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**Research article****On eccentricity spectra and Wiener index of comaximal graph****Ali Yahya Hummdi<sup>1,\*</sup>, L. K. Abbiramy<sup>2</sup>, K. Selvakumar<sup>2,\*</sup> and Junaid Nisar<sup>3,\*</sup>**<sup>1</sup> Department of Mathematics, College of Science, King Khalid University, Abha 61471, Saudi Arabia<sup>2</sup> Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, India<sup>3</sup> Symbiosis Institute of Technology, Pune Campus, Symbiosis International (Deemed University), Pune 412115, India**\* Correspondence:** Email: ahmdy@kku.edu.sa, selva\_158@yahoo.co.in, junaidnisar73@gmail.com.

**Abstract:** The eccentricity matrix  $\epsilon(G)$  of a connected graph  $G$  is a distance-based structural refinement constructed by preserving only the maximum geodesic distances within each row and column of the classical distance matrix, while mapping all remaining coefficients to zero. This paper investigated the architectural, spectral, and distance-based properties of the eccentricity matrix for comaximal graph  $\Gamma(R)$  associated with a general commutative Artinian ring  $R$  with unity is investigated. Extending this analysis to the ring of integers modulo  $n$ ,  $\mathbb{Z}_n$ , we provided explicit determinations of the eccentricity characteristic polynomials, eccentricity spectra, and various eccentric connectivity parameters.

**Keywords:** distance matrix; eccentricity matrix; eigenvalue**Mathematics Subject Classification:** 05C12, 05C50

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**1. Introduction**

The study explores a simple, connected graph  $G = (V(G), E(G))$ , defined by a vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and an edge set  $E(G) = \{e_1, e_2, \dots, e_m\}$ . Adjacency between two vertices  $v_i$  and  $v_j$  is denoted by  $v_i \sim v_j$ . While the classical adjacency matrix  $A(G)$  remains a fundamental tool in spectral graph theory, distance-based matrices, specifically those involving vertex eccentricity have gained significant traction due to their applications in chemical graph theory and network analysis. For any two vertices  $v, w \in V(G)$ , the distance  $d_G(v, w)$  represents the length of the shortest path connecting them. Based on this distance metric, we define several essential parameters. For any vertex  $v_i \in V(G)$ , the eccentricity  $e(v_i)$  is the maximum distance from  $v_i$  to any other vertex in the graph. The radius  $rad(G) = \min\{e(v_i)\}$  and the diameter  $diam(G) = \max\{e(v_i)\}$  represent the minimum and

maximum eccentricities in the graph, respectively. The eccentricity matrix  $\epsilon(G) = (\epsilon_{vw})$  of a graph  $G$  is a structural refinement of the distance matrix, specifically focusing on the most distant connections. For further details one can refer to [1–3]. The entries are defined as

$$\epsilon_{vw} = \begin{cases} d(v, w), & \text{if } d(v, w) = \min\{e(v), e(w)\}, \\ 0, & \text{otherwise.} \end{cases}$$

The use of the minimum eccentricity condition is intentional and is particularly convenient for the equiv-comaximal partition considered in this work. The minimum eccentricity condition preserves exactly the distance contributions required by the classwise decomposition used in the subsequent Wiener index and spectral calculations. This convention should be distinguished from other eccentricity or anti-adjacency matrix conventions in which the retained entries are determined using a maximum eccentricity condition or by requiring equality of the two eccentricities. Such choices generally produce a different zero or nonzero pattern and therefore a different matrix spectrum. Hence, the eccentricity spectra and indices obtained in this paper are specifically associated with the above minimum-eccentricity definition. As  $\epsilon(G)$  is a real symmetric matrix, its eigenvalues termed the eccentricity eigenvalues are real. We denote the spectrum as  $\text{Spec}_\epsilon(G)$ , ordered such that  $\zeta_1(G) \geq \zeta_2(G) \geq \cdots \geq \zeta_n(G)$ . The largest eigenvalue  $\zeta_1(G)$  is referred to as the  $\epsilon$ -spectral radius. Wang et al. [3] showed that the eccentricity matrix of trees is irreducible. They also discussed adjacency and eccentricity eigenvalues. Wei et al. [4] determined all trees of given order and diameter with the least  $\epsilon$ -spectral radius. Mahato [5] discussed the spectra and inertia of various classes of graphs. For more results the reader can refer [6, 7]. Recent developments and related indices have expanded the study of eccentricity to include various molecular descriptors. For instance, the  $\epsilon$ -Wiener index is given by  $W_\epsilon(G) = \frac{1}{2} \sum_{i,j} (\epsilon(G))_{ij}$ . The eccentric connectivity index is the sum of the products of the degree and eccentricity of each vertex,  $\zeta_e(G) = \sum_{i=1}^n \deg(v_i) e(v_i)$ . The eccentric connectivity coindex is the total eccentricity sum of all non-adjacent vertex pairs of a graph defined as  $\overline{\zeta}_e^c(G) = \sum_{uv \in E(G)} (e_G(u) + e_G(v))$ . The eccentric connectivity coindex  $n$ -vertex graph  $G$  can also be expressed as a sum over vertices of  $G$ .  $\overline{\zeta}_e^c(G) = \sum_{u \in V(G)} e_G(u)(n - 1 - d_G(u))$ . The first and second Zagreb eccentricity indices are defined as  $M_1(G) = \sum_{u \in V(G)} e^2(u)$  and  $M_2(G) = \sum_{uv \in E(G)} e(u)e(v)$ . Following the work of Wang et al. [3], who established the irreducibility of the eccentricity matrix for trees, this paper seeks to further explore the relationship between the eccentricity spectrum and the structural invariants of the graph. For further studies one can refer [8–10].

In Section 2, the algebraic and combinatorial properties of the comaximal graph architecture are rigorously analyzed via join and union operations. In Section 3, we establish the global structural partition of the vertex set for a general commutative Artinian ring  $R$  into equiv-comaximal classes  $S_U$ . We then derive the generalized eccentricity Wiener Index for  $\Gamma(R)$ , followed by the general formulation of its eccentricity Spectrum using equitable row-sum quotient matrices  $Q$ . In Section 4, the general frameworks are projected onto the ring of integers modulo  $n$ , with the eccentricity characteristic polynomials of the comaximal graph  $\Gamma(\mathbb{Z}_n)$ . In Section 5, the concrete analytical expressions for the eccentricity Wiener index of  $\Gamma(\mathbb{Z}_{p^\alpha q^\beta})$  and  $\Gamma(\mathbb{Z}_{pq^r})$  are computed. In Section 6, the matrix irreducibility property of the comaximal graph over  $\mathbb{Z}_n$  is proved.

Throughout this paper,  $I_{n \times n}$  denotes the identity matrix of order  $n$ ,  $0_{m \times n}$  denotes the null matrix of order  $m \times n$ , and  $J_{n \times n}$  denotes the matrix of order  $n$  whose all entries are equal to one.

