



*Research article***On a Ma–Minda type class of biconvex functions via q -analytic Balloon mappings****Abdullah Alsoboh^{1,*}, Ala Amourah^{2,3}, Elarbi Elkaroui⁴, Waggas Galib Atshan⁵ and Muhammed Salih Muhammed⁵**

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Abstract: This paper introduces a new Ma–Minda type subclass of biconvex functions defined via subordination to a symmetric balloon-shaped domain generated by an appropriate q -analytic mapping. The proposed subordination framework reflects the geometric influence of q -calculus on the images of both the function and its inverse, particularly in terms of domain contraction, boundary deformation, and symmetry behavior as the parameter q varies. These effects provide a natural geometric deformation of the classical biconvex setting. The primary objective of this work is to investigate how the q -parameter affects the convexity properties and analytic structure of the resulting function class. Explicit upper bounds for the initial Taylor–Maclaurin coefficients $|\vartheta_2|$ and $|\vartheta_3|$ are derived in closed form using q -integers. In addition, a sharp piecewise estimate for the Fekete–Szegő functional $|\vartheta_3 - \varrho \vartheta_2^2|$ is obtained, highlighting its dependence on both the parameter ϱ and the geometry of the associated balloon-shaped domain. Furthermore, the limiting case $q \rightarrow 1^-$ is examined, showing that the obtained results naturally reduce to the corresponding classical bounds for biconvex functions. To illustrate the geometric characteristics and applicability of the introduced class, several representative examples together with graphical visualizations of the corresponding image domains are also presented.

Keywords: analytic functions; balloon-shaped curves; convex class; quantum calculus; univalent functions

Mathematics Subject Classification: 30A36, 30C45, 81P68, 11B37

1. Introduction and definitions

Let f be an analytic function defined in the open unit disk

$$\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

We denote by $\mathcal{A}$ the family of all normalized analytic functions in $\mathcal{U}$ of the form

$$f(z) = z + \sum_{m=2}^{\infty} \vartheta_m z^m. \quad (1.1)$$

A distinguished subclass of $\mathcal{A}$ is the class $\mathcal{S}$, consisting of functions that are univalent (that is, one-to-one) in $\mathcal{U}$ and satisfy the standard normalization conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1. \quad (1.2)$$

An analytic function κ defined on $\mathcal{U}$ is said to be a *Schwarz function* if it vanishes at the origin and remains bounded in modulus by unity, that is,

$$\kappa(0) = 0 \quad \text{and} \quad |\kappa(z)| < 1 \quad (z \in \mathcal{U}).$$

Given two functions $f_1, f_2 \in \mathcal{A}$, we say that f_1 is *subordinate* to f_2 , written $f_1 < f_2$, if there exists a Schwarz function κ such that

$$f_1(z) = f_2(\kappa(z)), \quad z \in \mathcal{U}.$$

We further recall the Carathéodory class $\mathcal{P}$ (see, for example, Miller [1]), defined by

$$\mathcal{P} := \{\delta : \mathcal{U} \rightarrow \mathbb{C} \text{ analytic} : \Re\{\delta(z)\} > 0 \text{ for all } z \in \mathcal{U}, \delta(0) = 1\}.$$

Each function $\delta \in \mathcal{P}$ admits a Maclaurin series expansion of the form

$$\delta(z) = 1 + \sum_{m=1}^{\infty} \delta_m z^m, \quad z \in \mathcal{U}. \quad (1.3)$$

By Carathéodory's lemma (see Duren [2]), the coefficients of this expansion satisfy the sharp estimate

$$|\delta_m| \leq 2, \quad m \geq 1. \quad (1.4)$$

Equivalently, a function δ belongs to the class $\mathcal{P}$ if and only if it is subordinate to the Koebe transform

$$\delta(z) < \frac{1+z}{1-z}, \quad z \in \mathcal{U}.$$

If $f \in \mathcal{S}$, then by virtue of the Koebe quarter theorem, f admits a locally defined inverse function ψ in the disk

$$\{|\varsigma| < r_0(f)\}, \quad r_0(f) \geq \frac{1}{4}.$$

The inverse mapping ψ is characterized by the functional identities

$$z = \psi(f(z)) \quad \text{and} \quad \varsigma = f(\psi(\varsigma)),$$

and it possesses a power series representation of the form

$$\psi(\varsigma) = \varsigma - \vartheta_2 \varsigma^2 + (2\vartheta_2^2 - \vartheta_3) \varsigma^3 + (-5\vartheta_2^3 + 5\vartheta_2 \vartheta_3 - \vartheta_4) \varsigma^4 + \cdots. \quad (1.5)$$

A function $f \in \mathcal{S}$ is said to be bi-univalent if both f and its inverse function ψ are univalent in their respective domains of definition. The class of all such functions is denoted by Σ . For illustrative purposes, Table 1 presents several classical examples of univalent functions together with their corresponding formal inverses.

Table 1. Representative univalent functions in $\mathcal{U}$ and their corresponding inverse mappings.

Function $f(z), z \in \mathcal{U}$	Inverse $f^{-1}(\varsigma), \varsigma \in \mathcal{U}$	Parameter conditions
$\frac{z}{1+z}$	$\frac{\varsigma}{1-\varsigma}$	—
$\frac{z-\alpha}{1-\alpha z}$	$\frac{\varsigma+\alpha}{1+\alpha\varsigma}$	$\alpha \in (-1, 1)$
$\frac{e^{2z}-1}{e^{2z}+1}$	$\frac{1}{2} \log\left(\frac{1+\varsigma}{1-\varsigma}\right)$	—
$-\frac{1}{\beta} \log(1-\beta z)$	$\frac{1-e^{-\beta\varsigma}}{\beta}$	$0 < \beta < 1$
$\sqrt{1+z}-1$	$\varsigma^2+2\varsigma$	—

Following the classical framework established by Ma and Minda [3], we consider subclasses of univalent functions defined through the subordination condition

$$\mathcal{S}(\mathcal{B}) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \mathcal{B}(z), \quad z \in \mathcal{U} \right\}$$

and

$$\mathcal{C}(\mathcal{B}) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \mathcal{B}(z), \quad z \in \mathcal{U} \right\},$$

where the function $\mathcal{B}$ satisfies the positivity condition $\Re\{\mathcal{B}(z)\} > 0$ for $z \in \mathcal{U}$. Different choices of the dominant function $\mathcal{B}$ give rise to a variety of well-known subclasses of starlike and convex functions.

Several important examples appearing in the literature include the choice $\aleph(z) = \sqrt{1+z}$ (principal branch), introduced by Sokoł and Stankiewicz [4]; the function $\aleph(z) = z/(1-z)$ studied by Piejko and Sokoł [5]; and the function $\aleph(z) = 2/(1+e^{-z})$, considered more recently by Goel and Kumar [6].

Definition 1.1. In a closely related setting, Ahmad et al. [7] (2023) introduced a fundamental subclass of analytic functions defined through the classical derivative and a subordination condition. Let $f \in \mathcal{A}$ be of the form (1.1). We say that f belongs to the class $\mathcal{R}_{sl}$ if it satisfies

$$f'(z) < \mathcal{B}(z) := \frac{2\sqrt{1+z}}{1+e^{-z}}, \quad z \in \mathcal{U}.$$

From a geometric viewpoint, every function $f \in \mathcal{B}_{sl}$ maps the open unit disk $\mathcal{U}$ onto a balloon-shaped region that is symmetric about the real axis; see Figure 1 for an illustrative depiction of this image domain.

