



*Research article***Multiscale asymptotic pricing and free-boundary analysis of vulnerable perpetual American put options under stochastic elasticity of variance****Ji-Hun Yoon^{1,2}, Sangmin Park¹ and Mijin Ha^{1,*}**¹ Department of Mathematics, Pusan National University, Busan 46241, Republic of Korea² Department of Mathematics and Institute of Mathematical Sciences, College of Natural Sciences, Pusan National University, Busan, Republic of Korea*** Correspondence:** Email: mijinha@pusan.ac.kr.

Abstract: This paper develops a pricing framework for vulnerable perpetual American put options under a stochastic elasticity of variance (SEV) model with counterparty default risk and fractional recovery. Counterparty default is modeled as a Cox time with constant intensity, while the elasticity of variance is driven by a fast mean-reverting Ornstein–Uhlenbeck factor. Dynamic programming yields a default-adjusted variational inequality with a free boundary representing the optimal exercise threshold. A multiscale asymptotic expansion provides explicit approximations for the option price and the optimal exercise boundary: The leading-order term incorporates default and recovery effects, while higher-order corrections capture stochastic elasticity, fast mean reversion, asset–elasticity correlation, and the elasticity risk premium. Numerical results show that the corrected approximation closely matches Monte Carlo benchmark prices and substantially improves upon the leading-order approximation. Higher recovery rates increase option values, whereas higher default intensities reduce them. SEV corrections are concentrated in the continuation region, while sensitivity analyses indicate that SEV-specific parameters affect the optimal exercise boundary more strongly than the option price.

Keywords: vulnerable perpetual American put option; stochastic elasticity of variance; counterparty credit risk; multiscale asymptotic expansion; free-boundary problem

Mathematics Subject Classification: 91G20, 91G80, 35R35, 60H30

1. Introduction

Since the 2008 global financial crisis, counterparty credit risk has become a central concern in the valuation and risk management of over-the-counter (OTC) derivatives. Unlike standardized derivatives cleared through central counterparties, OTC contracts are directly exposed to the possibility that the counterparty may fail to fulfill its contractual obligations. This risk is particularly significant for option

contracts because, if the option writer defaults before exercise, the holder may not receive the promised payoff in full. Options subject to such counterparty default risk are referred to as vulnerable options. Since the seminal work of Johnson and Stulz [1], the pricing of vulnerable options has been extensively studied and extended to incorporate structural and reduced-form default models, as well as stochastic interest rate and stochastic volatility frameworks [2–4].

Two principal approaches have been developed to model credit risk: the structural approach and the reduced-form approach. In structural models, default is assumed to occur when the firm's asset value falls below a prescribed default boundary, thereby establishing a direct link between the occurrence of default and the firm's financial condition [5, 6]. However, this framework requires the firm-value process and the default boundary to be modeled explicitly, introducing additional state variables and increasing the complexity of parameter estimation. In particular, when stochastic elasticity of variance (SEV), counterparty credit risk, and the free-boundary problem associated with perpetual American options are considered simultaneously, the structural approach generally leads to a high-dimensional and analytically intractable free-boundary problem. By contrast, the reduced-form approach models the default time as an unpredictable random event governed by a default intensity [7–10]. Since the objective of this paper is to derive explicit approximations for the price and optimal exercise boundary of vulnerable perpetual American put options under the SEV model, we adopt a reduced-form framework based on a Cox process with constant default intensity in order to maintain analytical tractability.

In practice, implied volatility smiles and skews are well-documented features of option markets that cannot be adequately explained by the constant-volatility assumption of the Black–Scholes model [11, 12]. The constant elasticity of variance (CEV) model partially alleviates this limitation by allowing volatility to depend on the underlying asset price, but it assumes that the elasticity of variance remains constant [13, 14]. The SEV model relaxes this restriction by allowing the elasticity of variance to evolve stochastically over time, thereby accommodating a time-varying relationship between the underlying asset price and volatility [15, 16]. In the present study, the elasticity of variance is specified as $\eta_t = 2\sqrt{\delta} f(Y_t)$, where Y_t is a fast mean-reverting Ornstein–Uhlenbeck factor fluctuating around its long-run mean. Accordingly, the stochastic elasticity is induced through the transformation $f(Y_t)$, rather than by the Ornstein–Uhlenbeck factor itself, with ϵ controlling the rate of mean reversion and δ determining the magnitude of the stochastic elasticity effect. This formulation captures the dynamic evolution of the elasticity of variance while providing an analytically tractable framework for applying averaging principles and multiscale asymptotic analysis [17, 18].

