 0$. Write $\lambda = re^{i\theta}$ with $|\theta| \leq \pi/2$. Then $\operatorname{Re} f(\lambda) = r \cos \theta + kr^\delta \cos(\delta\theta) \geq 0 > \operatorname{Re} \mu$, since $\cos \theta \geq 0$ and $\cos(\delta\theta) > 0$ for $|\delta\theta| < \pi/2$.

Region 2: $\operatorname{Re} \lambda < 0$ and $\operatorname{Im} \lambda > 0$. Write $\lambda = re^{i\theta}$ with $\pi/2 < \theta < \pi$. Since $0 < \delta < 1$, we have $\delta\theta \in (\delta\pi/2, \delta\pi) \subset (0, \pi)$, so $\sin(\delta\theta) > 0$. Also $\sin \theta > 0$. Hence $\operatorname{Im} f(\lambda) = r \sin \theta + kr^\delta \sin(\delta\theta) > 0 \neq \operatorname{Im} \mu$.

Region 3: $\operatorname{Re} \lambda < 0$ and $\operatorname{Im} \lambda < 0$. By conjugation symmetry, $\operatorname{Im} f(\lambda) < 0 \neq 0 = \operatorname{Im} \mu$.

Combining the three regions, $f(\lambda) \neq \mu$ for all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$.

(iii) When $\gamma = \delta = 1$, the equation $\lambda^\gamma + k\lambda^\delta = \mu$ reduces to $(1+k)\lambda = \mu$, which has the unique root $\lambda = \mu/(1+k)$. $\square$

Remark 3.4. Proposition 3.3 is established for real $\mu < 0$. Consequently, its conclusion—that for $0 < \delta < \gamma \leq 1$ the characteristic equation has no roots in the cut plane $\mathbb{C} \setminus (-\infty, 0]$ —applies directly in the real-spectrum regime $\sigma(A) \subset (-\infty, 0]$ of Section 3, in particular in the self-adjoint setting of Section 4. For a general sectorial generator with a nonreal spectrum, the absence of poles in the cut plane is guaranteed only for $0 < \delta < \gamma < 1$ under the compatibility condition $\omega < \pi(1-\gamma)$; see Proposition 3.6 and Theorem 4.5. In these settings, the inverse Laplace transform of the resolvent family receives no pole contribution whatsoever—the entire time-domain representation comes from the branch cut integral. Only in the integer-order case $\gamma = \delta = 1$ does a pole (and hence exponential decay) appear. We emphasize that the case $\gamma = \delta = 1$ lies outside the standing assumptions $0 < \delta < \gamma \leq 1$ (which require the strict inequality $\delta < \gamma$) and is mentioned throughout this paper solely for comparison with the classical integer-order situation. This unified spectral structure has far-reaching consequences for the decay analysis in Section 4.

3.2. Spectral decomposition via the inverse Laplace transform

From Theorem 2.18, the Laplace transform of the resolvent family acting on $x \in X$ is

$$\mathcal{L}\{U(\cdot)x\}(\lambda) = R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})x, \quad \lambda > 0. \quad (3.2)$$

To recover $U(t)x$, we employ the inverse Laplace transform along a Hankel-type contour. The key insight is that the function $\lambda \mapsto R(\lambda^\gamma + k\lambda^\delta, A)$ has two types of singularities in the λ -plane:

- (1) **Poles:** For $0 < \delta < \gamma \leq 1$, there are no poles in the cut plane $\mathbb{C} \setminus (-\infty, 0]$ (by Proposition 3.3(i)–(ii), applied to real $\mu < 0$; in the real-spectrum regime $\sigma(A) \subset (-\infty, 0]$ this covers the whole spectrum, while in the general sectorial regime it is guaranteed by $\gamma < 1$ together with $\omega < \pi(1-\gamma)$, see Proposition 3.6). Only in the integer-order case $\gamma = \delta = 1$ does a pole appear (by Proposition 3.3(iii)).
- (2) **Branch cuts:** The functions λ^γ and λ^δ introduce branch cuts along the negative real axis $(-\infty, 0)$.

Theorem 3.5. Let $-A$ generate a bounded analytic semigroup on X with compact resolvent, and assume that the spectrum is real and negative, $\sigma(A) = \{\mu_n\}_{n=1}^\infty$ with $\mu_n < 0$ and $|\mu_n| \rightarrow \infty$ (this real-spectrum hypothesis is automatic in the self-adjoint setting of Section 4; for general sectorial generators with non-real spectrum, the absence of poles asserted below requires $0 < \delta < \gamma < 1$ and $\omega < \pi(1-\gamma)$, cf. Proposition 3.6). Then for $x \in X$ and $t > 0$:

For $0 < \delta < \gamma \leq 1$, the inverse Laplace transform is entirely given by the branch cut integral:

$$U(t)x = \frac{1}{2\pi i} \int_{\text{BC}} e^{t\lambda} R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})x d\lambda, \quad (3.3)$$

where BC denotes the contour wrapping around the branch cut on $(-\infty, 0)$. There is no pole contribution since the characteristic equation has no roots in $\mathbb{C} \setminus (-\infty, 0]$ by Proposition 3.3(i) and (ii).

For the integer-order case $\gamma = \delta = 1$, the equation reduces to $(1+k)u' = Au$, and $U(t) = e^{tA/(1+k)}$ decays exponentially.

Proof. We evaluate the inverse Laplace transform

$$U(t)x = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda t} R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})x \, d\lambda, \quad c > 0,$$

by deforming the Bromwich contour into the left half-plane. Since $-A$ generates a bounded analytic semigroup, $R(\lambda^\gamma + k\lambda^\delta, A)$ is well-defined and analytic for $\operatorname{Re}(\lambda^\gamma + k\lambda^\delta) > 0$, which includes a sector in the right half-plane.

For $0 < \delta < \gamma \leq 1$, by Proposition 3.3 (i) and (ii), the characteristic equation $\lambda^\gamma + k\lambda^\delta = \mu_n$ has no roots in $\mathbb{C} \setminus (-\infty, 0]$ for any $\mu_n < 0$. (Here, Proposition 3.3 applies because each μ_n is real and negative by hypothesis; this is precisely why the real-spectrum assumption is imposed, and it covers the case $\gamma = 1$ in the present setting.) Therefore, the integrand has no poles in the cut plane, and the Bromwich contour can be deformed directly to a keyhole contour wrapping around the branch cut $(-\infty, 0)$, consisting of the following:

- A large circular arc of radius $R \rightarrow \infty$ in the left half-plane;
- Two parallel lines hugging the branch cut $(-\infty, 0)$ from above and below.

Since $\operatorname{Re} \lambda \rightarrow -\infty$ on the large arc, the factor $e^{\lambda t}$ provides exponential decay that dominates the polynomial growth of the resolvent, so the contribution from the large circular arc vanishes as $R \rightarrow \infty$; cf. [13, Chapter 3]. The remaining integral along the branch cut gives (3.3).

For the integer-order case $\gamma = \delta = 1$, the equation reduces to $(1 + k)u' = Au$, and the result follows from classical semigroup theory. $\square$

We now analyze the integral along the branch cut in detail. Setting $\lambda = se^{\pm i\pi}$ for $s > 0$ (upper and lower edges of the cut), we have $\lambda^\alpha = s^\alpha e^{\pm i\pi\alpha}$ for $\alpha \in \{\gamma, \delta, \gamma - 1, \delta - 1\}$.

Proposition 3.6. *Let $-A$ generate a bounded analytic semigroup on X , so that $-A$ is sectorial of type $\omega \in (0, \pi/2)$ with resolvent bound $\|R(\mu, A)\| \leq C_\omega/|\mu|$ for μ in the sector $\{\mu \in \mathbb{C} : |\arg \mu| < \pi - \omega\}$. Assume $0 < \delta < \gamma < 1$ and the quantitative compatibility condition*

$$\omega < \pi(1 - \gamma), \quad (3.4)$$

which yields two facts used below and in Theorem 4.5. (i) No poles in the cut plane: for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ one has $|\arg(\lambda^\gamma + k\lambda^\delta)| < \pi\gamma < \pi - \omega$ (both summands have arguments of a modulus at most $\pi\gamma < \pi$) and $\lambda^\gamma + k\lambda^\delta \neq 0$ (since $\lambda^{\gamma-\delta} = -k$ has no solution with $|\arg \lambda| < \pi$); hence, the image of the cut plane under the symbol $\lambda \mapsto \lambda^\gamma + k\lambda^\delta$ is contained in the sector $\{|\arg \mu| < \pi - \omega\}$, which is disjoint from $\sigma(A) \setminus \{0\} \subset \{\mu \in \mathbb{C} : |\arg(-\mu)| \leq \omega\}$, i.e., $|\arg \mu| \geq \pi - \omega$. (ii) Resolvent bound along the cut: $\mu(s) = s^\gamma e^{\pm i\pi\gamma} + ks^\delta e^{\pm i\pi\delta}$ satisfies

$$\pi\delta \leq |\arg \mu(s)| \leq \pi\gamma < \pi - \omega \quad \text{for all } s > 0,$$

so $\mu(s)$ stays inside the resolvent sector $\{|\arg \mu| < \pi - \omega\}$ for all $s > 0$ —not merely for s near 0^+ —and the bound $\|R(\mu(s), A)\| \leq C_\omega/|\mu(s)|$ holds globally along the cut. Then the branch cut contribution to $U(t)x$ is given by the pair of edge integrals

$$U_{\text{BC}}(t)x = \frac{1}{2\pi i} (I_+(t)x + I_-(t)x), \quad (3.5)$$

where (with $d\lambda = -ds$ on both edges)

$$I_+(t)x = - \int_0^\infty e^{-st} R(s^\gamma e^{i\pi\gamma} + ks^\delta e^{i\pi\delta}, A)(s^{\gamma-1} e^{i\pi(\gamma-1)} + ks^{\delta-1} e^{i\pi(\delta-1)})x ds, \quad (3.6)$$

$$I_-(t)x = + \int_0^\infty e^{-st} R(s^\gamma e^{-i\pi\gamma} + ks^\delta e^{-i\pi\delta}, A)(s^{\gamma-1} e^{-i\pi(\gamma-1)} + ks^{\delta-1} e^{-i\pi(\delta-1)})x ds. \quad (3.7)$$

Moreover, if, in addition, $0 \in \rho(A)$ (equivalently, $A^{-1} \in \mathcal{L}(X)$), then a constant $C > 0$ (depending on $\gamma, \delta, k, \omega$ and $\|A^{-1}\|$ but not on t) such that

$$\|U_{\text{BC}}(t)A^{-1}\| \leq C(1+t)^{-\delta}, \quad t \geq 1. \quad (3.8)$$

We emphasize that the pointwise bound on the unregularized $U_{\text{BC}}(t)$ obtained from the triangle inequality behaves, near $s = 0$, like $\int_0^1 ds/s$ and is therefore vacuous; the meaningful estimate is the A^{-1} -regularized one above, which is what the decay analysis of Theorem 4.5 uses.

Proof. On the upper edge of the cut, $\lambda = se^{i\pi}$ traversed from $s = 0$ to $s = \infty$, so $d\lambda = e^{i\pi}ds = -ds$ and the contribution is $I_+(t)x$ as in (3.6). On the lower edge, $\lambda = se^{-i\pi}$ traversed from $s = \infty$ to $s = 0$, so $d\lambda = e^{-i\pi}ds = -ds$ and the contribution is $I_-(t)x$ as in (3.7). Formula (3.5) follows from the standard keyhole contour deformation (the large-arc contribution vanishes by the exponential factor e^{lt}).

For the estimate (3.8), set

$$\mu(s) := s^\gamma e^{i\pi\gamma} + ks^\delta e^{i\pi\delta}, \quad s > 0.$$

Since $0 < \delta < \gamma < 1$, both summands of $\mu(s)$ have arguments in $[\pi\delta, \pi\gamma]$, so

$$\pi\delta \leq |\arg \mu(s)| \leq \pi\gamma < \pi - \omega \quad \text{for all } s > 0,$$

by the compatibility condition (3.4). Hence $\mu(s)$ lies inside the resolvent sector $\{|\arg \mu| < \pi - \omega\}$ for all $s > 0$ —not merely for s near 0^+ —so $R(\mu(s), A)$ is defined for all $s > 0$; moreover, $|\operatorname{Im} \mu(s)| = s^\gamma \sin(\pi\gamma) + ks^\delta \sin(\pi\delta) > 0$, so $\mu(s)$ never lies on the real axis. By conjugation, the lower edge is handled identically (with $|\mu(s)|$ unchanged), so it suffices to estimate the upper edge.

Assume now $0 \in \rho(A)$.

Regime $0 < s \leq 1$: the set $K := \{\mu(s) : 0 < s \leq 1\} \cup \{0\}$ is compact and contained in $\rho(A)$ (each $\mu(s)$ lies in the resolvent sector, and $0 \in \rho(A)$ by hypothesis), and the map $\mu \mapsto R(\mu, A)A^{-1}$ is continuous on $\rho(A)$; hence

$$\sup_{0 < s \leq 1} \|R(\mu(s), A)A^{-1}\| \leq C_0.$$

Regime $s \geq 1$: the s^γ -term dominates, $|\mu(s)| \geq c s^\gamma$ for some $c > 0$, and submultiplicativity together with the sectorial bound gives

$$\|R(\mu(s), A)A^{-1}\| \leq \|R(\mu(s), A)\| \|A^{-1}\| \leq \frac{C_\omega \|A^{-1}\|}{c s^\gamma}, \quad s \geq 1.$$

Taking norms in (3.5) (the scalar factors $s^{\gamma-1} e^{i\pi(\gamma-1)} + ks^{\delta-1} e^{i\pi(\delta-1)}$ have modulus at most $s^{\gamma-1} + ks^{\delta-1}$), we obtain, for $t \geq 1$,

$$\|U_{\text{BC}}(t)A^{-1}\| \leq C \int_0^\infty e^{-st} \|R(\mu(s), A)A^{-1}\| (s^{\gamma-1} + ks^{\delta-1}) ds$$

$$\begin{aligned} &\leq C \left[\int_0^\infty e^{-st} (s^{\gamma-1} + ks^{\delta-1}) ds + \int_1^\infty e^{-st} (s^{-1} + ks^{\delta-1-\gamma}) ds \right] \\ &\leq C \left[(\Gamma(\gamma) + k\Gamma(\delta))t^{-\delta} + \int_1^\infty e^{-st} (s^{-1} + ks^{-1}) ds \right], \end{aligned}$$

where we used $t^{-\gamma} \leq t^{-\delta}$ for $t \geq 1$ (as $\gamma > \delta$) and $s^{\delta-1-\gamma} \leq s^{-1}$ for $s \geq 1$ (as $\delta - \gamma < 0$). Finally, the substitution $u = st$ yields

$$\int_1^\infty e^{-st} s^{-1} ds = \int_t^\infty \frac{e^{-u}}{u} du \leq \frac{e^{-t}}{t} \leq c_\delta (1+t)^{-\delta}, \quad t \geq 1,$$

with $c_\delta := \sup_{t \geq 1} e^{-t}(1+t)^\delta/t < \infty$. Combining the estimates yields (3.8). $\square$

Remark 3.7 (On the scalar form). *In the general Banach-space setting of Section 3, the two edge integrals (3.6) and (3.7) cannot, in general, be combined into a single “ $\text{Im}[\cdot]$ ” integral, because such a reduction requires an additional structural hypothesis on A (self-adjointness on a Hilbert space, or at least that A is a “real” operator commuting with conjugation, in which case, $R(\bar{\mu}, A) = \overline{R(\mu, A)}$ as operators). The self-adjoint case is treated separately in Section 4 (see (4.7)), where the scalar reduction to $\text{Im}[\cdot]$ is rigorously justified.*

Remark 3.8. *The branch cut integral (3.5) can be interpreted as a generalized ML type contribution. In the absence of damping ($k = 0$), it reduces to the classical result for the fractional evolution equation ${}^C D_t^\gamma u = Au$, where the branch cut produces a $t^{-\gamma}$ decay. The damping term $k {}^C D_t^\delta u$ modifies the branch cut structure and, as we shall see in Section 4, alters the polynomial decay rate.*

4. Decay rates for the linear equation

Having established the Laplace transform representation and spectral decomposition of the resolvent family in Section 3, we now turn to the central question of this paper: How fast does $U(t)$ decay as $t \rightarrow \infty$? The answer depends crucially on the interplay between the two fractional orders γ and δ . We first treat the self-adjoint case, where the spectral theorem enables a precise branch cut analysis via Watson’s lemma, yielding the polynomial rate $\mathcal{O}(t^{-\delta})$ which is sharp on the first (lowest-eigenvalue) mode. We then extend the results to general sectorial generators, in the restricted regime $0 < \delta < \gamma < 1$ under the compatibility condition $\omega < \pi(1 - \gamma)$ on the sectorial type (see Theorem 4.5), where a weaker estimate holds. The section concludes with a detailed analysis of how the damping parameters influence the decay behavior.

Notation. In the present section, where $\sigma(A) = \{-\lambda_n\}_{n=1}^\infty$ with $\lambda_n > 0$, we write λ_n^* for the unique real negative root of $\lambda + k\lambda^\delta = -\lambda_n$ (which exists only when $\gamma = \delta = 1$, the integer-order case), to distinguish it from the eigenvalue parameter λ_n of $-A$. For $0 < \delta < \gamma \leq 1$, there is no such root by Proposition 3.3(i) and (ii). We recall that the case $\gamma = \delta = 1$ lies outside the standing assumptions $0 < \delta < \gamma \leq 1$ (the strict inequality $\delta < \gamma$ fails) and is invoked solely for comparison with the classical integer-order damped equation.

4.1. Decay rates for the self-adjoint case

Throughout this subsection, we assume that $X = H$ is a Hilbert space and $-A$ is a self-adjoint positive definite operator with compact resolvent. Let $\sigma(A) = \{-\lambda_n\}_{n=1}^\infty$ where $0 < \lambda_1 \leq \lambda_2 \leq \dots$

and $\lambda_n \rightarrow \infty$, and let $\{e_n\}_{n=1}^\infty$ be the corresponding orthonormal eigenbasis. For each $n \geq 1$, the scalar function

$$y_n(t) := \langle U(t)e_n, e_n \rangle \quad (4.1)$$

satisfies the scalar Caputo fractional differential equation

$${}^C D_t^\gamma y_n(t) + k {}^C D_t^\delta y_n(t) = -\lambda_n y_n(t), \quad y_n(0) = 1, \quad (4.2)$$

and the Laplace transform of y_n is given by

$$\hat{y}_n(\lambda) = \mathcal{L}\{y_n\}(\lambda) = \frac{\lambda^{\gamma-1} + k\lambda^{\delta-1}}{\lambda^\gamma + k\lambda^\delta + \lambda_n}, \quad \lambda > 0. \quad (4.3)$$

The resolvent family then admits the spectral representation

$$U(t)x = \sum_{n=1}^{\infty} y_n(t) \langle x, e_n \rangle e_n, \quad x \in H, \quad (4.4)$$

and consequently

$$\|U(t)\| = \sup_{n \geq 1} |y_n(t)|. \quad (4.5)$$

Theorem 4.1. *Let $-A$ be a self-adjoint positive definite operator with compact resolvent on a Hilbert space H . Then the Caputo fractional (γ, δ, k) -resolvent family $\{U(t)\}_{t \geq 0}$ satisfies*

$$\|U(t)\| \leq M_{\text{lin}}(1+t)^{-\delta}, \quad t \geq 0, \quad (4.6)$$

for some constant $M_{\text{lin}} > 0$ depending on γ, δ, k , and λ_1 ; more precisely, the short-time bound entering M_{lin} is obtained via the generation constant M_{gen} of Theorem 2.18 (see Step 2 of the proof, where $M_0 = M_{\text{gen}}$), so M_{lin} depends on $(\gamma, \delta, k, \lambda_1)$ together with the uniform bound M_{gen} of the resolvent family, the latter being determined by the operator A .

