



Research article**Gauge transforms of metrics in fixed point theory: Contractive and nonexpansive mappings****Saud M. Alsulami* and Naseer Shahzad**

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* **Correspondence:** Email: alsulami@kau.edu.sa.**Abstract:** Let (Q, δ) be a metric space and let $\psi : [0, \infty) \rightarrow [0, \infty)$ be an admissible gauge. We study the transform

$$\delta_\psi(u, v) = \psi(\delta(u, v))$$

without assuming that δ_ψ is a metric. The class $\mathcal{G}$ preserves convergence to zero and Cauchy sequences, while the subclasses $\mathcal{G}(q)$ provide a local power lower estimate that is sufficient for summability of Picard increments.

We prove fixed point theorems for linear gauge contractions and nonlinear gauge comparison contractions in the original complete metric space. We also obtain a range-zero-separating transform theorem, which requires neither monotonicity nor differentiability of the scalar function and contains both the linear gauge theorem and the recent k -admissible transform theorem of Jleli and Samet. The global two-point inequalities give the fixed point property at the limit, so no continuity assumption on the mapping is needed. Every θ -hyperbolic sine distance is exactly a gauge transform with $\psi = \theta \circ \sinh$. For the regular subclass in which this function belongs to $\mathcal{G}(q)$, the corresponding regular-subclass versions of the hyperbolic-sine fixed point results follow directly; under these additional regularity assumptions, the sequential condition in the original formulation need not be imposed separately.

We also study the chain metric generated by a gauge transform. A local linear lower estimate guarantees that the chain distance is a metric; a root normalization gives a canonical equivalent-metric interpretation of every linear $\mathcal{G}(q)$ -gauge contraction. An integral operator is given, which is not a Banach contraction in the original metric. Finally, we construct a nonlinear Hutchinson operator, which has no uniform linear constant in the original Hausdorff metric or in the associated nonmetric gauge distance, and is not Banach contractive in the canonical chain metric generated by that gauge.

For admissible gauges that are continuous on $[0, \infty)$, we also treat nonexpansive mappings using proofs that do not use the contraction results or the chain-metric construction. Following the normal-structure method of Takahashi, we work directly with gauge balls of the possibly nonmetric function $D_\psi = \psi \circ \delta$. Gauge versions of convexity, ψ -property (C), normal structure, and a compact radius-reduction property are formulated for D_ψ . Fixed point results are obtained for ψ -nonexpansive

mappings, families with the invariant property, left amenable semigroups, commuting families, and one-parameter semigroups. The transformed distance is not assumed to satisfy the triangle inequality.

Keywords: metric transform; admissible gauge; transformed distance; altering distances; comparison contraction; chain metric; hyperbolic-sine distance; integral equation; iterated function system; nonexpansive mapping; normal structure; amenable semigroup; fixed point

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1. Introduction

The Banach contraction principle [1] remains the basic model for many fixed point theorems in nonlinear analysis. A large part of metric fixed point theory studies how far one can move away from the ordinary metric while retaining the Picard iteration argument. This has led to metric transforms, metric-preserving functions, altering distances, w -distances, equivalent metrics, admissible contractions, and related distance functions. For background and related results, see Borsik–Doboš [2], Doboš [3], Doboš–Piotrowski [4], Pokorný [5], Khan–Swaleh–Sessa [6], Kada–Suzuki–Takahashi [7], Kirk–Shahzad [8], Pongsriiam–Termwuttipong [9], Petruşel–Rus–Şerban [10], and Samet–Vetro–Vetro [11].

The present paper concerns transforms of the form

$$\delta_\psi(u, v) = \psi(\delta(u, v)), \quad (1.1)$$

where (Q, δ) is a metric space and $\psi : [0, \infty) \rightarrow [0, \infty)$. We do not assume that δ_ψ satisfies the triangle inequality. If $\psi \circ \delta$ is a metric for every metric δ , then ψ belongs to the classical theory of metric-preserving functions. Our setting is different: The transform may be nonmetric, and completeness is imposed on the original metric space.

A principal motivation is the θ -hyperbolic sine distance of Jleli and Samet [12],

$$\delta_\theta(u, v) = \theta(\sinh(\delta(u, v))).$$

This distance is exactly (1.1) with

$$\psi_\theta(r) = \theta(\sinh r).$$

Thus every hyperbolic-sine distance is a gauge transform. Conversely, because $\sinh$ is a bijection of $[0, \infty)$ onto itself, every scalar gauge transform can be written in hyperbolic-sine form by taking $\theta = \psi \circ \operatorname{arcsinh}$. The formal expressions are therefore equivalent; the mathematical distinction lies in the assumptions imposed on the scalar function. We prove that when θ is nondecreasing, continuous at zero, vanishes at zero, and satisfies the power lower estimate used in the hyperbolic-sine setting, the function ψ_θ belongs to $\mathcal{G}(q)$. The corresponding linear and nonlinear hyperbolic-sine fixed point results then follow from our gauge theorems. In this regular subclass, the global two-point inequality also removes the separate sequential condition used to identify the limit as a fixed point.

Jleli and Samet later introduced (ω, t_0) -Taylor–Lagrange distance functions [13], and Aydi, Baraket, and Felhi developed nonlinear fixed point results for that setting [14]. A Taylor–Lagrange distance is

not, in general, a scalar transform of the original metric: Its value may depend on the pair through derivative terms, even when two pairs have the same original distance. It belongs to the present setting precisely in the special case where it is constant on every level set of some underlying metric, so that it can be written as a fixed metric-and-gauge composition.

The assumptions used here have two distinct roles. If ψ is nondecreasing, positive away from zero, and continuous at zero, then δ and δ_ψ have the same convergent sequences and the same Cauchy sequences. A local power lower estimate is used as a convenient sufficient condition for converting decay of transformed Picard increments into summability of the original metric increments. It is not asserted to be necessary for every possible fixed point argument. We then consider a weaker, mapping-dependent condition: The scalar function need only separate zero and satisfy the local power lower estimate on the image-distance set $\Delta_T = \{\delta(Tu, Tv) : u, v \in Q\}$. The resulting theorem requires neither monotonicity nor differentiability and contains both our linear gauge theorem and the recent k -admissible transform theorem of Jleli and Samet [15].

A second issue is the chain distance generated by δ_ψ . A local linear lower estimate guarantees that this chain distance is a metric on every underlying metric space. An arbitrary superlinear power lower estimate does not give such a space-independent guarantee: Subdivision may force the chain distance to vanish. We also use the exact root normalization

$$\phi = \psi^{1/q}.$$

If $\psi \in \mathcal{G}(q)$, then $\phi \in \mathcal{G}_{\text{lin}}$. Consequently every linear $\mathcal{G}(q)$ -gauge contraction is a Banach contraction in the canonical chain metric generated by $\phi \circ \delta$. This metric interpretation is stated explicitly in order to delimit the linear theory. The nonlinear comparison theorem does not, in general, yield a uniform linear contraction constant.

The global inequalities used below also explain why no continuity of the mapping is needed. Once the Picard sequence converges in the original metric, the same two-point inequality applied to the limit shows that the limit is a fixed point. This differs from orbit-only conditions, for which continuity, closedness, or a sequential assumption may be required.

The paper remains close to altering-distance theory [6], but continuity is imposed only at zero; monotonicity and positivity away from zero remain global assumptions. We also give an integral operator, which is not Banach contractive in the original metric. The hyperspace example uses a summable nonlinear comparison function: Its Hutchinson operator has no uniform linear contraction constant in the original Hausdorff metric, and after the exponential gauge transform, it still has no uniform linear gauge constant; moreover, the transformed Hausdorff distance is not a metric. Finally, we give a sufficient condition under which a global Samet–Vetro–Vetro-type admissibility factor can be absorbed into an ordinary comparison function. Throughout the paper, $\log$ denotes the natural logarithm.

The final part of the paper treats nonexpansive mappings separately from contractions. It is motivated by Takahashi's normal-structure method [16]. Related Takahashi-type convexity methods have also been used in intuitionistic fuzzy and b -fuzzy metric spaces [17, 18], and convexity structures have been studied in quasi-metric and fuzzy quasi-metric settings [19, 20]. Our setting is different: The object is the scalar transform $D_\psi = \psi \circ \delta$ of an ordinary metric, and D_ψ need not be a metric. We require gauge balls to be convex with respect to an abstract convex structure and formulate the radius, center, ψ -property (C), and normal-structure conditions directly in terms of D_ψ . This yields fixed

point results for ψ -nonexpansive mappings, compact invariant families, left amenable semigroups, commuting families, and one-parameter semigroups. The proofs in this part use neither the chain-metric construction nor a strict contraction argument.

2. Gauges and comparison functions

We first state the assumptions on an admissible gauge. The notation is chosen to be short and neutral.

Definition 2.1. Let $\mathcal{G}$ denote the class of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying:

- 1) $\psi(0) = 0$ and $\psi(t) > 0$ for every $t > 0$;
- 2) ψ is nondecreasing;
- 3) ψ is continuous at 0.

Such a function will be called an admissible gauge. The terminology is only local to this paper; metric-transform terminology is classical [2–5, 9, 21].

Definition 2.2. Let $q > 0$. We say that $\psi \in \mathcal{G}(q)$ if $\psi \in \mathcal{G}$ and there exist constants $c > 0$ and $r_0 > 0$ such that

$$\psi(r) \geq cr^q \quad (0 \leq r \leq r_0). \quad (2.1)$$

If $q = 1$, we also write $\psi \in \mathcal{G}_{\text{lin}}$. Thus, $\mathcal{G}_{\text{lin}} = \mathcal{G}(1)$, and membership in $\mathcal{G}_{\text{lin}}$ means exactly that ψ satisfies a local linear lower estimate near zero.

Remark 2.3. The class $\mathcal{G}$ is used for convergence and Cauchy equivalence; this distinction follows the standard caution in working with distance spaces and metric transforms [2, 8, 9]. The lower estimate (2.1) is used here as a sufficient condition when Picard increments have to be summed. The special notation $\mathcal{G}_{\text{lin}}$ is used when the exponent is $q = 1$; it guarantees nondegeneracy of the chain construction on every metric space. A local linear lower estimate is not required for the topological part of the paper.

Definition 2.4. Let $q > 0$. A nondecreasing function $\eta : [0, \infty) \rightarrow [0, \infty)$ is called a comparison function of order q if

- 1) $\eta(t) < t$ for every $t > 0$;
- 2) for every $t > 0$,

$$\sum_{n=0}^{\infty} (\eta^n(t))^{1/q} < \infty, \quad (2.2)$$

where η^0 is the identity map and $\eta^{n+1} = \eta \circ \eta^n$.

For $q = 1$, condition (2.2) is the summable (c)-comparison condition. It is stronger than the Matkowski requirement $\eta^n(t) \rightarrow 0$, and it is distinct from the upper semicontinuous comparison condition of Boyd and Wong [22, 23]. The fact that $\eta : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, together with condition (1), implies that $\eta(0) = 0$.

Example 2.5. If $\eta(t) = \kappa t$ with $0 < \kappa < 1$, then η is a comparison function of order q for every $q > 0$, since

$$\sum_{n=0}^{\infty} (\eta^n(t))^{1/q} = t^{1/q} \sum_{n=0}^{\infty} \kappa^{n/q} < \infty.$$

Remark 2.6 (Normalization of the exponent). Let $q > 0$ and let $\psi \in \mathcal{G}(q)$. Put

$$\phi(t) = \psi(t)^{1/q} \quad (t \geq 0).$$

Then $\phi \in \mathcal{G}_{\text{lin}}$. If η is a comparison function of order q , define

$$\widehat{\eta}(t) = [\eta(t^q)]^{1/q} \quad (t \geq 0).$$

The function $\widehat{\eta}$ is a usual summable (c)-comparison function and

$$\widehat{\eta}^n(t) = [\eta^n(t^q)]^{1/q} \quad (n \geq 0).$$

Consequently,

$$\psi(\delta(Tu, Tv)) \leq \eta(\psi(\delta(u, v)))$$

is equivalent to

$$\phi(\delta(Tu, Tv)) \leq \widehat{\eta}(\phi(\delta(u, v))).$$

Thus the exponent q indicates the lower power estimate used in the direct proof, but it is not an order attached to the comparison condition itself.

3. Gauge transforms of metrics

Let (Q, δ) be a metric space. This section records the elementary separation and topological properties of scalar gauge transforms. It also relates this setting to the hyperbolic-sine construction.

Definition 3.1. Following the standard metric-transform viewpoint [2–5, 9, 21], let $\psi : [0, \infty) \rightarrow [0, \infty)$. The gauge transform of δ generated by ψ is

$$\delta_\psi : Q \times Q \rightarrow [0, \infty), \quad \delta_\psi(u, v) = \psi(\delta(u, v)).$$

If $\psi \in \mathcal{G}$, then δ_ψ will be called an admissible gauge transform of δ .

Proposition 3.2. Let (Q, δ) be a metric space and let $\psi \in \mathcal{G}$. Then, for all $u, v \in Q$,

- 1) $\delta_\psi(u, v) = 0$ if and only if $u = v$;
- 2) $\delta_\psi(u, v) = \delta_\psi(v, u)$.

Proof. These assertions follow directly from the corresponding properties of δ and from the assumptions $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$. $\square$

Theorem 3.3. Let (Q, δ) be a metric space and let $\psi \in \mathcal{G}$. Then the following assertions hold.

- 1) A sequence (u_n) converges to u in (Q, δ) if and only if $\delta_\psi(u_n, u) \rightarrow 0$.
- 2) A sequence (u_n) is Cauchy in (Q, δ) if and only if

$$\delta_\psi(u_n, u_m) \rightarrow 0 \quad (m, n \rightarrow \infty).$$

- 3) (Q, δ) is complete if and only if every δ_ψ -Cauchy sequence converges in (Q, δ) , where (u_n) is called δ_ψ -Cauchy when

$$\delta_\psi(u_n, u_m) \rightarrow 0 \quad (m, n \rightarrow \infty).$$

Proof. The assertions follow from the elementary fact that $t_n \rightarrow 0$ if and only if $\psi(t_n) \rightarrow 0$. Indeed, the forward implication follows from the continuity of ψ at 0. For the converse, if $t_n \not\rightarrow 0$, then some subsequence satisfies $t_{n_k} \geq \varepsilon > 0$, and hence

$$\psi(t_{n_k}) \geq \psi(\varepsilon) > 0,$$

contrary to $\psi(t_n) \rightarrow 0$. For the Cauchy assertion, apply the same argument to arbitrary sequences of pairs (m_k, n_k) with $m_k, n_k \rightarrow \infty$. $\square$

Example 3.4. Let $p > 0$ and set

$$\psi_p(r) = r^p \quad (r \geq 0).$$

Then $\psi_p \in \mathcal{G}(p)$. If $p > 1$, the transform $\psi_p \circ \delta$ need not be a metric. For example, on $\mathbb{R}$ with its usual metric, $|x - y|^p$ fails the triangle inequality. By contrast, if $0 < p \leq 1$, then $r \mapsto r^p$ is a standard metric-preserving function; see [2, 21].