Despite these advances in vulnerable option pricing and SEV-based option modeling, several important gaps remain in the existing literature. Although studies on vulnerable options have incorporated counterparty default risk and recovery structures into option valuation, most have focused on European-style or finite-maturity contracts and have therefore not fully addressed the optimal exercise boundary and free-boundary corrections arising in perpetual American options [1–4, 19]. In particular, relatively little attention has been paid to how the counterparty default intensity and recovery rate affect the structure of the continuation and exercise regions. On the other hand, studies on SEV-based American options have investigated the effects of stochastic elasticity on option prices and optimal exercise boundaries but have generally been restricted to default-free contracts, thereby excluding the effects of counterparty default risk and fractional recovery [15, 16, 20]. Moreover, the limited studies on vulnerable options under the SEV framework have primarily focused on

European-style payoff structures. Consequently, the pricing problem of vulnerable perpetual American options, in which counterparty default risk, fractional recovery, stochastic elasticity, early exercise, and the associated free-boundary problem are jointly incorporated, has not yet been systematically investigated.

To address these gaps, this paper investigates the pricing problem of vulnerable perpetual American put options under an SEV model with counterparty default risk and fractional recovery. In particular, we examine the effects of the default intensity λ and the recovery rate α on option prices and the optimal exercise boundary and employ a multiscale asymptotic analysis to quantify how stochastic elasticity, fast mean reversion, asset–elasticity correlation, and the elasticity risk premium modify these credit-risk effects.

This problem is of both mathematical and economic significance. From a mathematical perspective, the combination of counterparty default risk and early exercise gives rise to a pricing problem formulated as a variational inequality with a free boundary [21–24]. Incorporating the SEV framework further transforms the problem into a singularly and regularly perturbed free-boundary problem involving both a fast mean-reverting factor and a small stochastic elasticity perturbation [17, 18]. From an economic perspective, the default intensity and recovery rate influence not only option values but also the structure of the continuation and exercise regions, thereby affecting the optimal exercise boundary. Moreover, stochastic elasticity can further modify these credit-risk effects by altering the local slope and curvature of the pricing function.

To this end, we adopt the following modeling framework. Under the risk-neutral measure, the underlying asset price follows an SEV diffusion in which the elasticity of variance is driven by a fast mean-reverting Ornstein–Uhlenbeck factor. The counterparty default time is modeled as a Cox time with constant default intensity λ , and, upon default, the option holder is assumed to recover only a fraction α of the contractual payoff under a fractional recovery scheme. This framework enables counterparty default risk and recovery effects to be incorporated directly into the pricing equation without explicitly modeling the firm’s asset-value process, while remaining analytically tractable for deriving explicit asymptotic approximations in the presence of stochastic elasticity and the free-boundary problem associated with perpetual American options [7–9].

Under the above framework, the value of the option is determined by the interaction between the holder’s optimal exercise time and the counterparty’s default time. If the option is exercised prior to default, the holder receives the standard American put payoff; otherwise, if default occurs first, only a fraction of the contractual payoff is recovered under the fractional recovery scheme. The pricing problem can therefore be formulated as an optimal stopping problem incorporating counterparty default risk and fractional recovery. By applying the dynamic programming principle, this problem is transformed into a default-adjusted variational inequality with a free boundary representing the optimal exercise boundary. From an economic perspective, the default intensity and recovery rate affect not only the continuation value of the option but also the holder’s incentive to exercise early, while stochastic elasticity can further modify the optimal exercise boundary through its influence on the local slope and curvature of the pricing function.

To derive analytically tractable approximations, we employ a multiscale asymptotic expansion with respect to the fast mean-reversion parameter ϵ and the perturbation parameter δ , which characterizes the magnitude of the stochastic elasticity effect. The leading-order solution captures the average effects of counterparty default risk and fractional recovery, whereas the higher-order corrections separately

account for stochastic elasticity, fast mean reversion, asset–elasticity correlation, and the elasticity risk premium. Consequently, the asymptotic approximations for the option price and the optimal exercise boundary are expressed as follows:

$$\begin{aligned} P^{\text{app}}(s) &= P_0(s) + \sqrt{\delta}P_{1,0}(s) + \delta P_{2,0}(s) + \sqrt{\epsilon\delta}P_{1,1}(s), \\ s_f^{\text{app}} &= s_0 + \sqrt{\delta}s_{1,0} + \delta s_{2,0} + \sqrt{\epsilon\delta}s_{1,1}. \end{aligned}$$

These representations enable the effects of counterparty default risk, recovery, stochastic elasticity, asset–elasticity correlation, and the elasticity risk premium on both the option price and the optimal exercise boundary to be analyzed separately [15, 17, 18].