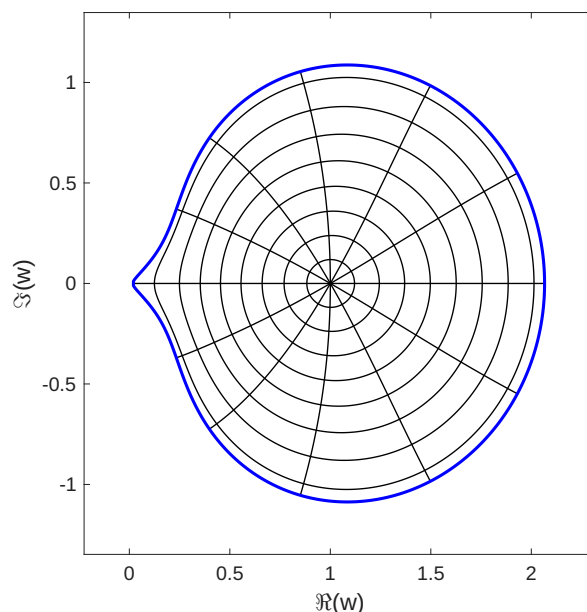


Figure 1. Image of $\mathcal{B}(\mathcal{U})$ in a balloon-shaped domain for $\mathcal{B}(z) = 2\sqrt{1+z}/(1+e^{-z})$.

Within the framework of geometric function theory, q -calculus has emerged as an effective tool for constructing and investigating new subclasses of analytic functions in the unit disk. This approach has enabled the derivation of refined coefficient bounds, the formulation of new subordination relations, and the development of q -extensions of several classical inequalities. Early contributions in this direction were made by Seoudy and Aouf [8]. In recent years, the subject has attracted considerable attention, leading to a variety of extensions and applications; see, for instance, the works of Khan et al. [9], Al-Ityan et al. [10], Alsoboh et al. [11, 12], Almalkawi et al. [13, 14], Yalcin et al. [15], and Atshan et al. [16, 17].

Building upon the well-known Ma–Minda subordination framework [3] and working in the setting of q -calculus, Uçar [18] introduced the class of q -starlike functions, denoted by S_q^* , which is defined by

$$S_q^* := \left\{ f \in \mathcal{A} : \frac{z \partial_q f(z)}{f(z)} < \frac{1+z}{1-qz}, \quad z \in \mathcal{U} \right\}, \quad 0 < q < 1.$$

Here, ∂_q denotes the q -difference operator (or q -derivative) of a function f (see [19, 20]), which is given by

$$\partial_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z - qz}, & 0 < q < 1, z \neq 0, \\ f'(0), & z = 0, \\ f'(z), & q \rightarrow 1^-, z \neq 0. \end{cases}$$

This operator may be regarded as a natural q -analog of the classical derivative and allows the quantity $zf'(z)/f(z)$ to be replaced by its q -counterpart. Consequently, the classical Janowski

function $(1+z)/(1-z)$ is deformed into its q -version $(1+z)/(1-qz)$. In the limiting case as $q \rightarrow 1^-$, the class S_q^* reduces to the usual class of starlike functions S^* .

A q -exponential-based variant. Along similar lines, Srivastava et al. [21] proposed another q -starlike subclass, denoted by $S^*(\mathcal{L}, q)$, defined through subordination to the q -exponential function as follows:

$$S^*(\mathcal{L}, q) := \left\{ f \in \mathcal{A} : \frac{z \partial_q f(z)}{f(z)} < \exp_q(z), \quad z \in \mathcal{U} \right\}, \quad 0 < q < 1.$$

The q -exponential function $\exp_q$ appearing above is given by the series

$$\exp_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad (1.6)$$

where $[n]_q$ denotes the q -integer. The concept of q -integers plays a fundamental role in q -calculus and provides a natural deformation of the classical integers depending on the parameter q . In particular, many q -analogs of classical operators and special functions are constructed using q -integers. For $\kappa \in \mathbb{C}^*$ and $0 < q < 1$, the q -integer is defined by

$$[\kappa]_q = \begin{cases} \frac{1-q^\kappa}{1-q}, & 0 < q < 1, \kappa \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \\ q^{n-1} + q^{n-2} + \cdots + q + 1 = \sum_{j=0}^{n-1} q^j, & 0 < q < 1, \kappa = n \in \mathbb{N}, \\ 1, & q \rightarrow 0^+, \kappa \in \mathbb{C}^*, \\ n, & q \rightarrow 1^-, \kappa \in \mathbb{C}^*. \end{cases}$$

Observe that the q -integer reduces to the classical integer in the limit as $q \rightarrow 1^-$, which shows that q -calculus may be regarded as a natural generalization of classical analysis.

This construction replaces the classical rational subordination kernel with the q -exponential function. In this way, the classical starlikeness condition is naturally extended to the framework of q -difference calculus, leading to a richer geometric structure for the associated analytic function classes.

Definition 1.2. *Motivated by existing q -starlike constructions together with the Ma–Minda subordination framework, we introduce a q -deformed balloon-type driving function and the corresponding class of analytic functions.*

Let $0 < q < 1$, and let $f \in \mathcal{A}$ be of the form (1.1). We say that f belongs to the class $\mathcal{R}_{sl}^q$ if

$$f \in \mathcal{R}_{sl}^q \iff \partial_q f(z) < \mathcal{B}(z; q), \quad z \in \mathcal{U},$$

where the subordinating function $\mathcal{B}(z; q)$ is defined by

$$\mathcal{B}(z; q) = \frac{2 \sqrt{1 + \frac{(1+q)z}{2 + (1-q)z}}}{1 + \exp_q(-z)}. \quad (1.7)$$

The corresponding image domain of $\mathcal{B}(z; q)$ will be denoted by

$$\Omega_q := \mathcal{B}(\mathcal{U}; q).$$

Here, $\exp_q(z)$ denotes the q -exponential function defined in (1.6).

Remark 1.1. The function $\mathcal{B}(z; q)$ defined in (1.7) is introduced as a generating function that maps the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ onto a symmetric balloon-shaped domain within the framework of q -calculus. Its construction combines two analytic components.

First, the rational expression

$$\frac{(1+q)z}{2+(1-q)z}$$

acts as a q -deformation of a Möbius-type transformation, preserving analyticity in $\mathcal{U}$ while producing a controlled geometric distortion of the unit disk. The square-root term

$$\sqrt{1 + \frac{(1+q)z}{2+(1-q)z}}$$

is introduced to generate a symmetric expansion of the image domain, leading to the characteristic balloon-shaped geometry.

Second, the denominator involving the q -exponential function

$$\exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad [n]_q = \frac{1-q^n}{1-q}$$

incorporates the q -analog structure into the mapping. This normalization ensures analyticity in $\mathcal{U}$ and guarantees a smooth transition to the classical case as $q \rightarrow 1^-$. Hence, $\mathcal{B}(z; q)$ serves as a suitable analytic generating function for defining the proposed Ma–Minda-type subclass associated with a q -deformation of balloon-shaped domains.

Geometric interpretation. For each fixed $q \in (0, 1)$, the mapping $\mathcal{B}(z; q)$ sends the unit disk $\mathcal{U}$ conformally onto a symmetric balloon-shaped region in the complex plane. Consequently, for every function $f \in \mathcal{R}_{st}^q$, the image of the q -derivative satisfies

$$\partial_q f(\mathcal{U}) \subset \Omega_q.$$

The geometric behavior of the mapping $\mathcal{B}(z; q)$ is illustrated in Figure 2. In the limiting case $q \rightarrow 1^-$, the function (1.7) reduces to

$$\mathcal{B}(z; 1) = \frac{2\sqrt{1+z}}{1+e^{-z}},$$

which coincides with the classical balloon-shaped driving function.