Proof. The proof proceeds in several steps, treating the cases $\gamma < 1$ and $\gamma = 1$ separately where necessary.

Step 1: Long-time asymptotics of $y_n(t)$ via branch cut analysis. From the inverse Laplace transform representation,

$$y_n(t) = \frac{1}{2\pi i} \int_{\text{Br}} e^{\lambda t} \frac{\lambda^{\gamma-1} + k\lambda^{\delta-1}}{\lambda^\gamma + k\lambda^\delta + \lambda_n} d\lambda,$$

where Br denotes the Bromwich contour. Deforming the contour as in Section 3:

Case $\gamma < 1$: By Proposition 3.3(i), the characteristic equation $\lambda^\gamma + k\lambda^\delta = -\lambda_n$ has no roots in $\mathbb{C} \setminus (-\infty, 0]$, so there is no pole contribution. The entire inverse Laplace transform is given by the branch cut integral:

$$y_n(t) = y_n^{\text{BC}}(t) = -\frac{1}{\pi} \int_0^\infty e^{-st} \operatorname{Im} \left[\frac{s^{\gamma-1} e^{i\pi(\gamma-1)} + k s^{\delta-1} e^{i\pi(\delta-1)}}{s^\gamma e^{i\pi\gamma} + k s^\delta e^{i\pi\delta} + \lambda_n} \right] ds. \quad (4.7)$$

Case $\gamma = 1$: By Proposition 3.3(ii), the characteristic equation $\lambda + k\lambda^\delta = -\lambda_n$ has no roots in the cut plane for $0 < \delta < 1$, so there is no pole contribution. (This application of Proposition 3.3(ii) is

legitimate because $\sigma(A) = \{-\lambda_n\}_{n=1}^\infty \subset (-\infty, 0)$ is real in the present self-adjoint setting; for general sectorial generators with non-real spectrum, the case $\gamma = 1$ is excluded from Theorem 4.5.) The entire inverse Laplace transform is given by the branch cut integral (4.7) with $\gamma = 1$.

In both cases, the asymptotic behavior for large t is governed by the small- s region of the branch cut integral. For $s \rightarrow 0^+$, the integrand behaves as

$$\operatorname{Im}\left[\frac{k s^{\delta-1} e^{i\pi(\delta-1)}}{\lambda_n}\right] = \frac{k \sin(\pi(\delta-1))}{\lambda_n} s^{\delta-1} = -\frac{k \sin(\pi\delta)}{\lambda_n} s^{\delta-1}.$$

Since $\sin(\pi\delta) > 0$ for $0 < \delta < 1$, the imaginary part of the integrand is negative; combined with the minus sign in the coefficient $-1/\pi$ of (4.7), the overall asymptotic is positive. By Watson's lemma (see, e.g., [22]),

$$y_n^{\text{BC}}(t) \sim +\frac{k \sin(\pi\delta)}{\pi \lambda_n} \int_0^\infty e^{-st} s^{\delta-1} ds = +\frac{k \sin(\pi\delta)\Gamma(\delta)}{\pi \lambda_n} t^{-\delta}.$$

Using the reflection formula $\Gamma(\delta)\Gamma(1-\delta) = \pi/\sin(\pi\delta)$, we obtain $\Gamma(\delta)\sin(\pi\delta)/\pi = 1/\Gamma(1-\delta)$, and therefore

$$y_n^{\text{BC}}(t) \sim +\frac{k}{\lambda_n \Gamma(1-\delta)} t^{-\delta} \quad \text{as } t \rightarrow \infty. \quad (4.8)$$

For all $0 < \delta < \gamma \leq 1$, this is the complete asymptotics of $y_n(t)$. The absolute value satisfies

$$|y_n(t)| \sim \frac{k}{\lambda_n \Gamma(1-\delta)} t^{-\delta} \quad \text{as } t \rightarrow \infty. \quad (4.9)$$

Step 2: Uniform bound for all $t \geq 0$. For short times, we use the series representation of $y_n(t)$. From the resolvent equation and the properties of the (γ, δ, k) -resolvent family (cf. Theorem 2.18), the solution $y_n(t)$ can be expressed in terms of a generalized ML type function. By [17, Theorem 3.2], the resolvent family satisfies $\|U(t)\| \leq M_{\text{gen}}$ for all $t \geq 0$, so, taking $M_0 := M_{\text{gen}}$, the following holds

$$|y_n(t)| \leq M_0, \quad 0 \leq t \leq 1, \quad n \geq 1. \quad (4.10)$$

Indeed, since $y_n(t) = \langle U(t)e_n, e_n \rangle$ by the spectral representation (4.4), the Cauchy–Schwarz inequality gives $|y_n(t)| \leq \|U(t)e_n\| \|e_n\| = \|U(t)e_n\| \leq \|U(t)\| \leq M_{\text{gen}}$, which is uniform in n and $t \in [0, 1]$.

For intermediate and long times, we estimate the branch cut integral directly. For all $0 < \delta < \gamma \leq 1$, there is no pole contribution by Proposition 3.3(i) and (ii), so $y_n(t) = y_n^{\text{BC}}(t)$. The key estimate is on the branch cut integral. We split the integration range into $s \in [0, \varepsilon]$ and $s \in [\varepsilon, \infty)$ for a fixed small $\varepsilon > 0$ (chosen independently of n ; specifically, $\varepsilon = \min\{(\lambda_1/4k)^{1/\delta}, (\lambda_1/4)^{1/\gamma}\}$, so that $s^\gamma \leq \lambda_1/4$ and $ks^\delta \leq \lambda_1/4$ for $s \leq \varepsilon$, ensuring $|\mu(s)| \leq \lambda_1/2 \leq \lambda_n/2$):

$$|y_n^{\text{BC}}(t)| \leq \underbrace{\frac{1}{\pi} \int_0^\varepsilon e^{-st} \frac{C_1 s^{\delta-1}}{\lambda_n} ds}_{I_1} + \underbrace{\frac{1}{\pi} \int_\varepsilon^\infty e^{-st} \frac{C_2 (s^{\gamma-1} + ks^{\delta-1})}{|\mu(s) + \lambda_n|} ds}_{I_2}.$$

For I_1 , we apply Watson's lemma directly: $I_1 \leq C_1 \Gamma(\delta)/(\pi \lambda_n) \cdot t^{-\delta}$, where the constant C_1 depends on ε but is independent of n . To justify the n -independence of C_1 rigorously: The point of Watson's lemma is that the short-time (small- s) behavior $f(s) = O(s^{\delta-1})$ determines the leading large- t asymptotic. Since ε is chosen as $\varepsilon = \min\{(\lambda_1/4k)^{1/\delta}, (\lambda_1/4)^{1/\gamma}\}$, it depends only on λ_1 and is a strictly positive

constant; on $[0, \varepsilon]$ the expansion of the integrand in powers of s has leading coefficient $k \sin(\pi\delta)/\lambda_n \leq k \sin(\pi\delta)/\lambda_1$ and higher-order coefficients $c_j(\gamma, \delta, k)/\lambda_n^{j+1} \leq c_j(\gamma, \delta, k)/\lambda_1^{j+1}$, all uniformly bounded in n . The remainder in the Taylor expansion is $O(s^\delta)$ with an O -constant depending on $(\gamma, \delta, k, \varepsilon, \lambda_1)$ but not on n . Thus the Watson bound for I_1 is $C_1\Gamma(\delta)/(\pi\lambda_n)t^{-\delta} \leq C_1\Gamma(\delta)/(\pi\lambda_1)t^{-\delta}$ with $C_1 = C_1(\gamma, \delta, k, \varepsilon, \lambda_1)$ only. (A fully quantified version of Watson's lemma with an explicit remainder bound, as in [22, Ch. 4], yields the same conclusion.) For I_2 , we need a lower bound on $|\mu(s) + \lambda_n|$. For $s \leq s_0(\lambda_n)$ (where s_0 is chosen so that $|\mu(s)| \leq \lambda_n/2$), we have $|\mu(s) + \lambda_n| \geq \lambda_n - |\mu(s)| \geq \lambda_n/2$; for $s > s_0(\lambda_n)$, we use $|\mu(s) + \lambda_n| \geq |\operatorname{Im} \mu(s)| = s^\gamma \sin(\pi\gamma) + ks^\delta \sin(\pi\delta) > 0$. Since $\lambda_n \geq \lambda_1$, $s_0(\lambda_n)$ is monotonically increasing in λ_n , so $\varepsilon \leq s_0(\lambda_1) \leq s_0(\lambda_n)$ for all $n \geq 1$. Since $s \geq \varepsilon > 0$, the bound $|\operatorname{Im} \mu(s)| \geq c(\varepsilon) > 0$ holds uniformly in n , and the exponential factor $e^{-st} \leq e^{-\varepsilon t}$ provides rapid decay. Thus $I_2 \leq C'_2 e^{-\varepsilon t}$ where C'_2 is independent of n . Combining,

$$|y_n^{\text{BC}}(t)| \leq \frac{C_1\Gamma(\delta)}{\pi\lambda_n} t^{-\delta} + C'_2 e^{-\varepsilon t}. \quad (4.11)$$

Since $C_1/\lambda_n \leq C_1/\lambda_1$ uniformly in n , and $e^{-\varepsilon t} \leq C(1+t)^{-\delta}$, we obtain

$$|y_n(t)| \leq C'(1+t)^{-\delta}, \quad t \geq 1, \quad (4.12)$$

where C' depends on γ, δ, k , and λ_1 (through ε and C_1), but is independent of $n \geq 1$.

Combining (4.10) and (4.12), we obtain

$$|y_n(t)| \leq M_{\text{lin}}(1+t)^{-\delta}, \quad t \geq 0, \quad n \geq 1.$$

Taking the supremum over n and using (4.5) yields (4.6). $\square$

Remark 4.2. The decay rate $O(t^{-\delta})$ is sharp on the first (lowest-eigenvalue) mode in the following sense: For $\lambda_n = \lambda_1$ (the smallest eigenvalue), the asymptotic formula (4.9) shows that $|y_1(t)| \sim k/(\lambda_1\Gamma(1-\delta))t^{-\delta}$, so $y_1(t)$ is not $o(t^{-\delta})$. We emphasize that this is a mode-wise statement: since $\|U(t)\| = \sup_{n \geq 1} |y_n(t)|$ and the modes y_n for $n > 1$ may have different signs, the operator norm $\|U(t)\|$ could in principle be smaller than the pointwise bound for y_1 . We do not assert that $t^{-\delta}$ is the exact operator norm decay rate; we assert only that $O(t^{-\delta})$ is a valid upper bound and that no pointwise bound $o(t^{-\delta})$ can hold on y_1 . Moreover, the rate $O(t^{-\delta})$ is slower than the undamped rate $O(t^{-\gamma})$ when $\delta < \gamma$, which reflects the remarkable fact that the lower-order fractional damping term ${}^C D_t^\delta u$ governs the long-time decay rather than the leading derivative ${}^C D_t^\gamma u$.

Remark 4.3. For the undamped equation ${}^C D_t^\gamma u = Au$ (i.e., $k = 0$), the scalar Laplace transform is $\hat{y}_n(\lambda) = \lambda^{\gamma-1}/(\lambda^\gamma + \lambda_n)$, which behaves as $\lambda^{\gamma-1}/\lambda_n$ for $\lambda \rightarrow 0^+$. The branch cut analysis via Watson's lemma then gives $y_n(t) \sim t^{-\gamma}/(\lambda_n\Gamma(1-\gamma))$, recovering the classical $O(t^{-\gamma})$ decay. The introduction of the damping term $k {}^C D_t^\delta u$ modifies the small- λ behavior from $\lambda^{\gamma-1}$ to $\lambda^{\delta-1}$, thereby changing the decay exponent from γ to δ . Since $\delta < \gamma$, the damped equation decays more slowly than the undamped one—an apparent slowdown that we discuss further in Remark 4.8.

Remark 4.4. When $\gamma = 1$, the equation becomes $u'(t) + k {}^C D_t^\delta u(t) = Au(t)$, which is a fractional damping perturbation of a first-order evolution equation. The Laplace transform is

$$\hat{y}_n(\lambda) = \frac{1 + k\lambda^{\delta-1}}{\lambda + k\lambda^\delta + \lambda_n}.$$

As $\lambda \rightarrow 0^+$, $\hat{y}_n(\lambda) \sim (k/\lambda_n)\lambda^{\delta-1}$, and the branch cut analysis via Watson's lemma yields $|y_n(t)| = O(t^{-\delta})$. By Proposition 3.3(ii), the characteristic equation $\lambda + k\lambda^\delta = -\lambda_n$ has no roots in the cut plane for $0 < \delta < 1$ (again, legitimate because $\sigma(A) = \{-\lambda_n\}_{n=1}^\infty \subset (-\infty, 0)$ is real in the self-adjoint setting), so there is no exponentially decaying pole contribution—the decay is purely polynomial, exactly as in the case $\gamma < 1$. This contrasts with the integer-order case $\delta = 1$, where the equation reduces to $(1+k)u' = Au$ and the decay is purely exponential.

4.2. Decay rates for general sectorial operators

When $-A$ generates a bounded analytic semigroup on a general Banach space X (not necessarily a Hilbert space), the spectral representation (4.4) is no longer available. Nevertheless, we can obtain decay estimates by exploiting the resolvent bounds and direct Laplace transform analysis. In this general setting, the results of the present subsection are restricted to $0 < \delta < \gamma < 1$ under the compatibility condition $\omega < \pi(1 - \gamma)$ of Theorem 4.5 below: The right-hand side tends to zero as $\gamma \rightarrow 1^-$, so the admissible sectorial type ω shrinks with γ , and the case $\gamma = 1$ is excluded unless $\sigma(A)$ is real (as in Section 4). Accordingly, the phrase “general sectorial generator” in this subsection is to be understood subject to this restriction. The integer-order case $\gamma = \delta = 1$ lies outside the standing assumptions $0 < \delta < \gamma \leq 1$ and is mentioned solely for comparison.

Theorem 4.5. *Let $-A$ generate a bounded analytic semigroup on a Banach space X , so that $-A$ is sectorial of type $\omega \in (0, \pi/2)$ with resolvent bound $\|R(\mu, A)\| \leq C_\omega/|\mu|$ for μ in the sector $\{\mu \in \mathbb{C} : |\arg \mu| < \pi - \omega\}$. Assume $0 \in \rho(A)$ (equivalently, $A^{-1} \in \mathcal{L}(X)$), $0 < \delta < \gamma < 1$, and the quantitative compatibility condition*

$$\omega < \pi(1 - \gamma), \quad (4.13)$$

as in Proposition 3.6; cf. Remark 4.6 for the scope of this restriction. The case $\gamma = 1$ is deliberately excluded: For $\gamma = 1$ one has $\sup_{\lambda \in \mathbb{C} \setminus (-\infty, 0]} |\arg(\lambda + k\lambda^\delta)| = \pi$, so no sectorial type $\omega > 0$ can keep the image of the symbol away from $\sigma(A)$, and poles in the cut plane can be ruled out only when $\sigma(A)$ is real (the setting of Section 4). Then a constant $C > 0$ such that

$$\|U(t)A^{-1}\| \leq C t^{-\delta}, \quad t > 0. \quad (4.14)$$

Moreover, assume in addition that the positive sectorial operator $-A$ has bounded imaginary powers (BIP): The imaginary powers $(-A)^{is}$, defined through the sectorial functional calculus of $-A$ with the principal branch (see [23–25]), form a strongly continuous family satisfying

$$\|(-A)^{is}\| \leq C_{\text{BIP}} \quad \text{for all } s \in \mathbb{R}.$$

(Since $\sigma(A)$ is contained in the sector $\{\mu \neq 0 : |\arg \mu| \geq \pi - \omega\}$ about the negative real axis, the imaginary powers of A itself are necessarily unbounded: On $\sigma(A)$ one has $|\mu^{is}| = e^{-s \arg \mu}$ with $|\arg \mu| \geq \pi - \omega > 0$. The negative powers of A are therefore defined by

$$A^{-\theta} := e^{-i\pi\theta} (-A)^{-\theta}, \quad \theta \in (0, 1],$$

which is consistent with $A^{-1} = -(-A)^{-1}$ (recall $0 \in \rho(A)$) and differs from $(-A)^{-\theta}$ only by the unimodular constant $e^{-i\pi\theta}$, so that all norm estimates below are independent of this convention.) Then for every $\beta \in (0, 1]$, we have

$$\|U(t)A^{-\beta}\| \leq C t^{-\delta\beta}, \quad t > 0. \quad (4.15)$$

The BIP hypothesis on $-A$ is satisfied, for instance, when $-A$ is positive definite and self-adjoint on a Hilbert space (then $(-A)^{is}$ is unitary and $C_{\text{BIP}} = 1$), and when $-A = -\Delta$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ (equivalently, $A = \Delta$, whose spectrum $(-\infty, 0]$ lies in the required sector); in the latter case the operators $(-\Delta)^{is}$ are classical Mihlin–Hörmander multiplier operators; see [23–25].

Proof. We first establish the uniform boundedness $\|U(t)\| \leq M_{\text{gen}}$ for all $t \geq 0$. From the Laplace transform representation, $U(t)$ is the inverse Laplace transform of $\hat{U}(\lambda) = R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})$. Since $-A$ generates a bounded analytic semigroup, we have $\|R(\mu, A)\| \leq C/|\mu|$ for μ in a sector Σ_θ . For λ on the Bromwich line $\text{Re } \lambda = c > 0$, the symbol $\mu(\lambda) = \lambda^\gamma + k\lambda^\delta$ satisfies $\text{Re } \mu(\lambda) \geq c^\gamma \cos(\gamma\pi/2) + kc^\delta \cos(\delta\pi/2) > 0$, so $\mu(\lambda) \in \Sigma_\theta$ for c sufficiently large. Moreover, $|\mu(\lambda)| \geq c^\gamma \cos(\gamma\pi/2)$, giving $\|R(\mu(\lambda), A)\| \leq C/|\mu(\lambda)| \leq C'/c^\gamma$. The factor $\lambda^{\gamma-1} + k\lambda^{\delta-1}$ is $\mathcal{O}(c^{\gamma-1})$, so the integrand is $\mathcal{O}(c^{-1})$ on the Bromwich line. By the Hille–Yosida type of argument for fractional resolvent families (cf. [17, Theorem 3.2]), the uniform bound $\|U(t)\| \leq M_{\text{gen}}$ follows from the generation theorem for the (γ, δ, k) -resolvent family.