Example 3.5. Let $p > 1$ and define

$$\psi(r) = r + r^p \quad (r \geq 0).$$

Then $\psi \in \mathcal{G}_{\text{lin}}$, because $\psi(r) \geq r$ for every $r \geq 0$. The transformed distance

$$\delta_\psi(u, v) = \delta(u, v) + \delta(u, v)^p$$

usually fails the triangle inequality, but it preserves convergence and Cauchy sequences by Theorem 3.3.

Example 3.6. Define

$$\psi(r) = r(1 + \log(1 + r)) \quad (r \geq 0).$$

Then $\psi \in \mathcal{G}_{\text{lin}}$ because $\psi(r) \geq r$ for every $r \geq 0$. This gives a nonhyperbolic transform, which agrees with the original metric at first order near zero and gives a larger weight to moderate distances.

Example 3.7. Let $0 < \alpha \leq 1$ and define

$$\psi(r) = \min\{1, e^{r^\alpha} - 1\}, \quad r \geq 0.$$

Then

$$\psi \in \mathcal{G}(\alpha) \cap \mathcal{G}_{\text{lin}}.$$

Indeed, put

$$r_0 = (\log 2)^{1/\alpha}.$$

Since $0 < \log 2 < 1$ and $1/\alpha \geq 1$, we have $0 < r_0 < 1$. If $0 \leq r \leq r_0$, then $r^\alpha \leq \log 2$, and hence $e^{r^\alpha} - 1 \leq 1$. Therefore

$$\psi(r) = e^{r^\alpha} - 1 \geq r^\alpha.$$

Moreover, since $r_0 < 1$ and $0 < \alpha \leq 1$, we have $r^\alpha \geq r$ on $[0, r_0]$ and hence $\psi \in \mathcal{G}_{\text{lin}}$.

The transformed distance δ_ψ is bounded by 1, even when δ is unbounded.

Definition 3.8. Let $\theta : [0, \infty) \rightarrow [0, \infty)$ and define

$$\psi_\theta(r) = \theta(\sinh r) \quad (r \geq 0).$$

For a metric space (Q, δ) , the corresponding transform is

$$\delta_{\psi_\theta}(u, v) = \theta(\sinh(\delta(u, v))).$$

This is exactly the θ -hyperbolic sine distance of Jleli and Samet [12].

Proposition 3.9. Let (Q, δ) be a metric space, let $q > 0$, and let $\theta : [0, \infty) \rightarrow [0, \infty)$. Put

$$\psi_\theta(r) = \theta(\sinh r).$$

Then, without any additional assumption on θ ,

$$\delta_\theta(u, v) = \theta(\sinh(\delta(u, v))) = \delta_{\psi_\theta}(u, v).$$

Suppose, in addition, that

- 1) $\theta(0) = 0$;
- 2) θ is nondecreasing and continuous at 0;
- 3) there is a constant $c > 0$ such that

$$\theta(s) \geq cs^q \quad (s \geq 0).$$

Then $\psi_\theta \in \mathcal{G}(q)$ and, in fact,

$$\psi_\theta(r) \geq cr^q \quad (r \geq 0).$$

Proof. The identity $\delta_\theta = \delta_{\psi_\theta}$ follows directly from the definition. The first two assumptions, together with the continuity and monotonicity of $\sinh$, show that $\psi_\theta(0) = 0$, where ψ_θ is nondecreasing and continuous at 0. The lower estimate gives positivity away from zero. Moreover, since $\sinh r \geq r$ for $r \geq 0$,

$$\psi_\theta(r) = \theta(\sinh r) \geq c(\sinh r)^q \geq cr^q.$$

Thus $\psi_\theta \in \mathcal{G}(q)$. □

Remark 3.10. The converse algebraic representation also holds. If $\delta_\psi = \psi \circ \delta$, then

$$\delta_\psi(u, v) = \theta(\sinh(\delta(u, v))), \quad \theta = \psi \circ \operatorname{arcsinh}.$$

Hence gauge and hyperbolic-sine transforms have the same formal scalar shape. The content of Proposition 3.9 is the inclusion of the regular hyperbolic-sine subclass in the admissible class used by the fixed point theorems.

Example 3.11. Let $Q = \{0, 1, 2\}$ with $\delta(i, j) = |i - j|$ and take $\psi(r) = r + r^3$. Then

$$\delta_\psi(0, 1) = 2, \quad \delta_\psi(1, 2) = 2, \quad \delta_\psi(0, 2) = 10.$$

Thus

$$\delta_\psi(0, 2) > \delta_\psi(0, 1) + \delta_\psi(1, 2).$$

So, δ_ψ need not be a metric even for a very simple admissible gauge. This example is included only to make explicit the distinction from metric-preserving functions [2, 21].

4. Fixed point theorems

We now prove fixed point theorems for mappings contractive in the gauge-transformed distance δ_ψ . The proofs use completeness of the original metric δ .

Definition 4.1. Let (Q, δ) be a metric space and let $\psi \in \mathcal{G}$ be an admissible gauge. A mapping $T : Q \rightarrow Q$ is called a ψ -gauge contraction if there exists $\kappa \in (0, 1)$ such that

$$\delta_\psi(Tu, Tv) \leq \kappa \delta_\psi(u, v) \quad (u, v \in Q). \quad (4.1)$$

Theorem 4.2. Let (Q, δ) be a complete metric space. Let $q > 0$, let $\psi \in \mathcal{G}(q)$ be an admissible gauge, and let $T : Q \rightarrow Q$ be a ψ -gauge contraction. Then T has a unique fixed point $w^* \in Q$, and for every $w_0 \in Q$, the Picard sequence $w_{n+1} = Tw_n$ converges to w^* in (Q, δ) .

Proof. Fix $w_0 \in Q$ and put $w_n = T^n w_0$. If $w_0 = w_1$, the conclusion follows after the uniqueness argument below. Assume otherwise. From (4.1),

$$\delta_\psi(w_{n+1}, w_{n+2}) \leq \kappa \delta_\psi(w_n, w_{n+1}) \quad (n \geq 0).$$

By induction,

$$\delta_\psi(w_n, w_{n+1}) \leq \kappa^n \delta_\psi(w_0, w_1) \quad (n \geq 0). \quad (4.2)$$

Thus $\delta_\psi(w_n, w_{n+1}) \rightarrow 0$, and by Theorem 3.3,

$$\delta(w_n, w_{n+1}) \rightarrow 0.$$

Choose $c > 0$ and $r_0 > 0$ such that

$$\psi(r) \geq cr^q \quad (0 \leq r \leq r_0).$$

There exists N such that $\delta(w_n, w_{n+1}) \leq r_0$ for $n \geq N$. For such n , (4.2) gives

$$c\delta(w_n, w_{n+1})^q \leq \delta_\psi(w_n, w_{n+1}) \leq \kappa^n \delta_\psi(w_0, w_1).$$

Hence

$$\delta(w_n, w_{n+1}) \leq \left(\frac{\delta_\psi(w_0, w_1)}{c} \right)^{1/q} \kappa^{n/q} \quad (n \geq N). \quad (4.3)$$

Since $\sum_{n=0}^{\infty} \kappa^{n/q} < \infty$, it follows that (w_n) is Cauchy in (Q, δ) . Completeness gives a point $w^* \in Q$ such that

$$w_n \rightarrow w^* \quad \text{in } (Q, \delta).$$

We now prove that w^* is a fixed point. By Theorem 3.3, $\delta_\psi(w_n, w^*) \rightarrow 0$. Applying the contractive inequality to (w_n, w^*) , we obtain

$$\delta_\psi(w_{n+1}, Tw^*) = \delta_\psi(Tw_n, Tw^*) \leq \kappa \delta_\psi(w_n, w^*) \rightarrow 0.$$

Again by Theorem 3.3, $w_{n+1} \rightarrow Tw^*$ in (Q, δ) . But also $w_{n+1} \rightarrow w^*$, and limits in a metric space are unique. Hence $Tw^* = w^*$.

If z^* is another fixed point, then

$$\delta_\psi(w^*, z^*) = \delta_\psi(Tw^*, Tz^*) \leq \kappa \delta_\psi(w^*, z^*).$$

Since $0 < \kappa < 1$, this forces $\delta_\psi(w^*, z^*) = 0$, and therefore $w^* = z^*$. $\square$

Corollary 4.3. *Let (Q, δ) be complete, let $q > 0$, and let $\theta : [0, \infty) \rightarrow [0, \infty)$ satisfy the three assumptions in Proposition 3.9. Suppose that $T : Q \rightarrow Q$ satisfies*

$$\delta_\theta(Tu, Tv) \leq \kappa \delta_\theta(u, v) \quad (u, v \in Q),$$

where $0 < \kappa < 1$. Then T has a unique fixed point, and every Picard sequence converges to it in (Q, δ) .

Proof. By Proposition 3.9, $\psi_\theta = \theta \circ \sinh$ belongs to $\mathcal{G}(q)$ and $\delta_\theta = \delta_{\psi_\theta}$. The conclusion follows from Theorem 4.2. $\square$

Remark 4.4. Corollary 4.3 contains the regular-subclass version of the linear θ -hyperbolic contraction theorem of Jleli and Samet [12]. Their main theorem does not assume continuity of the mapping; instead, it imposes a sequential condition ensuring that the limit of a Picard orbit is a fixed point. Continuity is used only in their subsequent corollary as a sufficient condition for that sequential hypothesis.

Under the additional assumptions that $\theta(0) = 0$, where θ is nondecreasing and continuous at zero, the transformed function $\theta \circ \sinh$ belongs to $\mathcal{G}(q)$. In this regular subclass, the global contractive inequality itself gives the fixed point property at the limit, so the sequential condition need not be imposed separately.

Remark 4.5. No continuity assumption is needed in Theorem 4.2. The global two-point inequality (4.1) gives the fixed point property at the limit. This is a standard and useful distinction: Continuity or closedness assumptions become necessary mainly for weaker orbit-only conditions.

Definition 4.6. Let (Q, δ) be a metric space, let $\psi \in \mathcal{G}$ be an admissible gauge, and let $\eta : [0, \infty) \rightarrow [0, \infty)$ be nondecreasing. A mapping $T : Q \rightarrow Q$ is called a (ψ, η) -gauge comparison contraction if

$$\delta_\psi(Tu, Tv) \leq \eta(\delta_\psi(u, v)) \quad (u, v \in Q). \quad (4.4)$$

Theorem 4.7. *Let (Q, δ) be a complete metric space. Let $q > 0$, let $\psi \in \mathcal{G}(q)$ be an admissible gauge, and let η be a comparison function of order q . If $T : Q \rightarrow Q$ is a (ψ, η) -gauge comparison contraction, then T has a unique fixed point $w^* \in Q$, and for every $w_0 \in Q$, the Picard sequence $w_{n+1} = Tw_n$ converges to w^* in (Q, δ) .*

Proof. Let $w_n = T^n w_0$. From (4.4),

$$\delta_\psi(w_{n+1}, w_{n+2}) \leq \eta(\delta_\psi(w_n, w_{n+1})) \quad (n \geq 0).$$

By induction and monotonicity of η ,

$$\delta_\psi(w_n, w_{n+1}) \leq \eta^n(\delta_\psi(w_0, w_1)) \quad (n \geq 0). \quad (4.5)$$

The summability condition in Definition 2.4 implies

$$\eta^n(\delta_\psi(w_0, w_1)) \rightarrow 0.$$

Hence $\delta(w_n, w_{n+1}) \rightarrow 0$ by Theorem 3.3.

Choose $c > 0$ and $r_0 > 0$ such that $\psi(r) \geq cr^q$ for $0 \leq r \leq r_0$. For all sufficiently large n ,

$$\delta(w_n, w_{n+1}) \leq r_0,$$

so (4.5) gives

$$\delta(w_n, w_{n+1}) \leq c^{-1/q} (\eta^n(\delta_\psi(w_0, w_1)))^{1/q}.$$

The right side is summable by assumption. Thus (w_n) is Cauchy in (Q, δ) , and completeness gives $w_n \rightarrow w^*$ for some $w^* \in Q$.

As in the proof of Theorem 4.2, $\delta_\psi(w_n, w^*) \rightarrow 0$. Since (4.4) is global,

$$\delta_\psi(w_{n+1}, Tw^*) = \delta_\psi(Tw_n, Tw^*) \leq \eta(\delta_\psi(w_n, w^*)) \leq \delta_\psi(w_n, w^*) \rightarrow 0.$$

Thus $w_{n+1} \rightarrow Tw^*$ in (Q, δ) , while also $w_{n+1} \rightarrow w^*$. Hence $Tw^* = w^*$.

If z^* is another fixed point and $\delta_\psi(w^*, z^*) > 0$, then

$$\delta_\psi(w^*, z^*) = \delta_\psi(Tw^*, Tz^*) \leq \eta(\delta_\psi(w^*, z^*)) < \delta_\psi(w^*, z^*),$$

which is impossible. Hence $\delta_\psi(w^*, z^*) = 0$, and $w^* = z^*$. □

Corollary 4.8. *Let (Q, δ) be complete, let $\psi \in \mathcal{G}_{\text{lin}}$ be an admissible gauge, and let η be a usual (c)-comparison function, that is,*

$$\sum_{n=0}^{\infty} \eta^n(t) < \infty \quad (t > 0).$$

Then every (ψ, η) -gauge comparison contraction has a unique fixed point, and the Picard iteration converges to it from every initial point.