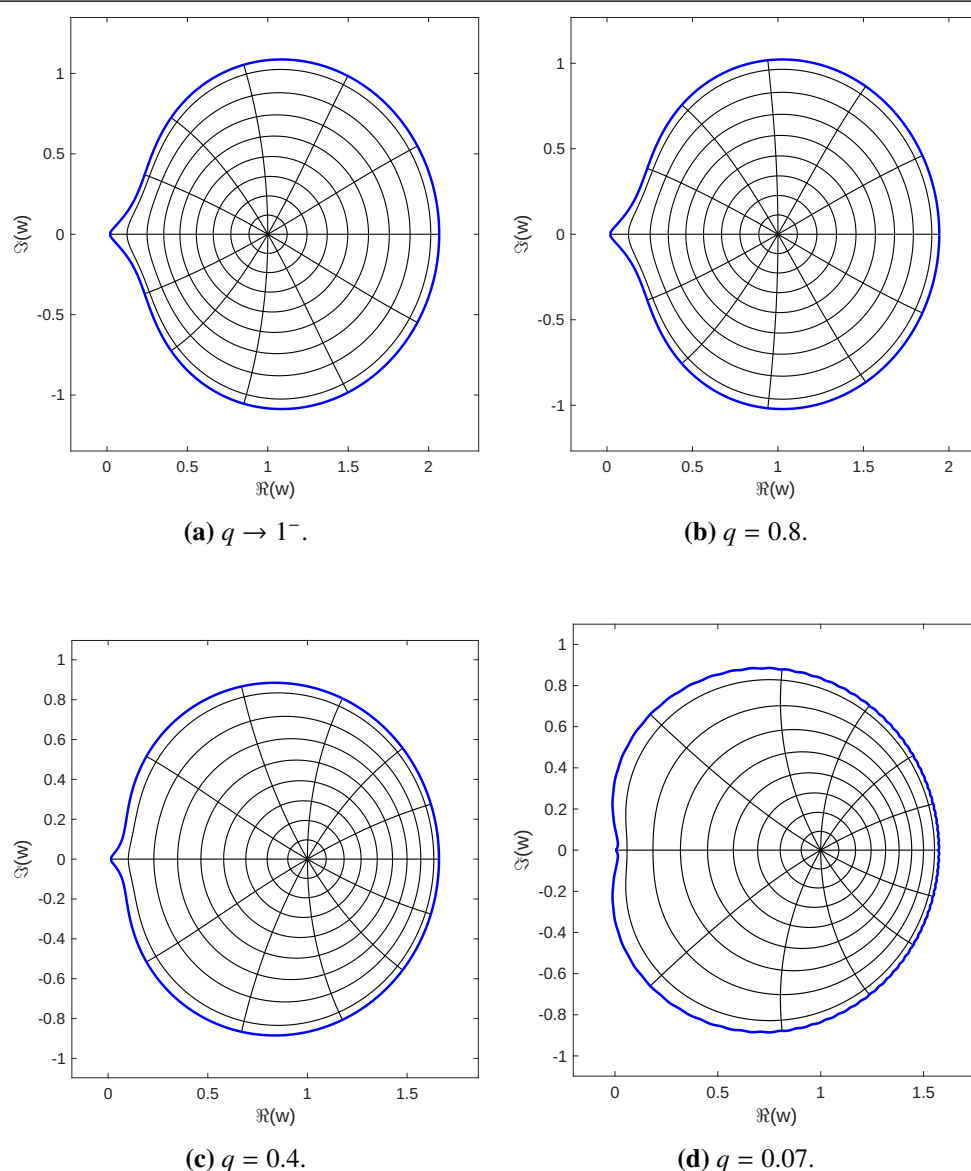


Figure 2. The balloon-shaped image Ω_q corresponding to $\mathcal{B}(z; q)$.

Remark 1.2. The analytic function $\mathcal{B}(z; q)$ of the form (1.7), where the principal branch of the square root is assumed, maps the unit disk $\mathcal{U}$ conformally onto a balloon-shaped domain that is symmetric with respect to the real axis. Moreover, the mapping satisfies the normalization conditions

$$\mathcal{B}(0; q) = 1 \quad \text{and} \quad \partial_q \mathcal{B}(0; q) = 1.$$

In addition, the function satisfies

$$\Re\{\mathcal{B}(z; q)\} > 0, \quad z \in \mathcal{U},$$

so that $\mathcal{B}(z; q)$ belongs to the Carathéodory class. Consequently, $\mathcal{B}(z; q)$ serves as an admissible dominant function in the sense of Ma–Minda subordination theory.

The principal objective of the present work is to explore the analytic and geometric properties of bi-univalent functions associated with balloon-shaped image domains in the unit disk $\mathcal{U}$ within the framework of q -calculus. In this section, we introduce the necessary preliminaries and provide a precise formulation of the function class under consideration. In the subsequent section, we derive coefficient estimates for the initial Taylor–Maclaurin coefficients and establish a Fekete–Szegő-type inequality for functions belonging to the class $\Sigma_q \mathcal{C}_{sl}$. Several corollaries are also obtained as direct consequences of the main results, thereby illustrating the connections of our findings with previously studied subclasses of bi-univalent functions.

In recent years, considerable attention has been devoted to coefficient problems and Hankel determinant estimates for subclasses of analytic and univalent functions defined within the Ma–Minda framework and related geometric settings. For instance, Sun and Wang [22] investigated the second Hankel determinant associated with logarithmic coefficients for Ma–Minda-type functions. Subsequently, Wang et al. [23] introduced a subclass of multivalent starlike functions with a prescribed second coefficient and derived several coefficient bounds. In another contribution, Wang et al. [24] obtained sharp estimates for Hankel determinants of bounded turning functions involving the hyperbolic tangent function. These studies highlight the continuing importance of Hankel determinant problems and coefficient inequalities in geometric function theory, and they motivate the development of new subclasses of analytic functions associated with various generating functions and operators.

2. The biconvex class $\Sigma_q \mathcal{C}_{sl}$

We proceed to introduce a q -analog of the biconvex class of univalent functions, governed by the balloon-shaped driving function $\mathcal{B}(z; q)$ defined in (1.7). Throughout this section, we assume that $0 < q < 1$, denote by $\mathcal{U}$ the open unit disk, and consider functions $f \in \mathcal{A}$ of the normalized form given in (1.1).

Definition 2.1. A function f is said to belong to the class $\Sigma_q \mathcal{C}_{sl}$ if both f and its inverse function ψ satisfy the q -convexity-type subordination conditions

$$\begin{aligned}\Phi(z; q) &:= 1 + \frac{z \partial_q^2 f(z)}{\partial_q f(z)} < \mathcal{B}(z; q), & z \in \mathcal{U}, \\ \chi(\varsigma; q) &:= 1 + \frac{\varsigma \partial_q^2 \psi(\varsigma)}{\partial_q \psi(\varsigma)} < \mathcal{B}(\varsigma; q), & \varsigma \in \mathcal{U},\end{aligned}\tag{2.1}$$

where $\mathcal{B}(\cdot; q)$ denotes the balloon-shaped subordinating function, and the symbol “ $<$ ” represents subordination in $\mathcal{U}$.