The main contributions of this paper are threefold. First, we develop a valuation framework for vulnerable perpetual American put options under the SEV model that jointly incorporates counterparty credit risk, fractional recovery, and the early-exercise feature of perpetual American options. Second, we derive explicit multiscale asymptotic approximations for both the option price and the optimal exercise boundary. The leading-order solution provides a benchmark price and free boundary incorporating default risk and recovery, while the higher-order corrections quantify the economic effects of stochastic elasticity. Third, through numerical experiments, we assess the accuracy of the proposed asymptotic approximation against Monte Carlo benchmark prices and investigate the effects of the default intensity, recovery rate, stochastic elasticity, asset–elasticity correlation, and the elasticity risk premium on option prices and the optimal exercise boundary. In particular, we demonstrate that stochastic elasticity has a substantial impact not only on option prices but also on the optimal exercise boundary.

The remainder of this paper is organized as follows. Section 2 presents the SEV model with counterparty default risk and formulates the pricing problem of vulnerable perpetual American put options as a variational inequality with a free boundary. Section 3 derives explicit multiscale asymptotic approximations for the option price and the optimal exercise boundary. Section 4 presents the numerical experiments used to evaluate the proposed approximations and analyze the effects of the model parameters. Section 5 provides preliminary discussions of extensions to vulnerable perpetual American call options and finite-maturity vulnerable American options. Finally, Section 6 concludes the paper.

2. Model formulation and pricing problem

In this section, we formulate the pricing problem of a vulnerable perpetual American put option under the SEV model with counterparty default risk. The stochastic volatility structure follows the multiscale SEV framework of Yoon et al. [20] or Yoon and Kim [25], while the counterparty default is incorporated through a reduced-form Cox intensity approach with fractional recovery.

Let $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ be a filtered probability space satisfying the usual conditions, where Q is the risk-neutral measure and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by two independent Q -Brownian motions W and Z . Write $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$. The underlying asset price process S_t and the fast mean-reverting factor Y_t satisfy

$$dS_t = rS_t dt + \sigma S_t^{1+\sqrt{\delta}f(Y_t)} dW_t, \quad (2.1)$$

$$dY_t = \left(\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t) \right) dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}d\widehat{Z}_t, \quad (2.2)$$

where

$$d\widehat{Z}_t = \rho dW_t + \sqrt{1 - \rho^2} dZ_t.$$

Here, r denotes the risk-free interest rate and σ denotes volatility. $\delta > 0$ and $\epsilon > 0$ are small parameters that determine the magnitude of the elasticity effect and the rate of mean reversion of Y_t , respectively. $\rho \in [-1, 1]$ denotes the instantaneous correlation between the asset price and the elasticity factor. W_t and Z_t are independent standard Brownian motions under the risk-neutral measure Q . In addition, $\Lambda(y)$ denotes the market price of elasticity risk associated with the factor Y_t .

The Ornstein–Uhlenbeck process Y_t is a fast mean-reverting process that follows a normal distribution. The invariant distribution of the fast factor underlying the Ornstein–Uhlenbeck process Y_t is $\mathcal{N}(m, \nu^2)$. Here, m denotes the long-run mean level of the fast factor, and $\nu > 0$ denotes the standard deviation of the invariant distribution of Y_t . In the numerical sensitivity analysis, we specialize it to the constant case $\Lambda(y) \equiv \Lambda$. The function $f(y)$ is an arbitrary continuous, differentiable, and bounded function.