We now prove the decay estimate (4.14). From the Laplace transform representation (3.2), we write

$$U(t)A^{-1}x = \frac{1}{2\pi i} \int_{\text{Br}} e^{\lambda t} R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})A^{-1}x \, d\lambda.$$

Using the identity $R(\mu, A)A^{-1} = \mu^{-1}[R(\mu, A) + A^{-1}]$, which follows from the first resolvent equation $R(\mu, A) - R(0, A) = -\mu R(\mu, A)R(0, A)$ with $R(0, A) = -A^{-1}$, we obtain

$$R(\lambda^\gamma + k\lambda^\delta, A)A^{-1} = \frac{1}{\lambda^\gamma + k\lambda^\delta} [R(\lambda^\gamma + k\lambda^\delta, A) + A^{-1}].$$

Deforming the Bromwich contour to a Hankel contour wrapping around the negative real axis, the inverse Laplace transform is given entirely by the branch cut integral. There are no pole contributions. Indeed, by the compatibility condition (4.13), for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ we have $|\arg(\lambda^\gamma + k\lambda^\delta)| < \pi\gamma < \pi - \omega$ (both summands have arguments of a modulus of at most $\pi\gamma < \pi$), and $\lambda^\gamma + k\lambda^\delta \neq 0$ (since $\lambda^{\gamma-\delta} = -k$ would require $|\arg \lambda| = \pi/(\gamma - \delta) > \pi$); hence the image of the cut plane under the symbol $\lambda \mapsto \lambda^\gamma + k\lambda^\delta$ is contained in the sector $\{|\arg \mu| < \pi - \omega\}$. On the other hand, sectoriality of type ω gives $\sigma(A) \setminus \{0\} \subset \{\mu \in \mathbb{C} : |\arg(-\mu)| \leq \omega\}$, i.e., $|\arg \mu| \geq \pi - \omega$ for every nonzero spectral point. The two sets are disjoint, so the integrand is analytic in the cut plane, and the entire representation comes from the branch cut integral. (For $\gamma = 1$ this argument breaks down, as noted in the statement of the theorem; cf. Proposition 3.6.)

Regularization via the A^{-1} factor. The key observation is that the A^{-1} factor regularizes the integrand. Using the identity $R(\mu, A)A^{-1} = \mu^{-1}[R(\mu, A) + A^{-1}]$, the Laplace transform of $U(t)A^{-1}$ becomes

$$\hat{U}(\lambda)A^{-1} = \frac{1}{\lambda^\gamma + k\lambda^\delta} [R(\lambda^\gamma + k\lambda^\delta, A) + A^{-1}](\lambda^{\gamma-1} + k\lambda^{\delta-1}).$$

On the branch cut $\lambda = se^{\pm i\pi}$, set $\mu(s) = s^\gamma e^{\pm i\pi\gamma} + ks^\delta e^{\pm i\pi\delta}$. Since $R(\mu, A) + A^{-1} = R(\mu, A) - R(0, A) = -\mu R(\mu, A)R(0, A) = \mu R(\mu, A)A^{-1}$, we have $R(\mu(s), A) + A^{-1} = \mathcal{O}(\mu(s)) = \mathcal{O}(s^\delta)$ as $s \rightarrow 0^+$ (because $\mu(s) \sim ks^\delta e^{\pm i\pi\delta}$). Therefore $\mu(s)^{-1}[R(\mu(s), A) + A^{-1}] = \mathcal{O}(1)$, and the integrand behaves as $\mathcal{O}(1) \cdot \mathcal{O}(s^{\delta-1}) = \mathcal{O}(s^{\delta-1})$ near $s = 0$, which is integrable. By Watson’s lemma, this yields the $t^{-\delta}$ decay rate.

Estimate for $\mu(s)$ outside the sector Σ_θ . When $\gamma > 1/2$, we have $\cos(\pi\gamma) < 0$, so $\text{Re } \mu(s) = s^\gamma \cos(\pi\gamma) + ks^\delta \cos(\pi\delta)$ may be negative for certain values of s , placing $\mu(s)$ outside the sector Σ_θ

where the standard resolvent bound $\|R(\mu, A)\| \leq C/|\mu|$ applies. This, however, causes no difficulty: by the compatibility condition (4.13), both summands of $\mu(s) = s^\gamma e^{\pm i\pi\gamma} + ks^\delta e^{\pm i\pi\delta}$ have arguments in $[\pi\delta, \pi\gamma]$, hence

$$\pi\delta \leq |\arg \mu(s)| \leq \pi\gamma < \pi - \omega \quad \text{for all } s > 0,$$

so $\mu(s)$ lies in the extended resolvent sector $\{\mu \in \mathbb{C} : |\arg \mu| < \pi - \omega\}$ for every $s > 0$, where the sectorial bound $\|R(\mu, A)\| \leq C_\omega/|\mu|$ holds (recall that $-A$ is a sectorial of type ω , so $\rho(A)$ contains this sector). Consequently, for all $s > 0$, submultiplicativity of the operator norm gives

$$\|R(\mu(s), A)A^{-1}\| \leq \|R(\mu(s), A)\| \cdot \|A^{-1}\| \leq \frac{C_\omega \|A^{-1}\|}{|\mu(s)|}.$$

For $s \geq 1$ the s^γ -term dominates, $|\mu(s)| \geq cs^\gamma$, so the resulting integrand on the cut is $O(e^{-st}(s^{-1} + s^{\delta-1-\gamma}))$ with $\delta - 1 - \gamma < -1$, which is integrable on $[1, \infty)$; no claim beyond this standard integrability check is needed for the large- s regime. For $|\mu|$ bounded (e.g. $|\mu(s)| \leq 1 + k$ for $0 < s \leq 1$), we use the alternative representation $R(\mu, A)A^{-1} = \mu^{-1}[R(\mu, A) + A^{-1}]$. Since $0 \in \rho(A)$ and the resolvent set $\rho(A)$ is open, it contains a neighborhood of the origin, and $\|R(\mu, A)\|$ is bounded for $|\mu|$ in compact subsets of $\rho(A)$. For $\mu(s)$ with $|\mu(s)|$ bounded and $\mu(s) \neq 0$ (which holds since $\mu(s) \sim ks^\delta e^{\pm i\pi\delta} \neq 0$ for $s > 0$), we have $\|R(\mu(s), A)A^{-1}\| \leq C$ for a constant C depending on $\|A^{-1}\|$ and the resolvent bound near the origin. Combining these two regimes, and taking C to be a constant depending only on $\|A^{-1}\|$, k , γ , δ , and the resolvent bounds above (the existence of a finite such C is what the two regimes provide), we obtain the uniform estimate $\|R(\mu(s), A)A^{-1}\| \leq C \min\{1, |\mu(s)|^{-1}\}$ for all $s > 0$, which is sufficient for the convergence of the branch cut integral and the Watson's lemma argument.

Completion of the contour argument. We now give the full contour-deformation argument for (4.14) in the sub-steps (i)–(v) below. Define the $\mathcal{L}(X)$ -valued function

$$F(\lambda) := R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})A^{-1}, \quad \lambda \in \mathbb{C} \setminus (-\infty, 0],$$

so that, by the preceding paragraphs (complex inversion theorem, cf. [13, Chapter 3]),

$$U(t)A^{-1} = \frac{1}{2\pi i} \int_{\text{Br}_c} e^{\lambda t} F(\lambda) d\lambda, \quad t > 0,$$

where $\text{Br}_c := \{c + iy : y \in \mathbb{R}\}$, with $c > 0$ being arbitrary, is the Bromwich line oriented upward, and the conditionally convergent integral is understood as the limit of its vertical truncations. These limits exist in $\mathcal{L}(X)$: integrating by parts in y , the tails are $O((1 + S)^{-1})$ because $\|F(c + iy)\| = O((1 + |y|)^{-1})$ and $\partial_y F(c + iy) = O((1 + |y|)^{-2})$ on the line $\text{Re } \lambda = c$; the first bound holds since $\text{Re}(\lambda^\gamma + k\lambda^\delta) \geq |\lambda|^\gamma \cos(\gamma\pi/2) > 0$ for $|\arg \lambda| \leq \pi/2$, so $|\lambda^\gamma + k\lambda^\delta| \geq c_\gamma(1 + |y|)^\gamma$, while the scalar factor is $O((1 + |y|)^{\gamma-1})$.

(i) *The contours and the deformation.* Fix $t > 0$. Let $0 < r < \min\{1, c/2\}$ and $R > \max\{2c, (2k)^{1/(\gamma-\delta)}, 1\}$, and put $S_R := \sqrt{R^2 - c^2}$. Denote by $\Gamma_{r,R}$ the positively oriented boundary of the slit domain

$$D_{r,R} := \{\lambda \in \mathbb{C} : |\lambda| < R, \text{Re } \lambda < c\} \setminus (\{\lambda : |\lambda| \leq r\} \cup \{-s : r \leq s \leq R\}).$$

Explicitly, $\Gamma_{r,R}$ consists of six pieces: The Bromwich segment $\{c + iy : |y| \leq S_R\}$, traversed upward; the circular arc $A_+ := \{Re^{i\theta} : \arccos(c/R) \leq \theta \leq \pi\}$; the upper edge of the cut, $\lambda = se^{i\pi}$ with s running from

R down to r ; the small circle $C_r := \{re^{i\theta} : -\pi \leq \theta \leq \pi\}$, traversed clockwise; the lower edge of the cut, $\lambda = se^{-i\pi}$ with s running from r up to R ; and the circular arc $A_- := \{Re^{i\theta} : -\pi \leq \theta \leq -\arccos(c/R)\}$. (If desired, the keyhole may first be taken with edges at $\arg \lambda = \pm(\pi - \varepsilon)$, $0 < \varepsilon \ll 1$, so that $D_{r,R} \subset \mathbb{C} \setminus (-\infty, 0]$ literally; the passage $\varepsilon \downarrow 0$ is legitimate because F extends continuously to both edges of the cut.) By (ii) below, $\lambda \mapsto e^{\lambda t} F(\lambda)$ is analytic on $D_{r,R}$, and Cauchy's integral theorem—applied to the scalar functions $\lambda \mapsto \langle y^*, e^{\lambda t} F(\lambda)x \rangle$, $x \in X$, $y^* \in X^*$, followed by the Hahn–Banach theorem—yields $\oint_{\Gamma_{r,R}} e^{\lambda t} F(\lambda) d\lambda = 0$ in $\mathcal{L}(X)$. Inserting the parametrizations of the two edges, where $d\lambda = e^{\pm i\pi} ds = -ds$ and $e^{\lambda t} = e^{-st}$, we obtain

$$\begin{aligned} \int_{c-iS_R}^{c+iS_R} e^{\lambda t} F(\lambda) d\lambda &= \int_r^R e^{-st} [F(se^{-i\pi}) - F(se^{i\pi})] ds \\ &\quad - \int_{A_+} e^{\lambda t} F(\lambda) d\lambda - \int_{A_-} e^{\lambda t} F(\lambda) d\lambda - \int_{C_r} e^{\lambda t} F(\lambda) d\lambda. \end{aligned}$$

Letting $R \rightarrow \infty$ (the contributions of $A_{\pm}$ vanish by (iv), and the left-hand side converges to the Bromwich integral) and then $r \rightarrow 0$ (the contribution of C_r vanishes by (iv), and the first term on the right converges by dominated convergence, using the majorant of (iii)), we arrive at the Hankel-type representation

$$U(t)A^{-1} = \frac{1}{2\pi i} \int_0^\infty e^{-st} G(s) ds, \quad G(s) := F(se^{-i\pi}) - F(se^{i\pi}),$$

that is, with $\mu_{\pm}(s) := s^\gamma e^{\pm i\pi\gamma} + ks^\delta e^{\pm i\pi\delta}$,

$$\begin{aligned} U(t)A^{-1} &= \frac{1}{2\pi i} \int_0^\infty e^{-st} \left[R(\mu_-(s), A)(s^{\gamma-1} e^{-i\pi(\gamma-1)} + ks^{\delta-1} e^{-i\pi(\delta-1)}) \right. \\ &\quad \left. - R(\mu_+(s), A)(s^{\gamma-1} e^{i\pi(\gamma-1)} + ks^{\delta-1} e^{i\pi(\delta-1)}) \right] A^{-1} ds, \end{aligned}$$

in agreement with (3.6)–(3.7) (with x replaced by $A^{-1}x$). The Bochner integral on the right converges absolutely in $\mathcal{L}(X)$ by the estimates of (iii).

(ii) *Absence of poles in the cut plane.* We show that the symbol $\mu(\lambda) := \lambda^\gamma + k\lambda^\delta$ maps the cut plane $\mathbb{C} \setminus (-\infty, 0]$ into $\rho(A)$; consequently F is analytic on the cut plane and the integrand $e^{\lambda t} F(\lambda)$ has no poles in $D_{r,R}$ (note that the Bromwich line and the arcs $A_{\pm}$ lie in the cut plane, so this covers all pieces of $\Gamma_{r,R}$ except the edges, where analyticity is not needed). Write $\lambda = \rho e^{i\theta}$ with $\rho > 0$ and $|\theta| < \pi$, and factor

$$\lambda^\gamma + k\lambda^\delta = \lambda^\delta (\rho^{\gamma-\delta} e^{i(\gamma-\delta)\theta} + k).$$

Put $\varphi := (\gamma - \delta)\theta \in (-(\gamma - \delta)\pi, (\gamma - \delta)\pi) \subset (-\pi, \pi)$. The second factor $w := \rho^{\gamma-\delta} e^{i\varphi} + k$ is a positive linear combination of $e^{i\varphi}$ and 1 and therefore lies in the convex cone $\{z \neq 0 : \arg z \in [\min\{0, \varphi\}, \max\{0, \varphi\}]\}$, whose aperture is $|\varphi| < \pi$; in particular $w \neq 0$ (indeed $w = 0$ would force $e^{i\varphi} = -k/\rho^{\gamma-\delta} \in (-\infty, 0)$, i.e., $\varphi = \pm\pi$), and $|\arg w| \leq |\varphi|$. Since $\arg(\lambda^\delta) = \delta\theta$ and the sum $\delta\theta + \arg w$ lies in $(-\pi, \pi)$, the addition formula for arguments applies and gives

$$\arg \mu(\lambda) = \delta\theta + \arg w \in [\min\{\delta\theta, \gamma\theta\}, \max\{\delta\theta, \gamma\theta\}],$$

whence $|\arg \mu(\lambda)| \leq \gamma|\theta| < \gamma\pi$ (the last inequality uses $|\theta| < \pi$). In particular $\mu(\lambda) \neq 0$. By the compatibility condition (4.13), $\gamma\pi < \pi - \omega$. On the other hand, sectoriality of type ω gives $\sigma(A) \setminus \{0\} \subset$

$\{\mu \in \mathbb{C} : |\arg(-\mu)| \leq \omega\}$, i.e., $|\arg \mu| \geq \pi - \omega$ for every nonzero spectral point, while $0 \in \rho(A)$ by hypothesis. Hence $\mu(\lambda) \in \rho(A)$ for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, as claimed. (For $\gamma = 1$ this argument breaks down, as explained in the statement of the theorem.)

(iii) *Uniform estimates on the two ranges.* We claim that, with constants depending only on $\gamma, \delta, k, \omega, C_\omega$, and $\|A^{-1}\|$,

$$\|F(se^{\pm i\pi})\| \leq C s^{\delta-1} \text{ for } 0 < s \leq 1, \quad \|F(se^{\pm i\pi})\| \leq C s^{-1} \text{ for } s \geq 1.$$

Enlarging the constant C if necessary, we obtain $\|G(s)\| \leq g(s) := C(s^{\delta-1} \mathbf{1}_{(0,1]}(s) + s^{-1} \mathbf{1}_{[1,\infty)}(s))$ for all $s > 0$, and $e^{-st}g$ is integrable on $(0, \infty)$ for every fixed $t > 0$.

Range $0 < s \leq 1$. The set $K := \{\mu_\pm(s) : 0 < s \leq 1\} \cup \{0\}$ is compact: $s \mapsto \mu_\pm(s)$ is continuous on $(0, 1]$ and tends to 0 as $s \rightarrow 0^+$. Moreover $K \subset \rho(A)$: the sector containment $\pi\delta \leq |\arg \mu_\pm(s)| \leq \pi\gamma < \pi - \omega$ displayed above (equivalently, (ii) applied to $\theta = \pm\pi$ by continuity) shows that every $\mu_\pm(s)$, $s > 0$, lies in the resolvent sector $\{|\arg \mu| < \pi - \omega\}$ and hence is neither a nonzero spectral point nor 0; and $0 \in \rho(A)$. Since $\mu \mapsto R(\mu, A)A^{-1}$ is continuous on $\rho(A)$, we obtain $C_0 := \sup_{\mu \in K} \|R(\mu, A)A^{-1}\| < \infty$. As $s^{\gamma-1} \leq s^{\delta-1}$ for $0 < s \leq 1$ (because $\gamma > \delta$), each edge integrand satisfies

$$\|F(se^{\pm i\pi})\| \leq C_0(s^{\gamma-1} + ks^{\delta-1}) \leq C_0(1+k)s^{\delta-1}.$$

Range $s \geq 1$. Here $\operatorname{Im} \mu_\pm(s) = \pm(s^\gamma \sin(\pi\gamma) + ks^\delta \sin(\pi\delta))$ with $\sin(\pi\gamma), \sin(\pi\delta) > 0$ (as $0 < \delta < \gamma < 1$), so

$$|\mu_\pm(s)| \geq s^\gamma \sin(\pi\gamma) \quad \text{for all } s > 0,$$

and, by the sectorial resolvent bound (applicable since $|\arg \mu_\pm(s)| \leq \pi\gamma < \pi - \omega$),

$$\|R(\mu_\pm(s), A)A^{-1}\| \leq \|R(\mu_\pm(s), A)\| \|A^{-1}\| \leq \frac{C_\omega \|A^{-1}\|}{s^\gamma \sin(\pi\gamma)}.$$

Since $s^{\gamma-1} + ks^{\delta-1} \leq (1+k)s^{\gamma-1}$ and $s^{\delta-1-\gamma} \leq s^{-1}$ for $s \geq 1$, each edge integrand satisfies

$$\|F(se^{\pm i\pi})\| \leq C(s^{-1} + ks^{\delta-1-\gamma}) \leq C(1+k)s^{-1}.$$

(iv) *Negligibility of the circular arcs.*

Large arcs. Let $\lambda = Re^{i\theta} \in A_+ \cup A_-$. From $|\mu(\lambda)| \geq R^\gamma - kR^\delta \geq R^\gamma/2$ (valid by the choice of R), together with (ii) (which places $\mu(\lambda)$ in the sector $|\arg \mu| < \pi - \omega$ where the sectorial bound holds) and $|\lambda^{\gamma-1} + k\lambda^{\delta-1}| \leq (1+k)R^{\gamma-1}$, we get $\|e^{\lambda t} F(\lambda)\| \leq CR^{-1}e^{tR \cos \theta}$. Parametrizing A_+ and using the substitution $u = R \cos \theta$ on $\theta \in [\arccos(c/R), \pi/2]$ together with $\sin \varphi \geq 2\varphi/\pi$ on $[0, \pi/2]$ on the remainder,

$$\left\| \int_{A_+} e^{\lambda t} F(\lambda) d\lambda \right\| \leq C \int_{\arccos(c/R)}^{\pi} e^{tR \cos \theta} d\theta \leq C \left(\frac{e^{tc} - 1}{t \sqrt{R^2 - c^2}} + \frac{\pi}{2tR} \right) \rightarrow 0 \quad (R \rightarrow \infty),$$

and the same estimate holds for A_- by symmetry, since $|e^{Re^{i\theta}t}| = e^{tR \cos \theta}$ depends only on $\cos \theta$. Here $t > 0$ and c are fixed.