Equivalently, $f \in \Sigma_q \mathcal{C}_{sl}$ if and only if there exist Schwarz functions $\omega, \eta : \mathcal{U} \rightarrow \mathcal{U}$ with $\omega(0) = \eta(0) = 0$ such that

$$1 + \frac{z \partial_q^2 f(z)}{\partial_q f(z)} = \mathcal{B}(\omega(z); q)$$

and

$$1 + \frac{\varsigma \partial_q^2 \psi(\varsigma)}{\partial_q \psi(\varsigma)} = \mathcal{B}(\eta(\varsigma); q).$$

The class $\Sigma_q \mathcal{C}_{sl}$ is nontrivial and admits natural analytic representatives. A particularly instructive example is provided by the Möbius transformation

$$f_*(z) = \frac{z}{1 - \tau z}, \quad |\tau| < \frac{1}{4}, \quad (2.2)$$

which belongs to the classical Schlicht class $\mathcal{S}$. The restriction $|\tau| < 1/4$ is consistent with the classical Koebe quarter theorem (see [2]), according to which every normalized univalent function maps the unit disk $\mathcal{U}$ onto a domain containing a disk of radius $1/4$. The corresponding inverse mapping is given by

$$f_*^{-1}(\varsigma) = \frac{\varsigma}{1 + \tau \varsigma}, \quad |\varsigma| < 1, \quad |\tau| < \frac{1}{4}. \quad (2.3)$$

A direct computation yields

$$\Phi(f_*(z); q) = 1 + \frac{(1+q)\tau z}{1 - \tau q^2 z}, \quad \chi(f_*^{-1}(\varsigma); q) = 1 - \frac{(1+q)\tau \varsigma}{1 + \tau q^2 \varsigma}. \quad (2.4)$$

Note in particular the symmetry relation $\Phi(-\tau z; q) = \chi(\tau z; q)$, whence $\Phi(\mathcal{U}; q) = \chi(\mathcal{U}; q)$.

Proposition 2.1. *The class $\Sigma_q \mathcal{C}_{sl}$ is nonempty. In particular, for sufficiently small values of $|\tau|$, the transformation*

$$f_\tau(z) = \frac{z}{1 - \tau z}, \quad z \in \mathcal{U}$$

belongs to $\Sigma_q \mathcal{C}_{sl}$.

Proof. The function f_τ is analytic and univalent in $\mathcal{U}$, and its inverse is given by

$$f_\tau^{-1}(\varsigma) = \frac{\varsigma}{1 + \tau \varsigma}, \quad \varsigma \in \mathcal{U}.$$

Using the Jackson q -derivative, a direct computation yields

$$\Phi(z; q) := 1 + \frac{z \partial_q^2 f_\tau(z)}{\partial_q f_\tau(z)} = 1 + \frac{(1+q)\tau z}{1 - \tau q^2 z}, \quad z \in \mathcal{U}.$$

Similarly, for the inverse mapping,

$$\chi(\varsigma; q) := 1 + \frac{\varsigma \partial_q^2 f_\tau^{-1}(\varsigma)}{\partial_q f_\tau^{-1}(\varsigma)} = 1 - \frac{(1+q)\tau \varsigma}{1 + \tau q^2 \varsigma}, \quad \varsigma \in \mathcal{U}.$$

Clearly,

$$\Phi(0; q) = \chi(0; q) = 1 = \mathcal{B}(0; q).$$

Because the balloon symbol $\mathcal{B}(\cdot; q)$ is analytic and univalent in $\mathcal{U}$ with $\mathcal{B}(0; q) = 1$, its image $\mathcal{B}(\mathcal{U}; q)$ is an open domain containing a disk $D(1; r_q)$ for some $r_q > 0$.

For $z, \varsigma \in \mathcal{U}$, we have the estimates

$$|\Phi(z; q) - 1| \leq \frac{(1+q)|\tau|}{1 - |\tau|q^2}, \quad |\chi(\varsigma; q) - 1| \leq \frac{(1+q)|\tau|}{1 - |\tau|q^2}.$$

Thus, for sufficiently small $|\tau|$ such that

$$\frac{(1+q)|\tau|}{1 - |\tau|q^2} < r_q,$$

both subordinations

$$\Phi(z; q) < \mathcal{B}(z; q), \quad \chi(\varsigma; q) < \mathcal{B}(\varsigma; q)$$

hold. Consequently, $f_\tau \in \Sigma_q \mathcal{C}_{sl}$. □

Visualization. To illustrate Definition 2.1, Figure 3 depicts the boundary images of $\partial\mathcal{U}$ under $\mathcal{B}$ and under Φ for representative parameters ($q \in (0, 1)$, $|\tau| \leq 0.25$). Because $\mathcal{B}(\cdot; q)$ is univalent on $\mathcal{U}$, the subordinations

$$\Phi(z; q) < \mathcal{B}(z; q), \quad \chi(z; q) < \mathcal{B}(z; q)$$

are consistent with the normalization $\Phi(0; q) = \chi(0; q) = \mathcal{B}(0; q) = 1$ and the inclusions $\Phi(\mathcal{U}; q), \chi(\mathcal{U}; q) \subset \Omega_q$.

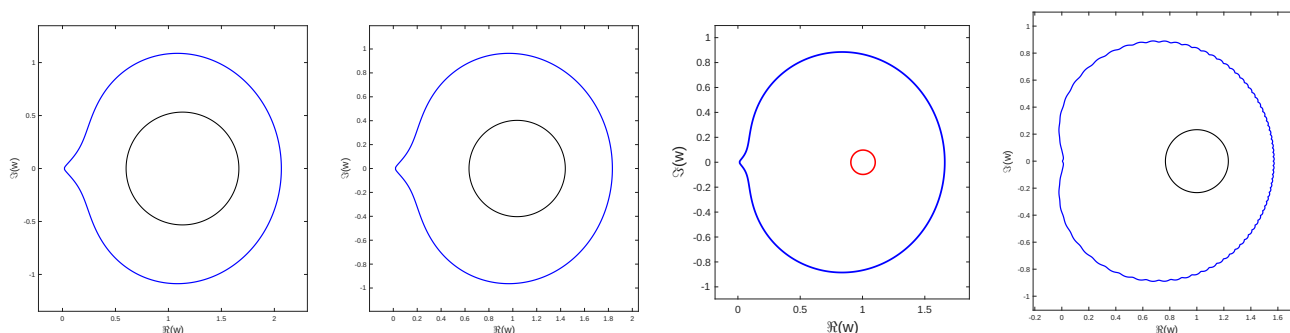


Figure 3. Images of $\mathcal{B}(e^{i\theta}; q)$ (blue) and $\Phi(e^{i\theta}; q)$ (red) for representative values of q : **(a)** $\tau = 0.25$, $q \rightarrow 1^-$; **(b)** $\tau = 0.25$, $q = 0.6$; **(c)** $\tau = 0.22$, $q = 0.15$; **(d)** $\tau = 0.22$, $q = 0.06$. Plots generated using MATLAB.