Proof. This is Theorem 4.7 with $q = 1$. □

Corollary 4.9. *Let (Q, δ) be complete. Suppose that $\theta : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, continuous at 0, satisfies $\theta(0) = 0$, and*

$$\theta(s) \geq cs \quad (s \geq 0)$$

for some $c > 0$. Let η be a summable (c)-comparison function. If

$$\delta_\theta(Tu, Tv) \leq \eta(\delta_\theta(u, v)) \quad (u, v \in Q),$$

then T has a unique fixed point, and every Picard sequence converges to it.

Proof. Proposition 3.9 gives $\theta \circ \sinh \in \mathcal{G}_{\text{lin}}$. The assertion follows from Corollary 4.8. □

Remark 4.10. Theorem 4.2 is also obtained from Theorem 4.7 by taking $\eta(t) = \kappa t$. Its separate proof is retained because it gives the explicit geometric estimate (4.3).

4.1. A fixed point theorem for range-zero-separating transforms

The preceding results assume that the scalar transform is nondecreasing. A recent theorem of Jleli and Samet [15] permits a nonmonotone transform, but imposes differentiability and the global separation condition

$$\inf_{t \geq \varepsilon} \varphi(t) > 0 \quad (\varepsilon > 0).$$

The next result uses only the properties needed along the distances produced by the mapping. It contains both the linear gauge theorem above and the main theorem of Jleli and Samet, while requiring neither monotonicity nor differentiability.

Let (Q, δ) be a metric space and let $T : Q \rightarrow Q$. Put

$$\Delta_T = \{\delta(Tu, Tv) : u, v \in Q\}.$$

Definition 4.11. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is said to be *zero-separating on Δ_T* if, for every sequence $(t_n) \subset \Delta_T$, the condition $\psi(t_n) \rightarrow 0$ implies $t_n \rightarrow 0$. Equivalently, for every $\varepsilon > 0$,

$$\inf\{\psi(t) : t \in \Delta_T, t \geq \varepsilon\} > 0,$$

where the infimum of the empty set is understood to be $+\infty$.

Theorem 4.12. Let (Q, δ) be a complete metric space and let $T : Q \rightarrow Q$. Suppose that $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfies the following conditions:

- 1) $\psi(0) = 0$ and ψ is continuous at 0;
- 2) ψ is zero-separating on Δ_T ;
- 3) there exist $q, c, r_0 > 0$ such that

$$\psi(t) \geq ct^q \quad (t \in \Delta_T \cap [0, r_0]); \quad (4.6)$$

- 4) there exists $\kappa \in (0, 1)$ such that

$$\psi(\delta(Tu, Tv)) \leq \kappa\psi(\delta(u, v)) \quad (u, v \in Q). \quad (4.7)$$

Then T has a unique fixed point $w^* \in Q$, and $T^n w_0 \rightarrow w^*$ in (Q, δ) for every $w_0 \in Q$.

Proof. We first prove uniqueness. Let w^* and z^* be fixed points. Then $\delta(w^*, z^*) = \delta(Tw^*, Tz^*) \in \Delta_T$, and

$$\psi(\delta(w^*, z^*)) \leq \kappa\psi(\delta(w^*, z^*)).$$

Thus $\psi(\delta(w^*, z^*)) = 0$. Applying condition (2) to the constant sequence with value $\delta(w^*, z^*)$ gives $\delta(w^*, z^*) = 0$. Hence a fixed point, if it exists, is unique.

Fix $w_0 \in Q$ and put $w_n = T^n w_0$. If $w_n = w_{n+1}$ for some n , then w_n is the unique fixed point, and the orbit is constant from that index onward. We may therefore suppose that $w_n \neq w_{n+1}$ for all n . Iterating (4.7) gives

$$\psi(\delta(w_n, w_{n+1})) \leq \kappa^n \psi(\delta(w_0, w_1)) \quad (n \geq 0). \quad (4.8)$$

For $n \geq 1$,

$$\delta(w_n, w_{n+1}) = \delta(Tw_{n-1}, Tw_n) \in \Delta_T.$$

Condition (2) and (4.8) therefore imply

$$\delta(w_n, w_{n+1}) \rightarrow 0.$$

Choose $N \geq 1$ such that $\delta(w_n, w_{n+1}) \leq r_0$ for $n \geq N$. By (4.6) and (4.8),

$$\delta(w_n, w_{n+1}) \leq \left(\frac{\psi(\delta(w_0, w_1))}{c} \right)^{1/q} \kappa^{n/q} \quad (n \geq N).$$

Hence (w_n) is Cauchy. Completeness yields $w_n \rightarrow w^*$ for some $w^* \in Q$.

Continuity of ψ at 0 gives

$$\psi(\delta(w_n, w^*)) \rightarrow 0.$$

Using (4.7),

$$\psi(\delta(w_{n+1}, Tw^*)) = \psi(\delta(Tw_n, Tw^*)) \leq \kappa\psi(\delta(w_n, w^*)) \rightarrow 0.$$

Since $\delta(w_{n+1}, Tw^*) \in \Delta_T$, condition (2) gives $\delta(w_{n+1}, Tw^*) \rightarrow 0$. But $w_{n+1} \rightarrow w^*$, so uniqueness of limits gives $Tw^* = w^*$. The fixed point is unique by the first paragraph. $\square$

Remark 4.13. Theorem 4.12 contains the following two results.

First, if $\psi \in \mathcal{G}(q)$, then monotonicity and positivity away from zero imply that ψ is zero-separating on every subset of $[0, \infty)$. The local estimate in Definition 2.2 implies (4.6). Thus Theorem 4.2 is recovered.

Second, let φ be a k -admissible function in the sense of Jleli and Samet [15]. Their condition

$$\inf_{t \geq \varepsilon} \varphi(t) > 0 \quad (\varepsilon > 0)$$

implies zero separation on Δ_T . If j is the order of the first nonzero derivative of φ at zero, Taylor's formula gives $\varphi(t) \geq ct^j$ near zero. Hence their main fixed point theorem is obtained from Theorem 4.12 by taking $\psi = \varphi$ and $q = j$.

For a fixed mapping T , these hypotheses can be strictly weaker than either global scalar-function hypothesis: The new theorem requires neither monotonicity nor differentiability and controls zero separation and the lower power estimate only on Δ_T . Example 4.14 shows that both relaxations can occur simultaneously.

There is also an exact equivalent-metric interpretation of the Jleli–Samet theorem. Proposition 5.6 below shows that, after taking the j th root of their k -admissible function, the mapping is a Banach contraction in the equivalent complete chain metric constructed there.

Example 4.14. We strengthen the three-point construction in Example 2.17 of Jleli and Samet [15] by adjoining pairs of points whose mutual distances are unbounded. This prevents the failure of their global separation condition from being caused merely by redefining the scalar function outside the distance set of the space.

Let

$$Q = \{0, 1, 2\} \cup \{a_n, b_n : n \geq 1\}.$$

Equip Q with the shortest-path metric δ of the weighted tree having center 1, with edges $[0, 1]$ and $[1, 2]$ of length 1, and edges $[1, a_n]$ and $[1, b_n]$ of length 3^n , for $n \geq 1$. Thus, for example,

$$\delta(0, 1) = \delta(1, 2) = 1, \quad \delta(1, a_n) = \delta(1, b_n) = 3^n,$$

and

$$\delta(a_n, b_n) = 2 \cdot 3^n.$$

More generally, the distance between two distinct vertices is the sum of the lengths of the edges in the unique path joining them. Every two distinct points have distance at least 1, so (Q, δ) is complete.

Define $T : Q \rightarrow Q$ by

$$T1 = 2, \quad Tx = 0 \quad (x \neq 1).$$

Then

$$\Delta_T = \{0, 2\}.$$

Define $\psi : [0, \infty) \rightarrow [0, \infty)$ first by

$$\psi(t) = t(1 + 2(t - 2)^2) \quad (0 \leq t \leq 2)$$

and

$$\psi(t) = t \quad (2 \leq t \leq 3).$$

For every integer $n \geq 1$, define ψ linearly on the two intervals $[3^n, 2 \cdot 3^n]$ and $[2 \cdot 3^n, 3^{n+1}]$ by

$$\psi(t) = 3 - \frac{3 - 1/n}{3^n}(t - 3^n) \quad (3^n \leq t \leq 2 \cdot 3^n)$$

and

$$\psi(t) = \frac{1}{n} + \frac{3 - 1/n}{3^n}(t - 2 \cdot 3^n) \quad (2 \cdot 3^n \leq t \leq 3^{n+1}).$$

These formulas agree at their endpoints, so ψ is continuous on $[0, \infty)$. The function is not differentiable at $t = 3$. Indeed, its left derivative at 3 is 1, whereas its right derivative is

$$-\frac{3 - 1}{3} = -\frac{2}{3}.$$

Thus $\psi \notin C^1([0, \infty))$ and is not k -admissible for any positive integer k . Moreover,

$$\psi(0) = 0, \quad \psi(1) = 3, \quad \psi(2) = 2, \quad \psi(3^n) = 3, \quad \psi(2 \cdot 3^n) = \frac{1}{n}.$$

In particular, ψ is not nondecreasing, because $\psi(1) > \psi(2)$, and hence it is not an admissible gauge in the sense of Definition 2.1. Also,

$$\inf_{t \geq \varepsilon} \psi(t) = 0 \quad (\varepsilon > 0),$$

because the distances $t_n = \delta(a_n, b_n) = 2 \cdot 3^n$ satisfy $\psi(t_n) = 1/n \rightarrow 0$. Thus the global separation condition of Jleli and Samet fails on the actual distance set of (Q, δ) , not merely outside it.

Nevertheless, ψ is zero-separating on $\Delta_T = \{0, 2\}$, and (4.6) holds with $q = c = 1$ and $r_0 = 2$. Take $\kappa = 3/4$. If exactly one of u, v is the point 1, then $\delta(Tu, Tv) = 2$, while

$$\delta(u, v) \in \{1\} \cup \{3^n : n \geq 1\}.$$

Hence

$$\psi(\delta(Tu, Tv)) = \psi(2) = 2 \leq \frac{3}{4} 3 = \kappa \psi(\delta(u, v)).$$

For every other pair, $Tu = Tv$, so the left-hand side of (4.7) is zero. Therefore (4.7) holds for all $u, v \in Q$. Theorem 4.12 applies and gives the unique fixed point 0.

Finally,

$$\delta(T0, T1) = \delta(0, 2) = 2 > 1 = \delta(0, 1),$$

so T is not a Banach contraction in the original metric. Therefore, neither Theorem 4.2 nor the theorem of Jleli and Samet applies to this example: ψ is not nondecreasing and does not satisfy their global zero-separation condition.

5. Chain metrics associated with gauge transforms

A nonmetric transformed distance may still generate a genuine metric by a chain construction. This is related to the classical metrization idea of Frink [24]. The next result gives a sufficient condition that prevents the chain distance from degenerating.

Definition 5.1. Let (Q, δ) be a metric space and let $\psi \in \mathcal{G}$ be an admissible gauge. Put

$$D(u, v) = \delta_\psi(u, v) = \psi(\delta(u, v)).$$

For $u, v \in Q$, define

$$\rho_D(u, v) = \inf \left\{ \sum_{i=0}^{m-1} D(x_i, x_{i+1}) : m \geq 1, x_0 = u, x_m = v \right\}. \quad (5.1)$$

Proposition 5.2. Let (Q, δ) be a metric space and let $\psi \in \mathcal{G}_{\text{lin}}$ be an admissible gauge. Then ρ_D is a metric on Q and

$$\rho_D(u, v) \leq D(u, v) \quad (u, v \in Q). \quad (5.2)$$

Moreover, if $\psi(r) \geq cr$ for $0 \leq r \leq r_0$, then

$$\rho_D(u, v) \geq c \min\{\delta(u, v), r_0\} \quad (u, v \in Q). \quad (5.3)$$

Proof. Symmetry and $\rho_D(u, u) = 0$ are immediate. The triangle inequality follows by concatenating chains. The upper estimate (5.2) follows by taking the one-step chain u, v .

We prove nondegeneracy. Let $u \neq v$, and let

$$u = x_0, x_1, \dots, x_m = v$$

be any finite chain. If $\delta(x_i, x_{i+1}) \leq r_0$, then

$$D(x_i, x_{i+1}) \geq c\delta(x_i, x_{i+1}).$$

If $\delta(x_i, x_{i+1}) > r_0$, then by monotonicity,

$$D(x_i, x_{i+1}) \geq \psi(r_0) \geq cr_0.$$

Therefore

$$\sum_{i=0}^{m-1} D(x_i, x_{i+1}) \geq c \sum_{i=0}^{m-1} \min\{\delta(x_i, x_{i+1}), r_0\} \geq c \min \left\{ \sum_{i=0}^{m-1} \delta(x_i, x_{i+1}), r_0 \right\}.$$

The triangle inequality for δ gives

$$\sum_{i=0}^{m-1} \delta(x_i, x_{i+1}) \geq \delta(u, v).$$

Hence every chain has length at least $c \min\{\delta(u, v), r_0\}$. Taking the infimum over all chains gives (5.3). Since $u \neq v$, the right side is positive. Thus ρ_D is a metric. $\square$

Proposition 5.3. *Let (Q, δ) be a metric space and let $\psi \in \mathcal{G}_{\text{lin}}$ be an admissible gauge. Then:*

- 1) $u_n \rightarrow u$ in (Q, δ) if and only if $\rho_D(u_n, u) \rightarrow 0$;
- 2) (u_n) is Cauchy in (Q, δ) if and only if it is Cauchy in (Q, ρ_D) ;
- 3) (Q, δ) is complete if and only if (Q, ρ_D) is complete.