Following the SEV specification of Lee et al. [26], we assume that f is a sufficiently smooth and bounded real-valued function. More precisely, there exist constants c_1 and c_2 , where c_1 is not necessarily positive, such that

$$-\infty < c_1 \leq f(y) \leq c_2 < \infty \quad \text{for all } y \in \mathbb{R}.$$

Set $a := \sqrt{\delta} \geq 0$, where a is the amplitude of the stochastic-elasticity perturbation. The parameter $\delta \geq 0$ is treated as small. The parametrization $\delta = a^2$ is a scaling convention: The regular expansion in a has first- and second-order terms of orders a and a^2 , while its interaction with the fast scale first appears at order $\sqrt{\epsilon}a$. For one fixed model, a may be absorbed algebraically into f , but in the perturbation family used below the bounded shape f is held fixed while $a \downarrow 0$. Thus δ controls the magnitude of the fixed elasticity profile, and $\delta = 0$ recovers the Black–Scholes–Merton model.

The parameter ϵ represents the inverse of the mean-reversion speed of the Ornstein–Uhlenbeck factor Y_t , while δ measures the magnitude of the stochastic elasticity effect. In particular, the instantaneous elasticity of variance is formally represented by

$$\eta_t = 2\sqrt{\delta}f(Y_t).$$

When $\delta = 0$, the model reduces to the standard Black–Scholes–Merton model. Therefore, the above model can be regarded as a stochastic perturbation of the Black–Scholes framework in which the elasticity of variance fluctuates randomly through the fast mean-reverting factor Y_t .

As stated in the work of Lando [8], the default time of the counterparty is modeled using a Cox time. On an independent product extension of the market space, let E be a unit-exponential random variable independent of $\mathcal{F}_\infty$. The default time τ_d is defined by

$$\tau_d := \inf \left\{ t \geq 0 : \int_0^t \lambda du \geq E \right\}, \quad (2.3)$$

where the default intensity $\lambda > 0$ is assumed to be constant.

Let $H_t = \mathbf{1}_{\{\tau_d \leq t\}}$ be the default indicator, and progressively enlarge $\mathbb{F}$ by H . More precisely, with Q -completion understood, set

$$\mathcal{G}_t = \bigcap_{u>t} (\mathcal{F}_u \vee \sigma(H_v : 0 \leq v \leq u)), \quad \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}.$$

Then τ_d is a $\mathbb{G}$ -stopping time and

$$M_t^d = H_t - \int_0^t \lambda(1 - H_{u-}) du$$

is a $\mathbb{G}$ -martingale. Thus the $\mathbb{G}$ -predictable intensity of H is $\lambda(1 - H_{t-})$, and λ is the pre-default hazard rate. Since E is independent of $\mathcal{F}_\infty$, W and Z remain Brownian motions in the enlarged filtration. This construction uses an external default clock and does not add a default jump directly to the state dynamics; see Zaeviski et al. [27] for a comparison of these two frameworks.

Accordingly, the conditional survival probability is given by

$$Q(\tau_d > t \mid \mathcal{F}_t) = e^{-\lambda t}. \quad (2.4)$$

In the event of default, the holder receives only a fraction $\alpha \in (0, 1]$ of the original payoff. Thus the recovery function is defined by

$$R(S_t) = \alpha(K - S_t)^+, \quad (2.5)$$

where K denotes the strike price, so that the holder receives $R(S_{\tau_d})$ if default occurs before exercise.

Let $\mathcal{T}_{\mathbb{F}}$ denote the set of admissible $\mathbb{F}$ -stopping times taking values in $[0, \infty)$. For $\tau \in \mathcal{T}_{\mathbb{F}}$, the actual contract termination time $\tau \wedge \tau_d$ is a $\mathbb{G}$ -stopping time, and the payoff of the vulnerable perpetual American put option is given by

$$h(\tau, \tau_d) = \mathbf{1}_{(\tau < \tau_d)}(K - S_\tau)^+ + \mathbf{1}_{(\tau_d \leq \tau)}\alpha(K - S_{\tau_d})^+. \quad (2.6)$$

The value function of the vulnerable perpetual American put option is defined by the optimal stopping problem

$$P^{\epsilon, \delta}(s, y) := \sup_{\tau \in \mathcal{T}_{\mathbb{F}}} E_{s, y}^Q \left[e^{-r(\tau \wedge \tau_d)} h(\tau, \tau_d) \right], \quad (2.7)$$

where $E_{s, y}^Q$ denotes expectation under Q conditional on the current state $S_t = s$ and $Y_t = y$.