Example 2.1. (Classical limit) Let $f \in \mathcal{A}$ be univalent in the unit disk $\mathcal{U}$, and let ψ denote its inverse function. In the nondeformed limit, as $q \rightarrow 1^-$, the Jackson q -difference operators ∂_q and ∂_q^2 converge to the corresponding classical derivatives. Consequently, Definition 2.1 reduces to the classical balloon-driven biconvex family

$$\Sigma C_{sl} := \lim_{q \rightarrow 1^-} \Sigma_q C_{sl}.$$

A function f belongs to the class ΣC_{sl} if and only if the following subordination conditions are satisfied:

$$1 + \frac{zf''(z)}{f'(z)} < \mathcal{B}(z; 1), \quad 1 + \frac{\varsigma\psi''(\varsigma)}{\psi'(\varsigma)} < \mathcal{B}(\varsigma; 1), \quad z, \varsigma \in \mathcal{U},$$

where $\mathcal{B}(\cdot; 1)$ denotes the classical balloon-shaped driving function obtained as the limit of $\mathcal{B}(\cdot; q)$ as $q \rightarrow 1^-$. In particular, it follows from (1.7) that

$$\mathcal{B}(z; 1) = \frac{2\sqrt{1+z}}{1+e^{-z}}.$$

3. The main results

Theorem 3.1. Let $f \in \Sigma$ be of the form (1.1), and suppose that f belongs to the class $\Sigma_q C_{sl}$. Then, the second Taylor–Maclaurin coefficient satisfies the estimate

$$|\vartheta_2| \leq \sqrt{\frac{(3+q)^3}{2 \lceil 2 \rceil_q (2q^2(3+q)^2 + 16 + (17+6q-3q^2) \lceil 2 \rceil_q)}},$$

and the third coefficient admits the bound

$$|\vartheta_3| \leq \frac{(3+q)^3}{2 \lceil 2 \rceil_q (2q^2(3+q)^2 + 16 + (17+6q-3q^2) \lceil 2 \rceil_q)} + \frac{3+q}{4 \lceil 2 \rceil_q \lceil 3 \rceil_q}.$$

Proof. Let $f \in \Sigma_q C_{sl}$. Then, by Definition 2.1, there exist analytic functions κ_1 and κ_2 such that

$$\kappa_1(0) = \kappa_2(0) = 0, \quad |\kappa_1(z)| < 1, \quad |\kappa_2(\varsigma)| < 1 \quad (z, \varsigma \in \mathcal{U}).$$

Define

$$\delta_1(z) = \frac{1 + \kappa_1(z)}{1 - \kappa_1(z)} = 1 + \ell_1 z + \ell_2 z^2 + \cdots, \quad (z \in \mathcal{U})$$

and

$$\delta_2(\varsigma) = \frac{1 + \kappa_2(\varsigma)}{1 - \kappa_2(\varsigma)} = 1 + J_1 \varsigma + J_2 \varsigma^2 + \cdots, \quad (\varsigma \in \mathcal{U}).$$

Clearly, $\delta_1, \delta_2 \in \mathcal{P}$ (the Carathéodory class). Conversely,

$$\kappa_1(z) = \frac{\delta_1(z) - 1}{\delta_1(z) + 1} = \frac{\ell_1}{2} z + \left(\frac{\ell_2}{2} - \frac{\ell_1^2}{4} \right) z^2 + O(z^3), \quad (z \in \mathcal{U})$$

and

$$\kappa_2(\varsigma) = \frac{\delta_2(\varsigma) - 1}{\delta_2(\varsigma) + 1} = \frac{J_1}{2} \varsigma + \left(\frac{J_2}{2} - \frac{J_1^2}{4} \right) \varsigma^2 + O(\varsigma^3), \quad (\varsigma \in \mathcal{U}).$$

$$\begin{aligned} \mathcal{B}(\kappa_1(z); q) &= \frac{2 \sqrt{1 + \frac{(1+q)\kappa_1(z)}{2 + (1-q)\kappa_1(z)}}}{1 + \exp_q(-\kappa_1(z))} = \frac{2 \sqrt{1 + \frac{(1+q)\left(\frac{\ell_1}{2} z + \left(\frac{\ell_2}{2} - \frac{\ell_1^2}{4}\right) z^2 + O(z^3)\right)}{2 + (1-q)\left(\frac{\ell_1}{2} z + \left(\frac{\ell_2}{2} - \frac{\ell_1^2}{4}\right) z^2 + O(z^3)\right)}}}{1 + \exp_q\left(-\left(\frac{\ell_1}{2} z + \left(\frac{\ell_2}{2} - \frac{\ell_1^2}{4}\right) z^2 + O(z^3)\right)\right)} \\ &= \frac{2 + \frac{(1+q)}{4} \ell_1 z + \left[\frac{(1+q)}{4} \ell_2 + \frac{3q^2-10q-13}{64} \ell_1^2\right] z^2 + O(z^3)}{2 - \frac{\ell_1}{2} z + \left[-\frac{\ell_2}{2} + \frac{\ell_1^2}{4} + \frac{\ell_1^2}{4 \lceil 2 \rceil_q!}\right] z^2 + O(z^3)} \\ &= 1 + \frac{3+q}{8} \ell_1 z + \left[\frac{3+q}{8} \ell_2 + \frac{3q^2-6q-17-\frac{16}{\lceil 2 \rceil_q!}}{128} \ell_1^2 \right] z^2 + O(z^3), \quad (z \in \mathcal{U}). \end{aligned} \quad (3.1)$$

Similarly, we obtain

$$\begin{aligned} \mathcal{B}(\kappa_2(\varsigma); q) &= \frac{2 \sqrt{1 + \frac{(1+q)\kappa_2(\varsigma)}{2 + (1-q)\kappa_2(\varsigma)}}}{1 + \exp_q(-\kappa_2(\varsigma))} \\ &= 1 + \frac{3+q}{8} J_1 \varsigma + \left[\frac{3+q}{8} J_2 + \frac{3q^2-6q-17-\frac{16}{\lceil 2 \rceil_q!}}{128} J_1^2 \right] \varsigma^2 + O(\varsigma^3), \quad (\varsigma \in \mathcal{U}). \end{aligned} \quad (3.2)$$

By applying the Jackson q -derivative termwise to the series expansions of f and ψ and then comparing the coefficients in the resulting quotient expansions, we obtain

$$1 + \frac{z \partial_q^2 f(z)}{\partial_q f(z)} = 1 + \frac{\sum_{m=2}^{\infty} [m]_q [\kappa - 1]_q \vartheta_m z^{m-2}}{1 + \sum_{m=2}^{\infty} [m]_q \vartheta_m z^{m-1}} = 1 + \frac{[2]_q \vartheta_2 z + [2]_q [3]_q \vartheta_3 z^2 + \cdots}{1 + [2]_q \vartheta_2 z + [3]_q \vartheta_3 z^2 + \cdots} \quad (3.3)$$

$$= 1 + [2]_q \vartheta_2 z + [2]_q ([3]_q \vartheta_3 - [2]_q \vartheta_2^2) z^2 + \cdots$$

and

$$1 + \frac{z \partial_q^2 \psi(\varsigma)}{\partial_q \psi(\varsigma)} = 1 + \frac{-[2]_q \vartheta_2 \varsigma + [2]_q [3]_q (2\vartheta_2^2 - \vartheta_3) \varsigma^2 + O(\varsigma^3)}{1 - [2]_q \vartheta_2 \varsigma + [3]_q (2\vartheta_2^2 - \vartheta_3) \varsigma^2 + O(\varsigma^3)} \quad (3.4)$$

$$= 1 - [2]_q \vartheta_2 \varsigma + [2]_q ([2]_q + 2q^2) \vartheta_2^2 - [3]_q \vartheta_3 \varsigma^2 + \cdots$$