Proof. The estimates

$$\rho_D(u, v) \leq \psi(\delta(u, v))$$

and

$$\rho_D(u, v) \geq c \min\{\delta(u, v), r_0\}$$

show that convergence in one metric is equivalent to convergence in the other. The same estimates applied to pairs (u_n, u_m) give the Cauchy equivalence. Completeness follows immediately. $\square$

Proposition 5.4. *Let (Q, δ) be a metric space, let $\psi \in \mathcal{G}_{\text{lin}}$ be an admissible gauge, and let ρ_D be the chain metric associated with $D = \psi \circ \delta$. If a mapping $T : Q \rightarrow Q$ satisfies*

$$D(Tu, Tv) \leq \kappa D(u, v) \quad (u, v \in Q)$$

for some $\kappa \in (0, 1)$, then

$$\rho_D(Tu, Tv) \leq \kappa \rho_D(u, v) \quad (u, v \in Q).$$

Proof. Let $u = x_0, x_1, \dots, x_m = v$ be any finite chain from u to v . Then $Tu = Tx_0, Tx_1, \dots, Tx_m = Tv$ is a finite chain from Tu to Tv , and

$$\sum_{i=0}^{m-1} D(Tx_i, Tx_{i+1}) \leq \kappa \sum_{i=0}^{m-1} D(x_i, x_{i+1}).$$

Taking the infimum over all finite chains from u to v gives the assertion. $\square$

Corollary 5.5. *Let (Q, δ) be a metric space, let $q > 0$, and let $\psi \in \mathcal{G}(q)$. Define*

$$\phi(t) = \psi(t)^{1/q}$$

and let ρ_ϕ be the chain metric generated by

$$D_\phi(u, v) = \phi(\delta(u, v)).$$

Then ρ_ϕ is equivalent to δ . If

$$\psi(\delta(Tu, Tv)) \leq \kappa \psi(\delta(u, v)), \quad 0 < \kappa < 1,$$

then

$$\rho_\phi(Tu, Tv) \leq \kappa^{1/q} \rho_\phi(u, v).$$

If (Q, δ) is complete, then (Q, ρ_ϕ) is complete.

Proof. By Remark 2.6, $\phi \in \mathcal{G}_{\text{lin}}$ and

$$\phi(\delta(Tu, Tv)) \leq \kappa^{1/q} \phi(\delta(u, v)).$$

The assertions follow from Propositions 5.3 and 5.4. $\square$

Proposition 5.6. *Let (Q, δ) be a metric space, and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be k -admissible in the sense of Jleli and Samet [15]. Let j be the order of the first nonzero derivative of φ at zero, and put*

$$\phi(t) = \varphi(t)^{1/j}, \quad D_\phi(u, v) = \phi(\delta(u, v)).$$

Define ρ_ϕ by the chain formula (5.1), with D_ϕ in place of D . Then ρ_ϕ is a metric equivalent to δ , and the two metrics have the same Cauchy sequences. In particular, completeness of (Q, δ) implies completeness of (Q, ρ_ϕ) .

By k -admissibility, φ is continuous at zero, the order j is finite, the coefficient of its first nonzero Taylor term is positive, and the global separation condition used below holds.

If $T : Q \rightarrow Q$ satisfies

$$\varphi(\delta(Tu, Tv)) \leq \kappa \varphi(\delta(u, v)), \quad u, v \in Q,$$

for some $\kappa \in (0, 1)$, then

$$\rho_\phi(Tu, Tv) \leq \kappa^{1/j} \rho_\phi(u, v), \quad u, v \in Q.$$

Thus their main fixed point theorem is also an application of Banach's contraction principle in this canonical equivalent chain metric.

Proof. Their Taylor argument gives numbers $a, r_0 > 0$ such that

$$\varphi(t) \geq a^j t^j \quad (0 \leq t \leq r_0).$$

Hence

$$\phi(t) = \varphi(t)^{1/j} \geq at \quad (0 \leq t \leq r_0).$$

Their global separation condition, applied with $\varepsilon = r_0$, gives

$$\mu := \inf_{t \geq r_0} \varphi(t) > 0.$$

Since $s \mapsto s^{1/j}$ is increasing on $[0, \infty)$,

$$m := \inf_{t \geq r_0} \phi(t) = \mu^{1/j} > 0.$$

With $b = \min\{a, m/r_0\}$, we therefore have

$$\phi(t) \geq b \min\{t, r_0\} \quad (t \geq 0).$$

The chain argument used in Proposition 5.2 now gives

$$\rho_\phi(u, v) \geq b \min\{\delta(u, v), r_0\}, \quad \rho_\phi(u, v) \leq \phi(\delta(u, v)).$$

The first estimate proves nondegeneracy and shows that ρ_ϕ -convergence or the ρ_ϕ -Cauchy property implies the corresponding property for δ . Conversely, continuity of φ at zero gives $\phi(t) \rightarrow 0$ as $t \rightarrow 0$, so the upper estimate gives the reverse implications. This proves equivalence and completeness.

Taking j th roots in the contractive inequality yields

$$D_\phi(Tu, Tv) \leq \kappa^{1/j} D_\phi(u, v).$$

Applying this estimate to every edge of a finite chain and taking the infimum proves the asserted contraction inequality for ρ_ϕ . $\square$

Remark 5.7. Thus, when $\psi \in \mathcal{G}_{\text{lin}}$, every linear ψ -gauge contraction is an ordinary Banach contraction in the associated chain metric ρ_D . Corollary 5.5 gives the corresponding root-chain representation for every $\psi \in \mathcal{G}(q)$. These observations do not make δ_ψ a metric and do not make the mapping contractive in the original metric δ ; they identify the precise equivalent-metric content of the linear gauge condition.

Example 5.8. As a space-independent sufficient condition, the local linear lower estimate in Proposition 5.2 cannot be replaced by an arbitrary power lower estimate with an exponent greater than one. The subdivision argument below is standard in discussions of the limits of Frink-type chain constructions; compare [24, 25]. Let $Q = [0, 1]$ with the usual metric and let $\psi(t) = t^p$, where $p > 1$. Then for the chain

$$0, \frac{1}{n}, \frac{2}{n}, \dots, 1,$$

one has

$$\sum_{i=0}^{n-1} \psi\left(\frac{1}{n}\right) = n \left(\frac{1}{n}\right)^p = n^{1-p} \rightarrow 0.$$

Hence the chain distance between 0 and 1 is zero. Thus the chain construction collapses and does not give a metric. This is the main reason why the chain-metric section is stated under the local linear lower estimate.

Proposition 5.9. Let ρ_D be the chain metric from Proposition 5.2. Define

$$\rho_1(u, v) = \frac{\rho_D(u, v)}{1 + \rho_D(u, v)}$$

and

$$\rho_2(u, v) = \log(1 + \rho_D(u, v)).$$

Then ρ_1 and ρ_2 are metrics on Q . The metric ρ_1 is bounded, while ρ_2 need not be bounded. Both have the same convergent sequences and the same Cauchy sequences as ρ_D , and therefore as δ .

Proof. The functions

$$h_1(t) = \frac{t}{1+t} \quad \text{and} \quad h_2(t) = \log(1+t)$$

are standard metric-transform functions; see Borsík and Doboš [2] and Corazza [21]. In particular, they are increasing, concave, and vanish only at 0, and their composition with any metric is again a metric. Hence

$$\rho_1 = h_1 \circ \rho_D \quad \text{and} \quad \rho_2 = h_2 \circ \rho_D$$

are metrics on Q .

Since h_1 and h_2 are continuous at 0 and positive away from 0, a sequence of nonnegative numbers (t_n) converges to 0 if and only if $h_i(t_n) \rightarrow 0$, $i = 1, 2$. Therefore ρ_1 and ρ_2 have the same convergent sequences and the same Cauchy sequences as ρ_D . The conclusion now follows from Proposition 5.3. $\square$

Definition 5.10. Let (Q, δ) be a metric space, let $\psi \in \mathcal{G}_{\text{lin}}$ be an admissible gauge, and let ρ be one of the metrics ρ_D, ρ_1, ρ_2 above. We say that $T : Q \rightarrow Q$ is a contraction on (Q, ρ) if there exists $\lambda \in (0, 1)$ such that

$$\rho(Tu, Tv) \leq \lambda \rho(u, v) \quad (u, v \in Q).$$

Corollary 5.11. Let (Q, δ) be complete, let $\psi \in \mathcal{G}_{\text{lin}}$ be an admissible gauge, and let $\rho \in \{\rho_D, \rho_1, \rho_2\}$. If $T : Q \rightarrow Q$ is a contraction on (Q, ρ) , then T has a unique fixed point $w^* \in Q$, and $T^n x \rightarrow w^*$ in both (Q, ρ) and (Q, δ) for every $x \in Q$.

Proof. By Propositions 5.3 and 5.9, the metric space (Q, ρ) is complete and has the same convergent sequences as (Q, δ) . Banach's contraction principle [1] applies in (Q, ρ) . $\square$

6. An integral equation example not covered in the original metric

The preceding fixed point theorem should not be illustrated only by a usual Lipschitz integral equation with a constant less than one. In that case, Banach's contraction principle applies in the original metric. The next example shows a different situation. The operator comes from an integral equation. It is not a contraction in the metric defined below, but it becomes contractive after applying an admissible gauge. The transformed distance is not a metric.

For each integer $n \geq 0$, let $u_n \in C([0, 1])$ be the constant function

$$u_n(t) = n \quad (0 \leq t \leq 1).$$

Put

$$Q = \{u_n : n \geq 0\}.$$

Define a metric δ on Q by

$$\delta(u_m, u_n) = \begin{cases} 0, & m = n, \\ m + n, & m \neq n. \end{cases}$$

This is a metric. The triangle inequality is immediate if two of the three points are equal. If m, n, k are distinct, then $\delta(u_m, u_n) = m + n \leq (m + k) + (k + n) = \delta(u_m, u_k) + \delta(u_k, u_n)$. Thus (Q, δ) is complete, because every Cauchy sequence in Q is eventually constant.

Consider the integral operator $T : Q \rightarrow Q$ defined by

$$(Tu)(t) = \int_0^1 (u(s) - 1)^+ ds, \quad 0 \leq t \leq 1,$$

where $r^+ = \max\{r, 0\}$. For the constant functions above,

$$Tu_0 = u_0, \quad Tu_n = u_{n-1} \quad (n \geq 1).$$

Thus the fixed point equation $u = Tu$ is the integral equation

$$u(t) = \int_0^1 (u(s) - 1)^+ ds, \quad 0 \leq t \leq 1, \quad (6.1)$$

considered on the complete metric space (Q, δ) .

We show that Banach's contraction principle does not apply in (Q, δ) . If T were a contraction in (Q, δ) with constant $\lambda < 1$, then, for $n \geq 2$,

$$\frac{\delta(Tu_n, Tu_0)}{\delta(u_n, u_0)} = \frac{\delta(u_{n-1}, u_0)}{\delta(u_n, u_0)} = \frac{n-1}{n} \leq \lambda.$$

Letting $n \rightarrow \infty$ gives $1 \leq \lambda$, a contradiction. Hence T is not a Banach contraction on (Q, δ) .

Now take

$$\psi(r) = e^r - 1 \quad (r \geq 0).$$

Then $\psi \in \mathcal{G}_{\text{lin}}$, since $e^r - 1 \geq r$ for $r \geq 0$. Moreover, δ_ψ is not a metric. For instance,

$$\delta_\psi(u_1, u_2) = e^3 - 1 > (e^1 - 1) + (e^2 - 1) = \delta_\psi(u_1, u_0) + \delta_\psi(u_0, u_2).$$

Thus Banach's theorem cannot be applied directly to δ_ψ , since δ_ψ is not a metric.

Let $m \neq n$. From the definition of T and δ , one has

$$\delta(Tu_m, Tu_n) \leq \delta(u_m, u_n) - 1.$$

Therefore

$$\delta_\psi(Tu_m, Tu_n) = e^{\delta(Tu_m, Tu_n)} - 1 \leq e^{\delta(u_m, u_n) - 1} - 1 \leq e^{-1}(e^{\delta(u_m, u_n)} - 1).$$

Hence

$$\delta_\psi(Tu_m, Tu_n) \leq e^{-1}\delta_\psi(u_m, u_n) \quad (m, n \geq 0).$$

So T is a ψ -gauge contraction with constant e^{-1} . By Theorem 4.2, T has a unique fixed point in Q , and the Picard sequence converges to it from every initial point. The fixed point is u_0 , and hence (6.1) has the unique solution u_0 in Q .

This example shows the direct role of the gauge theorem. The mapping is not a contraction in the original metric, and the transformed distance is not a metric. Nevertheless the gauge contractive inequality gives convergence of the Picard iteration. Since $\psi \in \mathcal{G}_{\text{lin}}$, Proposition 5.4 also shows that T is a Banach contraction in the associated chain metric ρ_D . Thus the example is not outside every metric reduction; its point is that Banach's theorem does not apply directly in either the original metric or the transformed nonmetric distance.

Corollary 6.1. *Let $\theta(r) = r$. Then the hyperbolic-sine gauge*

$$\psi_\theta(r) = \sinh r$$

also gives an example not covered by Banach's contraction principle in the original metric. More precisely, for the same space (Q, δ) and the same mapping T as above, T is a ψ_θ -gauge contraction with constant e^{-1} .

Proof. Put

$$\psi(r) = \sinh r \quad (r \geq 0).$$

Then $\psi \in \mathcal{G}_{\text{lin}}$, since $\sinh r \geq r$ for all $r \geq 0$.

We first note a simple estimate. If $r \geq 1$, then

$$\frac{\sinh(r-1)}{\sinh r} = e^{-1} \frac{1 - e^{-2(r-1)}}{1 - e^{-2r}} \leq e^{-1}.$$

Indeed, $r-1 \leq r$, and hence

$$1 - e^{-2(r-1)} \leq 1 - e^{-2r}.$$

Now let $m, n \geq 0$. If $m = n$, then

$$\delta_\psi(Tu_m, Tu_n) = 0 = e^{-1} \delta_\psi(u_m, u_n).$$

Assume that $m \neq n$, and put

$$r = \delta(u_m, u_n).$$

Then $r \geq 1$. From the definition of T and δ , we have

$$\delta(Tu_m, Tu_n) \leq r - 1.$$

Indeed, if one of m, n is zero, then this inequality is an equality; if $m, n \geq 1$, then

$$\delta(Tu_m, Tu_n) = \delta(u_{m-1}, u_{n-1}) = m + n - 2 = r - 2 \leq r - 1.$$

Since $\sinh$ is increasing, it follows that

$$\delta_\psi(Tu_m, Tu_n) = \sinh(\delta(Tu_m, Tu_n)) \leq \sinh(r-1) \leq e^{-1} \sinh r = e^{-1} \delta_\psi(u_m, u_n).$$

Thus

$$\delta_\psi(Tu_m, Tu_n) \leq e^{-1} \delta_\psi(u_m, u_n) \quad (m, n \geq 0).$$

Therefore T is a ψ -gauge contraction.