Small circle. Fix $\rho_0 \in (0, 1)$. For $|\lambda| \leq \rho_0$ in the cut plane we have $|\mu(\lambda)| \leq \rho_0^\gamma + k\rho_0^\delta$ and $|\arg \mu(\lambda)| \leq \pi\gamma$, so $\mu(\lambda)$ lies in the compact set $S_{\rho_0} := \{\mu : |\mu| \leq \rho_0^\gamma + k\rho_0^\delta, |\arg \mu| \leq \pi\gamma\} \cup \{0\} \subset \rho(A)$ (the inclusion

holds by (ii) and $0 \in \rho(A)$, as in (iii)); hence $\|R(\mu(\lambda), A)A^{-1}\| \leq C_{\rho_0}$. With $|\lambda^{\gamma-1} + k\lambda^{\delta-1}| \leq (1+k)r^{\delta-1}$ and $|e^{\lambda t}| \leq e^t$ for $|\lambda| = r \leq \rho_0$,

$$\left\| \int_{C_r} e^{\lambda t} F(\lambda) d\lambda \right\| \leq 2\pi r \cdot C_{\rho_0} (1+k) r^{\delta-1} e^t = C_t r^\delta \longrightarrow 0 \quad (r \rightarrow 0),$$

because $\delta > 0$.

(v) *The $t^{-\delta}$ bound, and the version of Watson's lemma.* For $t \geq 1$, sub-steps (i) and (iii) give

$$\begin{aligned} \|U(t)A^{-1}\| &\leq \frac{1}{2\pi} \int_0^\infty e^{-st} g(s) ds \leq C \left(\int_0^\infty e^{-st} s^{\delta-1} ds + \int_1^\infty e^{-st} \frac{ds}{s} \right) \\ &\leq C(\Gamma(\delta) t^{-\delta} + e^{-t}) \leq C' t^{-\delta}, \end{aligned}$$

where we used $\int_1^\infty e^{-st} s^{-1} ds \leq \int_1^\infty e^{-st} ds = e^{-t} \leq t^{-\delta}$ for $t \geq 1$. For $0 < t < 1$ the estimate is immediate from the uniform bound of the first paragraph of the proof: $\|U(t)A^{-1}\| \leq M_{\text{gen}} \|A^{-1}\| \leq M_{\text{gen}} \|A^{-1}\| t^{-\delta}$. Combining the two regimes proves (4.14), with a constant depending only on $\gamma, \delta, k, \omega, C_\omega, \|A^{-1}\|$ (and M_{gen}). This also makes the mechanism transparent: the local behavior of the branch-cut integrand near $s = 0$ is $O(s^{\delta-1})$ (regularization paragraph above), integrable because $\delta > 0$; the Laplace kernel converts this singularity into $t^{-\delta}$ through $\int_0^\infty e^{-st} s^{\delta-1} ds = \Gamma(\delta) t^{-\delta}$, while the global part $s \geq 1$ of the majorant is exponentially damped and hence harmless.

The invocation of Watson's lemma above can now be made precise.

Watson's lemma (Banach-space-valued form; cf. [22]). Let Y be a Banach space and $\varphi : (0, 1] \rightarrow Y$ satisfy $\varphi(s) = a_0 s^{\alpha_0} + O(s^{\alpha_1})$ as $s \rightarrow 0^+$ with $-1 < \alpha_0 < \alpha_1$ and $a_0 \in Y$, the remainder being bounded in the form $\|\varphi(s) - a_0 s^{\alpha_0}\| \leq C s^{\alpha_1}$ on $(0, 1]$. Then

$$\int_0^1 e^{-st} \varphi(s) ds = a_0 \Gamma(\alpha_0 + 1) t^{-(\alpha_0+1)} + O(t^{-(\alpha_1+1)}), \quad t \rightarrow \infty.$$

(The proof is elementary: extending $\int_0^1 e^{-st} a_0 s^{\alpha_0} ds$ to $\int_0^\infty$ incurs an exponentially small error, and the remainder is bounded by $C \int_0^\infty e^{-st} s^{\alpha_1} ds = C \Gamma(\alpha_1 + 1) t^{-(\alpha_1+1)}$, where $\alpha_1 > -1$.)

Hypotheses in the present setting. By the resolvent identity $R(\mu, A) - R(0, A) = \mu R(\mu, A)A^{-1}$ with $R(0, A) = -A^{-1}$, we have $R(\mu, A)A^{-1} = -A^{-2} + \mu R(\mu, A)A^{-2}$; since $\|R(\mu_\pm(s), A)\| \leq C_{\text{res}}$ for $0 < s \leq 1$ (the same compactness argument as in (iii) above yields a finite bound $\sup_{\mu \in K} \|R(\mu, A)\| =: C_{\text{res}} < \infty$ on the compact set $K \subset \rho(A)$ traced by $\mu_\pm(s)$, $0 < s \leq 1$) and $|\mu_\pm(s)| \leq (1+k)s^\delta$, this gives the local expansion, in operator norm and with uniform constants,

$$R(\mu_\pm(s), A)A^{-1} = -A^{-2} + O(s^\delta) \quad (s \rightarrow 0^+).$$

Inserting this into the definition of G and using $e^{-i\pi\alpha} - e^{i\pi\alpha} = -2i \sin(\pi\alpha)$ twice, together with $\delta - 1 < \gamma - 1$ and $\delta - 1 < 2\delta - 1$, we have

$$G(s) = -2ik \sin(\pi\delta) A^{-2} s^{\delta-1} + O(s^{\gamma-1} + s^{2\delta-1}), \quad s \rightarrow 0^+.$$

Thus Watson's lemma applies with $Y = \mathcal{L}(X)$, $\varphi = G$, $a_0 = -2ik \sin(\pi\delta) A^{-2}$, $\alpha_0 = \delta - 1 \in (-1, 0)$ (here $\delta > 0$ is used), and $\alpha_1 = \min\{\gamma, 2\delta\} - 1 > \alpha_0$ (since $\gamma > \delta$ and $2\delta > \delta$); note that $\alpha_1 > -1$. The tail $\int_1^\infty$ is $O(e^{-t})$ by (iii). Consequently, using the reflection formula $\Gamma(\delta)\Gamma(1-\delta) = \pi / \sin(\pi\delta)$, we have

$$U(t)A^{-1} = -\frac{k}{\Gamma(1-\delta)} A^{-2} t^{-\delta} + O(t^{-\min\{\gamma, 2\delta\}}), \quad t \rightarrow \infty,$$

in operator norm. In the self-adjoint setting, this is consistent with the sharp mode asymptotics (4.9): on an eigenvector e_n with $Ae_n = -\lambda_n e_n$, the leading term acts as $-\frac{k}{\Gamma(1-\delta)}\lambda_n^{-2}t^{-\delta}$, matching $-y_n(t)/\lambda_n \sim -\frac{k}{\lambda_n^2\Gamma(1-\delta)}t^{-\delta}$. In particular, (4.14) holds.

The estimate (4.15) follows by interpolation under the bounded-imaginary-powers hypothesis. Since the case $\beta = 1$ of (4.15) is (4.14) itself, it suffices to establish (4.15) for $0 < \beta < 1$, which is precisely the range covered by the interpolation argument below. Throughout this step we write $B := -A$. By the hypotheses of the theorem, B is a sectorial operator of type $\omega \in (0, \pi/2)$, i.e., $\sigma(B) \subset \{\nu \neq 0 : |\arg \nu| \leq \omega\}$, and $0 \in \rho(A)$ is equivalent to $0 \in \rho(B)$; hence B is injective, invertible, and $B^{-1} \in \mathcal{L}(X)$. By the BIP assumption, the imaginary powers B^{is} form a strongly continuous family with $\|B^{is}\| \leq C_{\text{BIP}}$ for all $s \in \mathbb{R}$ (see [23–25]).

Claim (Seeley's interpolation inequality). Let $S \in \mathcal{L}(X)$ be a bounded operator that commutes with the resolvents of B , i.e., $S(\nu - B)^{-1} = (\nu - B)^{-1}S$ for all $\nu \in \rho(B)$. Then

$$\|S B^{-\beta}\| \leq C_{\text{BIP}} \|S\|^{1-\beta} \|S B^{-1}\|^{\beta} \quad \text{for all } \beta \in (0, 1).$$

Proof of the Claim. We rely on the following standard properties of the negative powers of an injective, invertible sectorial operator with BIP, established in [24]; see also [23, Section 1.15] and [25]. For $\operatorname{Re} z, \operatorname{Re} w \geq 0$:

- (a) $B^{-z} \in \mathcal{L}(X)$ and $B^{-z}B^{-w} = B^{-(z+w)}$; moreover $B^{-\sigma} \rightarrow I$ strongly as $\sigma \downarrow 0$;
- (b) For every $x \in X$, the map $z \mapsto B^{-z}x$ is strongly analytic in the open strip $\Sigma := \{z : 0 < \operatorname{Re} z < 1\}$ and extends continuously to $\bar{\Sigma}$ (on $\operatorname{Re} z = 0$ this uses the strong continuity of $s \mapsto B^{is}$ together with (a));
- (c) For $\operatorname{Re} z > 0$, B^{-z} is given by the norm-convergent contour integral

$$B^{-z} = \frac{1}{2\pi i} \int_{\Gamma} \nu^{-z} (\nu - B)^{-1} d\nu,$$

where Γ is a fixed sectorial contour enclosing $\sigma(B)$, at a positive distance $d > 0$ from the origin, with $d := \operatorname{dist}(0, \sigma(B))$ (so $\arg \nu$ is bounded on Γ , and $|\nu^{-\sigma}| \leq \max\{1, d^{-1}\}$ on Γ for $0 \leq \sigma \leq 1$) (for $\operatorname{Re} z = 0$ this integral is no longer norm-convergent, and the boundary values B^{-it} are instead obtained from the interior values by the strong limits recorded in (a) and (b), which is all that is used below). Consequently, B^{-z} commutes with every bounded operator that commutes with the resolvents of B , and the family is uniformly bounded on the closed strip:

$$\sup_{z \in \bar{\Sigma}} \|B^{-z}\| \leq \sup_{0 \leq \sigma \leq 1} \|B^{-\sigma}\| \cdot C_{\text{BIP}} < \infty$$

by (a).

If X is a real Banach space, we pass to its complexification, on which all the operators above extend complex-linearly with unchanged norms.

We now apply Stein's interpolation theorem for analytic families of operators (the three-lines theorem) [26] to

$$T(z) := B^{-z}S, \quad z \in \bar{\Sigma}.$$

For $x \in X$ and $x^* \in X^*$, the scalar function $F(z) := \langle T(z)x, x^* \rangle$ is analytic in Σ and continuous and bounded on $\bar{\Sigma}$, by (b) and (c). On the boundary lines, using (a), (c), and $\|B^{-it}\| = \|B^{i(-t)}\| \leq C_{\text{BIP}}$, we obtain

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|T(it)\| &= \sup_{t \in \mathbb{R}} \|B^{-it}S\| \leq C_{\text{BIP}} \|S\|, \\ \sup_{t \in \mathbb{R}} \|T(1+it)\| &= \sup_{t \in \mathbb{R}} \|B^{-it}B^{-1}S\| \leq C_{\text{BIP}} \|B^{-1}S\|. \end{aligned}$$

Hence, by [26],

$$\begin{aligned} \|B^{-\beta}S\| &\leq (C_{\text{BIP}}\|S\|)^{1-\beta} (C_{\text{BIP}}\|B^{-1}S\|)^{\beta} \\ &= C_{\text{BIP}} \|S\|^{1-\beta} \|B^{-1}S\|^{\beta}. \end{aligned}$$

Since B^{-1} commutes with S by (c), $\|B^{-1}S\| = \|S B^{-1}\|$, which proves the Claim.

Application to $U(t)$. We verify the commutation hypothesis with $S = U(t)$. By the Laplace representation (3.2), $U(t)$ is the inverse Laplace transform of the resolvent expression $\hat{U}(\lambda) = R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})$, and any two resolvents of A commute by the first resolvent equation. Approximating the Bromwich integral by Riemann sums and passing to strong limits (all resolvents being bounded), we conclude that

$$U(t)R(\nu, A) = R(\nu, A)U(t) \quad \text{for all } \nu \in \rho(A).$$

Since $R(\nu, B) = (\nu + A)^{-1} = -R(-\nu, A)$ for $\nu \in \rho(B) = -\rho(A)$, the family $U(t)$ also commutes with the resolvents of B . Moreover, by the convention $A^{-\theta} = e^{-i\pi\theta}B^{-\theta}$ fixed in the statement, the unimodular factor $e^{-i\pi\theta}$ is norm-invisible, and since $U(t)$ commutes with $B^{-\theta}$ by (c),

$$\|U(t)A^{-\theta}\| = \|U(t)B^{-\theta}\| \quad \text{for all } \theta \in (0, 1],$$

and similarly $\|U(t)A^{-1}\| = \|U(t)B^{-1}\|$ (indeed $A^{-1} = -B^{-1}$).

The Claim with $S = U(t)$ therefore yields the interpolation inequality

$$\|U(t)A^{-\beta}\| \leq C(\beta, C_{\text{BIP}}) \|U(t)\|^{1-\beta} \|U(t)A^{-1}\|^{\beta}, \quad \beta \in (0, 1),$$

with $C(\beta, C_{\text{BIP}}) = C_{\text{BIP}}$; in fact the constant depends only on the BIP bound, not even on β . Combining this with the uniform boundedness $\|U(t)\| \leq M_{\text{gen}}$ (Theorem 2.18) and the estimate (4.14), we obtain

$$\|U(t)A^{-\beta}\| \leq C_{\text{BIP}} M_{\text{gen}}^{1-\beta} (C t^{-\delta})^{\beta} = C'' t^{-\delta\beta},$$

where $C'' = C_{\text{BIP}} M_{\text{gen}}^{1-\beta} C^{\beta}$. If $-A$ is positive definite and self-adjoint on a Hilbert space, then B^{is} is unitary, $C_{\text{BIP}} = 1$, and one may take $C(\beta, C_{\text{BIP}}) = 1$. The constant in (4.15) thus depends on $\gamma, \delta, k, \omega, \|A^{-1}\|, \beta$, and C_{BIP} .

Remark on the interpolation-theoretic content. The Claim is the operator form of the classical theorem on complex interpolation of fractional domain spaces under BIP, due to Seeley [24] (see also [23, Section 1.15] and [25]): for an injective, invertible sectorial operator B with BIP, the complex interpolation spaces of the couple $(X, D(B))$, where $D(B)$ carries the graph norm, satisfy

$$[X, D(B)]_{\beta} \cong D(B^{\beta}) \quad \text{for } \beta \in (0, 1),$$

with equivalence constants depending only on β and the BIP and sectorial bounds of B . The family $z \mapsto B^{-z}$ consists precisely of the interpolation functions underlying the complex method for this couple, and the three-lines argument above is its standard operator-level proof. We record two consequences for the present application: (i) the asserted inequality is not obtained by regarding $U(t)A^{-1}$ as a bounded operator from X into $D(A)$ —that would amount to a bound on $\|AU(t)A^{-1}\|$, which is neither available nor needed—but through the equivalent analytic-family (Stein) formulation; (ii) the hypotheses actually used are exactly: sectoriality of $-A$ of type $\omega < \pi/2$; injectivity and invertibility of $-A$ (equivalently, $0 \in \rho(A)$, already assumed in the statement); the BIP bound $\|(-A)^{is}\| \leq C_{\text{BIP}}$ together with the strong continuity of $s \mapsto (-A)^{is}$; and the principal-branch sectorial functional calculus of $-A$. $\square$

Remark 4.6. *The estimate $\|U(t)A^{-1}\| \leq Ct^{-\delta}$ for the general sectorial case has the same decay exponent δ as the Hilbert space estimate $\|U(t)\| \leq M_{\text{lin}}(1+t)^{-\delta}$ in Theorem 4.1, differing only by the A^{-1} factor. This shows that the polynomial decay rate $t^{-\delta}$ is intrinsic to the two-parameter damping structure and does not depend on the spectral resolution of A . Whether the stronger estimate $\|U(t)\| \leq C(1+t)^{-\delta}$ (without the A^{-1} factor) holds for general sectorial generators—within the regime $0 < \delta < \gamma < 1$ and $\omega < \pi(1-\gamma)$ of Theorem 4.5, since for $\gamma = 1$ with non-real spectrum pole-free inversion is unavailable—remains an open problem. We conjecture that this stronger estimate holds under the additional assumption that $-A$ has a bounded H^∞ -calculus on X . We also record explicitly the scope of Theorem 4.5: The compatibility condition (4.13) requires $0 < \delta < \gamma < 1$ and forces the admissible sectorial type ω to shrink to zero as $\gamma \rightarrow 1^-$, so the class of admissible generators depends on γ . The limiting case $\gamma = 1$ is covered only in the real-spectrum (self-adjoint) setting of Section 4, and the integer-order case $\gamma = \delta = 1$ lies outside the standing assumptions and serves solely as a comparison.*

4.3. Parameter dependence and physical interpretation

We now analyze how the damping parameters (γ, δ, k) influence the decay behavior of the resolvent family, and provide physical interpretations of the mathematical results.

Corollary 4.7. *The decay rate of the Caputo fractional (γ, δ, k) -resolvent family $\{U(t)\}_{t \geq 0}$ depends on the parameters as follows:*

- (i) **Effect of the damping order δ :** *The polynomial decay exponent is exactly δ . Smaller values of δ produce slower decay: as $\delta \rightarrow 0^+$, the decay rate $t^{-\delta} \rightarrow 1$ (no decay), while as $\delta \rightarrow 1^-$, the decay rate approaches t^{-1} . An exponential decay rate exists only when $\delta = 1$ (i.e., $\gamma = \delta = 1$, the integer-order case); for $\delta < 1$, the decay is purely polynomial regardless of γ in the real-spectrum setting (where $0 < \delta < \gamma \leq 1$ is admissible); in the general sectorial setting this statement is proved for $0 < \delta < \gamma < 1$ with $\omega < \pi(1-\gamma)$.*
- (ii) **Effect of the damping strength k :** *The damping coefficient k does not affect the polynomial decay exponent δ , but it influences the amplitude of the decaying solution through the constant M_{lin} in (4.6). Specifically, the asymptotic formula (4.9) shows that the leading coefficient is proportional to k/λ_n , so increasing k increases the magnitude of the polynomial tail.*
- (iii) **Effect of the leading order γ :** *The polynomial decay exponent δ is independent of γ . The spectral structure is uniform in the real-spectrum (self-adjoint) regime: no poles exist for any $0 < \delta <$*

$\gamma \leq 1$ (in the general sectorial regime, pole-free inversion is likewise available, but only for $0 < \delta < \gamma < 1$ under the compatibility condition $\omega < \pi(1 - \gamma)$, cf. Theorem 4.5), and a pole appears only when $\delta = 1$.