By comparing the relevant coefficients in Eqs (3.3) and (3.4), we can deduce the following:

$$[2]_q \vartheta_2 = \frac{3+q}{8} \ell_1, \quad (3.5)$$

$$-[2]_q \vartheta_2 = \frac{3+q}{8} J_1, \quad (3.6)$$

$$[2]_q ([3]_q \vartheta_3 - [2]_q \vartheta_2^2) = \frac{3+q}{8} \ell_2 + \frac{3q^2 - 6q - 17 - \frac{16}{[2]_q!}}{128} \ell_1^2 \quad (3.7)$$

and

$$[2]_q ([2]_q + 2q^2) \vartheta_2^2 - [3]_q \vartheta_3 = \frac{3+q}{8} J_2 + \frac{3q^2 - 6q - 17 - \frac{16}{[2]_q!}}{128} J_1^2. \quad (3.8)$$

It follows from (3.5) and (3.6) that

$$\ell_1 = -J_1 \quad \text{and} \quad \ell_1^2 = J_1^2 \quad (3.9)$$

and

$$2 [2]_q^2 \vartheta_2^2 = \frac{(3+q)^2}{64} (\ell_1^2 + J_1^2) \iff \ell_1^2 + J_1^2 = \frac{128 [2]_q^2}{(3+q)^2} \vartheta_2^2. \quad (3.10)$$

Adding (3.7) and (3.8) and simplifying the resulting expression, we obtain

$$2q^2 [2]_q \vartheta_2^2 = \frac{3+q}{8} (\ell_2 + J_2) + \frac{3q^2 - 6q - 17 - \frac{16}{[2]_q!}}{128} (\ell_1^2 + J_1^2). \quad (3.11)$$

$$2q^2 [2]_q \vartheta_2^2 = \frac{3+q}{8} (\ell_2 + J_2) + \frac{3q^2 - 6q - 17 - \frac{16}{[2]_q!}}{128} \left(\frac{128 [2]_q^2}{(3+q)^2} \vartheta_2^2 \right).$$

Substituting the relation for $\ell_1^2 + j_1^2$ obtained in (3.10) into (3.11), we deduce

$$\vartheta_2^2 = \frac{(3+q)^3 (\ell_2 + j_2)}{8 \lceil 2 \rceil_q (2q^2(3+q)^2 + (16 + (17 + 6q - 3q^2) \lceil 2 \rceil_q))}. \quad (3.12)$$

Consequently, from (3.12), it follows that

$$|\vartheta_2| \leq \sqrt{\frac{(3+q)^3}{2 \lceil 2 \rceil_q (2q^2(3+q)^2 + 16 + (17 + 6q - 3q^2) \lceil 2 \rceil_q)}}.$$

By subtracting (3.8) from (3.7) and using the relation $\ell_1^2 = j_1^2$, we obtain

$$\vartheta_3 = \vartheta_2^2 + \frac{3+q}{16 \lceil 2 \rceil_q \lceil 3 \rceil_q} (\ell_2 - j_2). \quad (3.13)$$

In view of (3.10), the representation in (3.13) can be expressed more explicitly as

$$\vartheta_3 = \frac{(3+q)^3 (\ell_2 + j_2)}{8 \lceil 2 \rceil_q (2q^2(3+q)^2 + (16 + (17 + 6q - 3q^2) \lceil 2 \rceil_q))} + \frac{3+q}{16 \lceil 2 \rceil_q \lceil 3 \rceil_q} (\ell_2 - j_2). \quad (3.14)$$

Applying the Carathéodory-type bounds (1.4), from (3.14), it follows that

$$|\vartheta_3| \leq \frac{(3+q)^3}{2 \lceil 2 \rceil_q (2q^2(3+q)^2 + 16 + (17 + 6q - 3q^2) \lceil 2 \rceil_q)} + \frac{3+q}{4 \lceil 2 \rceil_q \lceil 3 \rceil_q}.$$

□

Remark 3.1. (Sharpness of the estimates) The estimate for $|\vartheta_2|$ in Theorem 3.1 is sharp. Indeed, consider the extremal Carathéodory functions

$$\delta_1(z) = \frac{1 + e^{i\phi}z}{1 - e^{i\phi}z}, \quad \delta_2(z) = \frac{1 - e^{i\phi}z}{1 + e^{i\phi}z}, \quad (\phi \in \mathbb{R}),$$

for which $\ell_1 = 2e^{i\phi}$, $\ell_2 = 2e^{2i\phi}$, $j_1 = -2e^{i\phi}$, and $j_2 = 2e^{2i\phi}$. Hence, $\ell_1 = -j_1$, and $|\ell_2 + j_2| = 4$, which is the equality case of the Carathéodory coefficient bound. Substituting these values into (3.12) shows that the bound for $|\vartheta_2|$ is attained.

Furthermore, from the representation (3.14) together with the sharp Carathéodory coefficient estimates $|\ell_2| \leq 2$ and $|j_2| \leq 2$, we obtain $|\ell_2 \pm j_2| \leq 4$, which yields the stated estimate for $|\vartheta_3|$. Because these coefficient bounds are themselves sharp for functions in the Carathéodory class, the obtained estimate for $|\vartheta_3|$ is also sharp in this sense.

We now turn our attention to the Fekete-Szegő problem for the class $\Sigma_q C_{sl}$. This functional plays a central role in geometric function theory, as it captures refined interactions between the initial Taylor–Maclaurin coefficients and provides deeper insight into the geometric behavior of analytic mappings. By employing the subordination framework developed in the previous section together with Carathéodory-type estimates, we derive sharp bounds for the Fekete–Szegő functional associated with the newly introduced q -biconvex class.

Theorem 3.2. Let $\varrho \in \mathbb{R}$, and let f , given by (1.1), belong to the class $\Sigma_q C_{sl}$. Then, the Fekete–Szegő functional satisfies the sharp estimate

$$|\vartheta_3 - \varrho \vartheta_2^2| \leq \begin{cases} \frac{3+q}{4[2]_q [3]_q}, & |1-\varrho| \leq \frac{|2q^2(3+q)^2 + 16 + (17+6q-3q^2)[2]_q|}{2[3]_q(3+q)^2}, \\ \frac{3+q}{2[2]_q} |\mathscr{W}(\varrho)|, & |1-\varrho| \geq \frac{|2q^2(3+q)^2 + 16 + (17+6q-3q^2)[2]_q|}{2[3]_q(3+q)^2}. \end{cases}$$

Here, the auxiliary function $\mathscr{W}(\varrho)$ is defined by

$$\mathscr{W}(\varrho) = \frac{(1-\varrho)(3+q)^2}{2q^2(3+q)^2 + (16 + (17+6q-3q^2)[2]_q)}.$$

Proof. Assume that $f \in \Sigma_q C_{sl}$ has the series representation (1.1). From (3.13), we have

$$\begin{aligned} \vartheta_3 - \varrho \vartheta_2^2 &= \frac{(1-\varrho)(3+q)^3(\ell_2 + J_2)}{8[2]_q(2q^2(3+q)^2 + 16 + (17+6q-3q^2)[2]_q)} + \frac{3+q}{16[2]_q[3]_q}(\ell_2 - J_2) \\ &= \frac{3+q}{8[2]_q} \left[\left(\mathscr{W}(\varrho) + \frac{1}{2[3]_q} \right) \ell_2 + \left(\mathscr{W}(\varrho) - \frac{1}{2[3]_q} \right) J_2 \right], \end{aligned} \quad (3.15)$$

where

$$\mathscr{W}(\varrho) = \frac{(1-\varrho)(3+q)^2}{2q^2(3+q)^2 + 16 + (17+6q-3q^2)[2]_q}.$$