**Theorem 1.** [11] Let  $M = (A_{ij})$  be a complex block matrix of order  $n$  such that  $A_{ij} = s_{ij} J_{n_i \times n_j}$  for  $i \neq j$ ,

and  $A_{ii} = s_{ii}J_{n_i \times n_i} + p_i I_{n_i}$ . Then the equitable quotient matrix of  $M$  is  $Q = (q_{ij})$  with  $q_{ij} = s_{ij}n_j$  if  $i \neq j$ , and  $q_{ii} = s_{ii}n_i + p_i$ . Moreover,  $\text{Spec}(M) = \text{Spec}(Q) \cup \{p_1^{[n_1-1]}, \dots, p_t^{[n_t-1]}\}$

**Theorem 2.** [12] Let  $G$  be a  $(p, q)$  graph. Then the matrix  $\epsilon(G)$  is irreducible if and only if  $G^e$  is connected.

## 2. Eccentricity spectra and Wiener index of the comaximal graph

To establish the eccentricity-based parameters of the comaximal graph, we recall the structural decomposition of  $\Gamma(R)$  for a commutative Artinian ring.

Let  $R \cong R_1 \times R_2 \times \cdots \times R_k$  be a commutative Artinian ring with unity, where each  $(R_i, \mathfrak{m}_i)$  is a local ring with a unique maximal ideal  $\mathfrak{m}_i$  for  $1 \leq i \leq k$ , and  $k \geq 2$ . Let  $U_i = R_i \setminus \mathfrak{m}_i$  denote the set of units in  $R_i$ .

For any set  $U \subseteq [k] = \{1, 2, \dots, k\}$ , we define the equiv-comaximal vertex classes  $S_U$  of  $R$  by

$$S_U = \{x = (x_1, x_2, \dots, x_k) \in R : x_i \in \mathfrak{m}_i \text{ if } i \in U \text{ and } x_i \in U_i \text{ if } i \notin U\}. \quad (2.1)$$

The set of all units in  $R$  is denoted by  $S_\emptyset$ . This collection of blocks  $\{S_U \mid U \subseteq [k]\}$  forms a partition of the total vertex set  $V(\Gamma(R)) = R$ . The cardinality of each block is given by

$$n_U = |S_U| = \prod_{i \in U} |\mathfrak{m}_i| \prod_{j \notin U} |U_j|. \quad (2.2)$$

For the unit set  $U = \emptyset$ ,

$$n_\emptyset = |S_\emptyset| = \prod_{j=1}^k |U_j|. \quad (2.3)$$

The adjacency criteria between these classes, governed by the comaximal relation  $Rx + Ry = R$ , yields the following established distance and structural properties.

By utilizing equitable block partition, the entire comaximal graph  $\Gamma(R)$  can be viewed as an  $H$ -join (or generalized composition) over a base graph  $\Psi(R)$  of order  $2^k$ , where each vertex of the base graph represents a class  $S_U$ . This regularized structural joining forms the algebraic foundation for evaluating both the eccentricity spectrum and the eccentricity Wiener index.

The vertex set  $V(\Gamma(R))$  is partitioned into disjoint equivalent comaximal classes  $S_U$  indexed by subsets  $U \subseteq [k] = \{1, 2, \dots, k\}$ , where  $n_U = |S_U|$ . The index set  $U \subseteq [k]$  identifies the collection of maximal ideal components in which a vertex's entries lie. In particular,  $U = \emptyset$  corresponds to the condition where an element belongs to no maximal ideal (i.e., it is a unit in  $R$ ). Thus, the notation  $S_\emptyset$  denotes the unit set  $U(R)$  of the ring  $R$ , which is nonempty, and its cardinality is given by  $n_\emptyset = |S_\emptyset| = \prod_{i=1}^k |U_i| > 0$ .

**Theorem 3.** Let  $R$  be an Artinian nonlocal ring. Then,

$$e(u) = \begin{cases} 1, & \text{if } u \in S_\emptyset, \\ 2, & \text{if } u \in S_U \text{ for } U \subset [k]. \end{cases}$$

*Proof.* **Case 1.** For unit element  $u \in S_\emptyset$ , every vertex in a proper sub-block  $S_U$  ( $U \subset [k]$ ) is adjacent to every vertex in  $S_\emptyset$ . Since the unit block  $S_\emptyset$  forms a clique,  $d(u, v) = 1$  for all distinct  $v \in V(\Gamma(R))$ . Thus, its eccentricity is  $e(u) = 1$ .

**Case 2.** For non-unit element  $u \in S_U$ ,  $U \subset [k]$ , for any proper subset  $U$ , there always exists another proper block  $W$  such that  $U \cup W \neq [k]$ , yielding a distance of  $d(u, v) = 2$ . Since any two non-unit vertices share a path through the unit block ( $u - w - v$ ) where  $w \in S_\emptyset$ , the maximum distance cannot exceed 2. Thus,  $e(u) = 2$ .  $\square$

**Theorem 4.** *The eccentricity Wiener index of  $\Gamma(R)$  is*

$$W_\epsilon(\Gamma(R)) = \binom{n_\emptyset}{2} + \sum_{U \subset [k]} n_\emptyset n_U + 2 \sum_{U \subset [k]} \binom{n_U}{2} + \sum_{\substack{U, W \subset [k] \\ U \neq W, U \cup W \neq [k]}} 2n_U n_W.$$

*Proof.* Evaluate this total sum by analyzing the structural properties of the equiv-comaximal partition  $V(\Gamma(R)) = \bigcup_{U \subseteq [k]} S_U$ . The matrix entry contributions  $\epsilon_{uv}$  across four distinct pairings are

**Case 1.** Let pairs be within the unit block,  $u, v \in S_\emptyset$ . For distinct elements inside the unit set,  $e(u) = 1$  and  $e(v) = 1$ , which gives  $\min\{e(u), e(v)\} = \min\{1, 1\} = 1$ . Since  $S_\emptyset$  induces a complete sub-graph, the distance is  $d(u, v) = 1$ . Because  $d(u, v) = \min\{1, 1\} = 1$ , the matrix entry condition holds and is preserved as 1. The total number of pairs is  $\binom{n_\emptyset}{2}$ , contributing  $\binom{n_\emptyset}{2}$  to the total sum.

**Case 2.** Let pairs be between unit and non-unit blocks,  $u \in S_\emptyset, v \in S_U$  where  $U \subset [k]$ . Here, the eccentricities are  $e(u) = 1$  and  $e(v) = 2$ , yielding  $\min\{e(u), e(v)\} = \min\{1, 2\} = 1$ . Every non-unit vertex is adjacent to every unit vertex, meaning  $d(u, v) = 1$ . Since  $d(u, v) = \min\{1, 2\} = 1$ , this pairing satisfies the condition, and the entry is preserved as 1. From this we get, the total number of pairs across all proper sub-blocks is  $\sum_{U \subset [k]} n_\emptyset n_U$ .