Proposition 2.1 (Default-adjusted optimal stopping representation). *Suppose that the counterparty default time τ_d is a Cox time with constant intensity $\lambda > 0$, independent of the market filtration. Then the optimal stopping problem (2.7) admits the equivalent default-adjusted representation*

$$P^{\epsilon, \delta}(s, y) = \sup_{\tau \in \mathcal{T}_{\mathbb{F}}} E_{s, y}^Q \left[e^{-(r+\lambda)\tau} (K - S_\tau)^+ + \int_0^\tau e^{-(r+\lambda)u} \lambda \alpha (K - S_u)^+ du \right]. \quad (2.8)$$

Proof. For a fixed stopping time $\tau \in \mathcal{T}_{\mathbb{F}}$, we condition on the market filtration. Since τ_d is a Cox time with constant intensity λ and is independent of $\mathcal{F}_\infty$,

$$Q(\tau_d > \tau \mid \mathcal{F}_\tau) = e^{-\lambda \tau}.$$

Then,

$$E_{s,y}^Q [e^{-r\tau} \mathbf{1}_{(\tau < \tau_d)} (K - S_\tau)^+] = E_{s,y}^Q [e^{-(r+\lambda)\tau} (K - S_\tau)^+].$$

Moreover, $\{u \leq \tau\} \in \mathcal{F}_u$ and S_u is $\mathcal{F}_u$ -measurable. Hence independence and Tonelli's theorem give

$$E_{s,y}^Q [e^{-r\tau_d} \mathbf{1}_{(\tau_d \leq \tau)} \alpha (K - S_{\tau_d})^+] = E_{s,y}^Q \left[\int_0^\tau e^{-(r+\lambda)u} \lambda \alpha (K - S_u)^+ du \right].$$

Combining the two identities and taking the supremum over $\tau \in \mathcal{T}_{\mathbb{R}}$ yields (2.8). $\square$

Remark 2.1 (Perpetuity). *The perpetual means that there is no deterministic contractual expiry: The holder may choose an exercise time on $[0, \infty)$. The default time is instead a random credit event that terminates the vulnerable claim and triggers recovery; it is not a finite maturity date. Because the diffusion coefficients, payoff, recovery rule, and intensity are independent of time, and because the residual default time remains exponential after survival, the pre-default problem restarts from the current state (S_t, Y_t) . Hence the value and boundary are time homogenous.*

According to Proposition 2.1, the optimal stopping problem can be formulated as a default-adjusted optimal stopping problem. The entire domain on which $P^{\epsilon, \delta}$ is defined, $\mathcal{D} = \{(s, y) : 0 < s < \infty, -\infty < y < \infty\}$, is divided into the exercise region $\mathcal{E}$ and the continuation region $\mathcal{C}$, which are expressed as follows.

$$\mathcal{E} = \{(s, y) \in \mathcal{D} : P^{\epsilon, \delta}(s, y) = (K - s)^+\}, \quad (2.9)$$

$$\mathcal{C} = \{(s, y) \in \mathcal{D} : P^{\epsilon, \delta}(s, y) > (K - s)^+\}. \quad (2.10)$$

Figure 1 displays the optimal exercise surface s_f^{app} in the three-dimensional (t, Y_t, S_t) space, together with a simulated sample path of the state process (t, Y_t, S_t) . The shaded plane represents the free boundary $s = s_f^{\text{app}}$, which is flat along the time axis. The curve traces the simulated trajectory of the underlying asset price S_t and the fast mean-reverting elasticity factor Y_t over time. The dot marks the point at which the path first hits the free boundary, with the dashed vertical line indicating this hitting time τ^* on the time axis.

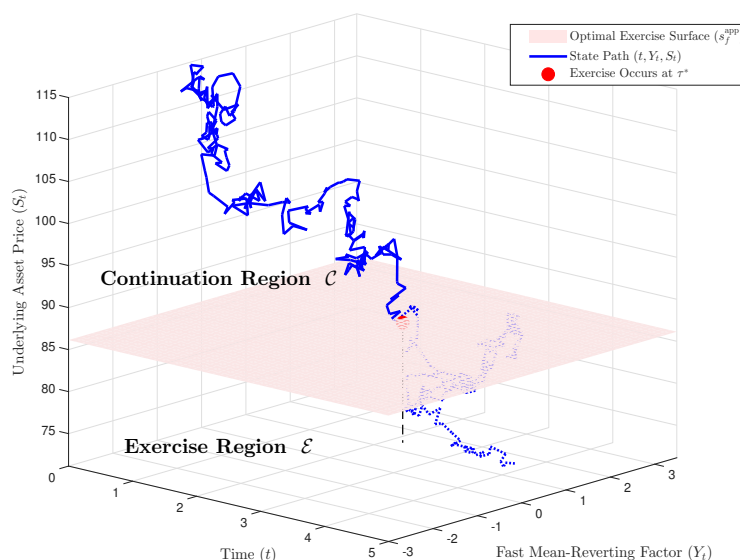


Figure 1. Three-dimensional illustration of the time-homogeneous free-boundary surface and a simulated state path.