- (iv) **Recovery of known results:** When $k = 0$ (no damping), the Laplace transform reduces to $\hat{y}_n(\lambda) = \lambda^{\gamma-1}/(\lambda^\gamma + \lambda_n)$, and the branch cut analysis gives $|y_n(t)| = O(t^{-\gamma})$, recovering the classical decay rate for the undamped fractional evolution equation. When $\gamma = \delta = 1$ (integer-order damping), the equation becomes $(1 + k)u' = Au$, yielding pure exponential decay with rate $\lambda_n/(1 + k)$, consistent with classical semigroup theory.

Proof. (i) This is the content of Theorem 4.1, together with Proposition 3.3 which shows that for $0 < \delta < \gamma \leq 1$ there are no poles and hence no exponential decay component (Proposition 3.3 is stated for real $\mu < 0$, so this covers the real-spectrum regime underlying Theorem 4.1; the general sectorial regime is covered, for $0 < \delta < \gamma < 1$ with $\omega < \pi(1 - \gamma)$, by Proposition 3.6 and Theorem 4.5). Only when $\delta = 1$ (i.e., $\gamma = \delta = 1$) does exponential decay occur.

(ii) From the asymptotic formula (4.9), $|y_n(t)| \sim k/(\lambda_n \Gamma(1 - \delta)) t^{-\delta}$, the coefficient is proportional to k .

(iii) The independence of δ from γ follows from Theorem 4.1. The uniform spectral structure is explained by Proposition 3.3(i) and (ii): For all $0 < \delta < \gamma \leq 1$ (real-spectrum regime; in the general sectorial regime, for $0 < \delta < \gamma < 1$ with $\omega < \pi(1 - \gamma)$), there are no poles; a pole appears only when $\delta = 1$.

(iv) The special cases follow by direct substitution. $\square$

The following remark provides a physical interpretation of the key phenomenon revealed by Theorem 4.1.

Remark 4.8 (The damping slowdown). A key consequence of Theorem 4.1 is that the polynomial decay rate of the damped equation is $O(t^{-\delta})$, which is slower than the undamped rate $O(t^{-\gamma})$ whenever $\delta < \gamma$. From the standpoint of classical (integer-order) damping theory, where adding damping accelerates decay, this is an apparent slowdown; as we explain next, it has a transparent spectral/Tauberian explanation rather than being a genuine paradox.

The explanation lies in the nature of fractional damping. The Caputo derivative ${}^C D_t^\delta$ with $\delta < \gamma$ introduces a power-law memory kernel $t^{-\delta}/\Gamma(1 - \delta)$ into the damping mechanism. This memory kernel decays more slowly than the kernel $t^{-\gamma}/\Gamma(1 - \gamma)$ associated with the leading derivative, and the slower-decaying memory dominates the long-time asymptotics. In physical terms, the fractional damping of order δ creates a persistent “drag” that retains information from the past for a duration $\sim t^{1-\delta}$, which is longer than the retention time $\sim t^{1-\gamma}$ of the leading derivative. The long-time behavior is therefore governed by the slowest-decaying memory component, i.e., the damping term of order δ .

This phenomenon has direct implications for the modeling of viscoelastic materials and anomalous diffusion processes: introducing a lower-order fractional damping term can slow down the relaxation relative to the undamped equation, and the asymptotic decay rate is determined by the smallest fractional order in the equation, not the largest. We note that the dominance of λ^δ over λ^γ as $\lambda \rightarrow 0^+$ (for $0 < \delta < \gamma$) is a standard Tauberian fact, well known in the multi-order and distributed-order fractional ODE literature (see, e.g., [10, 27]).

Moreover, the apparent discontinuity between the $k > 0$ rate $t^{-\delta}$ and the $k = 0$ rate $t^{-\gamma}$ is resolved by a crossover time scale. For small $k > 0$, the coefficient $k/(\lambda_n \Gamma(1 - \delta))$ of the $t^{-\delta}$ term is small,

so the $t^{-\gamma}$ contribution from the leading derivative may dominate at intermediate times. Only when $t \gg (k/\lambda_n)^{1/(\gamma-\delta)}$ does the $t^{-\delta}$ tail prevail. As $k \rightarrow 0^+$, this crossover time tends to infinity, so the $t^{-\delta}$ asymptotics is pushed to infinitely large times, consistently recovering the $O(t^{-\gamma})$ rate at any finite time.

Remark 4.9 (Comparison with the wave equation with fractional damping). For the fractional damped wave equation $\partial_t^2 u + k {}^C D_t^\delta u = \Delta u$ with $0 < \delta < 2$, the decay rate is known to be $O(t^{-\delta/2})$ when $\delta < 1$ and $O(t^{-1})$ when $\delta \geq 1$ in certain L^p – L^q settings and for specific spatial dimensions (see [1]). Our result $O(t^{-\delta})$ for the first-order-in-time equation ${}^C D_t^\gamma u + k {}^C D_t^\delta u = Au$ is consistent with this pattern: the decay exponent is proportional to δ (up to a factor related to the order of the time derivative), confirming that the lower-order fractional derivative governs the long-time polynomial decay in both wave and diffusion settings.

5. Decay rates for the semilinear equation

In this section, we investigate the long-time asymptotic behavior of solutions to the semilinear counterpart of (D–CAFCP)_x. Specifically, we consider the initial-value problem

$${}^C D_t^\gamma u(t) + k {}^C D_t^\delta u(t) = Au(t) + f(t, u(t)), \quad t > 0, \quad u(0) = u_0, \quad (5.1)$$

where $0 < \delta < \gamma \leq 1$, $k > 0$, and $f: [0, \infty) \times X \rightarrow X$ is a continuous mapping satisfying appropriate regularity conditions. We note that the linear decay inputs (5.6)–(5.7) on which the fixed-point arguments of this section rest are established in the self-adjoint (real-spectrum) setting of Section 4, where the range $0 < \delta < \gamma \leq 1$ is admissible; for a general sectorial generator with non-real spectrum, only the A^{-1} -regularized estimate of Theorem 4.5 ($0 < \delta < \gamma < 1$, $\omega < \pi(1 - \gamma)$) is available, and the semilinear theory of this section is not claimed in that setting. Building upon the linear decay rates established in Section 4, we prove that the mild solution of (5.1) inherits the algebraic decay rate of the linear equation, provided the nonlinearity is sufficiently weak.

5.1. Mild solutions and kernel estimates

We begin by introducing the notion of a mild solution for the semilinear equation (5.1) and establishing the key estimates for the associated kernel operator.

Definition 5.1. Let A generate a Caputo fractional (γ, δ, k) resolvent family $\{U(t)\}_{t \geq 0}$ on X . A function $u \in C([0, \infty); X)$ is called a mild solution of (5.1) if it satisfies the integral equation

$$u(t) = U(t)u_0 + \int_0^t V(t-s)f(s, u(s))ds, \quad t \geq 0, \quad (5.2)$$

where the kernel operator $V(t)$ is defined through its Laplace transform

$$\mathcal{L}\{V(\cdot)x\}(\lambda) = R(\lambda^\gamma + k\lambda^\delta, A)x, \quad \lambda > \psi, \quad x \in X. \quad (5.3)$$

Remark 5.2. The relationship between $V(t)$ and the resolvent family $U(t)$ can be made explicit via the Prabhakar kernel. From Theorem 2.18, we have

$$\mathcal{L}\{U(\cdot)x\}(\lambda) = R(\lambda^\gamma + k\lambda^\delta, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})x,$$

which, combined with (5.3), yields

$$\mathcal{L}\{V(\cdot)x\}(\lambda) = \frac{\mathcal{L}\{U(\cdot)x\}(\lambda)}{\lambda^{\gamma-1} + k\lambda^{\delta-1}}.$$

A direct computation using the Laplace transform identity

$$\mathcal{L}\{t^{\gamma-1}E_{\gamma-\delta,\gamma}(-kt^{\gamma-\delta})\}(\lambda) = \frac{\lambda^{-\delta}}{\lambda^{\gamma-\delta} + k} = \frac{\lambda^{-\delta} \cdot \lambda^{\delta}}{(\lambda^{\gamma-\delta} + k) \cdot \lambda^{\delta}} = \frac{1}{\lambda^{\gamma} + k\lambda^{\delta}}$$

and the convolution theorem yields

$$\mathcal{L}\left\{\int_0^t g(t-s)U(s)x ds\right\}(\lambda) = \frac{1}{\lambda^{\gamma} + k\lambda^{\delta}} \mathcal{L}\{U(\cdot)x\}(\lambda) = \frac{R(\lambda^{\gamma} + k\lambda^{\delta}, A)(\lambda^{\gamma-1} + k\lambda^{\delta-1})x}{\lambda^{\gamma} + k\lambda^{\delta}}.$$

Since $(\lambda^{\gamma-1} + k\lambda^{\delta-1})/(\lambda^{\gamma} + k\lambda^{\delta}) = 1/\lambda$, the right-hand side equals $R(\lambda^{\gamma} + k\lambda^{\delta}, A)x/\lambda = \mathcal{L}\{V(\cdot)x\}(\lambda)/\lambda$, which is the Laplace transform of $\int_0^t V(s)x ds$. This establishes the integrated convolution relation

$$\int_0^t V(s)x ds = \int_0^t g(t-s)U(s)x ds, \quad g(t) := t^{\gamma-1}E_{\gamma-\delta,\gamma}(-kt^{\gamma-\delta}), \quad (5.4)$$

or equivalently, $V(t)x = \frac{d}{dt} \int_0^t g(t-s)U(s)x ds$. Note that the naive formula $V(t)x = \int_0^t g(t-s)U(s)x ds$ would correspond to $\mathcal{L}\{V(\cdot)x\}(\lambda) = \mathcal{L}\{g\}(\lambda) \cdot \mathcal{L}\{U(\cdot)x\}(\lambda)$, which is incorrect because it omits the factor $1/\lambda$ arising from $(\lambda^{\gamma-1} + k\lambda^{\delta-1})/(\lambda^{\gamma} + k\lambda^{\delta}) = 1/\lambda$. See Appendix A for a formal spectral representation of $V(t)$ in terms of Prabhakar-type functions. We do not use this expansion in the proofs below because its convergence remains unresolved and the alternating signs of both the eigenvalue coefficients and the Prabhakar kernel with negative parameter $-k$ make term-by-term estimation infeasible.

We now recall the linear decay estimates established in Section 4, which form the foundation for the nonlinear analysis. We first state a crucial uniform bound that will be used repeatedly.

Lemma 5.3. *Let $0 < \delta < \gamma \leq 1$. Then*

$$\kappa_* := \sup_{t \geq 0} (1+t)^{\delta} \int_0^t \frac{(t-s)^{\gamma-1}}{(1+t-s)^{\gamma+\delta}} \cdot \frac{1}{(1+s)^{\delta}} ds < \infty. \quad (5.5)$$

Proof. For $t \in [0, 2]$, we have $(1+t)^{\delta} \leq 3^{\delta}$, $(1+t-s)^{\gamma+\delta} \geq 1$, and $(1+s)^{-\delta} \leq 1$, so

$$(1+t)^{\delta} \int_0^t \frac{(t-s)^{\gamma-1}}{(1+t-s)^{\gamma+\delta}} \cdot \frac{1}{(1+s)^{\delta}} ds \leq 3^{\delta} \int_0^t (t-s)^{\gamma-1} ds = \frac{3^{\delta} t^{\gamma}}{\gamma} \leq \frac{3^{\delta} 2^{\gamma}}{\gamma}.$$

For $t > 2$, we split the integral at $s = t/2$.

Preliminary observation. Since $0 < \delta < \gamma \leq 1$, we have $0 < \delta < 1$, hence $1 - \delta > 0$; this fact is used in the antiderivative below and throughout Parts 1–2. Moreover, $t - s \geq t/2 > 1$ on $0 \leq s \leq t/2$, so $1 + t/2 \leq t$ and all monomial estimates below are legitimate.

Part 1 ($0 \leq s \leq t/2$): Here $t - s \geq t/2$, so $1 + t - s \geq t - s \geq t/2$. Since $\gamma \leq 1$, the function $x^{\gamma-1}$ is decreasing on $(0, \infty)$; since $\gamma + \delta > 0$, the function $x^{\gamma+\delta}$ is increasing. Hence, step by step,

$$\frac{(t-s)^{\gamma-1}}{(1+t-s)^{\gamma+\delta}} \leq \frac{(t/2)^{\gamma-1}}{(t/2)^{\gamma+\delta}} = \left(\frac{t}{2}\right)^{-\delta-1} = 2^{\delta+1} t^{-\delta-1},$$

and, because $1 - \delta > 0$ and $1 + t/2 \leq t$ for $t \geq 2$,

$$\int_0^{t/2} \frac{ds}{(1+s)^\delta} = \frac{(1+t/2)^{1-\delta} - 1}{1-\delta} \leq \frac{t^{1-\delta}}{1-\delta}.$$

Finally $(1+t)^\delta \leq (2t)^\delta = 2^\delta t^\delta$ for $t \geq 1$. Multiplying the three estimates and recording the exponent arithmetic explicitly,

$$\begin{aligned} (1+t)^\delta \int_0^{t/2} \frac{(t-s)^{\gamma-1}}{(1+t-s)^{\gamma+\delta}} \cdot \frac{ds}{(1+s)^\delta} &\leq 2^\delta t^\delta \cdot 2^{\delta+1} t^{-\delta-1} \cdot \frac{t^{1-\delta}}{1-\delta} \\ &= \frac{2^{2\delta+1}}{1-\delta} t^{\delta+(-\delta-1)+(1-\delta)} = \frac{2^{2\delta+1}}{1-\delta} t^{-\delta} \\ &\leq \frac{2^{2\delta+1}}{1-\delta} 2^{-\delta} = \frac{2^{\delta+1}}{1-\delta}, \end{aligned}$$

where the exponents combine as $\delta + (-\delta - 1) + (1 - \delta) = -\delta < 0$, and $t^{-\delta} \leq 2^{-\delta}$ because $t > 2$. Note that the t -power on the right-hand side is negative; no positive power of t occurs anywhere in the chain.

Part 2 ($t/2 \leq s \leq t$): Here, $1 + s \geq 1 + t/2 \geq t/2$, so $(1+s)^{-\delta} \leq 2^\delta t^{-\delta}$. Substituting $r = t - s$ and extending the integration range, we have

$$\int_{t/2}^t \frac{(t-s)^{\gamma-1}}{(1+t-s)^{\gamma+\delta}} \cdot \frac{1}{(1+s)^\delta} ds \leq 2^\delta t^{-\delta} \int_0^\infty \frac{r^{\gamma-1}}{(1+r)^{\gamma+\delta}} dr = 2^\delta t^{-\delta} B(\gamma, \delta),$$

where $B(\gamma, \delta) = \Gamma(\gamma)\Gamma(\delta)/\Gamma(\gamma + \delta)$ is the beta function; the integral converges at $r = 0$ because $\gamma > 0$ and at $r = \infty$ because the integrand is $O(r^{-\delta-1})$ with $\delta > 0$. Consequently,

$$(1+t)^\delta \int_{t/2}^t \frac{(t-s)^{\gamma-1}}{(1+t-s)^{\gamma+\delta}} \cdot \frac{1}{(1+s)^\delta} ds \leq 2^\delta \cdot 2^\delta B(\gamma, \delta) = 2^{2\delta} B(\gamma, \delta).$$

Combining the two parts (and the case $t \in [0, 2]$) gives

$$\kappa_* \leq \max\left\{\frac{3^\delta 2^\gamma}{\gamma}, \frac{2^{\delta+1}}{1-\delta} + 2^{2\delta} B(\gamma, \delta)\right\} < \infty,$$

where $1 - \delta > 0$ because $0 < \delta < \gamma \leq 1$. □

Remark on the formulation of Lemma 5.3. The weight $(1+t)^\delta$ and the exponent $\gamma + \delta$ in (5.5) are dictated by the kernel estimate (5.7) below and by the weighted norm of the space Y_δ used in the fixed-point arguments of Theorems 5.7 and 5.8: the exponent must compensate the factor $(1+t)^\delta$ arising in those proofs. For large t the integral in (5.5) is $O(t^{-\delta})$ (the dominant contribution comes from s close to t), so the weighted supremum κ_* is finite. By contrast, with the exponent $\gamma - \delta$ instead of $\gamma + \delta$ the integral would converge, as $t \rightarrow \infty$, to the positive limit $B(\delta, 1 - \delta) = \pi / \sin(\pi\delta)$, and the corresponding weighted supremum $\sup_{t \geq 0} (1+t)^\delta \int_0^t (\cdots) ds$ would be infinite; the exponent $\gamma + \delta$ is therefore unavoidable.

Proposition 5.4. *Let H be a Hilbert space and let $-A$ be a self-adjoint positive definite operator with compact resolvent on H . Then the Caputo fractional (γ, δ, k) resolvent family $\{U(t)\}_{t \geq 0}$ generated by A satisfies*

$$\|U(t)\| \leq \frac{M_{\text{lin}}}{(1+t)^\delta}, \quad t \geq 0, \quad (5.6)$$

and the kernel operator $V(t)$ satisfies

$$\|V(t)\| \leq M_V \frac{t^{\gamma-1}}{(1+t)^{\gamma+\delta}}, \quad t > 0, \quad (5.7)$$

where M_{lin} is the constant of Theorem 4.1 and $M_V > 0$ is a constant depending on γ , δ , k , and λ_1 (equivalently, on the operator A).

Proof. The estimate (5.6) follows from a spectral decomposition argument combined with the asymptotic analysis of the scalar ML function. For each eigenvalue $\lambda_n > 0$ of $-A$, the scalar solution $y_n(t)$ of

$${}^C D_t^\gamma y + k {}^C D_t^\delta y + \lambda_n y = 0, \quad y(0) = 1,$$

has Laplace transform $\hat{y}_n(\lambda) = (\lambda^{\gamma-1} + k\lambda^{\delta-1})/(\lambda^\gamma + k\lambda^\delta + \lambda_n)$. As $\lambda \rightarrow 0^+$, one finds $\hat{y}_n(\lambda) \sim k\lambda^{\delta-1}/\lambda_n$, and the branch cut analysis via Watson's lemma yields $y_n(t) \sim \frac{k}{\lambda_n \Gamma(1-\delta)} t^{-\delta}$ as $t \rightarrow \infty$. The uniform bound $|y_n(t)| \leq C(1+t)^{-\delta}$ for all $t \geq 0$ is obtained by combining the short-time series representation with the long-time asymptotic; see [20] for the precise estimate of ML functions.