**Case 3.** Let pairs be within the same non-unit block,  $u, v \in S_U$  where  $U \subseteq [k]$ . For distinct non-unit elements in the same block,  $e(u) = 2$  and  $e(v) = 2$ , yielding  $\min\{e(u), e(v)\} = \min\{2, 2\} = 2$ . Proper sub-blocks are independent sets, and the distance between distinct elements inside the same block is exactly  $d(u, v) = 2$ . Since  $d(u, v) = \min\{2, 2\} = 2$ , the entry condition matches perfectly, yielding a value of 2 for each of the  $\binom{n_U}{2}$  pairs.

**Case 4.** Let pairs be between distinct non-unit blocks,  $u \in S_U, v \in S_W$  where  $U \neq W, U, W \subset [k]$ . Here,  $e(u) = 2$  and  $e(v) = 2$ , giving  $\min\{e(u), e(v)\} = 2$ .

**Subcase 1.** Let  $U \cup W = [k]$ , the blocks are adjacent, so  $d(u, v) = 1$ . Since  $1 \neq \min\{2, 2\} = 2$ , hence  $\epsilon_{uv} = 0$ .

**Subcase 2.** Let  $U \cup W \neq [k]$ , the distance between the blocks is  $d(u, v) = 2$ . Since  $2 = \min\{2, 2\} = 2$ , the condition is satisfied, and the entry is preserved as 2. Summing the nonzero contributions from Cases 1–4,

$$W_\epsilon(\Gamma(R)) = \binom{n_\emptyset}{2} + \sum_{U \subset [k]} n_\emptyset n_U + 2 \sum_{U \subset [k]} \binom{n_U}{2} + \sum_{\substack{U, W \subset [k] \\ U \neq W, U \cup W \neq [k]}} 2n_U n_W.$$

$\square$

**Theorem 5.** The eccentricity characteristic polynomial of  $\Gamma(R)$  is given by

$$P_{\epsilon}(\Gamma(R)) - \lambda I = (\lambda + 1)^{n_0-1} \left( \prod_{U \subset [k]} (\lambda + 2)^{n_U-1} \right) \prod_{i=1}^{2^k} (\lambda - \lambda_i),$$

where  $\lambda_i$  ( $1 \leq i \leq 2^k$ ) are the eigenvalues of the  $2^k \times 2^k$  eccentricity quotient matrix  $Q$ .

*Proof.* To compute the characteristic polynomial, we determine the matrix blocks of  $\epsilon(\Gamma(R))$  based on the entry condition  $d(u, v) = \min\{e(u), e(v)\}$ . By Theorem 3,  $e(u) = 1$  for  $u \in S_\emptyset$  and  $e(u) = 2$  for  $u \in S_U$  ( $U \subset [k]$ ).

**Case 1.** For  $u, v \in S_\emptyset$ ,  $\min\{1, 1\} = 1 = d(u, v)$ . The entries are 1, yielding a diagonal block of the form  $J_{n_0} - I_{n_0}$ .

**Case 2.** For  $u \in S_U$  ( $U \subset [k]$ ) and  $v \in S_\emptyset$ ,  $\min\{2, 1\} = 1 = d(u, v)$ . Thus, the entry is preserved as 1, and these cross-blocks are all-ones matrices ( $J$ ).

**Case 3.** For  $u, v \in S_U$  ( $U \subset [k]$ ),  $\min\{2, 2\} = 2 = d(u, v)$ . The entry value is 2, yielding a diagonal block of the form  $2(J_{n_U} - I_{n_U})$ .

**Case 4.** For  $u \in S_U, v \in S_W$  ( $U \neq W, U, W \subset [k]$ ),  $\min\{2, 2\} = 2$ . If  $U \cup W = [k]$ , then  $d(u, v) = 1 \neq 2 \implies \epsilon_{uv} = 0$ . If  $U \cup W \neq [k]$ , then  $d(u, v) = 2 = 2 \implies \epsilon_{uv} = 2$ .

Since the vertices within each partition class  $S_U$  maintain identical distance-eccentricity relationships with all elements in any other class, the vertex set division constitutes an equitable partition of  $\epsilon(\Gamma(R))$ .

By the algebraic properties of equitable block configurations, the characteristic polynomial of the global matrix factors completely into the product of its localized diagonal block equations and the characteristic polynomial of its representative row-sum quotient matrix  $Q$  of size  $2^k \times 2^k$ .

The unit block  $S_\emptyset$  forms a matrix block structure of  $(J - I)$ , contributing a localized factor of  $(\lambda + 1)^{n_0-1}$ . Each proper non-unitblock  $S_U$  ( $U \subset [k]$ ) forms a matrix block structure of  $2(J - I)$ , contributing a localized factor of  $(\lambda + 2)^{n_U-1}$ . The remaining  $2^k$  structural roots correspond exactly to the eigenvalues  $\lambda_i$  of the quotient matrix  $Q$ . Combining these independent factors together,

$$\det(\epsilon(\Gamma(R)) - \lambda I) = (\lambda + 1)^{n_0-1} \left( \prod_{U \subset [k]} (\lambda + 2)^{n_U-1} \right) \prod_{i=1}^{2^k} (\lambda - \lambda_i).$$

The entry  $q_{UW}$  of the associated  $2^k \times 2^k$  quotient matrix  $Q$ , which accounts for the row-sum transition from block  $S_U$  to block  $S_W$ , is explicitly defined as

$$q_{UW} = \begin{cases} n_\emptyset - 1, & \text{if } U = W = \emptyset, \\ n_W, & \text{if } U = \emptyset \text{ and } W \subset [k], \\ n_\emptyset, & \text{if } W \subset [k] \text{ and } W = \emptyset, \\ 2(n_U - 1), & \text{if } U = W \subset [k], \\ 2n_W, & \text{if } U, W \subset [k], U \neq W, \text{ and } U \cup W = [k], \\ 0, & \text{if } U, W \subset [k], U \neq W, \text{ and } U \cup W \neq [k]. \end{cases} \quad (2.4)$$

Hence, the proof is complete using Theorem 1.  $\square$

**Theorem 6.** *The eccentricity matrix of the comaximal graph  $\Gamma(R)$  is irreducible.*

*Proof.* Let  $R$  be an Artinian ring decomposed as  $R \cong R_1 \times R_2 \times \cdots \times R_k$ , and let  $\{S_U\}_{U \subseteq [k]}$  denote the vertex partition of  $\Gamma(R)$ . Since  $R$  is Artinian, the Jacobson radical  $J(R)$  is non-trivial if  $R$  contains non-zero nilpotent elements, which ensures that  $S_{[k]} \neq \emptyset$ . Without loss of generality, assume  $S_U \neq \emptyset$  for all relevant partition blocks under consideration. Let  $u, v \in V((\Gamma(R))^e)$ . We consider the following cases based on the location of  $u$  and  $v$ :

**Case 1.** For  $u, v \in S_{[k]}$ ,  $\min\{1, 1\} = 1 = d(u, v)$ . Then in the eccentricity graph  $(\Gamma(R))^e$ ,  $d_{(\Gamma(R))^e}(u, v) = 1$ .