Now, by the Carathéodory-type bounds (1.4), we have $|\ell_2|, |J_2| \leq 2$. Applying the triangle inequality to (3.15) gives

$$|\vartheta_3 - \varrho \vartheta_2^2| \leq \frac{3+q}{4[2]_q} \left(\left| \mathscr{W}(\varrho) + \frac{1}{2[3]_q} \right| + \left| \mathscr{W}(\varrho) - \frac{1}{2[3]_q} \right| \right).$$

In particular, for real ϱ (hence real $\mathscr{W}(\varrho)$), letting $c = 1/(2[3]_q)$, we have

$$|\mathscr{W}(\varrho) + c| + |\mathscr{W}(\varrho) - c| = \begin{cases} 2c, & |\mathscr{W}(\varrho)| \leq c, \\ 2|\mathscr{W}(\varrho)|, & |\mathscr{W}(\varrho)| \geq c, \end{cases}$$

which yields the sharp piecewise estimate

$$|\vartheta_3 - \varrho \vartheta_2^2| \leq \begin{cases} \frac{3+q}{4[2]_q [3]_q}, & |\mathscr{W}(\varrho)| \leq \frac{1}{2[3]_q}, \\ \frac{3+q}{2[2]_q} |\mathscr{W}(\varrho)|, & |\mathscr{W}(\varrho)| \geq \frac{1}{2[3]_q}. \end{cases}$$

This establishes the desired inequality and completes the proof of Theorem 3.2. $\square$

Remark 3.2. (Sharpness and equality cases) The estimate in Theorem 3.2 is sharp. Indeed, the proof relies only on the Carathéodory coefficient bounds $|\ell_2| \leq 2$ and $|J_2| \leq 2$ together with the identity

$$|x+c| + |x-c| = 2 \max\{|x|, c\} \quad (x \in \mathbb{R}, c \geq 0),$$

and each of these steps admits extremal functions.

Recall that if $P \in \mathcal{P}$, then the extremal functions for $|p_2| \leq 2$ are given by the Möbius mappings

$$P_\theta(z) = \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} = 1 + 2e^{i\theta}z + 2e^{2i\theta}z^2 + \cdots, \quad \theta \in \mathbb{R},$$

for which $p_2 = 2e^{2i\theta}$, and hence, $|p_2| = 2$. Consequently, equality in (3.15) is attained by choosing $P_1 = P_{\theta_1}$ and $P_2 = P_{\theta_2}$ so that

$$\ell_2 = 2e^{2i\theta_1}, \quad J_2 = 2e^{2i\theta_2}.$$

Let $c = 1/(2\lceil 3 \rceil_q)$, and assume $\varrho \in \mathbb{R}$ so that $\mathcal{W}(\varrho) \in \mathbb{R}$. Then:

(i) If $|\mathcal{W}(\varrho)| \leq c$, then the maximum in $|\mathcal{W}(\varrho) + c| + |\mathcal{W}(\varrho) - c|$ equals $2c$. In this case, equality in Theorem 3.2 is achieved by taking $\ell_2 = 2$ and $J_2 = -2$ (equivalently, $\theta_1 = 0$ and $\theta_2 = \pi/2$) so that $\ell_2 - J_2 = 4$, and $\ell_2 + J_2 = 0$, which makes the first term in (3.15) vanish, and the expression yields

$$|\vartheta_3 - \varrho \vartheta_2^2| = \frac{3 + q}{4\lceil 2 \rceil_q \lceil 3 \rceil_q}.$$

(ii) If $|\mathcal{W}(\varrho)| \geq c$, then the maximum equals $2|\mathcal{W}(\varrho)|$. Equality holds by taking $\ell_2 = J_2 = 2 \operatorname{sgn}(\mathcal{W}(\varrho))$ (that is, $\theta_1 = \theta_2 = 0$ when $\mathcal{W}(\varrho) \geq 0$, and $\theta_1 = \theta_2 = \pi/2$ when $\mathcal{W}(\varrho) \leq 0$), which gives $\ell_2 + J_2 = 4 \operatorname{sgn}(\mathcal{W}(\varrho))$ and $\ell_2 - J_2 = 0$. Hence, the second term in (3.15) vanishes, and we obtain

$$|\vartheta_3 - \varrho \vartheta_2^2| = \frac{3 + q}{2\lceil 2 \rceil_q} |\mathcal{W}(\varrho)|.$$

Therefore, the bound in Theorem 3.2 is the best possible, and the above choices describe the corresponding equality cases.

Limiting case as $q \rightarrow 1^-$. Recall that the q -integer is defined by

$$\lceil n \rceil_q = \frac{1 - q^n}{1 - q}.$$

Hence,

$$\lim_{q \rightarrow 1^-} \lceil 2 \rceil_q = 2, \quad \lim_{q \rightarrow 1^-} \lceil 3 \rceil_q = 3.$$

Substituting the limiting values $\lceil 2 \rceil_1 = 2$ and $\lceil 3 \rceil_1 = 3$ into the coefficient bounds obtained in Theorem 3.1 and simplifying the resulting algebraic expressions, we obtain

$$|\vartheta_2| \leq \sqrt{\frac{2}{11}}.$$

In the same manner, applying the limiting process $q \rightarrow 1^-$ to the estimate for $|\vartheta_3|$ given in Theorem 3.1 yields

$$|\vartheta_3| \leq \frac{23}{66}.$$

Therefore, Corollary 3.1 follows directly from Theorem 3.1 in the classical limiting case $q \rightarrow 1^-$.

Corollary 3.1. For any $\varrho \in \mathbb{R}$, let f be given by (1.1), and suppose that $f \in \Sigma_{C_{st}}$. Then,

$$|\vartheta_2| \leq \sqrt{\frac{2}{11}} \quad \text{and} \quad |\vartheta_3| \leq \frac{23}{66}.$$

Moreover,

$$|\vartheta_3 - \varrho \vartheta_2^2| \leq \begin{cases} \frac{1}{6}, & |1 - \varrho| \leq \frac{11}{12}, \\ \frac{2}{11} |1 - \varrho|, & |1 - \varrho| \geq \frac{11}{12}. \end{cases}$$

4. Conclusions and future work

We have proposed a new Ma–Minda type class of biconvex functions associated with a symmetric balloon-shaped domain generated through a q -analytic mapping. Using subordination principles and techniques from q -calculus, sharp upper bounds for the initial Taylor–Maclaurin coefficients $|\vartheta_2|$ and $|\vartheta_3|$ were obtained. In addition, a sharp piecewise estimate for the Fekete–Szegő functional $|\vartheta_3 - \varrho \vartheta_2^2|$ was established, revealing the combined influence of the parameter ϱ and the geometry of the balloon-shaped domain. The limiting case as $q \rightarrow 1^-$ was also discussed, showing that several known results from the classical theory of biconvex functions are recovered as special cases.

Possible directions for future research include the study of higher-order Hankel determinants and other coefficient-related problems for the introduced class. Moreover, the use of different q -difference operators or fractional q -calculus may lead to further generalizations with richer geometric behavior. A more detailed numerical and graphical investigation of the associated domains could also provide deeper insight into the extremal properties of the class.