For the kernel estimate (5.7), we use a direct Laplace transform contour analysis rather than the (incorrect) pointwise convolution $V(t)x = \int_0^t g(t-s)U(s)x ds$; the correct relation is the integrated convolution (5.4). For each eigenfunction e_n of $-A$ with the eigenvalue $\lambda_n > 0$, define $z_n(t) := \langle V(t)e_n, e_n \rangle$. From (5.3), the Laplace transform of z_n is

$$\mathcal{L}\{z_n\}(\lambda) = \frac{1}{\lambda^\gamma + k\lambda^\delta + \lambda_n}.$$

By the same contour deformation as in Theorem 4.1, $z_n(t) = z_n^{\text{pole}}(t) + z_n^{\text{BC}}(t)$, where the pole contribution $z_n^{\text{pole}}(t)$ is exponentially decaying (present only when $\delta = 1$, i.e., $\gamma = \delta = 1$; for $0 < \delta < \gamma \leq 1$ there is no pole contribution by Proposition 3.3(i) and (ii), again legitimate because $\sigma(A) = \{-\lambda_n\}_{n=1}^\infty \subset (-\infty, 0)$ is real in the present self-adjoint setting) and the branch cut contribution is

$$z_n^{\text{BC}}(t) = -\frac{1}{\pi} \int_0^\infty e^{-st} \operatorname{Im} \left[\frac{1}{s^\gamma e^{i\pi\gamma} + ks^\delta e^{i\pi\delta} + \lambda_n} \right] ds.$$

Writing

$$a_n(s) := \lambda_n + s^\gamma \cos(\pi\gamma) + ks^\delta \cos(\pi\delta), \quad b(s) := s^\gamma \sin(\pi\gamma) + ks^\delta \sin(\pi\delta) > 0,$$

and $\mu(s) := s^\gamma e^{i\pi\gamma} + ks^\delta e^{i\pi\delta}$, so that $a_n(s) + ib(s) = \lambda_n + \mu(s)$ and $a_n(s)^2 + b(s)^2 = |\lambda_n + \mu(s)|^2$, we have $|\operatorname{Im}[1/(a_n + ib)]| = b/(a_n^2 + b^2)$ and therefore

$$|z_n^{\text{BC}}(t)| = \frac{1}{\pi} \int_0^\infty e^{-st} \frac{b(s)}{a_n(s)^2 + b(s)^2} ds.$$

We note that the crude bound $b/(a_n^2 + b^2) \leq 1/b$ is too lossy for our purposes: since $b(s) \geq ks^\delta \sin(\pi\delta)$, it yields only $|z_n^{\text{BC}}(t)| \leq \Gamma(1-\delta) t^{\delta-1}/(\pi k \sin(\pi\delta))$, which is both singular at $t = 0$ and non-decaying, whereas the true large-time decay of $z_n(t)$ is $t^{-1-\delta}$ (for fixed n , the small- s expansion $b(s)/(a_n^2 + b^2) =$

$\frac{k \sin(\pi\delta)}{\lambda_n^2} s^\delta + O(s^\gamma + s^{2\delta})$ together with Watson's lemma gives $z_n^{\text{BC}}(t) \sim \frac{k \sin(\pi\delta) \Gamma(1+\delta)}{\pi \lambda_n^2} t^{-1-\delta}$ as $t \rightarrow \infty$. To capture this decay uniformly in n , we split the integral at the n -independent threshold

$$s_1 := \min\left\{\left(\frac{\lambda_1}{8}\right)^{1/\gamma}, \left(\frac{\lambda_1}{8k}\right)^{1/\delta}\right\} > 0,$$

so that $s^\gamma \leq \lambda_1/8$ and $ks^\delta \leq \lambda_1/8$, hence $|\mu(s)| \leq s^\gamma + ks^\delta \leq \lambda_1/4 \leq \lambda_n/4$, for all $0 < s \leq s_1$ and all $n \geq 1$ (recall that $\lambda_n \geq \lambda_1 > 0$).

Region 1 ($0 < s \leq s_1$): Here $|\lambda_n + \mu(s)| \geq \lambda_n - |\mu(s)| \geq \frac{3}{4}\lambda_n \geq \frac{3}{4}\lambda_1$, while $b(s) \leq s^\gamma + ks^\delta$. Hence

$$\frac{b(s)}{a_n(s)^2 + b(s)^2} \leq \frac{16}{9\lambda_1^2} (s^\gamma + ks^\delta),$$

and consequently, for $t \geq 1$,

$$\begin{aligned} \frac{1}{\pi} \int_0^{s_1} e^{-st} \frac{b(s)}{a_n(s)^2 + b(s)^2} ds &\leq \frac{16}{9\pi\lambda_1^2} \int_0^\infty e^{-st} (s^\gamma + ks^\delta) ds \\ &= \frac{16}{9\pi\lambda_1^2} (\Gamma(1+\gamma) t^{-1-\gamma} + k\Gamma(1+\delta) t^{-1-\delta}). \end{aligned}$$

Since $t^{-1-\gamma} \leq t^{-1-\delta}$ for $t \geq 1$, the right-hand side is bounded by $C_1 t^{-1-\delta}$ with $C_1 := 16(\Gamma(1+\gamma) + k\Gamma(1+\delta))/(9\pi\lambda_1^2) > 0$ independent of n .

Region 2 ($s > s_1$): Here $b(s) \geq ks^\delta \sin(\pi\delta) \geq ks_1^\delta \sin(\pi\delta) =: c_1 > 0$, so $b/(a_n^2 + b^2) \leq 1/c_1$ and

$$\frac{1}{\pi} \int_{s_1}^\infty e^{-st} \frac{b(s)}{a_n(s)^2 + b(s)^2} ds \leq \frac{1}{\pi c_1} \int_{s_1}^\infty e^{-st} ds = \frac{e^{-s_1 t}}{\pi c_1 t} \leq C_2 t^{-1-\delta}, \quad t \geq 1,$$

with $C_2 := \pi^{-1} c_1^{-1} \sup_{t \geq 1} t^\delta e^{-s_1 t} < \infty$ again independent of n .

Combining the two regions gives the uniform long-time estimate

$$|z_n^{\text{BC}}(t)| \leq (C_1 + C_2) t^{-1-\delta}, \quad t \geq 1, \quad n \geq 1.$$

Step 3: Short-time bound, uniform in n . We now prove the short-time bound by a direct contour (branch-cut) argument that is uniform in n ; note that the relation $\hat{z}_n(\lambda) \sim \lambda^{-\gamma}$ as $\lambda \rightarrow \infty$ alone does not imply a pointwise bound on $z_n(t)$ (the order of Laplace transforms is not the order of the originals), so a genuine estimate is required. Since $0 < \delta < \gamma \leq 1$ entails $0 < \delta < 1$, Proposition 3.3(i) and (ii) gives no pole contribution, so $z_n(t) = z_n^{\text{BC}}(t)$; moreover $b(s) > 0$ for $s > 0$ in both cases below, so the branch-cut integrand is nonnegative, hence $z_n(t) \geq 0$ and $|z_n(t)| = z_n(t)$ for all $t > 0$.

Case $0 < \delta < \gamma < 1$. Here $\sin(\pi\gamma) > 0$ and $\sin(\pi\delta) > 0$, so $b(s) = s^\gamma \sin(\pi\gamma) + ks^\delta \sin(\pi\delta) \geq s^\gamma \sin(\pi\gamma) > 0$ for $s > 0$. Consequently,

$$\frac{b(s)}{a_n(s)^2 + b(s)^2} \leq \frac{1}{b(s)} \leq \frac{1}{s^\gamma \sin(\pi\gamma)},$$

and therefore, for every $t > 0$ and every $n \geq 1$,

$$|z_n(t)| = \frac{1}{\pi} \int_0^\infty e^{-st} \frac{b(s)}{a_n(s)^2 + b(s)^2} ds \leq \frac{1}{\pi \sin(\pi\gamma)} \int_0^\infty e^{-st} s^{-\gamma} ds$$

$$= \frac{\Gamma(1-\gamma)}{\pi \sin(\pi\gamma)} t^{\gamma-1} = \frac{t^{\gamma-1}}{\Gamma(\gamma) \sin^2(\pi\gamma)} = \frac{C_\gamma t^{\gamma-1}}{\Gamma(\gamma)},$$

where the Euler reflection formula $\Gamma(\gamma)\Gamma(1-\gamma) = \pi/\sin(\pi\gamma)$ was applied to $\Gamma(1-\gamma)$, and $C_\gamma := 1/\sin^2(\pi\gamma) \geq 1$ depends only on γ . Note that the bound is completely independent of n (and of λ_n); in fact it is valid for all $t > 0$, although only $0 < t \leq 1$ is needed below. The order $t^{\gamma-1}$ is optimal: For each fixed n , the symbol $\lambda \mapsto \lambda^\gamma + k\lambda^\delta + \lambda_n$ is a Bernstein function, so $\hat{z}_n(\lambda) = 1/(\lambda^\gamma + k\lambda^\delta + \lambda_n)$ is a Stieltjes function, and therefore z_n is completely monotone (in particular, it is non-negative and non-increasing); since $\hat{z}_n(\lambda) \sim \lambda^{-\gamma}$ as $\lambda \rightarrow \infty$, Karamata's Tauberian theorem yields $z_n(t) \sim t^{\gamma-1}/\Gamma(\gamma)$ as $t \rightarrow 0^+$; the uniform bound above has the same order with the constant $C_\gamma \geq 1$ in place of 1.

Case $\gamma = 1$ (so $0 < \delta < 1$). Here $\sin(\pi\gamma) = 0$, and

$$a_n(s) = \lambda_n - \psi(s), \quad b(s) = k \sin(\pi\delta) s^\delta, \quad \psi(s) := s - k \cos(\pi\delta) s^\delta,$$

so that

$$z_n(t) = \frac{1}{\pi} \int_0^\infty e^{-st} \frac{k \sin(\pi\delta) s^\delta}{(\lambda_n - \psi(s))^2 + k^2 \sin^2(\pi\delta) s^{2\delta}} ds \geq 0.$$

We claim that $\sup_{n \geq 1} \int_0^\infty F_n(s) ds$ is finite, where F_n denotes the s -integrand above; since $e^{-st} \leq 1$, this yields a bound on $z_n(t)$ uniform in both n and t . Put $s_k := \max\{(2k)^{1/(1-\delta)}, 1\}$. For $s \geq s_k$ we have $ks^\delta \leq s/2$, hence

$$\frac{s}{2} \leq \psi(s) \leq \frac{3s}{2}, \quad \psi'(s) = 1 - k\delta \cos(\pi\delta) s^{\delta-1} \in \left[\frac{1}{2}, \frac{3}{2}\right],$$

so ψ is strictly increasing and C^1 on $[s_k, \infty)$. We split $[0, \infty)$ into four regions.

Region I ($0 < s \leq s_k$): From $F_n(s) \leq 1/(k \sin(\pi\delta) s^\delta)$ (the elementary inequality $y/(x^2 + y^2) \leq 1/y$ for $x \in \mathbb{R}$, $y > 0$, with $y = k \sin(\pi\delta) s^\delta$), we have

$$\int_0^{s_k} F_n(s) ds \leq \frac{1}{k \sin(\pi\delta)} \int_0^{s_k} s^{-\delta} ds = \frac{s_k^{1-\delta}}{(1-\delta) k \sin(\pi\delta)} =: K_I.$$

Region II ($s \geq s_k$, $\psi(s) \leq \lambda_n/2$): Then $\lambda_n - \psi(s) \geq \lambda_n/2$ and $s \leq 2\psi(s) \leq \lambda_n$, so

$$F_n(s) \leq \frac{k \sin(\pi\delta) s^\delta}{(\lambda_n/2)^2} = \frac{4k \sin(\pi\delta) s^\delta}{\lambda_n^2},$$

and therefore

$$\begin{aligned} \int_{\text{II}} F_n ds &\leq \frac{4k \sin(\pi\delta)}{\lambda_n^2} \int_0^{\lambda_n} s^\delta ds = \frac{4k \sin(\pi\delta)}{1+\delta} \lambda_n^{\delta-1} \\ &\leq \frac{4k \sin(\pi\delta)}{1+\delta} \lambda_1^{\delta-1} =: K_{\text{II}}, \end{aligned}$$

using $\delta - 1 < 0$ and $\lambda_n \geq \lambda_1 > 0$.

Region III ($s \geq s_k$, $\psi(s) \geq 3\lambda_n/2$): Then $\lambda_n \leq \frac{2}{3}\psi(s)$, so $|\lambda_n - \psi(s)| = \psi(s) - \lambda_n \geq \psi(s)/3 \geq s/6$, and, since $\psi(s) \leq \frac{3}{2}s$ implies $s \geq \frac{2}{3}\psi(s) \geq \lambda_n \geq \lambda_1$ on this region, we have

$$F_n(s) \leq \frac{k \sin(\pi\delta) s^\delta}{(s/6)^2} = 36k \sin(\pi\delta) s^{\delta-2},$$

and therefore

$$\int_{\text{III}} F_n ds \leq 36k \sin(\pi\delta) \int_{\lambda_1}^{\infty} s^{\delta-2} ds = \frac{36k \sin(\pi\delta)}{1-\delta} \lambda_1^{\delta-1} =: K_{\text{III}},$$

the integral converging because $\delta - 2 < -1$.

Region IV ($s \geq s_k$, $\lambda_n/2 \leq \psi(s) \leq 3\lambda_n/2$): Here $\frac{2}{3}\psi(s) \leq s \leq 2\psi(s)$, hence $s \in [\lambda_n/3, 3\lambda_n]$, and therefore $Y(s) := k \sin(\pi\delta) s^\delta$ satisfies $Y(s) \in [a_*, A_*]$ on this region, where $a_* := k \sin(\pi\delta)(\lambda_n/3)^\delta$ and $A_* := k \sin(\pi\delta)(3\lambda_n)^\delta$. Substituting $u = \psi(s)$ (so $du = \psi'(s) ds$ with $\psi' \geq 1/2$, and $F_n = Y/((u - \lambda_n)^2 + Y^2)$),

$$\int_{\text{IV}} F_n ds = \int_{\lambda_n/2}^{3\lambda_n/2} \frac{Y(u)}{(u - \lambda_n)^2 + Y(u)^2} \frac{du}{\psi'(s(u))} \leq 2 \int_{\lambda_n/2}^{3\lambda_n/2} \min\left\{\frac{1}{a_*}, \frac{A_*}{(u - \lambda_n)^2}\right\} du,$$

where $Y(u) \in [a_*, A_*]$ was used in both branches of the minimum. The integrand is even in $u - \lambda_n$; writing $r = |u - \lambda_n| \in [0, \lambda_n/2]$ and using $\min\{a_*^{-1}, A_* r^{-2}\} \leq a_*^{-1}$ for $r \leq \sqrt{a_* A_*}$ and $\leq A_* r^{-2}$ for $r \geq \sqrt{a_* A_*}$,

$$\int_{\text{IV}} F_n ds \leq 2 \cdot 2 \left(\frac{\sqrt{a_* A_*}}{a_*} + \frac{A_*}{\sqrt{a_* A_*}} \right) = 8 \sqrt{\frac{A_*}{a_*}} = 8 \cdot 3^\delta =: K_{\text{IV}} \leq 24.$$

Combining the four regions, $\sup_{n \geq 1} \int_0^\infty F_n(s) ds \leq K_{\text{I}} + K_{\text{II}} + K_{\text{III}} + K_{\text{IV}} =: C_* < \infty$, where C_* depends only on δ, k , and λ_1 . Consequently $z_n(t) \leq C_*/\pi$ for all $t \geq 0$ and $n \geq 1$; in the claimed form, $|z_n(t)| \leq C t^{\gamma-1}/\Gamma(\gamma)$ for $0 < t \leq 1$ with $C := C_*/\pi$ (recall $\gamma = 1$, $\Gamma(1) = 1$).

Step 4: From the mode-wise bounds to the operator norm: $V(t)$ is a spectral multiplier. The scalar quantities z_n completely determine $V(t)$. The argument is placed here because it relies on the uniform bounds of Steps 2–3. Let $(e_n)_{n \geq 1}$ be the orthonormal basis of eigenvectors of $-A$ with $-Ae_n = \lambda_n e_n$ (compact resolvent), and define, for $t > 0$,

$$Z(t)x := \sum_{n=1}^{\infty} z_n(t) \langle x, e_n \rangle e_n, \quad x \in H.$$

(i) *Well-definedness, norm, and spectral-multiplier property.* By Steps 2–3, $\sup_{n \geq 1} |z_n(t)|$ is finite for every $t > 0$ (short-time bound for $0 < t \leq 1$, long-time bound for $t \geq 1$). Hence the series converges in H and $\|Z(t)x\|^2 = \sum_n |z_n(t)|^2 |\langle x, e_n \rangle|^2$, which is at most $(\sup_n |z_n(t)|)^2 \|x\|^2$, so $Z(t)$ is a bounded diagonal operator with

$$\|Z(t)\| = \sup_{n \geq 1} |z_n(t)|.$$

Let g_t be the bounded Borel function on $\mathbb{R}$ with $g_t(\lambda_n) := z_n(t)$ and $g_t := 0$ if and only if $\sigma(-A) = \{\lambda_n : n \geq 1\}$ (a countable closed set, since $-A$ has compact resolvent). By the spectral theorem, $Z(t) = g_t(-A)$: both bounded operators agree on the orthonormal basis (e_n) . In particular, $Z(t)$ is a spectral multiplier of $-A$ and commutes with A : for $x \in D(A)$, $AZ(t)x = -\sum_n \lambda_n z_n(t) \langle x, e_n \rangle e_n = Z(t)Ax$, where $Z(t)x \in D(A)$ because $\sum_n \lambda_n^2 |z_n(t)|^2 |\langle x, e_n \rangle|^2 \leq (\sup_n |z_n(t)|)^2 \|Ax\|^2 < \infty$.

(ii) Z has the defining transform of V . We claim $\mathcal{L}\{Z(\cdot)x\}(\lambda) = R(\lambda^\gamma + k\lambda^\delta, A)x$ for every $\lambda > 0$ and $x \in H$. For the partial sums $S_N(t, x) := \sum_{n \leq N} z_n(t) \langle x, e_n \rangle e_n$ we have $\|Z(t)x - S_N(t, x)\| \rightarrow 0$ for each t (as shown in (i)) and

$$e^{-\lambda t} \|S_N(t, x)\| \leq e^{-\lambda t} \sup_{m \geq 1} |z_m(t)| \|x\|,$$

$$\int_0^\infty e^{-\lambda t} \sup_{m \geq 1} |z_m(t)| dt \leq C_0 \int_0^1 t^{\gamma-1} dt + (C_1 + C_2) \int_1^\infty t^{-1-\delta} dt < \infty,$$

by Steps 2–3 and $\gamma, \delta > 0$. Dominated convergence permits the termwise transform,

$$\mathcal{L}\{Z(\cdot)x\}(\lambda) = \lim_{N \rightarrow \infty} \sum_{n \leq N} \mathcal{L}\{z_n\}(\lambda) \langle x, e_n \rangle e_n = \sum_{n=1}^\infty \frac{\langle x, e_n \rangle}{\lambda^\gamma + k\lambda^\delta + \lambda_n} e_n = R(\lambda^\gamma + k\lambda^\delta, A)x,$$

where the series converges because $|\lambda^\gamma + k\lambda^\delta + \lambda_n|^{-1} \leq (\lambda^\gamma + \lambda_1)^{-1}$ for $\lambda > 0$, and the last equality holds because $\sigma(A) = \{-\lambda_n\}$, so $\mu := \lambda^\gamma + k\lambda^\delta > 0$ lies in $\rho(A)$ and $R(\mu, A)e_n = (\mu + \lambda_n)^{-1}e_n$.