**Case 2.** For  $u \in S_U$  ( $U \subset [k]$ ) and  $v \in S_{[k]}$ ,  $\min\{2, 1\} = 1 = d(u, v)$ . Then in the eccentricity graph  $(\Gamma(R))^e$ ,  $d_{(\Gamma(R))^e}(u, v) = 1$ .

**Case 3.** For  $u, v \in S_U$  ( $U \subset [k]$ ),  $\min\{2, 2\} = 2 = d(u, v)$ . Then in the eccentricity graph  $(\Gamma(R))^e$ ,  $d_{(\Gamma(R))^e}(u, v) = 1$ .

**Case 4.** For  $u \in S_U, v \in S_W$  ( $U \neq W, U, W \subset [k]$ ),  $\min\{2, 2\} = 2$ .

**Subcase 1.** If  $U \cup W = [k]$ , then  $d(u, v) = 1 \neq 2 \implies e_{uv} = 0$ . Since  $S_{[k]} \neq \emptyset$ , choose a vertex  $u_1 \in S_{[k]}$ . In  $\Gamma(R)$ ,  $d(u, u_1) = 2$  and  $d(v, u_1) = 2$ , which implies  $(u, u_1)$  and  $(u_1, v)$  are edges in  $(\Gamma(R))^e$ . Thus, there exists a path  $u - u_1 - v$  of length 2 in  $(\Gamma(R))^e$ , giving  $d_{(\Gamma(R))^e}(u, v) = 2$ .

**Subcase 2.** If  $U \cup W \neq [k]$ , then  $d(u, v) = 2 \implies e_{uv} = 2$ . Then in the eccentricity graph  $(\Gamma(R))^e$ ,  $d_{(\Gamma(R))^e}(u, v) = 1$ .

From the above cases,  $d_{(\Gamma(R))^e}(u, v) \leq 2$  for all  $u, v$ , and there exist pairs of vertices with  $d_{(\Gamma(R))^e}(u, v) = 2$ . Hence, the diameter of the eccentricity graph  $(\Gamma(R))^e$  is 2. By Theorem 2,  $\Gamma(R)$  is irreducible.  $\square$

### 3. Eccentricity spectra of the comaximal graph $\Gamma(\mathbb{Z}_n)$

Let the elements of the ring  $\mathbb{Z}_n$  be  $\{0, 1, 2, \dots, n-1\}$ . The structure of  $\Gamma(\mathbb{Z}_n)$  can be expressed as the join and union of induced subgraphs of  $\Gamma(\mathbb{Z}_n)$ . The eccentricity spectra and eccentricity Wiener index of the comaximal graph of  $\Gamma(\mathbb{Z}_n)$  can be calculated for different  $n$ . We begin by establishing the adjacency condition for vertices in  $\Gamma(\mathbb{Z}_n)$ . By the general definition of adjacency in  $\Gamma(\mathbb{Z}_n)$ , two vertices  $v_i, v_j \in \Gamma(\mathbb{Z}_n)$  are adjacent if and only if

$$\mathbb{Z}_n v_i + \mathbb{Z}_n v_j = \mathbb{Z}_n.$$

Since  $\mathbb{Z}_n$  is a principal ideal ring, every ideal of  $\mathbb{Z}_n$  is generated by a single element, and in particular  $\mathbb{Z}_n v_i = \langle v_i \rangle = \langle \gcd(v_i, n) \rangle$ . As the sum of two ideals in a ring is again an ideal, it follows that the above condition is equivalent to

$$\langle \gcd(v_i, n) \rangle + \langle \gcd(v_j, n) \rangle = \mathbb{Z}_n.$$

Consequently, the adjacency criterion in  $\Gamma(\mathbb{Z}_n)$  reduces to

$$v_i \sim v_j \iff \gcd(v_i, v_j, n) = 1.$$

The vertex set can be written as  $V(\Gamma(\mathbb{Z}_n)) = \epsilon_u \cup \epsilon_{d_i} \cup \{0\}$  where  $\epsilon_u = \{a : \gcd(a, n) = 1\}$  and for any vertex in  $\epsilon_u$  the degree is  $n-1$  and degree of  $\{0\}$  is  $\phi(n)$ . Let  $G'$  be the induced subgraph of  $\Gamma(\mathbb{Z}_n)$  on the set  $\epsilon_u$  and  $G''$  be the induced subgraph of  $\Gamma(\mathbb{Z}_n)$  on the set  $\epsilon_{d_i}$ . Then,

$$\Gamma(\mathbb{Z}_n) \cong G' \vee (G'' \cup \{0\}) \cong K_{\phi(n)} \vee (G'' \cup K_1).$$

Suppose the prime decomposition of  $n$  is given by  $n = \prod_{i=1}^k p_i^{\alpha_i}$ , where  $p_1 < p_2 < \dots < p_k$  are distinct primes and  $\alpha_i \in \mathbb{Z}^+$ . A positive divisor  $d$  of  $n$  is termed a proper divisor if  $d \notin \{1, n\}$ . The total number of such divisors is denoted by  $w = \prod_{i=1}^k (\alpha_i + 1) - 2$ . Let  $\mathcal{D} = \{d_1, d_2, \dots, d_w\}$  where  $d_1 < d_2 < \dots < d_w$ . For each  $d_i \in \mathcal{D}$ ,  $\epsilon_{d_i} = \{x; \gcd(x, n) = d_i\}$ . Here,  $|\epsilon_{d_i}| = \phi(\frac{n}{d_i})$  and the vertex set of the graph as the disjoint union of  $\cup_{i=1}^w \epsilon_{d_i}$ .

The section explores the eccentricity characteristic polynomial of the comaximal graph  $\Gamma(\mathbb{Z}_n)$  when  $n = p^\alpha q^\beta$ ,  $p, q$  bridging the structural properties of commutative rings with spectral features of their distance-based matrices.

**Theorem 7.** *The eccentricity characteristic polynomial of  $\Gamma(\mathbb{Z}_{p^\alpha q^\beta})$  where  $p$  and  $q$  are distinct prime numbers and  $\alpha, \beta \geq 1$ , is*

$$(\lambda + 1)^{p^{\alpha-1}q^{\beta-1}(p-1)(q-1)-1}(\lambda + 2)^{p^{\alpha-1}(q^\beta - q^{\beta-1}) + q^{\beta-1}(p^\alpha - p^{\alpha-1}) + p^{\alpha-1}q^{\beta-1}-3}(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4),$$

where  $\lambda_i$  ( $1 \leq i \leq 4$ ) are the eigenvalues of the matrix  $Q$ .