As shown above, T is not a Banach contraction on (Q, δ) . Also, δ_ψ is not a metric. For example,

$$\delta_\psi(u_1, u_2) = \sinh 3 > \sinh 1 + \sinh 2 = \delta_\psi(u_1, u_0) + \delta_\psi(u_0, u_2).$$

Hence Banach's contraction principle cannot be applied directly either in (Q, δ) or in (Q, δ_ψ) . The conclusion follows from Theorem 4.2. Since $\psi \in \mathcal{G}_{\text{lin}}$, Proposition 5.4 also yields a Banach contraction in the associated chain metric. $\square$

7. A nonlinear Hutchinson operator beyond the classical contraction condition

Iterated function systems and compact attractors were developed systematically by Hutchinson [26]. Weakly contractive iterated function systems are also well established; see Leśniak, Snigireva, and Strobin [27]. The example below is included for a precise purpose. Its Hutchinson operator has no uniform linear contraction constant in the original Hausdorff metric. After a nonmetric exponential

gauge transform, it still has no uniform linear gauge constant. The convergence is instead governed by the nonlinear comparison theorem.

Let

$$Q = [0, 1]$$

with the usual metric, and define

$$\eta(t) = \frac{t}{(1 + \sqrt{t})^2}, \quad t \geq 0.$$

Then η is increasing, $\eta(t) < t$ for $t > 0$, and

$$\eta^n(t) = \frac{t}{(1 + n\sqrt{t})^2} \quad (t \geq 0, n \geq 0).$$

Indeed,

$$\eta(t)^{-1/2} = t^{-1/2} + 1 \quad (t > 0).$$

Consequently,

$$\sum_{n=0}^{\infty} \eta^n(t) < \infty \quad (t > 0),$$

so η is a comparison function of order 1.

Define two maps $S_0, S_1 : Q \rightarrow Q$ by

$$S_0x = \eta(x), \quad S_1x = \frac{3}{4} + \frac{x}{8}.$$

The derivative

$$\eta'(t) = \frac{1}{(1 + \sqrt{t})^3} \quad (t > 0)$$

is decreasing, so η is concave on $[0, \infty)$. Since $\eta(0) = 0$, it is subadditive. Indeed, for $a, b > 0$, concavity gives

$$\eta(a) \geq \frac{a}{a+b} \eta(a+b), \quad \eta(b) \geq \frac{b}{a+b} \eta(a+b).$$

Hence $\eta(a+b) \leq \eta(a) + \eta(b)$. Therefore

$$|S_0x - S_0y| = |\eta(x) - \eta(y)| \leq \eta(|x - y|).$$

Also, for $0 \leq t \leq 1$,

$$\frac{\eta(t)}{t} = \frac{1}{(1 + \sqrt{t})^2} \geq \frac{1}{4},$$

and therefore

$$|S_1x - S_1y| = \frac{|x - y|}{8} \leq \eta(|x - y|).$$

Let $\mathcal{K}(Q)$ denote the family of all nonempty compact subsets of Q , equipped with the Hausdorff metric δ_H . Since Q is compact, $(\mathcal{K}(Q), \delta_H)$ is complete [28]. Define the Hutchinson operator

$$\mathcal{H}(A) = S_0(A) \cup S_1(A).$$

For $A, B \in \mathcal{K}(Q)$,

$$\delta_H(\mathcal{H}(A), \mathcal{H}(B)) \leq \eta(\delta_H(A, B)). \quad (7.1)$$

To see this, let $\varepsilon > 0$. For each $a \in A$, choose $b \in B$ with

$$|a - b| \leq \delta_H(A, B) + \varepsilon.$$

For $i = 0, 1$,

$$|S_i a - S_i b| \leq \eta(\delta_H(A, B) + \varepsilon).$$

The same argument with A and B interchanged gives

$$\delta_H(\mathcal{H}(A), \mathcal{H}(B)) \leq \eta(\delta_H(A, B) + \varepsilon).$$

Since η is continuous, letting $\varepsilon \downarrow 0$ proves (7.1).

Inequality (7.1) already gives the attractor conclusion from Theorem 4.7 with the identity gauge. The exponential-gauge calculation below does not produce a different attractor; its purpose is to show that the same nonlinear comparison remains valid after passing to a nonmetric admissible gauge transform and still cannot be replaced by a uniform linear estimate.

We now apply a gauge whose transform is genuinely nonmetric to the Hausdorff distance:

$$\psi(r) = e^r - 1.$$

Then $\psi \in \mathcal{G}_{\text{lin}}$. Put

$$\delta_{H,\psi}(A, B) = e^{\delta_H(A, B)} - 1.$$

This transformed distance is not a metric. Indeed, for

$$A = \{0\}, \quad B = \left\{\frac{1}{2}\right\}, \quad C = \{1\},$$

we have

$$\delta_{H,\psi}(A, C) = e - 1 > 2(e^{1/2} - 1) = \delta_{H,\psi}(A, B) + \delta_{H,\psi}(B, C).$$

Define

$$\eta_\psi(t) = \exp(\eta(\log(1 + t))) - 1, \quad t \geq 0.$$

Equivalently,

$$\eta_\psi = \psi \circ \eta \circ \psi^{-1},$$

where

$$\psi^{-1}(t) = \log(1 + t).$$

Consequently,

$$\eta_\psi^n(t) = \exp(\eta^n(\log(1 + t))) - 1, \quad n \geq 0.$$

The function η_ψ is nondecreasing. Moreover, if $t > 0$ and

$$r = \log(1 + t),$$

then $r > 0$ and $\eta(r) < r$. Hence

$$\eta_\psi(t) = e^{\eta(r)} - 1 < e^r - 1 = t.$$

We next verify the summability condition. Since

$$0 \leq \eta^n(r) \leq r$$

and the function $s \mapsto e^s - 1$ is convex and vanishes at zero,

$$e^{\eta^n(r)} - 1 \leq \frac{\eta^n(r)}{r}(e^r - 1).$$

Therefore

$$\sum_{n=0}^{\infty} \eta_\psi^n(t) \leq \frac{e^r - 1}{r} \sum_{n=0}^{\infty} \eta^n(r) < \infty.$$

Thus η_ψ is a comparison function of order 1.

From (7.1),

$$\begin{aligned} \delta_{H,\psi}(\mathcal{H}(A), \mathcal{H}(B)) &= e^{\delta_H(\mathcal{H}(A), \mathcal{H}(B))} - 1 \\ &\leq e^{\eta(\delta_H(A, B))} - 1 \\ &= \eta_\psi(\delta_{H,\psi}(A, B)). \end{aligned}$$

Thus $\mathcal{H}$ is a (ψ, η_ψ) -gauge comparison contraction. By Theorem 4.7, there is a unique compact set $K_* \subset [0, 1]$ such that

$$K_* = S_0(K_*) \cup S_1(K_*),$$

and

$$\mathcal{H}^n(A) \longrightarrow K_*$$

in the Hausdorff metric for every $A \in \mathcal{K}(Q)$.

It remains to show that this is not a hidden linear Hutchinson contraction in either of the two distances used above. For $0 < \varepsilon \leq 1$, let

$$A_\varepsilon = \{0\}, \quad B_\varepsilon = \{\varepsilon\}.$$

The two ranges

$$S_0(Q) \subset \left[0, \frac{1}{4}\right], \quad S_1(Q) \subset \left[\frac{3}{4}, \frac{7}{8}\right]$$

are separated by a gap of at least $1/2$. The displacements within the first and second branches are $\eta(\varepsilon) \leq 1/4$ and $\varepsilon/8 \leq 1/8$, respectively. Thus, in the Hausdorff distance, each point is matched with the point in the same branch. Hence

$$\delta_H(\mathcal{H}(A_\varepsilon), \mathcal{H}(B_\varepsilon)) = \max\left\{\eta(\varepsilon), \frac{\varepsilon}{8}\right\} = \eta(\varepsilon).$$

It follows that

$$\frac{\delta_H(\mathcal{H}(A_\varepsilon), \mathcal{H}(B_\varepsilon))}{\delta_H(A_\varepsilon, B_\varepsilon)} = \frac{1}{(1 + \sqrt{\varepsilon})^2} \longrightarrow 1.$$

Thus no $\lambda < 1$ satisfies the classical Hutchinson inequality

$$\delta_H(\mathcal{H}(A), \mathcal{H}(B)) \leq \lambda \delta_H(A, B) \quad (A, B \in \mathcal{K}(Q)).$$

Moreover,

$$\frac{\delta_{H,\psi}(\mathcal{H}(A_\varepsilon), \mathcal{H}(B_\varepsilon))}{\delta_{H,\psi}(A_\varepsilon, B_\varepsilon)} = \frac{e^{\eta(\varepsilon)} - 1}{e^\varepsilon - 1} \longrightarrow 1.$$

Therefore no $\kappa < 1$ makes $\mathcal{H}$ a linear ψ -gauge contraction.

We can say more about the canonical chain metric. Let ρ_ψ denote the chain metric generated by

$$D_\psi(A, B) = e^{\delta_H(A, B)} - 1.$$

Since $e^r - 1 \geq r$ for $r \geq 0$, every finite chain gives

$$\sum_i D_\psi(C_i, C_{i+1}) \geq \sum_i \delta_H(C_i, C_{i+1}) \geq \delta_H(C_0, C_m).$$

Hence

$$\rho_\psi(A, B) \geq \delta_H(A, B).$$

For $A_\varepsilon = \{0\}$ and $B_\varepsilon = \{\varepsilon\}$, consider the chain

$$E_j = \left\{ \frac{j\varepsilon}{N} \right\}, \quad 0 \leq j \leq N.$$

It follows that

$$\rho_\psi(A_\varepsilon, B_\varepsilon) \leq N(e^{\varepsilon/N} - 1).$$

Letting $N \rightarrow \infty$ and using the preceding lower estimate gives

$$\rho_\psi(A_\varepsilon, B_\varepsilon) = \varepsilon.$$

Similarly, set

$$F_j = \left\{ \frac{j}{N}\eta(\varepsilon), \frac{3}{4} + \frac{j\varepsilon}{8N} \right\}, \quad 0 \leq j \leq N.$$

For every j , the first point of F_j lies in $[0, 1/4]$, and the second lies in $[3/4, 7/8]$. Thus every cross-branch distance is at least $1/2$, whereas the two same-branch increments are at most $1/(4N)$ and $1/(8N)$. Hence nearest points are matched within their respective branches. Since $\eta(\varepsilon) \geq \varepsilon/4 > \varepsilon/8$, we obtain

$$\delta_H(F_{j-1}, F_j) = \frac{\eta(\varepsilon)}{N}.$$

Since $F_0 = \mathcal{H}(A_\varepsilon)$ and $F_N = \mathcal{H}(B_\varepsilon)$, we obtain

$$\rho_\psi(\mathcal{H}(A_\varepsilon), \mathcal{H}(B_\varepsilon)) \leq N(e^{\eta(\varepsilon)/N} - 1).$$

Letting $N \rightarrow \infty$ and using $\rho_\psi \geq \delta_H$ yields

$$\rho_\psi(\mathcal{H}(A_\varepsilon), \mathcal{H}(B_\varepsilon)) = \eta(\varepsilon).$$

Consequently,

$$\frac{\rho_\psi(\mathcal{H}(A_\varepsilon), \mathcal{H}(B_\varepsilon))}{\rho_\psi(A_\varepsilon, B_\varepsilon)} = \frac{\eta(\varepsilon)}{\varepsilon} = \frac{1}{(1 + \sqrt{\varepsilon})^2} \longrightarrow 1.$$

Thus $\mathcal{H}$ is not a Banach contraction in the original Hausdorff metric or in the canonical chain metric generated by the exponential gauge, and it has no uniform linear contraction constant in the nonmetric exponential gauge transform. The nonlinear estimate remains within the established class of weakly contractive iterated function systems. We make no claim concerning all other equivalent metrics.

8. Relation with known fixed point conditions

We now compare the gauge framework with several standard constructions. The discussion distinguishes exact special cases from results that require additional representability or uniformity assumptions.

Banach and nonlinear comparison contractions.

If $\psi(t) = t$ and $\eta(t) = \kappa t$, then (4.4) is the Banach contraction condition [1]. More generally, Corollary 5.5 shows that every linear $\mathcal{G}(q)$ -gauge contraction is Banach contractive in a canonical equivalent chain metric. This chain metric is complete whenever the original metric is complete. This statement delimits the linear theory. A nonlinear comparison function need not admit a uniform estimate $\eta(t) \leq \lambda t$ with $\lambda < 1$; the Hutchinson operator in Section 7 is an explicit example. When $\psi(t) = t$, Theorem 4.7 gives a summable Matkowski-type comparison contraction, stronger than the condition $\eta^n(t) \rightarrow 0$ and distinct from the Boyd–Wong hypothesis [22, 23].

Hyperbolic-sine distances.

For every function $\theta : [0, \infty) \rightarrow [0, \infty)$,

$$\delta_\theta = \delta_{\theta \circ \sinh}.$$

Hence every θ -hyperbolic sine distance is a gauge transform. If θ satisfies the regularity and lower power assumptions in Proposition 3.9, then $\theta \circ \sinh \in \mathcal{G}(q)$, and the corresponding regular-subclass versions of the fixed point theorems of Jleli and Samet [12] follow from Corollaries 4.3 and 4.9. This inclusion concerns the regular subclass. The original lower power condition by itself does not imply that $\theta(0) = 0$, that θ is monotone, or that it is continuous at zero.

Taylor–Lagrange distances.