(iii) *Identification* $V = Z$. Both $t \mapsto \int_0^t V(s)x ds$ and $t \mapsto \int_0^t Z(s)x ds$ are continuous on $[0, \infty)$ (Bochner primitives of locally integrable functions, vanishing at $t = 0$), of exponential order, and have equal Laplace transforms: By (ii) and (5.3), both transforms equal $R(\lambda^\gamma + k\lambda^\delta, A)x/\lambda$. By the uniqueness theorem for Laplace transforms [28], the two primitives coincide on $[0, \infty)$. Since Definition 5.1 specifies the kernel operator $V(\cdot)x$ through (5.3), i.e. up to equality a.e., and since $t \mapsto Z(t)x$ is continuous on $(0, \infty)$ (each z_n is continuous, and $\sup_m |z_m(t)|$ is locally bounded on $(0, \infty)$ by Steps 2–3, so dominated convergence applies in ℓ^2), we identify V with its continuous representative: $V(t) := Z(t)$ for all $t > 0$. (All appearances of V in Sections 5 and 6 are through (5.3) and through Bochner integrals $\int_0^t V(t-s)h(s) ds$, both insensitive to modifications on null sets.) Consequently, for every $t > 0$:

- $V(t) = g_t(-A)$ is a bounded Borel function of $-A$; in particular $V(t)$ commutes with A and with every spectral projection of $-A$;
- $V(t)e_n = z_n(t)e_n$ for all n (each e_n is an eigenvector of $V(t)$);
- the spectral expansion $V(t)x = \sum_{n=1}^\infty z_n(t) \langle x, e_n \rangle e_n$, $x \in H$;
- $\|V(t)\| = \sup_{n \geq 1} |z_n(t)|$.

Step 5: Conclusion. Finally, we combine the two regimes. For $0 < t \leq 1$ we have $(1+t)^{\gamma+\delta} \leq 2^{\gamma+\delta}$, and for $t \geq 1$ we have $(1+t)^{\gamma+\delta} \leq 2^{\gamma+\delta} t^{\gamma+\delta}$; hence, in both cases,

$$t^{\gamma-1} \leq \frac{2^{\gamma+\delta} t^{\gamma-1}}{(1+t)^{\gamma+\delta}} \quad \text{and} \quad t^{-1-\delta} = \frac{t^{\gamma-1}}{t^{\gamma+\delta}} \leq \frac{2^{\gamma+\delta} t^{\gamma-1}}{(1+t)^{\gamma+\delta}}.$$

By Steps 2–4,

$$\|V(t)\| = \sup_{n \geq 1} |z_n(t)| \leq \begin{cases} C_0 t^{\gamma-1}, & 0 < t \leq 1, \\ (C_1 + C_2) t^{-1-\delta}, & t \geq 1, \end{cases}$$

with $C_0 := C_\gamma/\Gamma(\gamma) = 1/(\Gamma(\gamma) \sin^2(\pi\gamma))$ if $0 < \delta < \gamma < 1$ (Step 3, first case; in fact valid for all $t > 0$) and $C_0 := C_*/\pi$ if $\gamma = 1$ (Step 3, second case). Therefore

$$\|V(t)\| \leq M_V \frac{t^{\gamma-1}}{(1+t)^{\gamma+\delta}}, \quad t > 0,$$

with $M_V := 2^{\gamma+\delta} \max\{C_0, C_1 + C_2\}$, where C_0 is the short-time constant above; M_V depends only on γ , δ , k , and λ_1 . This establishes (5.7).

We remark that $V(t)$ admits a formal Prabhakar-type eigenexpansion (see Appendix A), but this expansion is not used in the present proof for two reasons: (i) The convergence of the double series $\sum_{p=0}^{\infty} (-\lambda_n)^p e_{\gamma-\delta, \gamma(p+1), -k}^{p+1}(t)$ has not been established, and (ii) the alternating signs of the coefficients $(-\lambda_n)^p$ combined with the sign oscillations of the Prabhakar kernel $e_{\gamma-\delta, \gamma(p+1), -k}(t)$ make termwise asymptotic analysis intractable. The direct Laplace transform contour analysis employed above bypasses these difficulties. $\square$

Remark 5.5. The decay rate $(1+t)^{-\delta}$ reflects the dominant contribution of the branch cut in the Laplace inversion of the resolvent. The smaller fractional order δ governs the long-time behavior, which is consistent with the physical interpretation that the damping term $k {}^C D_t^\delta u$ introduces memory-dependent dissipation that slows down the relaxation process compared to the undamped case (where the decay rate would be $(1+t)^{-\gamma}$; see [13]). By (5.7), the kernel operator $V(t)$ decays at the faster rate $t^{-1-\delta}$ for large t ; this reflects the fact that its Laplace transform $\mathcal{L}\{V(\cdot)x\}(\lambda) = R(\lambda^\gamma + k\lambda^\delta, A)x$ remains bounded as $\lambda \rightarrow 0^+$, so that the singularity of the branch cut integrand is softened by one power of s compared to that of U .

5.2. Global existence and decay under Lipschitz condition

We first consider the case where f is globally Lipschitz continuous in u .

Theorem 5.6. Let A generate a Caputo fractional (γ, δ, k) resolvent family $\{U(t)\}_{t \geq 0}$ on X such that the estimates (5.6) and (5.7) hold. Assume that $f: [0, \infty) \times X \rightarrow X$ satisfies:

(A1) $f(\cdot, 0) \in L^\infty([0, \infty); X)$;

(A2) $L > 0$ such that $\|f(t, u) - f(t, v)\| \leq L\|u - v\|$ for all $t \geq 0$ and $u, v \in X$.

Then for every $u_0 \in X$, Eq (5.1) admits a unique global mild solution $u \in C([0, \infty); X)$.

Proof. For $T > 0$, define the mapping $\Phi: C([0, T]; X) \rightarrow C([0, T]; X)$ by

$$(\Phi u)(t) = U(t)u_0 + \int_0^t V(t-s)f(s, u(s)) ds.$$

From (5.7) and the Lipschitz condition (A2), we obtain

$$\begin{aligned} \|(\Phi u)(t) - (\Phi v)(t)\| &\leq \int_0^t \|V(t-s)\| \|f(s, u(s)) - f(s, v(s))\| ds \\ &\leq M_V L \int_0^t \frac{(t-s)^{\gamma-1}}{(1+t-s)^{\gamma+\delta}} ds \cdot \|u - v\|_{C([0, T]; X)} \\ &\leq M_V L \int_0^t (t-s)^{\gamma-1} ds \cdot \|u - v\|_{C([0, T]; X)} \leq \frac{M_V L T^\gamma}{\gamma} \|u - v\|_{C([0, T]; X)}, \end{aligned}$$

where we used $(1+t-s)^{\gamma+\delta} \geq 1$. Choosing a T sufficiently small so that $M_V L T^\gamma / \gamma < 1$, the mapping Φ is a contraction on $C([0, T]; X)$. The unique fixed point yields a local mild solution. Since the Lipschitz condition is global and the kernel $\|V(t)\|$ is integrable on every compact interval $[0, T]$, the solution extends to any time interval by a standard continuation argument. $\square$

We now establish the decay rate of the mild solution.

Theorem 5.7. *Under the assumptions of Theorem 5.6, suppose additionally that $f(t, 0) = 0$ for all $t \geq 0$. Let κ_* be the constant of Lemma 5.3 and M_V the constant of (5.7), and set*

$$L_0 := \frac{1}{2M_V\kappa_*} > 0.$$

Then, whenever $L \leq L_0$ (equivalently, whenever the smallness condition $M_V L \kappa_ \leq \frac{1}{2}$ holds), the mild solution of (5.1) satisfies*

$$\|u(t)\| \leq \frac{C}{(1+t)^\delta} \|u_0\|, \quad t \geq 0, \quad (5.8)$$

where $C > 0$ is a constant independent of u_0 .

Proof. Since $f(t, 0) = 0$ and f is globally Lipschitz with constant L , we have $\|f(t, u)\| \leq L\|u\|$ for all $t \geq 0$ and $u \in X$. Define the weighted space

$$Y_\delta := \{u \in C([0, \infty); X) : \|u\|_{Y_\delta} := \sup_{t \geq 0} (1+t)^\delta \|u(t)\| < \infty\}.$$

We first establish the nonlinear integral estimate in Y_δ , which is the core of the fixed-point argument. For $u, v \in Y_\delta$ and $t \geq 0$, using the Lipschitz condition (A2), the kernel estimate (5.7) (whose denominator carries the exponent $\gamma + \delta$), and $\|u(s) - v(s)\| \leq \|u - v\|_{Y_\delta} (1+s)^{-\delta}$, we obtain

$$\begin{aligned} & (1+t)^\delta \left\| \int_0^t V(t-s)[f(s, u(s)) - f(s, v(s))] ds \right\| \\ & \leq (1+t)^\delta \int_0^t \|V(t-s)\| L \|u(s) - v(s)\| ds \\ & \leq M_V L \|u - v\|_{Y_\delta} (1+t)^\delta \int_0^t \frac{(t-s)^{\gamma-1}}{(1+t-s)^{\gamma+\delta}} \frac{ds}{(1+s)^\delta} \\ & \leq M_V L \kappa_* \|u - v\|_{Y_\delta}, \end{aligned}$$

where the last step is precisely the content of Lemma 5.3: The supremum (5.5) bounds the weighted integral for every $t \geq 0$. Thus we have proved

$$(1+t)^\delta \left\| \int_0^t V(t-s)[f(s, u(s)) - f(s, v(s))] ds \right\| \leq M_V L \kappa_* \|u - v\|_{Y_\delta}, \quad t \geq 0, \quad u, v \in Y_\delta, \quad (5.9)$$

and this is exactly where the reformulated Lemma 5.3 and the exponent $\gamma + \delta$ in (5.7) are needed.

For $u \in Y_\delta$, applying (5.9) with $v = 0$ (note $f(s, 0) = 0$ and $0 \in Y_\delta$) and using (5.6) for the linear part,

$$\begin{aligned} (1+t)^\delta \|(\Phi u)(t)\| & \leq (1+t)^\delta \|U(t)u_0\| + (1+t)^\delta \left\| \int_0^t V(t-s)f(s, u(s)) ds \right\| \\ & \leq M_{\text{lin}} \|u_0\| + M_V L \kappa_* \|u\|_{Y_\delta}, \end{aligned}$$

and hence

$$\|\Phi u\|_{Y_\delta} \leq M_{\text{lin}} \|u_0\| + M_V L \kappa_* \|u\|_{Y_\delta}.$$

Set $L_0 := 1/(2M_V L_{\kappa_*})$. If $L \leq L_0$, then

$$M_V L_{\kappa_*} \leq \frac{1}{2},$$

so that

$$\|\Phi u\|_{Y_\delta} \leq M_{\text{lin}} \|u_0\| + \frac{1}{2} \|u\|_{Y_\delta}.$$

On the ball $B_R := \{u \in Y_\delta : \|u\|_{Y_\delta} \leq R\}$ with $R = 2M_{\text{lin}} \|u_0\|$, we have

$$\|\Phi u\|_{Y_\delta} \leq M_{\text{lin}} \|u_0\| + \frac{1}{2} \cdot 2M_{\text{lin}} \|u_0\| = 2M_{\text{lin}} \|u_0\| = R,$$

so Φ maps B_R into itself. Moreover, for $u, v \in B_R$, by (5.9),

$$\|\Phi u - \Phi v\|_{Y_\delta} \leq M_V L_{\kappa_*} \|u - v\|_{Y_\delta} \leq \frac{1}{2} \|u - v\|_{Y_\delta},$$

which shows that Φ is a contraction on B_R . By the Banach fixed-point theorem, Φ has a unique fixed point $u \in B_R \subset Y_\delta$. Substituting $u = \Phi u$ into $\|\Phi u\|_{Y_\delta} \leq M_{\text{lin}} \|u_0\| + \frac{1}{2} \|u\|_{Y_\delta}$ and absorbing the factor $\frac{1}{2}$ yields

$$\|u\|_{Y_\delta} \leq 2M_{\text{lin}} \|u_0\|,$$

that is, (5.8) holds with the constant

$$C = 2M_{\text{lin}},$$

which is consistent with the smallness condition $M_V L_{\kappa_*} \leq \frac{1}{2}$: The factor $\frac{1}{2}$ entering this condition is precisely the factor that is absorbed into the final constant $2M_{\text{lin}}$. $\square$

5.3. Small data global existence under local Lipschitz condition

For nonlinearities that are only locally Lipschitz, global existence with decay can still be ensured provided the initial data is sufficiently small.

Theorem 5.8. *Let the assumptions of Proposition 5.4 hold. Assume that $f: [0, \infty) \times X \rightarrow X$ satisfies:*

(B1) $f(t, 0) = 0$ for all $t \geq 0$;

(B2) For every $r > 0$, there exists $L(r) > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq L(r) \|u - v\|,$$

for all $t \geq 0$ and $\|u\|, \|v\| \leq r$;

(B3) $L(r) \rightarrow 0$ as $r \rightarrow 0^+$.

Then there exists $\rho > 0$ such that for every $u_0 \in X$ with $\|u_0\| \leq \rho$, Eq (5.1) admits a unique global mild solution satisfying

$$\|u(t)\| \leq \frac{C}{(1+t)^\delta} \|u_0\|, \quad t \geq 0. \quad (5.10)$$

Proof. We work in the weighted space Y_δ as in the proof of Theorem 5.7, and let Φ be the mapping from the proof of Theorem 5.6. For $R > 0$ consider the closed ball $B_R := \{u \in Y_\delta : \|u\|_{Y_\delta} \leq R\}$; every $u \in B_R$ satisfies $\|u(s)\| \leq R(1+s)^{-\delta} \leq R$, so the local Lipschitz condition (B2) gives $\|f(s, u(s)) - f(s, v(s))\| \leq L(R) \|u(s) - v(s)\|$ for $u, v \in B_R$.

Exactly as in the proof of Theorem 5.7 (the estimate (5.9) with L replaced by $L(R)$), for $u, v \in B_R$ and $t \geq 0$,

$$(1+t)^\delta \left\| \int_0^t V(t-s)[f(s, u(s)) - f(s, v(s))]ds \right\| \leq M_V L(R) \kappa_* \|u - v\|_{Y_\delta},$$

where (5.7) (the denominator exponent $\gamma + \delta$), (B2), and Lemma 5.3 are used in this order.

By condition (B3) ($L(r) \rightarrow 0$ as $r \rightarrow 0^+$), we can choose $R_0 > 0$ such that

$$M_V L(R_0) \kappa_* \leq \frac{1}{2}.$$

Set $\rho := R_0/(2M_{\text{lin}})$. Applying the preceding estimate with $v = 0$ (note $f(s, 0) = 0$ by (B1)) and using (5.6), for $\|u_0\| \leq \rho$ and $u \in B_{R_0}$,

$$\|\Phi u\|_{Y_\delta} \leq M_{\text{lin}} \|u_0\| + M_V L(R_0) \kappa_* \|u\|_{Y_\delta} \leq M_{\text{lin}} \rho + \frac{1}{2} R_0 = \frac{1}{2} R_0 + \frac{1}{2} R_0 = R_0,$$

so Φ maps B_{R_0} into itself. The contraction property follows from the same estimate with $u, v \in B_{R_0}$:

$$\|\Phi u - \Phi v\|_{Y_\delta} \leq M_V L(R_0) \kappa_* \|u - v\|_{Y_\delta} \leq \frac{1}{2} \|u - v\|_{Y_\delta}.$$

By the Banach fixed-point theorem, Φ has a unique fixed point $u \in B_{R_0}$. Substituting $u = \Phi u$ into $\|\Phi u\|_{Y_\delta} \leq M_{\text{lin}} \|u_0\| + \frac{1}{2} \|u\|_{Y_\delta}$ and absorbing the factor $\frac{1}{2}$ yields

$$\|u\|_{Y_\delta} \leq 2M_{\text{lin}} \|u_0\| \leq 2M_{\text{lin}} \rho = R_0,$$

which is consistent with $u \in B_{R_0}$, and implies (5.10) with

$$C = 2M_{\text{lin}} = R_0/\rho,$$

in agreement with the choice $\rho := R_0/(2M_{\text{lin}})$ and the smallness condition $M_V L(R_0) \kappa_* \leq \frac{1}{2}$: the factor $\frac{1}{2}$ in that condition is exactly the factor absorbed into the final constant $2M_{\text{lin}}$. $\square$

Remark 5.9. Under additional regularity assumptions on the nonlinearity—for instance, if f satisfies $\|f(t, u)\| \leq C\|u\|^{1+\sigma}$ for some $\sigma > 0$ —the decay rate of the mild solution can be shown to be strictly faster than $(1+t)^{-\delta}$. In this case, the nonlinear term acts as a higher-order perturbation that does not affect the leading-order asymptotics. We refer to [13] for analogous results in the undamped fractional setting.

6. An example: Fractional damped beam equation

As an application of the abstract theory developed in Sections 4 and 5, we consider the fractional damped beam equation—A model that describes the vibration of a viscoelastic beam with memory-dependent damping. This example illustrates how the parameters (γ, δ, k) influence the decay rate of solutions.

6.1. Model formulation and resolvent family

Consider the initial-boundary value problem

$$\begin{cases} {}^C D_t^\gamma u(t, x) + k {}^C D_t^\delta u(t, x) = -a^4 \frac{\partial^4 u}{\partial x^4}(t, x), & x \in (0, 1), t > 0, \\ u(t, 0) = u(t, 1) = 0, \quad u_{xx}(t, 0) = u_{xx}(t, 1) = 0, & t > 0, \\ u(0, x) = f(x), & x \in (0, 1), \end{cases} \quad (6.1)$$

where $0 < \delta < \gamma \leq 1$, $k > 0$ is the damping coefficient, and $a > 0$ is a material parameter such that $a^4 = EI/(\rho \mathcal{A})$, where E is the elastic modulus, I is the second moment of area, ρ is the density, and $\mathcal{A}$ is the cross-sectional area.