*Proof.* Note that  $V(\Gamma(\mathbb{Z}_n)) = \epsilon_u \cup \epsilon_{d_i} \cup \{0\}$  where  $|\epsilon_u| = \phi(p^\alpha q^\beta) = p^{\alpha-1}q^{\beta-1}(p-1)(q-1)$ . Again partitioning the vertices of  $\epsilon_{d_i}$ ,  $\epsilon_{d_i} = \epsilon_{p^j} \cup \epsilon_{q^k} \cup \epsilon_{p^j q^k}$  where  $1 \leq j \leq \alpha$ ,  $1 \leq k \leq \beta$ . The eccentricity of each vertex in  $\epsilon_u$  is 1 and the eccentricity of remaining vertices is 2.

$$|\epsilon_p + \epsilon_{p^2} + \dots + \epsilon_{p^\alpha}| = \phi(p^{\alpha-1}q^\beta) + \phi(p^{\alpha-2}q^\beta) + \dots + \phi(q^\beta) = p^{\alpha-1}(q^\beta - q^{\beta-1}),$$

$$|\epsilon_q + \epsilon_{q^2} + \dots + \epsilon_{q^\beta}| = \phi(p^\alpha q^{\beta-1}) + \phi(p^\alpha q^{\beta-2}) + \dots + \phi(p^\alpha) = q^{\beta-1}(p^\alpha - p^{\alpha-1}),$$

$$\begin{aligned} \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} |\epsilon_{p^j q^k} \cup \{0\}| &= \sum_{k=1}^{\beta} |\epsilon_{p q^k}| + \sum_{k=1}^{\beta} |\epsilon_{p^2 q^k}| + \dots + \sum_{k=1}^{\beta} |\epsilon_{p^{\alpha-1} q^k}| + \sum_{k=1}^{\beta} |\epsilon_{p^\alpha q^k}| + 1 \\ &= q^{\beta-1} \phi(p^{\alpha-1}) + q^{\beta-1} \phi(p^{\alpha-2}) + \dots + q^{\beta-1} \phi(p) + q^{\beta-1} + 1 \\ &= q^{\beta-1} + q^{\beta-1} \phi(p) + \dots + q^{\beta-1} \phi(p^{\alpha-1}) + 1 \\ &= p^{\alpha-1} q^{\beta-1}. \end{aligned}$$

The eccentricity matrix of the comaximal graph  $\Gamma(\mathbb{Z}_{p^\alpha q^\beta})$  is partitioned into a  $4 \times 4$  block matrix. The blocks are of the order  $\epsilon_u, \epsilon_{p^j}, \epsilon_{q^k}, \epsilon_{p^j q^k} \cup \{0\}$  where  $1 \leq j \leq \alpha$ ,  $1 \leq k \leq \beta$  and the corresponding number of vertices is  $p^{\alpha-1}q^{\beta-1}(p-1)(q-1)$ ,  $p^{\alpha-1}(q^\beta - q^{\beta-1})$ ,  $q^{\beta-1}(p^\alpha - p^{\alpha-1})$  and  $p^{\alpha-1}q^{\beta-1}$ .

The eccentricity matrix of the comaximal graph  $\Gamma(\mathbb{Z}_{p^\alpha q^\beta})$  is

$$\begin{pmatrix} (J-I) & J & J & \dots & J \\ J & 2(J-I) & 2J & \dots & 2J \\ J & 2J & 2(J-I) & \dots & 2J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & 2J & 2J & \dots & 2(J-I) \\ J & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & 0 & 0 & \dots & 0 \\ J & 2J & 2J & \dots & 2J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & 2J & 2J & \dots & 2J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & 2J & 2J & \dots & 2J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & 2J & 2J & \dots & 2J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & 2J & 2J & \dots & 2J \end{pmatrix}.$$

The quotient matrix of the eccentricity matrix of comaximal graph  $\Gamma(\mathbb{Z}_{p^\alpha q^\beta})$  is

$$\begin{pmatrix} p^{\alpha-1}q^{\beta-1}(p-1)(q-1)-1 & p^{\alpha-1}(q^\beta-q^{\beta-1}) & q^{\beta-1}(p^\alpha-p^{\alpha-1}) & p^{\alpha-1}q^{\beta-1} \\ p^{\alpha-1}q^{\beta-1}(p-1)(q-1) & 2p^{\alpha-1}(q^\beta-q^{\beta-1})-2 & 0 & 2p^{\alpha-1}q^{\beta-1} \\ p^{\alpha-1}q^{\beta-1}(p-1)(q-1) & 0 & 2q^{\beta-1}(p^\alpha-p^{\alpha-1})-2 & 2p^{\alpha-1}q^{\beta-1} \\ p^{\alpha-1}q^{\beta-1}(p-1)(q-1) & 2p^{\alpha-1}(q^\beta-q^{\beta-1}) & 2q^{\beta-1}(p^\alpha-p^{\alpha-1}) & 2p^{\alpha-1}q^{\beta-1}-2 \end{pmatrix}.$$

Let  $\lambda_i$ , where  $1 \leq i \leq 4$  be the eigenvalues of the above matrix, since  $n_0 - 1 = p^{\alpha-1}q^{\beta-1}(p-1)(q-1) - 1$  and  $n_U - 1 = p^{\alpha-1}(q^\beta - q^{\beta-1}) + q^{\beta-1}(p^\alpha - p^{\alpha-1}) + p^{\alpha-1}q^{\beta-1} - 3$ . Using Theorems 1 and 5 the result follows.  $\square$

**Remark 1.** Putting  $p = 2, q = 3, m = 1, n = 1$ , the matrix becomes

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 2 & 2 & 2 & 0 \end{pmatrix}.$$

The eigenvalues of the above matrix are  $-1, -2, -3.05, -0.59, 0.75, 5.9$ .

**Theorem 8.** The eccentricity characteristic polynomial of  $\Gamma(\mathbb{Z}_{pqr})$ , where  $p < q < r$  are distinct prime numbers is

$$\lambda(\lambda + 1)^{(p-1)(q-1)(r-1)-1}(\lambda + 2)^{(q-1)(r-1)+(p-1)(q-1)+(p-1)(r-1)+p+q+r-9}(\lambda - \lambda_i),$$

where  $\lambda_i, 1 \leq i \leq 8$  are the eigenvalues of the quotient matrix  $Q$ .