Write the Taylor–Lagrange distance in the form

$$d_{\omega, t_0}(x, y) = \omega(L_{t_0}(x, y)),$$

where

$$L_{t_0}(x, y) = \delta(t_0)(x, y) + t_0|\delta'(0)(x, y)| + \frac{t_0^2}{2}D_\delta^2(x, y).$$

Here

$$D_\delta^2(x, y) = \max_{0 \leq t \leq 1} |\delta''(t)(x, y)|.$$

Relative to the original metric $d_0 = \delta(0)$, there exists a scalar function ψ such that

$$d_{\omega, t_0}(x, y) = \psi(d_0(x, y))$$

if and only if the Taylor–Lagrange distance is constant on every level set of d_0 ; that is, whenever

$$d_0(x, y) = d_0(u, v),$$

one has

$$d_{\omega, t_0}(x, y) = d_{\omega, t_0}(u, v).$$

If ω is injective, this is equivalent to $L_{t_0} = h \circ d_0$ for some scalar function h , in which case $\psi = \omega \circ h$. In particular, when $t_0 = 0$,

$$d_{\omega,0} = \omega \circ d_0.$$

Thus Taylor–Lagrange distances are covered only in these representable cases; they are not gauge transforms of the original metric in general [13, 14].

Metric-preserving functions and equivalent metrics.

The theory of metric-preserving functions asks for conditions on f under which $f \circ d$ is a metric whenever d is a metric; see Borsik–Doboš [2], Doboš [3], Doboš–Piotrowski [4], Pokorný [5], Corazza [21], and Pongsriiam–Termwuttipong [9]. Here $\psi \circ \delta$ is not required to be a metric. The use of equivalent metrics in fixed point theory is also established [10]. Our chain metric gives an explicit metric generated by a possibly nonmetric transform and gives an explicit sufficient condition under which the linear gauge condition is Banach contractive after this canonical remetrization.

Altering distances and w -distances.

The altering-distance method of Khan–Swaleh–Sessa [6] also places an auxiliary scalar function in a contractive inequality. The present assumptions are formulated directly in terms of the transform $\psi \circ \delta$, require continuity only at zero, and distinguish topological equivalence from the lower estimates used for summability. The w -distance of Kada–Suzuki–Takahashi [7] is a different nonmetric device and is not generally of the form $\psi \circ \delta$. Recent work on presymmetric w -distances further illustrates the need for care when metric arguments are transferred to nonmetric functions [29].

Samet–Vetro–Vetro α - ψ contractions.

Samet, Vetro and Vetro [11] introduced the notion of an α - ψ contraction. Since the symbol ψ is already used in this paper for an admissible gauge, we denote the comparison function in their contractive condition by ϕ . Thus, in the notation of the present paper, their inequality has the form

$$\alpha(x, y)\delta(Tx, Ty) \leq \phi(\delta(x, y)), \quad x, y \in Q, \quad (8.1)$$

where $\alpha : Q \times Q \rightarrow [0, \infty)$ is an admissibility function and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a comparison function. The change from ψ to ϕ is only a change of notation; the notion remains the Samet–Vetro–Vetro α - ψ contraction.

If $\alpha(x, y) \geq 1$ on the relevant pairs, then (8.1) gives

$$\delta(Tx, Ty) \leq \phi(\delta(x, y))$$

on those pairs. In the usual admissible setting, this is sufficient along a Picard orbit provided that the orbit remains admissible. It does not give a global comparison estimate for all pairs unless additional assumptions are imposed.

We now give a global comparison result under a uniform lower bound on α .

Definition 8.1. Let (Q, δ) be a metric space and let $\alpha : Q \times Q \rightarrow (0, \infty)$. For $r > 0$, define

$$m_\alpha(r) = \inf\{\alpha(x, y) : 0 < \delta(x, y) \leq r\},$$

whenever the set is nonempty. If the set is empty, put $m_\alpha(r) = +\infty$.

The role of m_α is to remove the dependence of (8.1) on the individual pair. Indeed, if $r = \delta(x, y) > 0$, then

$$m_\alpha(r) \leq \alpha(x, y),$$

and hence

$$\delta(Tx, Ty) \leq \frac{\phi(r)}{\alpha(x, y)} \leq \frac{\phi(r)}{m_\alpha(r)}.$$

The supremum used below replaces this possibly nonmonotone scalar bound by a nondecreasing one.

Proposition 8.2. *Let (Q, δ) be a metric space, let $T : Q \rightarrow Q$, and assume that*

$$\alpha(x, y)\delta(Tx, Ty) \leq \phi(\delta(x, y)) \quad (x, y \in Q) \quad (8.2)$$

for some functions $\alpha : Q \times Q \rightarrow (0, \infty)$ and $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$. No monotonicity of ϕ is assumed. Suppose that $m_\alpha(r) > 0$ for every $r > 0$ for which the set in Definition 8.1 is nonempty. Assume that the following function is finite-valued, and define

$$\eta(r) = \begin{cases} 0, & r = 0, \\ \sup_{0 < s \leq r} \frac{\phi(s)}{m_\alpha(s)}, & r > 0, \end{cases}$$

where terms with $m_\alpha(s) = +\infty$ are read as zero. The function η is nondecreasing by construction, even if ϕ is not. If η is a comparison function of order 1, then

$$\delta(Tx, Ty) \leq \eta(\delta(x, y)) \quad (x, y \in Q).$$

Consequently, if (Q, δ) is complete, then T has a unique fixed point, and every Picard sequence converges to it.

Proof. Let $x, y \in Q$ and put $r = \delta(x, y)$. If $r = 0$, then $x = y$ and $\delta(Tx, Ty) = 0 = \eta(0)$. Suppose that $r > 0$. By definition of $m_\alpha(r)$,

$$m_\alpha(r) \leq \alpha(x, y).$$

Therefore (8.2) gives

$$\delta(Tx, Ty) \leq \frac{\phi(r)}{\alpha(x, y)} \leq \frac{\phi(r)}{m_\alpha(r)} \leq \eta(r),$$

where the last inequality follows from the definition of η as a supremum over $0 < s \leq r$. Thus T is a global comparison contraction in the original metric. The fixed point conclusion follows from Theorem 4.7 with $\psi(t) = t$ and $q = 1$. $\square$

Example 8.3. Let (Q, δ) be a complete metric space and suppose that $T : Q \rightarrow Q$ satisfies

$$\alpha(x, y)\delta(Tx, Ty) \leq \frac{q\delta(x, y)}{1 + \delta(x, y)} \quad (x, y \in Q),$$

where $0 < q < 1$ and

$$\alpha(x, y) = \frac{2}{1 + \delta(x, y)}.$$

For $r > 0$, we have

$$m_\alpha(r) \geq \frac{2}{1+r}.$$

Hence the function in Proposition 8.2 satisfies

$$\eta(r) \leq \frac{q}{2}r \quad (r \geq 0).$$

Therefore

$$\delta(Tx, Ty) \leq \frac{q}{2}\delta(x, y) \quad (x, y \in Q).$$

Thus T is a Banach contraction in (Q, δ) .

Example 8.4. Let

$$Q = \{u_n : n \geq 0\}$$

and define

$$\delta(u_m, u_n) = \begin{cases} 0, & m = n, \\ m + n, & m \neq n. \end{cases}$$

Then (Q, δ) is a complete metric space. Define $T : Q \rightarrow Q$ by

$$Tu_0 = u_0, \quad Tu_n = u_{n-1} \quad (n \geq 1).$$

We first note that T is not a Banach contraction on (Q, δ) . Indeed, for $n \geq 2$,

$$\frac{\delta(Tu_n, Tu_0)}{\delta(u_n, u_0)} = \frac{\delta(u_{n-1}, u_0)}{\delta(u_n, u_0)} = \frac{n-1}{n} \rightarrow 1.$$

Hence there is no $\lambda < 1$ such that

$$\delta(Tx, Ty) \leq \lambda\delta(x, y) \quad (x, y \in Q).$$

Now define

$$\alpha(x, y) = \frac{1}{1 + 2\delta(x, y)} \quad (x, y \in Q)$$

and

$$\phi(r) = \frac{\max\{r-1, 0\}}{1+2r} \quad (r \geq 0).$$

We claim that

$$\alpha(x, y)\delta(Tx, Ty) \leq \phi(\delta(x, y)) \quad (x, y \in Q).$$

The case $x = y$ is clear. Let $x = u_m$, $y = u_n$, and suppose that $m \neq n$. Put

$$r = \delta(u_m, u_n).$$

Then $r \geq 1$. From the definition of T and δ , we have

$$\delta(Tu_m, Tu_n) \leq r - 1.$$

Indeed, if one of m, n is zero, then the left-hand side is at most $r - 1$. If $m, n \geq 1$, then

$$\delta(Tu_m, Tu_n) = \delta(u_{m-1}, u_{n-1}) = m + n - 2 = r - 2 \leq r - 1.$$

Therefore

$$\alpha(u_m, u_n)\delta(Tu_m, Tu_n) \leq \frac{r-1}{1+2r} = \phi(r).$$

For $r > 0$, the function m_α from Definition 8.1 is given by

$$m_\alpha(r) = \begin{cases} +\infty, & 0 < r < 1, \\ \frac{1}{1+2[r]}, & r \geq 1. \end{cases}$$

In particular,

$$m_\alpha(r) \geq \frac{1}{1+2r} \quad (r > 0).$$

Thus the function in Proposition 8.2 satisfies

$$\eta(r) = \sup_{0 < s \leq r} \frac{\phi(s)}{m_\alpha(s)} \leq \sup_{0 < s \leq r} \max\{s-1, 0\} = \max\{r-1, 0\}.$$

Set

$$\chi(r) = \max\{r-1, 0\}.$$

Then, by induction,

$$\chi^n(t) = \max\{t-n, 0\} \quad (t \geq 0, n \geq 0),$$

and hence

$$\sum_{n=0}^{\infty} \chi^n(t) < \infty \quad (t > 0).$$

Thus χ is a comparison function of order 1. Since $0 \leq \eta \leq \chi$, in particular,

$$\eta(t) \leq \chi(t) < t \quad (t > 0),$$

and since both η and χ are nondecreasing, it follows by induction that

$$\eta^n(t) \leq \chi^n(t) \quad (t \geq 0, n \geq 0).$$

Therefore η is also a comparison function of order 1.

By Proposition 8.2, T has a unique fixed point, and every Picard sequence converges to it. The fixed point is u_0 .

The mapping T is not a Banach contraction in (Q, δ) , since

$$\frac{\delta(Tu_n, Tu_0)}{\delta(u_n, u_0)} \rightarrow 1.$$

The example instead leads to the nonlinear comparison function

$$\chi(r) = \max\{r-1, 0\}.$$

Thus Proposition 8.2 does not yield a Banach contraction in the original metric, although (8.1), written with ϕ in place of the original symbol ψ , is equivalent in this example to an ordinary nonlinear comparison estimate.

Remark 8.5. The condition $m_\alpha(r) > 0$ is global: For each $r > 0$, it gives a uniform positive lower bound for $\alpha(x, y)$ over all pairs satisfying $0 < \delta(x, y) \leq r$. This is different from the orbitwise condition $\alpha(x_n, x_{n+1}) \geq 1$ commonly used for Samet–Vetro–Vetro α – ψ contractions. The orbitwise condition may be sufficient to continue a Picard iteration along admissible pairs, but it does not provide a lower bound for arbitrary pairs. Consequently, it does not by itself allow (8.2) to be converted into a global comparison estimate. If α can be arbitrarily small on pairs with comparable distances, (8.2) gives no uniform control of $\delta(Tx, Ty)$ in terms of $\delta(x, y)$ alone.

Summary.

The preceding observations summarize the comparison. Linear gauge contractions have the root-chain Banach representation of Corollary 5.5. Nonlinear comparison contractions need not possess a uniform linear constant. Every hyperbolic-sine distance is algebraically a gauge transform, and the regular subclass belongs to $\mathcal{G}(q)$. Taylor–Lagrange distances are included only when they admit a fixed metric-and-gauge representation. Finally, a Samet–Vetro–Vetro α – ψ contraction, written here with ϕ in place of their comparison function ψ , yields a global comparison contraction only under the uniform positive lower control on the admissibility factor used in Proposition 8.2.

9. Nonexpansive mappings for gauge transforms

The fixed point theory of nonexpansive mappings is different from the contraction theory used earlier in the paper. In particular, completeness alone does not imply the existence of a fixed point. We therefore treat nonexpansive mappings independently. The basic idea is the normal-structure method of Takahashi [16], but the geometry below is imposed directly on the gauge transform rather than on the original metric.

Throughout this section, (Q, δ) is a metric space, $\psi \in \mathcal{G}$ is continuous on $[0, \infty)$, and

$$D_\psi(x, y) = \psi(\delta(x, y)).$$

The function D_ψ is symmetric and separates points, but it is not assumed to satisfy the triangle inequality. Thus D_ψ is a semimetric in the usual sense. Fixed point results for nonexpansive mappings have already been studied in abstract convexity, nonmetric, and semimetric settings; see, for example, Kirk [30], Kirk and Kang [31], Espínola [32], and Gabeleh [33]. The purpose of this section is not to claim a new abstract semimetric fixed point principle. Rather, we use the specific scalar representation

$$D_\psi = \psi \circ \delta$$

to identify a gauge convexity that remains meaningful for noninjective gauges, to explain why Takahashi’s weighted convexity is too strong in that case, and to obtain radius-reduction, family, and semigroup consequences directly in this gauge setting.

9.1. Gauge convexity, centers, and normal structure

This subsection introduces the ball-convexity structure used in the nonexpansive part. It then defines gauge radii, centers, ψ -property (C), and ψ -normal structure.

Definition 9.1. A mapping

$$W : Q \times Q \times [0, 1] \longrightarrow Q$$

is called a *convex structure for the gauge transform* D_ψ if

$$W(x, y, 0) = y, \quad W(x, y, 1) = x.$$

For every $x, y \in Q$, the map

$$[0, 1] \ni \lambda \longmapsto W(x, y, \lambda) \in (Q, \delta)$$

is continuous, and

$$D_\psi(u, W(x, y, \lambda)) \leq \max\{D_\psi(u, x), D_\psi(u, y)\} \quad (9.1)$$

for all $u, x, y \in Q$ and $\lambda \in [0, 1]$. A subset $C \subset Q$ is called *W-convex* if

$$W(x, y, \lambda) \in C \quad (x, y \in C, 0 \leq \lambda \leq 1).$$

Condition (9.1) says exactly that every closed D_ψ -ball is *W-convex*. The continuity requirement prevents the interpolation map from being purely set-theoretic and is satisfied by the standard convex structures used in the examples below. The ball-convexity condition is weaker than Takahashi's weighted convexity inequality when D_ψ is a metric. The next observation explains why we do not impose the weighted inequality on D_ψ .