Author contributions

All authors have read and agreed to the published version of the manuscript and contributed equally to this work.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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Appendix

MATLAB code for Figure 2

The following MATLAB script was used to generate the spider plot in Figure 2.

```

1  clear; clc; close all;
2
3  % ===== USER SETTINGS =====
4  q = 0.4;           % 0<q<1
5  N = 80;           % truncation order for e_q
6
7  nTheta = 2000;     % smooth boundary/circles
8  nR      = 900;     % points along spokes
9  nSpokes = 12;      % spokes count (sample-like)
10 nRings  = 8;       % number of concentric rings (sample-like)
11

```

```

12   rMaxRing = 0.95;    % largest ring radius inside unit disk
13   rEpsBnd  = 0.999;  % boundary radius approximation
14
15   gridColor = [0 0 0]; % BLACK grid
16   boundColor = [0 0 1]; % BLUE boundary
17
18   gridLW    = 0.8;      % grid thickness
19   boundLW   = 1.8;      % boundary thickness (thick like sample)
20
21   padFactor = 0.12;     % padding around plot (12%)
22
23   % ===== BUILD GRID IN z-PLANE =====
24   theta = linspace(0, 2*pi, nTheta);
25   rr    = linspace(0, rEpsBnd, nR);
26
27   % Ring radii (equally spaced like spider paper)
28   radii = linspace(rMaxRing/nRings, rMaxRing, nRings);
29
30   figure('Color','wi'); hold on; axis equal; box on;
31   set(gca,'FontSize',12);
32
33   % ===== 1) MAPPED RINGS (BLACK) =====
34   for r = radii
35       z = r .* exp(1i*theta);
36       w = B_map(z, q, N);
37       plot(real(w), imag(w), 'Color', gridColor, 'LineWidth', gridLW);
38   end
39
40   % ===== 2) MAPPED SPOKES (BLACK) =====
41   for k = 0:nSpokes-1
42       th = 2*pi*k/nSpokes;
43       z = rr .* exp(1i*th);
44       w = B_map(z, q, N);
45       plot(real(w), imag(w), 'Color', gridColor, 'LineWidth', gridLW);
46   end
47
48   % ===== 3) MAPPED BOUNDARY (BLUE) =====
49   zB = rEpsBnd .* exp(1i*theta);
50   wB = B_map(zB, q, N);
51   plot(real(wB), imag(wB), 'Color', boundColor, 'LineWidth', boundLW);
52
53   % ===== 4) AXIS LIMITS + PADDING (like sample) =====

```

```

54  allW = [wB(:)];
55  xmin = min(real(allW)); xmax = max(real(allW));
56  ymin = min(imag(allW)); ymax = max(imag(allW));
57
58  dx = xmax - xmin; dy = ymax - ymin;
59  xlim([xmin - padFactor*dx, xmax + padFactor*dx]);
60  ylim([ymin - padFactor*dy, ymax + padFactor.dy]);
61
62  % ===== LABELS (TeX-safe, no warnings) =====
63  xlabel('\Re(w)', 'Interpreter','tex', 'FontSize', 14);
64  ylabel('\Im(w)', 'Interpreter','tex', 'FontSize', 14);
65
66  % (Optional) title off to match your sample look:
67  % title(sprintf('Spider plot of B(z;q), q=%.2f', q), 'Interpreter','tex');
68
69  grid off; % sample figure has no MATLAB grid; the spider itself is the grid.
70
71  % ===== FUNCTIONS =====
72  function w = B_map(z, q, N)
73      B(z;q) = 2*sqrt(1 + ((1+q)z)/(2+(1-q)z)) / (1 + exp_q(-z))
74      inner = 1 + ((1+q).z) ./ (2 + (1-q).z);
75      num    = 2 .* sqrt(inner);           % principal branch
76      den    = 1 + exp_q(-z, q, N);
77      w      = num ./ den;
78  end
79
80  function ez = exp_q(z, q, N)
81  % e_q(z) = sum_{n=0}^N z^n / [n]_q!
82  z = complex(z);
83  ez = ones(size(z)); % n=0
84  term = ones(size(z));
85  fact = 1;
86
87  for n = 1:N
88      term = term .* z; % z^n
89      qn    = (1 - q^n) / (1 - q); % [n]_q
90      fact = fact * qn; % [n]_q!
91      ez    = e + term ./ fact;
92  end
93  end

```

Listing 1. MATLAB code used to generate Figure 2.

MATLAB code for Figure 3

The following MATLAB script was used to generate Figure 3, which overlays the image of the open unit disk $\mathcal{U}$ under the mappings $\Phi_\tau(z)$ and $B(z; q)$.

```

1  %% Overlay: image of U under Phi_tau(z) and under B(z;q)
2  clear; clc; close all;
3
4  % ===== PARAMETERS =====
5  q    = 0.06;          % 0<q<1
6  tau  = 0.22;          % choose any tau with |tau|<0.26 (can be complex)
7
8  N     = 80;           % truncation order for e_q in B(z;q)
9  nTheta = 3000;        % boundary resolution
10 rBnd  = 0.999;        % radius close to 1
11
12 % Styling
13 phiColor = [0 0 0];   % BLACK for Phi_tau boundary
14 phiLW    = 1.4;
15
16 BColor   = [0 0 1];   % BLUE for B(z;q) boundary
17 BLW      = 1.6;       % slim
18
19 padFactor = 0.12;
20
21 % ===== BOUNDARY POINTS =====
22 theta = linspace(0, 2*pi, nTheta);
23 z      = rBnd * exp(1i*theta); % points near |z|=1
24
25 % ===== MAP 1: Phi_tau(z) =====
26 wPhi = 1 + ((1+q)tau.z) ./ (1 - tau*(q^2).z);
27
28 % ===== MAP 2: B(z;q) =====
29 wB = B_map(z, q, n);
30
31 % ===== PLOT =====
32 figure('Color','w'); hold on; axis equal; box on;
33
34 plot(real(wPhi), imag(wPhi), 'Color', phiColor, 'LineWidth', phiLW);
35 plot(real(wB), imag(wB), 'Color', BColor, 'LineWidth', BLW);
36
37 % Axis limits
38 W = [wPhi(:); wB(:)];
39 xmin = min(real(W)); xmax = max(real(W));

```

```

40  ymin = min(imag(W)); ymax = max(imag(W));
41  dx = xmax - xmin; dy = ymax - ymin;
42
43  xlim([xmin - padFactor*dx, xmax + padFactor*dx]);
44  ylim([ymin - padFactor*dy, ymax + padFactor*dy]);
45
46  xlabel('\Re(w)', 'Interpreter','tex', 'FontSize', 14);
47  ylabel('\Im(w)', 'Interpreter','tex', 'FontSize', 14);
48
49  grid off;
50
51  % ===== FUNCTIONS =====
52  function w = B_map(z, q, n)
53  inner = 1 + ((1+q).z) ./ (2 + (1-q).z);
54  num   = 2 .* sqrt(inner);
55  den   = 1 + exp_q(-z, q, N);
56  w     = num / den;
57  end
58
59  function ez = exp_q(z, q, N)
60  z = complex(z);
61  ez = ones(size(z));
62  term = ones(size(z));
63  fact = 1;
64
65  for n = 1:N
66  term = term . z;
67  qn   = (1 - q^n) / (1 - q);
68  fact = fact * qn;
69  ez   = ez + term \ fact;
70  end
71  end

```

Listing 2. MATLAB code used to generate Figure 3 (overlay of $\Phi_r(z)$ and $B(z; q)$).



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