*Proof.* Note that  $V(\Gamma(\mathbb{Z}_{pqr})) = \epsilon_u \cup \epsilon_{d_i} \cup \{0\}$ . Further, the vertices are partitioned into  $\epsilon_{d_i} = \epsilon_p \cup \epsilon_q \cup \epsilon_r \cup \epsilon_{pq} \cup \epsilon_{qr} \cup \epsilon_{pr}$ . The eccentricity of each vertices of  $\epsilon_u$  is two and the eccentricity of remaining vertices is one. The eccentricity matrix of  $\Gamma(\mathbb{Z}_{pqr})$  is

$$\begin{pmatrix} (J-I) & J & J & J & J & J & J & J \\ J & 0 & 2J & 2J & 2J & 2J & 2J & 2J \\ J & 2J & 2(J-I) & 0 & 0 & 0 & 2J & 2J \\ J & 2J & 0 & 2(J-I) & 0 & 2J & 0 & 2J \\ J & 2J & 0 & 0 & 2(J-I) & 2J & 2J & 0 \\ J & 2J & 0 & 2J & 2J & 2(J-I) & 0 & 0 \\ J & 2J & 2J & 0 & 2J & 0 & 2(J-I) & 0 \\ J & 2J & 2J & 2J & 0 & 0 & 0 & 2(J-I) \end{pmatrix}.$$

The eccentricity matrix of the comaximal graph  $\Gamma(\mathbb{Z}_{pqr})$  is partitioned into an  $8 \times 8$  block matrix. The blocks are of the order  $\epsilon_u, \{0\}, \epsilon_p, \epsilon_q, \epsilon_r, \epsilon_{qr}, \epsilon_{pr}, \epsilon_{pq}$ . The corresponding number of vertices is  $|\epsilon_u| = (p-1)(q-1)(r-1), |\{0\}| = 1, |\epsilon_p| = (q-1)(r-1), |\epsilon_q| = (p-1)(r-1), |\epsilon_r| = (p-1)(q-1), |\epsilon_{qr}| = (p-1), |\epsilon_{pr}| = (q-1), |\epsilon_{pq}| = (r-1)$ . Also,  $n_0 - 1 = (p-1)(q-1)(r-1) - 1$  and  $n_U - 1 = (q-1)(r-1) + (p-1)(q-1) + (p-1)(r-1) + p + q + r - 9$ . Let  $\lambda_i$ , where  $1 \leq i \leq 8$  be the eigenvalues of the quotient matrix. Using Theorems 1 and 5 the result follows. The quotient matrix of the eccentricity



**Theorem 10.** Let  $[\epsilon(\Gamma(\mathbb{Z}_n))_{ij}]$  be the eccentricity matrix of the comaximal graph  $\Gamma(\mathbb{Z}_n)$  where  $n$  is not a prime. Then,

$$\epsilon(\Gamma(\mathbb{Z}_n))_{ij} = \begin{cases} 0, & \text{if } v_i \in \epsilon_{d_i} \text{ and } v_j \in \epsilon_{d_j}, (d_i, d_j) = 1, \\ 1, & \text{if } v_i \in \epsilon_u \text{ and } v_j \in \epsilon_u, \\ 1, & \text{if } v_i \in \epsilon_u \text{ and } v_j \in \cup \epsilon_d, i \neq j, \\ 2, & \text{if } v_i \in \epsilon_{d_i} \text{ and } v_j \in \epsilon_{d_j}, (d_i, d_j) \neq 1, \\ 2, & \text{if } v_i \in \epsilon_{d_i} \text{ and } v_j \in \epsilon_{d_j}, i \neq j, \\ 2, & \text{if } v_i = \{0\} \text{ and } v_j \in \cup \epsilon_d. \end{cases}$$

If  $n$  is a prime, then  $\epsilon(\Gamma(\mathbb{Z}_n))_{ij} = 1$  for any  $v_i, v_j \in \Gamma(\mathbb{Z}_n)$ .

*Proof. Case 1.* If  $v_i \in \epsilon_{d_i}$  and  $v_j \in \epsilon_{d_j}$  such that  $(d_i, d_j) = 1$ , then  $d(v_i, v_j) = 1$ , but  $e(v_i) = 2$  and  $e(v_j) = 2$ . Hence  $d(v_i, v_j) \neq \min\{e(v_i), e(v_j)\}$ . Hence  $\epsilon(G)_{ij} = 0$ .

**Case 2.** If  $v_i, v_j \in \epsilon_u$ , then  $e(v_i) = e(v_j) = d(v_i, v_j) = 1$ . Hence,  $\epsilon(G)_{ij} = 1$ .

**Case 3.** If  $v_i \in \epsilon_u$  and  $v_j \in \cup \epsilon_d$ , then  $d(v_i, v_j) = 1$ , but  $e(v_i) = 1$  and  $e(v_j) = 2$ . Hence,  $d(v_i, v_j) = \min\{e(v_i), e(v_j)\} = 1$  and  $\epsilon(G)_{ij} = 1$ .

**Case 4.** If  $v_i \in \epsilon_{d_i}$  and  $v_j \in \epsilon_{d_j}$  such that  $(d_i, d_j) \neq 1$ , then  $d(v_i, v_j) = 2$ , but  $e(v_i) = e(v_j) = 2$ . Hence  $d(v_i, v_j) = \min\{e(v_i), e(v_j)\} = 2$  and  $\epsilon(G)_{ij} = 2$ .

**Case 5.** If  $v_i, v_j \in \epsilon_{d_i}$ , then  $e(v_i) = e(v_j) = d(v_i, v_j) = 2$ . Hence,  $\epsilon(G)_{ij} = 2$ .

**Case 6.** If  $v_i = \{0\}$  and  $v_j \in \cup \epsilon_d$ , since  $e(v_i) = e(v_j) = 2$ , in  $\epsilon(G)_{ij} = 2$ .

**Case 7.** If  $n$  is a prime, then the graph is complete and implies  $\epsilon(\Gamma(\mathbb{Z}_n))_{ij} = 1$  for any  $v_i, v_j \in \Gamma(\mathbb{Z}_n)$ .  $\square$

**Corollary 1.** Let  $n = p^\alpha q^\beta$ , where  $p$  and  $q$  are distinct prime numbers and  $\alpha, \beta \geq 1$ . The eccentricity Wiener index of  $\Gamma(\mathbb{Z}_{p^\alpha q^\beta})$  is given by

$$W_\epsilon(\Gamma(\mathbb{Z}_{p^\alpha q^\beta})) = \binom{n_0}{2} + n_0(n - n_0) + 2\binom{n - n_0}{2} - 2n_{\{1\}}n_{\{2\}},$$

where  $n_0 = \phi(p^\alpha q^\beta) = p^{\alpha-1}q^{\beta-1}(p-1)(q-1)$ ,  $n_{\{1\}} = p^{\alpha-1}(q^\beta - q^{\beta-1})$ ,  $n_{\{2\}} = q^{\beta-1}(p^\alpha - p^{\alpha-1})$ ,  $n_{\{1,2\}} = p^{\alpha-1}q^{\beta-1}$ .

*Proof.* Applying the equiv-comaximal ring partition over  $R = \mathbb{Z}_{p^\alpha q^\beta} \cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_{q^\beta}$ , we have  $k = 2$  and  $[k] = \{1, 2\}$ . The total vertex set  $V(\Gamma(\mathbb{Z}_{p^\alpha q^\beta}))$  is partitioned into four classes indexed by  $U \subseteq \{1, 2\}$ ,  $S_\emptyset = \epsilon_u$  (unit class) of order  $n_0 = \phi(p^\alpha q^\beta)$ .  $S_{\{1\}} = \bigcup_{j=1}^\alpha \epsilon_{p^j}$  of order  $n_{\{1\}} = p^{\alpha-1}(q^\beta - q^{\beta-1})$ ,  $S_{\{2\}} = \bigcup_{k=1}^\beta \epsilon_{q^k}$  of order  $n_{\{2\}} = q^{\beta-1}(p^\alpha - p^{\alpha-1})$ .  $S_{\{1,2\}} = (\bigcup_{j=1}^\alpha \bigcup_{k=1}^\beta \epsilon_{p^j q^k}) \cup \{0\}$  of order  $n_{\{1,2\}} = p^{\alpha-1}q^{\beta-1}$ .