Proposition 9.2. Suppose that $\psi \in \mathcal{G}$ and that a mapping $W : Q \times Q \times [0, 1] \rightarrow Q$ satisfies

$$D_\psi(u, W(x, y, \lambda)) \leq \lambda D_\psi(u, x) + (1 - \lambda) D_\psi(u, y) \quad (9.2)$$

for all $u, x, y \in Q$ and $\lambda \in [0, 1]$. Then ψ is strictly increasing on the set of values taken by δ . Consequently, every self-mapping $T : Q \rightarrow Q$ satisfying

$$D_\psi(Tx, Ty) \leq D_\psi(x, y) \quad (x, y \in C)$$

is nonexpansive in the original metric δ .

Proof. The case $r = 0 < s$ follows from $\psi(0) = 0 < \psi(s)$. Let $0 < r < s$ be two values taken by δ and suppose, to the contrary, that $\psi(r) = \psi(s)$. Since ψ is nondecreasing, it is constant on $[r, s]$. Choose $x, y \in Q$ with $\delta(x, y) = s$, choose $\lambda_n \in (0, 1)$ such that $\lambda_n \downarrow 0$, and put

$$w_n = W(x, y, \lambda_n).$$

Taking $u = y$ in (9.2) gives

$$D_\psi(y, w_n) \leq \lambda_n D_\psi(y, x) \rightarrow 0.$$

By Theorem 3.3, $\delta(y, w_n) \rightarrow 0$. Hence

$$\delta(x, w_n) \rightarrow \delta(x, y) = s.$$

For all sufficiently large n , $\delta(x, w_n) > r$. If $\delta(x, w_n) \leq s$, then $r < \delta(x, w_n) \leq s$, hence $D_\psi(x, w_n) = \psi(s)$; if $\delta(x, w_n) > s$, monotonicity gives $D_\psi(x, w_n) \geq \psi(s)$. On the other hand, taking $u = x$ in (9.2) yields

$$D_\psi(x, w_n) \leq (1 - \lambda_n) D_\psi(x, y) < \psi(s),$$

a contradiction. Thus $\psi(r) < \psi(s)$ whenever $0 < r < s$ are both values taken by δ .

Now let T be ψ -nonexpansive. If $\delta(Tx, Ty) > \delta(x, y)$ for some x, y , both numbers are values taken by δ , so the first part gives

$$D_\psi(Tx, Ty) > D_\psi(x, y),$$

contrary to ψ -nonexpansiveness. Hence T is nonexpansive in δ . $\square$

Thus a direct weighted replacement of Takahashi's inequality would remove the intended distinction from ordinary nonexpansiveness. The ball-convexity condition (9.1), however, does not imply ordinary nonexpansiveness.

Example 9.3. Let $Q = [0, 4]$ and

$$\delta(x, y) = \sqrt{|x - y|}.$$

Then (Q, δ) is a metric space, but it is not a convex metric space in the sense of Takahashi. Indeed, for three distinct collinear points, the triangle inequality for the snowflake metric is strict, whereas Takahashi's Proposition 3 [16] forces equality along the points generated by a convex structure.

Define

$$\psi(t) = \begin{cases} e^{t^2} - 1, & 0 \leq t \leq 1, \\ e - 1, & 1 \leq t \leq 2, \\ e - 1 + (t - 2), & t \geq 2, \end{cases}$$

and let

$$W(x, y, \lambda) = \lambda x + (1 - \lambda)y.$$

Since

$$u - W(x, y, \lambda) = \lambda(u - x) + (1 - \lambda)(u - y),$$

we have

$$\begin{aligned} |u - W(x, y, \lambda)| &\leq \lambda|u - x| + (1 - \lambda)|u - y| \\ &\leq \max\{|u - x|, |u - y|\}. \end{aligned}$$

Consequently,

$$\delta(u, W(x, y, \lambda)) \leq \max\{\delta(u, x), \delta(u, y)\}.$$

Since ψ is nondecreasing,

$$\begin{aligned} D_\psi(u, W(x, y, \lambda)) &\leq \psi(\max\{\delta(u, x), \delta(u, y)\}) \\ &= \max\{D_\psi(u, x), D_\psi(u, y)\}. \end{aligned}$$

Thus condition (9.1) holds.

Notice that the stronger weighted inequality (9.2) does not hold in this example. Indeed, take

$$u = y = 0, \quad x = 4, \quad \lambda = \frac{1}{2}.$$

Then $W(x, y, \lambda) = 2$, and

$$D_\psi(0, 2) = D_\psi(0, 4) = e - 1, \quad D_\psi(0, 0) = 0.$$

Hence

$$D_\psi(0, 2) = e - 1 > \frac{1}{2}(e - 1) = \frac{1}{2}D_\psi(0, 4) + \frac{1}{2}D_\psi(0, 0).$$

Therefore the ball-convexity condition is strictly weaker than Takahashi's weighted convexity condition.

Moreover, D_ψ is not a metric. For

$$0, \quad \frac{1}{2}, \quad 1,$$

we obtain

$$D_\psi(0, 1) = e - 1 > 2(e^{1/2} - 1) = D_\psi\left(0, \frac{1}{2}\right) + D_\psi\left(\frac{1}{2}, 1\right).$$

The compact assumptions used later are also nonempty in this example. Restrict the preceding structure to

$$K = \left[0, \frac{3}{2}\right].$$

Let $M \subset K$ be nonempty and compact, put $a = \min M$, $b = \max M$, and $L = b - a$, and take $u = (a + b)/2$. Then $u \in \overline{\text{co}}_W(M)$. If $L > 0$, monotonicity of ψ gives

$$\Delta_\psi(M) = \psi(\sqrt{L}), \quad \sup_{x \in M} D_\psi(x, u) \leq \psi\left(\sqrt{L/2}\right).$$

If $0 < L \leq 1$, the right-hand side is $e^{L/2} - 1$, which is strictly smaller than $e^L - 1 = \Delta_\psi(M)$. If $1 < L \leq 3/2$, then $\Delta_\psi(M) = e - 1$, whereas $\sqrt{L/2} < 1$ and hence

$$\psi\left(\sqrt{L/2}\right) = e^{L/2} - 1 < e - 1.$$

Thus K has the ψ -radius-reduction property introduced below, even though ψ is not injective on the set of values taken by δ and D_ψ is not a metric.

The preceding example separates the present setting from Takahashi's metric convexity. It also shows that the gauge need not be injective. In a general metric space, this matters: If ψ has a flat interval, the condition

$$D_\psi(Tx, Ty) \leq D_\psi(x, y)$$

need not imply $\delta(Tx, Ty) \leq \delta(x, y)$. We do not claim that this strict separation occurs in every gauge-convex space; the fixed point results below are stated directly in the gauge geometry.

For a nonempty set $E \subset Q$, define

$$R_x^\psi(E) = \sup_{y \in E} D_\psi(x, y), \quad R_\psi(E) = \inf_{x \in E} R_x^\psi(E),$$

and

$$\Delta_\psi(E) = \sup\{D_\psi(x, y) : x, y \in E\}.$$

These quantities are understood in the extended-real sense; all later uses of them are for ψ -bounded sets. The ψ -center of E is

$$E_c^\psi = \{x \in E : R_x^\psi(E) = R_\psi(E)\}.$$

A point $x \in E$ is called ψ -diametral if

$$R_x^\psi(E) = \Delta_\psi(E).$$

We call E ψ -bounded if $\Delta_\psi(E) < \infty$.

Definition 9.4. We say that (Q, δ, W, ψ) has ψ -property (C) if every decreasing net of nonempty δ -closed, W -convex, ψ -bounded subsets of Q has a nonempty intersection.

Definition 9.5. A W -convex set C is said to have ψ -normal structure if every δ -closed, W -convex, ψ -bounded subset $E \subset C$ containing more than one point has a point which is not ψ -diametral. Equivalently,

$$R_\psi(E) < \Delta_\psi(E)$$

for every such E .

Proposition 9.6. Assume that W is a convex structure for D_ψ .

1) For every $x \in Q$ and $r \geq 0$, the set

$$B_\psi(x, r) = \{y \in Q : D_\psi(x, y) \leq r\}$$

is δ -closed and W -convex.

2) If (Q, δ, W, ψ) has ψ -property (C), then the ψ -center E_c^ψ of every nonempty δ -closed, W -convex, ψ -bounded set E is nonempty, δ -closed, and W -convex.

Proof. The closedness of $B_\psi(x, r)$ follows from the continuity of ψ . If $y, z \in B_\psi(x, r)$, then (9.1) gives

$$D_\psi(x, W(y, z, \lambda)) \leq \max\{D_\psi(x, y), D_\psi(x, z)\} \leq r.$$

Hence the ball is W -convex.

For the second assertion, for $n \geq 1$, put

$$C_n = \bigcap_{x \in E} \{y \in E : D_\psi(x, y) \leq R_\psi(E) + 1/n\}.$$

For each n , there exists $y_n \in E$ such that

$$R_{y_n}^\psi(E) < R_\psi(E) + 1/n.$$

By symmetry of D_ψ , this gives $y_n \in C_n$. Hence every C_n is nonempty. By the first part, each C_n is δ -closed and W -convex; it is also ψ -bounded because $C_n \subset E$. The sequence (C_n) is decreasing. Therefore ψ -property (C) gives

$$\bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$

This intersection is exactly E_c^ψ . □

9.2. A normal-structure fixed point theorem

We first define nonexpansiveness with respect to the gauge transform. The theorem that follows applies the preceding center construction and ψ -normal structure.

Definition 9.7. A mapping $T : C \rightarrow C$ is called ψ -nonexpansive if

$$D_\psi(Tx, Ty) \leq D_\psi(x, y) \quad (x, y \in C). \quad (9.3)$$

Theorem 9.8. Suppose that W is a convex structure for D_ψ and that (Q, δ, W, ψ) has ψ -property (C). Let $C \subset Q$ be nonempty, δ -closed, W -convex, and ψ -bounded, and suppose that C has ψ -normal structure. Then every ψ -nonexpansive mapping $T : C \rightarrow C$ has a fixed point.

Proof. Let $\mathcal{A}$ be the family of all nonempty δ -closed, W -convex subsets E of C such that $T(E) \subset E$. A decreasing chain in $\mathcal{A}$ has a nonempty intersection by ψ -property (C), and the intersection again belongs to $\mathcal{A}$. Hence Zorn's lemma gives a minimal element E .

Suppose that E contains more than one point. By Proposition 9.6, E_c^ψ is nonempty, δ -closed, and W -convex. Let $x \in E_c^\psi$. For every $y \in E$,

$$D_\psi(Tx, Ty) \leq D_\psi(x, y) \leq R_\psi(E),$$

so

$$T(E) \subset B_\psi(Tx, R_\psi(E)).$$

If $y \in E \cap B_\psi(Tx, R_\psi(E))$, then, since $x \in E_c^\psi$,

$$D_\psi(Tx, Ty) \leq D_\psi(x, y) \leq R_\psi(E).$$

Thus

$$T(E \cap B_\psi(Tx, R_\psi(E))) \subset E \cap B_\psi(Tx, R_\psi(E)).$$

Minimality of E yields

$$E \subset B_\psi(Tx, R_\psi(E)).$$

Consequently,

$$R_{Tx}^\psi(E) \leq R_\psi(E).$$

Since $Tx \in E$, the reverse inequality follows from the definition of $R_\psi(E)$. Hence $Tx \in E_c^\psi$, and therefore

$$T(E_c^\psi) \subset E_c^\psi.$$

By ψ -normal structure,

$$R_\psi(E) < \Delta_\psi(E).$$

For $z, w \in E_c^\psi$,

$$D_\psi(z, w) \leq R_z^\psi(E) = R_\psi(E),$$

and hence

$$\Delta_\psi(E_c^\psi) \leq R_\psi(E) < \Delta_\psi(E).$$

Thus E_c^ψ is a proper member of $\mathcal{A}$, contrary to the minimality of E . Therefore E is a singleton, and its point is fixed by T . $\square$

Remark 9.9. When $\psi(t) = t$, Takahashi's weighted convexity inequality implies the ball-convexity condition (9.1). Thus Theorem 9.8 is a gauge analogue of Takahashi's normal-structure theorem [16]; for convex structures which also satisfy the continuity requirement in Definition 9.1, it gives the same fixed point conclusion.

More abstract normal-structure fixed point principles are also available. Kirk [30] formulated an abstract fixed point theorem for nonexpansive mappings, Kirk and Kang [31] gave a nonmetric form

of Kirk's theorem, Espínola [32] used a semimetric version of Kirk's theorem in the study of relatively nonexpansive mappings, and Gabeleh [33] used seminormal structure in generalized semimetric spaces. These results are relevant because D_ψ is itself a semimetric. The point of the present formulation is more specific: The semimetric has the scalar form $D_\psi = \psi \circ \delta$, and this allows us to compare ball convexity with Takahashi's weighted convexity, to admit noninjective gauges such as Example 9.3, and to formulate the radius-reduction and semigroup results in terms of the same gauge used in the earlier sections.

Similar convexity and normal-structure ideas have also been developed in fuzzy metric settings [17, 18].

9.3. A compact radius-reduction condition

For the common fixed point results below, we need the compact radius reduction, which plays the role of Proposition 5 in Takahashi's paper. Rather than deriving it from a weighted convexity inequality, we state the exact gauge property, which is used in the proof.

Definition 9.10. A compact W -convex set K is said to have the ψ -radius-reduction property if, for every nonempty compact set $M \subset K$ with $\Delta_\psi(M) > 0$, there is a point

$$u \in \overline{\text{co}}_W(M)$$

such that

$$\sup_{x \in M} D_\psi(x, u) < \Delta_\psi(M), \quad (9.4)$$

where $\overline{\text{co}}_W(M)$ denotes the least δ -closed, W -convex subset of K containing M .

The next condition is a convenient sufficient condition for Definition 9.10.

Definition 9.11. A convex structure W for D_ψ is called *strict* if, for $x \neq y$, $u \in Q$, and $0 < \lambda < 1$,

$$D_\psi(u, W(x, y, \lambda)) < \max\{D_\psi(u, x), D_\psi(u, y)\}. \quad (9.5)$$

Proposition 9.12. Let K be a compact W -convex subset of Q . If W is a strict convex structure for D_ψ , then K has the ψ -radius-reduction property. In particular, K has ψ -normal structure.