Let $X = L^2(0, 1)$ and define the operator $A = -a^4 \partial^4 / \partial x^4$ with domain

$$D(A) = \{h \in W^{4,2}(0, 1) : h(0) = h(1) = 0, h''(0) = h''(1) = 0\}.$$

The operator $-A$ is positive self-adjoint on X with eigenvalues $\lambda_n = a^4 n^4 \pi^4$, $n \in \mathbb{N}$, and corresponding eigenfunctions $\varphi_n(x) = \sqrt{2} \sin(n\pi x)$. In particular, $-A$ is a self-adjoint positive definite operator with compact resolvent on the Hilbert space $X = L^2(0, 1)$, so the assumptions of Proposition 5.4 are satisfied.

As shown in [17, Example 1], the Caputo fractional (γ, δ, k) resolvent family generated by A is given by

$$(U(t)h)(x) = \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} (-a^4 n^4 \pi^4)^p e_{\gamma-\delta, \gamma p+1, -k}(t) h_n \sin(n\pi x), \quad (6.2)$$

where $h(x) = \sum_{n=1}^{\infty} h_n \sin(n\pi x)$ and $e_{\gamma-\delta, \gamma p+1, -k}(t) = t^{\gamma p} E_{\gamma-\delta, \gamma p+1}(-kt^{\gamma-\delta})$. Note that the coefficient $(-a^4 n^4 \pi^4)^p$ arises from the spectral decomposition $U(t)\varphi_n = y_n(t)\varphi_n$ with $A\varphi_n = -a^4 n^4 \pi^4 \varphi_n$, so that the p -th term involves $(-\lambda_n)^p = (-a^4 n^4 \pi^4)^p$. The convergence of this double series has not been established; it is presented here as a formal expansion for structural insight only. We note that we work with our sign convention $\sigma(A) \subset (-\infty, 0)$, which differs from the positive-spectrum convention of [17, Example 1]; the mode equation (6.3) and the decay analysis below are unaffected by this choice of convention, with λ_n denoting throughout the positive eigenvalue of $-A$.

6.2. Decay rate analysis

The action of $U(t)$ on the n -th eigenfunction is determined by the scalar function $y_n(t) = \langle U(t)\varphi_n, \varphi_n \rangle$, which is the solution of the scalar fractional ODE

$${}^C D_t^\gamma y_n(t) + k {}^C D_t^\delta y_n(t) + \lambda_n y_n(t) = 0, \quad y_n(0) = 1, \quad (6.3)$$

where $\lambda_n = a^4 n^4 \pi^4$ is the n -th eigenvalue of $-A$.

Since $-A$ is a self-adjoint positive definite operator with compact resolvent on the Hilbert space $X = L^2(0, 1)$ (see the discussion preceding (6.2)), the abstract decay theory of Section 4 applies directly. Specifically, Theorem 4.1 gives

$$|y_n(t)| \leq \frac{M_n}{(1+t)^\delta}, \quad t \geq 0,$$

where M_n depends on γ, δ, k and the eigenvalue λ_n . The asymptotic sharpness of Remark 4.2 applies on the first mode $n = 1$:

$$y_1(t) \sim \frac{k}{\lambda_1 \Gamma(1-\delta)} t^{-\delta} = \frac{k}{a^4 \pi^4 \Gamma(1-\delta)} t^{-\delta} \quad \text{as } t \rightarrow \infty. \quad (6.4)$$

Moreover, the uniform bound on the resolvent family (Proposition 5.4, equivalently (6.5) below) gives

$$\|U(t)\|_{\mathcal{L}(X)} \leq \frac{M}{(1+t)^\delta}, \quad t \geq 0, \quad (6.5)$$

where $M = M_{\text{lin}}$ is the constant of Theorem 4.1 and is independent of t . Since the spectral representation gives $\|U(t)\| = \sup_{n \geq 1} |y_n(t)|$ and $\lambda_n = a^4 n^4 \pi^4 \geq a^4 \pi^4 =: \lambda_1$ for all $n \geq 1$, the constant M_{lin} can be taken uniform in n . The detailed three-step proof (short-time uniform bound, long-time branch-cut estimate via Watson's lemma, and matching) is precisely the content of the proof of Theorem 4.1, so we do not repeat it here.

6.3. Influence of parameters on the decay rate

For each mode n , the associated scalar characteristic equation is

$$s^\gamma + k s^\delta + a^4 n^4 \pi^4 = 0. \quad (6.6)$$

Equation (6.6) provides insight into how the triplet (γ, δ, k) affects the qualitative behavior of solutions.

Effect of γ . As γ increases toward 1, the dominant derivative ${}^C D_t^\gamma$ approaches the first-order derivative. For all $0 < \delta < \gamma \leq 1$, the characteristic equation has no roots in the cut plane (by Proposition 3.3(i) and (ii); the beam operator is self-adjoint with real spectrum, so the real-spectrum regime applies here), so the decay is purely algebraic. Only in the integer-order case $\gamma = \delta = 1$ does the equation reduce to $(1+k)u' = Au$, yielding exponential decay.

Effect of δ . The fractional order δ of the damping term is the decisive parameter for the long-time decay rate. As $\delta \rightarrow \gamma^-$, the two fractional orders coincide and the equation effectively reduces to a single-term fractional equation with enhanced damping; the decay rate approaches $(1+t)^{-\gamma}$, which is the fastest rate achievable in this framework. Conversely, as $\delta \rightarrow 0^+$, the damping term $k {}^C D_t^\delta u$ approaches $k(u(t) - u(0))$ (since ${}^C D_t^0 u = u(t) - u(0)$ by the continuity of the Caputo derivative as $\delta \rightarrow 0^+$), and the decay rate slows to $(1+t)^{-\delta} \rightarrow 1$, indicating near-absence of effective dissipation.

Effect of k . The damping coefficient k affects the amplitude of the algebraic tail—as seen from (6.4), the tail amplitude is proportional to k —but does not alter the decay rate $t^{-\delta}$. This is a distinctive feature of fractional damping: increasing k amplifies the memory-dependent dissipation without accelerating the asymptotic decay. In contrast, for the integer-order case $\gamma = \delta = 1$, the equation reduces to $(1+k)u' = Au$, which yields exponential decay rather than algebraic decay.

Transition from undamped to damped. When $k = 0$, Eq (6.1) reduces to the undamped fractional beam equation ${}^C D_t^\gamma u = -a^4 \partial_x^4 u$, whose solution decays at the rate $(1+t)^{-\gamma}$ (cf. [13]). Introducing fractional damping ($k > 0$) slows the decay from $(1+t)^{-\gamma}$ to $(1+t)^{-\delta}$. This apparent slowdown has a clear physical and Tauberian explanation (see Remark 4.8): The fractional damping term $k {}^C D_t^\delta u$ introduces a power-law memory kernel that retains information about the entire deformation history,

thereby prolonging the relaxation process. The slowest memory process, governed by the order δ , ultimately determines the long-time behavior.

Concluding remarks. The results of this paper establish, for the first time, decay rates for Caputo-type fractional evolution equations with two-parameter damping. The central findings can be summarized as follows:

- (i) For the linear equation ${}^C D_t^\gamma u + k {}^C D_t^\delta u = Au$, the resolvent family decays at the algebraic rate $(1+t)^{-\delta}$, where the smaller fractional order δ governs the long-time behavior. The unregularized estimate $\|U(t)\| \leq M_{\text{lin}}(1+t)^{-\delta}$ is established in the self-adjoint, real-spectrum setting, where $0 < \delta < \gamma \leq 1$ is admissible (Theorem 4.1); for a general sectorial generator with possibly non-real spectrum, the corresponding result is the A^{-1} -regularized estimate $\|U(t)A^{-1}\| \leq Ct^{-\delta}$, valid only in the restricted regime $0 < \delta < \gamma < 1$ under the compatibility condition $\omega < \pi(1-\gamma)$ (Theorem 4.5).
- (ii) For the semilinear equation, the mild solution preserves the linear decay rate under a global Lipschitz condition with sufficiently small Lipschitz constant (Theorem 5.7), or under a local Lipschitz condition with sufficiently small initial data (Theorem 5.8).
- (iii) The application to the fractional damped beam equation demonstrates that the parameter δ is the critical factor determining the decay rate, while γ and k influence the transient behavior and the amplitude of the algebraic tail, respectively.

These results have clear physical significance. In viscoelastic materials modeled by fractional derivatives, the damping order δ characterizes the intensity of the memory effect: A smaller δ corresponds to longer-range memory and slower energy dissipation. The algebraic decay $t^{-\delta}$ observed in our analysis is consistent with the power-law relaxation observed experimentally in polymers and biological tissues [11, 12, 20].

Several directions for future research present themselves. First, the non-autonomous case, where the operator A or the damping coefficient k depends on time, requires the development of an evolution family framework for fractional equations with damping. Second, the inclusion of stochastic perturbations—in particular, Lévy noise or fractional Brownian motion—would extend the model to random environments and necessitate moment estimates for the stochastic convolution. Third, beyond the single-mode experiment of Section 6.4, a systematic numerical verification across parameter regimes, especially exponential decay for $\gamma = \delta = 1$ versus algebraic decay for $0 < \delta < \gamma \leq 1$ in the real-spectrum setting, would provide valuable insight into the practical implications of the theory. These extensions are the subject of ongoing work.

6.4. Numerical verification of the decay rate

We provide a numerical experiment that confirms the theoretical decay rate $O(t^{-\delta})$ established in Theorem 4.1. Consider the scalar mode $n = 1$ of the beam equation (6.1) with the parameters

$$\gamma = 0.9, \quad \delta = 0.5, \quad k = 0.5, \quad \lambda_1 = \pi^4 \approx 97.41,$$

and initial condition $y_1(0) = 1$. The solution $y_1(t)$ is computed in two independent ways:

- (a) By the fully implicit L1 time-stepping scheme for the Caputo derivatives ${}^C D_t^\gamma$ and ${}^C D_t^\delta$, with step size $h = 0.01$ on $[0, 1000]$ (i.e., $N = 10^5$ uniform steps): evaluated at t_{n+1} , each fractional

derivative is approximated by

$$D^\alpha y(t_{n+1}) \approx \frac{1}{h^\alpha \Gamma(2-\alpha)} \sum_{j=0}^n b_{n+1-j}^{(\alpha)} (y_{j+1} - y_j), \quad b_m^{(\alpha)} = m^{1-\alpha} - (m-1)^{1-\alpha},$$

and the resulting scalar linear equation for y_{n+1} is solved exactly at every step;

(b) By numerical quadrature of the branch cut integral

$$y_1(t) = -\frac{1}{\pi} \int_0^\infty e^{-st} \operatorname{Im} \left[\frac{s^{\gamma-1} e^{i\pi(\gamma-1)} + ks^{\delta-1} e^{i\pi(\delta-1)}}{s^\gamma e^{i\pi\gamma} + ks^\delta e^{i\pi\delta} + \lambda_1} \right] ds,$$

evaluated with adaptive Gauss–Kronrod quadrature after the substitution $s = u/t$ (with F denoting the bracketed fraction above),

$$y_1(t) = -\frac{1}{\pi t} \int_0^\infty e^{-u} \operatorname{Im} \left[F\left(\frac{u}{t}\right) \right] du,$$

which isolates the exponentially decaying factor e^{-u} (independent of t) and thereby makes the quadrature accurate and reproducible uniformly in t , particularly for a large t .

Both computations are fully deterministic: No random numbers are involved, the L1 recursion is a fixed arithmetic procedure (its implicit relation is solved exactly at each step), and the quadrature is tolerance-driven; rerunning the code therefore reproduces the reported values to the printed precision. Since the history sum at step n has a length n , the total cost of the L1 scheme is $O(N^2)$ (about 10^{10} floating-point operations for $N = 10^5$), which completes in roughly one minute on a standard desktop computer. The complete Python source code (NumPy/SciPy/Matplotlib, with adaptive Gauss–Kronrod tolerances $\varepsilon_{\text{abs}} = 10^{-13}$ and $\varepsilon_{\text{rel}} = 10^{-11}$) is provided as supplementary material. The two methods agree to within 0.5% relative error for $t \gtrsim 1$, providing an independent cross-check: the relative error is 0.32% at $t = 1$, 0.03% at $t = 10$, and below 0.01% for $t \geq 100$. For $t \lesssim 1$ the L1 scheme is less accurate, owing to the weak initial singularity of the solution ($y_1'(t) \sim t^{\gamma-1}$ as $t \rightarrow 0^+$, which locally degrades the first-order L1 approximation): the relative error is $\approx 2.1\%$ at $t = 0.2$ and $\approx 6.5\%$ at $t = 0.1$. Figure 1 shows $|y_1(t)|$ (computed by both methods) together with the theoretical asymptotic

$$y_{\text{asympt}}(t) = \frac{k}{\lambda_1 \Gamma(1-\delta)} t^{-\delta} \approx 2.90 \times 10^{-3} t^{-0.5}.$$

The numerical solution converges to the theoretical curve as $t \rightarrow \infty$: at $t = 10$ the ratio $|y_1(t)|/y_{\text{asympt}}(t) \approx 1.15$, at $t = 100$ the ratio is ≈ 1.06 , and at $t = 1000$ it is ≈ 1.02 , confirming the predicted $t^{-\delta}$ rate.

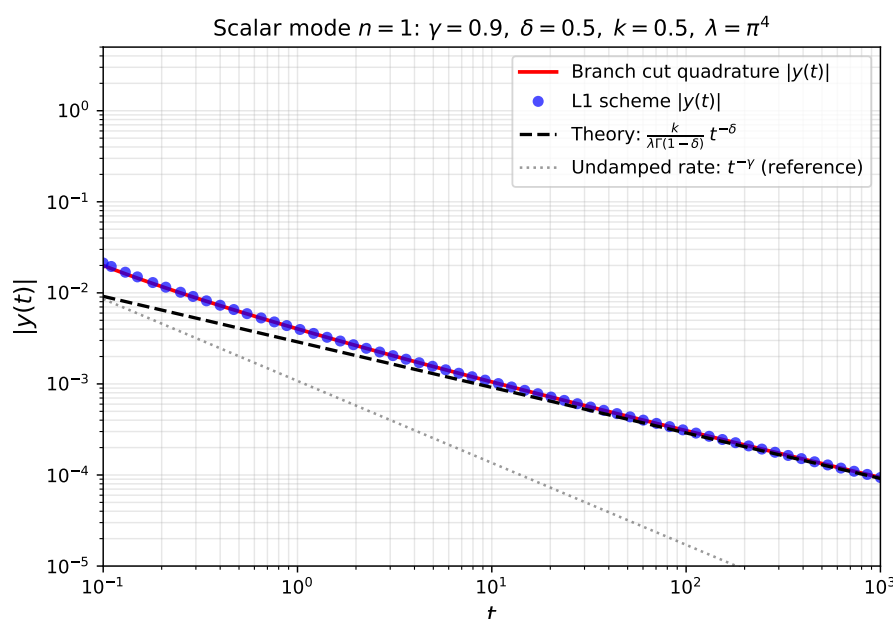


Figure 1. Numerical verification of the $t^{-\delta}$ decay rate for the first mode $n = 1$ of the scalar equation ${}^C D_t^{0.9} y + 0.5 {}^C D_t^{0.5} y + \pi^4 y = 0$, $y(0) = 1$. The L1 time-stepping solution (blue dots) and the branch cut quadrature (red solid) agree to within 0.5% relative error for $t \gtrsim 1$ (the error grows to $\approx 2.1\%$ at $t = 0.2$, reflecting the reduced accuracy of the L1 scheme near the initial weak singularity). Both converge to the theoretical asymptotic $y_{\text{asyp}}(t) = \frac{k}{\lambda_1 \Gamma(1-\delta)} t^{-\delta}$ (black dashed) as $t \rightarrow \infty$; the undamped reference rate $t^{-\gamma}$ is shown in gray dotted. The slope $-0.5 = -\delta$ is clearly visible on the log–log scale.

7. Conclusions

In this paper, we have investigated the long-time decay behavior of solutions to the Caputo-type fractional evolution equation ${}^C D_t^\gamma u(t) + k {}^C D_t^\delta u(t) = Au(t)$ with $0 < \delta < \gamma \leq 1$ and $k > 0$. Building on the Caputo fractional (γ, δ, k) resolvent family introduced in [17], we performed a detailed Laplace transform analysis: since the characteristic equation $\lambda^\gamma + k\lambda^\delta = \mu$ ($\mu < 0$) admits no roots in the cut plane $\mathbb{C} \setminus (-\infty, 0]$, the resolvent family is represented entirely by a branch cut integral, whose small-frequency asymptotics, evaluated via Watson's lemma, determine the long-time behavior. For self-adjoint A with compact resolvent we established the polynomial decay estimate $\|U(t)\| \leq M_{\text{lin}}(1+t)^{-\delta}$ (Theorem 4.1), which is sharp on the lowest eigenmode; for sectorial generators satisfying the compatibility condition $\omega < \pi(1-\gamma)$, the same rate $\|U(t)A^{-1}\| \leq Ct^{-\delta}$ holds (Theorem 4.5). A central finding is that the decay exponent is governed by the lower-order derivative δ rather than γ , so that fractional damping slows the relaxation relative to the undamped rate $t^{-\gamma}$, while the coefficient k affects only the prefactor. For the semilinear equation, global mild solutions exist under a global Lipschitz condition, and when the Lipschitz constant is sufficiently small the solutions inherit the linear decay rate $(1+t)^{-\delta}$ (Theorems 5.6 and 5.7). As an application, explicit decay estimates were derived for a fractionally damped beam equation and confirmed by numerical experiments (Section 6, Figure 1). Future work includes relaxing the compatibility condition $\omega < \pi(1-\gamma)$, treating critical and non-Lipschitz nonlinearities, and extending the analysis to time-dependent or distributed-order damping.

Author contributions

Shi-you Lin: Conceptualization, methodology, validation, formal analysis and investigation, resources, writing–original draft preparation, writing–review and editing, supervision, project administration, funding acquisition; Ting-ting Hu: Validation, formal analysis and investigation, writing–review and editing; Zhi-chao Lu: Validation, formal analysis and investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

The original contributions presented in this study are included in the article. The Python source code generating the numerical results and Figure 1 of Section 6.4 is available as supplementary material. Further inquiries can be directed to the corresponding author.

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Conflicts of interest

The authors declare no conflict of interest.

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A. Formal spectral representation of $V(t)$

Remark A.1. *The operator $V(t)$ admits a formal eigenexpansion in terms of Prabhakar-type functions. Specifically, in the spectral setting with orthonormal eigenfunctions $\{\varphi_n\}$ of A and eigenvalues $\{-\lambda_n\}$, one may formally write $V(t)\varphi_n = \sum_{p=0}^{\infty} (-\lambda_n)^p e_{\gamma-\delta, \gamma(p+1), -k}^{p+1}(t)\varphi_n$. However, the convergence and exact form of this expansion require further investigation, as the Laplace transform of the right-hand side does not simplify to $1/(\lambda^\gamma + k\lambda^\delta + \lambda_n)$ when the damping term $k\lambda^\delta$ is present. The expansion is therefore not used in any of the proofs in this paper. Note that the coefficient $(-a^4 n^4 \pi^4)^p$ alternates in sign with p , and the Prabhakar kernel $e_{\gamma-\delta, \gamma(p+1), -k}(t)$ also exhibits sign oscillations due to the negative parameter $-k$. The convergence and sign behavior of the double series require further investigation.*



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