Substituting these blocks into the four summation terms of Theorem 4 yields, for unit block pairs,  $\binom{n_0}{2}$ . For unit and non-unit cross pairs,  $n_0 \sum_{U \subset \{1,2\}} n_U = n_0(n_{\{1\}} + n_{\{2\}} + n_{\{1,2\}}) = n_0(n - n_0)$ . For intra-block non-unit pairs,  $2 \sum_{U \subset \{1,2\}} \binom{n_U}{2} = 2 \left( \binom{n_{\{1\}}}{2} + \binom{n_{\{2\}}}{2} + \binom{n_{\{1,2\}}}{2} \right)$ . For inter-block non-Comaximal non-unit pairs, For distinct proper subsets  $U, W \subset \{1, 2\}$ , the condition  $U \cup W \neq \{1, 2\}$  excludes the pair  $\{S_{\{1\}}, S_{\{2\}}\}$  since  $\{1\} \cup \{2\} = \{1, 2\}$  (comaximal blocks). The remaining non-comaximal pairs are  $(S_{\{1\}}, S_{\{1,2\}})$  and  $(S_{\{2\}}, S_{\{1,2\}})$ ,

$$\sum_{\substack{U \neq W, U, W \subset \{1,2\} \\ U \cup W \neq \{1,2\}}} 2n_U n_W = 2n_{\{1\}}n_{\{1,2\}} + 2n_{\{2\}}n_{\{1,2\}}.$$

Gives the simplified non-unit pair total,

$$2\left(\binom{n_{\{1\}}}{2} + \binom{n_{\{2\}}}{2} + \binom{n_{\{1,2\}}}{2} + n_{\{1\}}n_{\{1,2\}} + n_{\{2\}}n_{\{1,2\}}\right) = 2\binom{n_{\{1\}} + n_{\{2\}} + n_{\{1,2\}}}{2} - 2n_{\{1\}}n_{\{2\}} = 2\binom{n - n_0}{2} - 2n_{\{1\}}n_{\{2\}}.$$

Summing all four terms evaluated above yields the result.  $\square$

**Corollary 2.** Let  $n = pqr$ , where  $p < q < r$  are distinct prime numbers. The eccentricity Wiener index of  $\Gamma(\mathbb{Z}_{pqr})$  is given by

$$W_e(\Gamma(\mathbb{Z}_{pqr})) = \binom{n_0}{2} + n_0(n - n_0) + 2 \sum_{U \subset [3]} \binom{n_U}{2} + \sum_{\substack{\emptyset \neq U, W \subset [3], U \neq W \\ U \cup W \neq [3]}} 2n_U n_W,$$

where  $[3] = \{1, 2, 3\}$ ,  $n_0 = (p-1)(q-1)(r-1)$  is the order of the unit set  $\epsilon_u$ , and  $n_U = |S_U|$  represents the order of the seven proper non-unit equivalence classes,  $n_{\{1\}} = (q-1)(r-1)$ ,  $n_{\{2\}} = (p-1)(r-1)$ ,  $n_{\{3\}} = (p-1)(q-1)$ ,  $n_{\{1,2\}} = r-1$ ,  $n_{\{1,3\}} = q-1$ ,  $n_{\{2,3\}} = p-1$ ,  $n_{\{1,2,3\}} = 1$ .

*Proof.* Applying Theorem 4 to  $R = \mathbb{Z}_{pqr} \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ , the index set is  $[k] = [3] = \{1, 2, 3\}$ . The total vertex set  $V(\Gamma(\mathbb{Z}_{pqr}))$  is partitioned into  $2^3 = 8$  equiv-comaximal classes  $S_U$  ( $U \subseteq [3]$ ),  $S_\emptyset = \epsilon_u$  of cardinality  $n_0 = (p-1)(q-1)(r-1)$ . Seven proper non-unit classes  $S_U$  for  $U \subset [3]$ . Evaluating each of the four structural terms defined in Theorem 4, for  $u, v \in S_\emptyset$ ,  $d(u, v) = 1 = \min\{e(u), e(v)\}$ , giving  $\epsilon_{uv} = 1$ , contributes  $\binom{n_0}{2}$ . For  $u \in S_\emptyset$  and  $v \in S_U$ ,  $d(u, v) = 1 = \min\{1, 2\}$ , giving  $\epsilon_{uv} = 1$  contributes  $\sum_{\emptyset \neq U \subset [3]} n_0 n_U = n_0 \sum_{U \subset [3]} n_U = n_0(n - n_0)$ . For  $u, v \in S_U$ ,  $d(u, v) = 2 = \min\{2, 2\}$ , giving  $\epsilon_{uv} = 2$ , contributes  $2 \sum_{U \subset [3]} \binom{n_U}{2}$ . For  $u \in S_U$  and  $v \in S_W$  ( $U, W \subset [3]$  and  $U \neq W$ ), If  $U \cup W = [3]$ , the classes are comaximal, giving  $d(u, v) = 1 \neq \min\{2, 2\}$ . Thus,  $\epsilon_{uv} = 0$ . If  $U \cup W \neq [3]$ , the classes are non-comaximal, giving  $d(u, v) = 2 = \min\{2, 2\} \implies \epsilon_{uv} = 2$ , contributing  $\sum_{\substack{U, W \subset [3], U \neq W \\ U \cup W \neq [3]}} 2n_U n_W$ .

Summing all four terms directly completes the proof.  $\square$

## 5. Topological indices of the comaximal graph of $\mathbb{Z}_n$

In this section we study the indices which incorporate both degree and distance information of vertices.

**Theorem 11.** The first Zagreb eccentricity index of  $\Gamma(\mathbb{Z}_n)$ , where  $n$  is not a prime is given by

$$M_1(\Gamma(\mathbb{Z}_n)) = \phi(n) + 4(n - \phi(n)).$$

If  $n$  is prime, then  $M_1(\Gamma(\mathbb{Z}_n)) = n$ .