Proof. Let $M \subset K$ be nonempty and compact with $\Delta_\psi(M) > 0$. Choose distinct $x, y \in M$ and put

$$u = W(x, y, 1/2).$$

For every $z \in M$, strict gauge convexity gives

$$D_\psi(z, u) < \max\{D_\psi(z, x), D_\psi(z, y)\} \leq \Delta_\psi(M).$$

The function $z \mapsto D_\psi(z, u)$ is continuous on the compact set M , so its maximum is attained and is strictly smaller than $\Delta_\psi(M)$. This proves (9.4).

If $E \subset K$ is non-singleton, δ -closed, and W -convex, then E is compact. Applying the radius-reduction property with $M = E$ gives a point $u \in E$ with

$$R_u^\psi(E) < \Delta_\psi(E).$$

Hence K has ψ -normal structure. □

Corollary 9.13. *Let K be a nonempty compact W -convex subset of Q with the ψ -radius-reduction property. If W is a convex structure for D_ψ , then every ψ -nonexpansive mapping $T : K \rightarrow K$ has a fixed point.*

Proof. With the restricted metric and convex structure, compactness of K gives ψ -property (C) on K , and the radius-reduction property gives ψ -normal structure. Apply Theorem 9.8. $\square$

9.4. Families with the invariant property and amenable semigroups

Following Takahashi [16], we now use an invariant-set condition. No contractive member of the family is assumed.

Definition 9.14. Let K be compact and W -convex, and let $\mathcal{F}$ be a family of ψ -nonexpansive self-mappings of K . We say that $\mathcal{F}$ has the *invariant property in K* if, whenever $E \subset K$ is nonempty, compact, and W -convex and

$$T(E) \subset E \quad (T \in \mathcal{F}),$$

there is a nonempty compact set $M \subset E$ such that

$$T(M) = M \quad (T \in \mathcal{F}).$$

The following argument is the gauge version of Takahashi's invariant-property theorem [16].

Theorem 9.15. *Let K be a nonempty compact W -convex subset of Q with the ψ -radius-reduction property, and suppose that W is a convex structure for D_ψ . If a family $\mathcal{F}$ of ψ -nonexpansive self-mappings of K has the invariant property in K , then*

$$\bigcap_{T \in \mathcal{F}} \text{Fix}(T) \neq \emptyset.$$

Proof. By compactness and Zorn's lemma, there is a minimal nonempty compact W -convex set $E \subset K$ such that

$$T(E) \subset E \quad (T \in \mathcal{F}).$$

By the invariant property, there is a nonempty compact set $M \subset E$ such that

$$T(M) = M \quad (T \in \mathcal{F}).$$

If M is a singleton, its point is a common fixed point.

Assume that M contains more than one point. By the ψ -radius-reduction property, there are $u \in \overline{\text{co}}_W(M) \subset E$ and

$$\rho = \sup_{x \in M} D_\psi(x, u) < \Delta_\psi(M).$$

Define

$$E_0 = E \cap \bigcap_{x \in M} B_\psi(x, \rho).$$

Since $D_\psi(x, u) \leq \rho$ for every $x \in M$, we have $u \in E_0$. Thus E_0 is nonempty, compact, and W -convex. It is a proper subset of E ; otherwise $M \subset E_0$, and all mutual D_ψ -distances in M would be at most ρ , contrary to $\rho < \Delta_\psi(M)$.

Let $T \in \mathcal{F}$, $y \in E_0$, and $x \in M$. Since $T(M) = M$, there is $z \in M$ such that $Tz = x$. Therefore

$$D_\psi(x, Ty) = D_\psi(Tz, Ty) \leq D_\psi(z, y) \leq \rho.$$

Thus $T(E_0) \subset E_0$ for every $T \in \mathcal{F}$, contradicting the minimality of E . Hence M is a singleton. $\square$

We use left amenability in the standard sense of Day [34]. Here the semigroup product is composition,

$$(ST)x = S(Tx),$$

and left amenability means that $\ell^\infty(\mathcal{S})$ admits a left invariant mean m , that is,

$$m(\ell_S g) = m(g), \quad (\ell_S g)(T) = g(ST),$$

for all $S \in \mathcal{S}$ and $g \in \ell^\infty(\mathcal{S})$. Takahashi studied amenable semigroups of nonexpansive mappings in [35]; the invariant-property formulation used here follows his later paper [16]. The argument below is the same invariant-mean argument; in the present setting, the gauge condition is used to obtain continuity of the mappings.

Theorem 9.16. *Under the standing assumptions of this section, let K be a nonempty compact W -convex subset of Q , and suppose that W is a convex structure for D_ψ . Let $\mathcal{S}$ be a left amenable semigroup of ψ -nonexpansive self-mappings of K . Then $\mathcal{S}$ has the invariant property in K .*

Proof. Every ψ -nonexpansive mapping is continuous with respect to δ . Indeed, if $x_n \rightarrow x$ in (Q, δ) , then

$$D_\psi(Tx_n, Tx) \leq D_\psi(x_n, x) \rightarrow 0.$$

By Theorem 3.3, it follows that $Tx_n \rightarrow Tx$ in (Q, δ) .

Let $E \subset K$ be nonempty, compact, and W -convex, and suppose that

$$T(E) \subset E \quad (T \in \mathcal{S}).$$

Consider the family of all nonempty compact subsets A of E such that

$$T(A) \subset A \quad (T \in \mathcal{S}),$$

ordered by inclusion. The intersection of every decreasing chain in this family is nonempty by compactness, and it is again compact and invariant under every member of $\mathcal{S}$. Hence Zorn's lemma gives a minimal nonempty compact set $M \subset E$ satisfying

$$T(M) \subset M \quad (T \in \mathcal{S}).$$

Let $C(M)$ denote the Banach space of all continuous real-valued functions on M , endowed with the supremum norm

$$\|f\|_\infty = \sup_{x \in M} |f(x)|.$$

Since $\mathcal{S}$ is left amenable, let m be a left invariant mean on $\ell^\infty(\mathcal{S})$. Fix $x_0 \in M$. For each $f \in C(M)$, define

$$g_f : \mathcal{S} \rightarrow \mathbb{R}, \quad g_f(T) = f(Tx_0).$$

Since M is compact and f is continuous, f is bounded on M . Therefore $g_f \in \ell^\infty(\mathcal{S})$. Define

$$L(f) = m(g_f).$$

Then L is linear and positive. If $\mathbf{1}$ denotes the constant function equal to 1 on M , then

$$g_1(T) = 1 \quad (T \in \mathcal{S}),$$

and hence

$$L(\mathbf{1}) = m(\mathbf{1}) = 1.$$

Thus L is a positive norm-one functional on $C(M)$.

For $S \in \mathcal{S}$, define

$$(U_S f)(x) = f(Sx), \quad x \in M.$$

This is well defined because $S(M) \subset M$ and S is continuous on M . For every $T \in \mathcal{S}$,

$$\begin{aligned} g_{U_S f}(T) &= (U_S f)(Tx_0) \\ &= f(S(Tx_0)) \\ &= f((ST)x_0) \\ &= g_f(ST). \end{aligned}$$

Hence

$$g_{U_S f} = \ell_S g_f.$$

By the left invariance of m ,

$$m(\ell_S g_f) = m(g_f),$$

and consequently

$$L(U_S f) = m(g_{U_S f}) = m(\ell_S g_f) = m(g_f) = L(f).$$

This is the usual invariant-mean argument of Day [34, 36].

By the Riesz representation theorem, there exists a regular Borel probability measure μ on M such that

$$L(f) = \int_M f d\mu \quad (f \in C(M)).$$

For every $S \in \mathcal{S}$ and $f \in C(M)$,

$$\int_M f(Sx) d\mu(x) = L(U_S f) = L(f) = \int_M f(x) d\mu(x).$$

Hence the push-forward measure $S_\# \mu$ equals μ . Equivalently,

$$\mu(S^{-1}(A)) = \mu(A)$$

for every Borel set $A \subset M$.

Let

$$P = \text{supp } \mu.$$

Then P is a nonempty compact subset of M . We show that it is invariant. Let $x \in P$, let $S \in \mathcal{S}$, and let V be an open neighborhood of Sx in M . Since S is continuous,

$$S^{-1}(V)$$

is an open neighborhood of x . Since $x \in \text{supp } \mu$,

$$\mu(S^{-1}(V)) > 0.$$

By the invariance of μ ,

$$\mu(V) = \mu(S^{-1}(V)) > 0.$$

Thus $Sx \in P$. Therefore

$$S(P) \subset P \quad (S \in \mathcal{S}).$$

By the minimality of M , we obtain

$$P = M.$$

Fix $S \in \mathcal{S}$. Since M is compact and S is continuous, $S(M)$ is compact and hence closed in M . Since S maps M into $S(M)$,

$$S^{-1}(S(M)) = M,$$

where the inverse image is taken with respect to the restriction $S : M \rightarrow M$. Therefore

$$\mu(S(M)) = \mu(S^{-1}(S(M))) = \mu(M) = 1.$$

Since $S(M)$ is a closed subset of M of full measure, it contains $\text{supp } \mu = M$. Hence

$$M \subset S(M).$$

Together with $S(M) \subset M$, this gives

$$S(M) = M.$$

Since $S \in \mathcal{S}$ is arbitrary,

$$T(M) = M \quad (T \in \mathcal{S}).$$

Thus $\mathcal{S}$ has the invariant property in K . □

Corollary 9.17. *Let K be a nonempty compact W -convex subset of Q with the ψ -radius-reduction property, and suppose that W is a convex structure for D_ψ . Then every left amenable semigroup of ψ -nonexpansive self-mappings of K has a common fixed point.*

Proof. Combine Theorems 9.16 and 9.15. □

Corollary 9.18. *Under the assumptions of Corollary 9.17, every commuting family of ψ -nonexpansive self-mappings of K has a common fixed point.*

Proof. The semigroup generated by a commuting family is commutative and hence left amenable. The composition of two ψ -nonexpansive mappings is again ψ -nonexpansive. Apply Corollary 9.17. □

Corollary 9.19. *Let K satisfy the assumptions of Corollary 9.17, and let $\{S_t : t \geq 0\}$ be a one-parameter semigroup of ψ -nonexpansive self-mappings of K :*

$$S_0 = I, \quad S_{s+t} = S_s S_t \quad (s, t \geq 0).$$

Then

$$\bigcap_{t \geq 0} \text{Fix}(S_t) \neq \emptyset.$$

Proof. The semigroup is commutative because addition on $[0, \infty)$ is commutative. Hence it is left amenable, and Corollary 9.17 applies. $\square$

9.5. Discussion and further questions

The results in this section use the gauge transform in a different way from the contraction results. No strict contraction coefficient and no chain metric are introduced. The normal-structure argument uses D_ψ -balls, ψ -radii, and ψ -diameters directly. The convexity condition (9.1) is deliberately stated in ball form. Proposition 9.2 shows that the weighted analogue would force every ψ -nonexpansive mapping to be ordinary nonexpansive in δ .

The radius-reduction property is the compact geometric condition used in the invariant-family proof. Strict gauge convexity is one simple sufficient condition, but it is not built into the main single-map theorem. It would be useful to find other verifiable conditions on (δ, ψ, W) , which imply radius reduction while allowing noninjective gauges on the realized distance set.

Further questions include gauge versions of uniform normal structure, asymptotically ψ -nonexpansive mappings, quasi- ψ -nonexpansive mappings, multivalued nonexpansive mappings, and iteration methods. For semigroups, extensions from compact domains to spaces satisfying ψ -property (C), and from left amenability to left reversibility or other invariant-mean hypotheses, also appear natural.

10. Concluding remarks

We studied transforms $\delta_\psi = \psi \circ \delta$ without assuming that the transformed distance is a metric. The assumptions were kept separate. Membership in $\mathcal{G}$ gives equivalence of convergence to zero and of Cauchy sequences. A local power lower estimate is used as a sufficient condition for summability of Picard increments. A local linear lower estimate guarantees that the associated chain distance is nondegenerate on every metric space.

Every θ -hyperbolic sine distance is exactly a gauge transform with $\psi_\theta = \theta \circ \sinh$. For the regular subclass in which $\psi_\theta \in \mathcal{G}(q)$, the linear and nonlinear fixed point results follow from the gauge theorems, and the additional sequential condition of the original formulation is not required. The Taylor–Lagrange construction is different in general and is included only when it is constant on level sets of an underlying metric.

The chain construction also gives a precise limitation of the linear theory. After the root normalization $\phi = \psi^{1/q}$, every linear $\mathcal{G}(q)$ -gauge contraction is Banach contractive in a canonical equivalent chain metric; this metric is complete whenever the original metric is complete. The same conclusion holds for the recent k -admissible theorem of Jleli and Samet after taking the root

determined by the first nonzero derivative at zero. The nonlinear comparison theorem is not covered by this uniform linear argument: The Hutchinson operator constructed above has no uniform linear constant in either the original Hausdorff metric or the nonmetric exponential gauge transform, and it is not Banach contractive in the canonical chain metric generated by that transform. The integral equation section likewise shows that the direct gauge inequality may apply when Banach's theorem does not apply in the metric originally assigned to the problem.

The global comparison result for the Samet–Vetro–Vetro α - ψ contraction, written here with ϕ in place of their comparison function ψ , identifies the stronger hypothesis under which an admissibility factor can be absorbed into an ordinary comparison function. These observations state both the scope and the limitations of the method and avoid treating a nonmetric notation as, by itself, a new metric structure.

The proofs in the nonexpansive part do not use the contraction or chain-metric results. They use gauge balls, ψ -normal structure, and a compact radius-reduction property directly for the possibly nonmetric transform D_ψ . This yields fixed and common fixed point results for ψ -nonexpansive mappings, invariant families, left amenable semigroups, commuting families, and one-parameter semigroups. The ball form of the convexity assumption allows noninjective gauges; Proposition 9.2 shows that the weighted analogue would reduce ψ -nonexpansiveness to ordinary nonexpansiveness.

Author contributions

Saud M Alsulami: Formal analysis, Investigation, Validation, Writing-review and editing; Naseer Shahzad: Conceptualization, Formal analysis, Investigation, Validation, Writing-original draft, writing-review and editing. Both authors read and approved the final version.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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