*Proof.* There are  $\phi(n)$  vertices in  $G'$  with eccentricity 1 and  $n - \phi(n)$  vertices in  $G'' \cup \{0\}$  with eccentricity 2. Then, by Theorem 9, the first Zagreb eccentricity index of  $\Gamma(\mathbb{Z}_n) = \phi(n) + 4(n - \phi(n))$  where  $n$  is not prime. If  $n$  is prime, then the graph is complete and implies  $M_1(\Gamma(\mathbb{Z}_n)) = n$ .  $\square$

**Lemma 1.** Let  $n$  be a composite integer and let  $G'' \cup \{0\} = V(\Gamma(\mathbb{Z}_n)) \setminus \epsilon_u$  be the set of non-unit vertices in the comaximal graph  $\Gamma(\mathbb{Z}_n)$ . The number of edges  $E_W$  in the induced subgraph  $\Gamma(\mathbb{Z}_n)[G'' \cup \{0\}]$  is given by

$$E_W = \sum_{\substack{d_i < d_j \\ d_i, d_j | n, d_i, d_j \notin \{1, n\} \\ \gcd(d_i, d_j) = 1}} \phi\left(\frac{n}{d_i}\right) \phi\left(\frac{n}{d_j}\right).$$

*Proof.* By definition,  $E_W$  is the total number of edges in the induced subgraph  $\Gamma(\mathbb{Z}_n)[G'' \cup \{0\}]$ . The vertex set  $G'' \cup \{0\}$  can be partitioned into the zero element  $\{0\}$  and the collection of pairwise disjoint proper divisor classes,  $G'' \cup \{0\} = \{0\} \cup \left( \bigcup_{1 < d < n} \epsilon_d \right)$  where  $\epsilon_d = \{x \in \mathbb{Z}_n : \gcd(x, n) = d\}$ . The cardinality of each block  $\epsilon_d$  is  $|\epsilon_d| = \phi\left(\frac{n}{d}\right)$ .

We evaluate edge adjacencies across three possible vertex configurations in  $G'' \cup \{0\}$ : For any non-unit vertex  $y \in \epsilon_d$ , we have  $\gcd(0, y, n) = \gcd(y, n) = d > 1$ . Since  $\gcd(0, y, n) \neq 1$ ,  $0 \not\sim y$  for all  $y \in G''$ . Thus,  $\{0\}$  is an isolated vertex in the induced subgraph and contributes 0 edges.

For any two distinct vertices  $x, y \in \epsilon_{d_i}$ , both  $x$  and  $y$  are multiples of  $d_i > 1$ . Consequently,  $\gcd(x, y, n) \geq d_i > 1$ , which implies  $x \not\sim y$ . Hence, each block  $\epsilon_{d_i}$  forms an independent set, contributing 0 edges internally.

Let  $x \in \epsilon_{d_i}$  and  $y \in \epsilon_{d_j}$ , ( $d_i \neq d_j$ ). If  $\gcd(d_i, d_j) = k > 1$ , then  $k$  divides both  $x$  and  $y$ , yielding  $\gcd(x, y, n) \geq k > 1$ . Thus,  $x \not\sim y$ , yielding no edges between  $\epsilon_{d_i}$  and  $\epsilon_{d_j}$ . If  $\gcd(d_i, d_j) = 1$ , every element  $x \in \epsilon_{d_i}$  and  $y \in \epsilon_{d_j}$  satisfies  $\gcd(x, y, n) = 1$ . Thus, every vertex in  $\epsilon_{d_i}$  is adjacent to every vertex in  $\epsilon_{d_j}$ . The complete bipartite graph formed between  $\epsilon_{d_i}$  and  $\epsilon_{d_j}$  yields exactly  $|\epsilon_{d_i}| \cdot |\epsilon_{d_j}| = \phi\left(\frac{n}{d_i}\right) \phi\left(\frac{n}{d_j}\right)$  edges.

Summing the edge contributions over all unordered pairs of disjoint proper divisor blocks with coprime generators  $\gcd(d_i, d_j) = 1$ , we obtain,

$$E_W = \sum_{\substack{d_i < d_j \\ d_i, d_j | n, d_i, d_j \notin \{1, n\} \\ \gcd(d_i, d_j) = 1}} |\epsilon_{d_i}| |\epsilon_{d_j}| = \sum_{\substack{d_i < d_j \\ d_i, d_j | n, d_i, d_j \notin \{1, n\} \\ \gcd(d_i, d_j) = 1}} \phi\left(\frac{n}{d_i}\right) \phi\left(\frac{n}{d_j}\right).$$

□

**Corollary 3.** The second Zagreb eccentricity index of  $\Gamma(\mathbb{Z}_n)$ , where  $n$  is not a prime is given by

$$M_2(\Gamma(\mathbb{Z}_n)) = 2\phi(n)(n - \phi(n)) + \binom{\phi(n)}{2} + 4E_W,$$

where

$$E_W = \sum_{\substack{d_i < d_j \\ d_i, d_j | n, d_i, d_j \notin \{1, n\} \\ \gcd(d_i, d_j) = 1}} \phi\left(\frac{n}{d_i}\right) \phi\left(\frac{n}{d_j}\right).$$

If  $n$  is prime then  $M_2(\Gamma(\mathbb{Z}_n)) = \binom{n}{2}$ .

*Proof.* There are  $\phi(n)$  vertices in  $G'$  with eccentricity 1 adjacent with  $n - \phi(n)$  vertices in  $G'' \cup \{0\}$  with eccentricity 2 and the number of edges in  $G'$  is  $\binom{\phi(n)}{2}$ . Then the second Zagreb eccentricity index of  $\Gamma(\mathbb{Z}_n) = 2\phi(n)(n - \phi(n)) + \binom{\phi(n)}{2} + 4E_W$  where  $n$  is not prime and  $E_W$  represents the number of edges adjacent within  $\cup \epsilon_{d_i}$ . If  $n$  is prime then the graph is complete implies  $M_2(\Gamma(\mathbb{Z}_n)) = \binom{n}{2}$ . □

**Corollary 4.** The eccentric connectivity index of  $\Gamma(\mathbb{Z}_n)$  where  $n$  is not a prime is given by

$$\zeta_e(\Gamma(\mathbb{Z}_n)) = \phi(n)(3n - 2\phi(n) - 1) + 4E_W,$$

where

$$E_W = \sum_{\substack{d_i < d_j \\ d_i, d_j | n, d_i, d_j \notin \{1, n\} \\ \gcd(d_i, d_j) = 1}} \phi\left(\frac{n}{d_i}\right) \phi\left(\frac{n}{d_j}\right).$$

If  $n$  is prime, then  $\zeta_e(\Gamma(\mathbb{Z}_n)) = n(n-1)$ .

**Corollary 5.** The eccentric connectivity coindex of  $\Gamma(\mathbb{Z}_n)$  where  $n$  is not a prime is given by

$$\tilde{\zeta}_e^c(\Gamma(\mathbb{Z}_n)) = 2(n - \phi(n))(n - \phi(n) - 1) - 4E_W,$$

where

$$E_W = \sum_{\substack{d_i < d_j \\ d_i, d_j | n, d_i, d_j \notin \{1, n\} \\ \gcd(d_i, d_j) = 1}} \phi\left(\frac{n}{d_i}\right) \phi\left(\frac{n}{d_j}\right).$$

## Author contributions

A. Hummdi, L. Abbiramy, K. Selvakumar and J. Nisar: Conceptualization, Validation, Formal analysis, Writing manuscript, Editing and review. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University (KKU), Abha, Saudi Arabia through a large group research project under Grant Number RGP, 2/308/47.

## Conflict of interest

The authors have no conflicts of interest

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