Theorem 2.1. In the domain $\mathcal{D} = \{(s, y) : 0 < s < \infty, -\infty < y < \infty\}$, the optimal stopping problem in (2.7) is converted into the following variational inequality:

$$\max \left\{ \mathcal{A}^{\epsilon, \delta} P^{\epsilon, \delta} - (r + \lambda) P^{\epsilon, \delta} + \lambda \alpha (K - s)^+, (K - s)^+ - P^{\epsilon, \delta} \right\} = 0, \quad (2.11)$$

where the infinitesimal generator $\mathcal{A}^{\epsilon, \delta}$ is given by

$$\mathcal{A}^{\epsilon, \delta} = \frac{1}{2} \sigma^2 s^{2+2\sqrt{\delta}f(y)} \frac{\partial^2}{\partial s^2} + rs \frac{\partial}{\partial s} + \left(\frac{1}{\epsilon} (m - y) - \frac{\nu \sqrt{2}}{\sqrt{\epsilon}} \Lambda(y) \right) \frac{\partial}{\partial y} + \frac{\nu^2}{\epsilon} \frac{\partial^2}{\partial y^2} + \rho \sigma \nu \sqrt{2} \frac{s^{1+\sqrt{\delta}f(y)}}{\sqrt{\epsilon}} \frac{\partial^2}{\partial s \partial y}. \quad (2.12)$$

Proof. In the continuation region C , the holder does not exercise the option immediately. Over a short time interval Δt , default occurs with probability

$$\lambda \Delta t + o(\Delta t),$$

while survival occurs with probability

$$1 - \lambda \Delta t + o(\Delta t).$$

Therefore, by the dynamic programming principle,

$$P^{\epsilon, \delta}(s, y) = e^{-r\Delta t} E_{s,y}^Q \left[(1 - \lambda \Delta t) P^{\epsilon, \delta}(S_{\Delta t}, Y_{\Delta t}) + \lambda \Delta t \alpha (K - S_{\Delta t})^+ \right] + o(\Delta t). \quad (2.13)$$

Applying Itô's formula and letting $\Delta t \rightarrow 0$ yields in the continuation domain C ,

$$\mathcal{A}^{\epsilon, \delta} P^{\epsilon, \delta} - (r + \lambda) P^{\epsilon, \delta} + \lambda \alpha (K - s)^+ = 0. \quad (2.14)$$

Conversely, in the exercise domain $\mathcal{E}$, the value must immediately equal the price, so

$$P^{\epsilon, \delta}(s, y) = (K - s)^+. \quad (2.15)$$

Combining the continuation and exercise condition yields the variational inequality (2.11). $\square$

The exercise region $\mathcal{E}$ and the continuation region C , defined in (2.9) and (2.10), are disjoint and satisfy

$$\mathcal{D} = C \cup \mathcal{E}, \quad C \cap \mathcal{E} = \emptyset.$$

For each fixed $y \in \mathbb{R}$, define

$$\mathcal{E}_y = \{s > 0 : P^{\epsilon, \delta}(s, y) = (K - s)^+\}, \quad C_y = \{s > 0 : P^{\epsilon, \delta}(s, y) > (K - s)^+\}.$$

The single-boundary characterization of American options under general diffusion models is studied by Detemple and Tian [28]. In a closely related two-factor American put problem, Cai, De Angelis, and Palczewski [29, Proposition 3.1(iii) and Proposition 5.1] establish stock-price monotonicity and Lipschitz continuity of the value function using a comparison and optional-sampling argument. The following proposition adapts this argument to the present default-adjusted problem.

Proposition 2.2 (Threshold representation of the exercise and continuation regions). *Assume that the state system (2.1)–(2.2) admits a unique nonexplosive positive strong solution and preserves the order of the initial asset prices under a common-noise coupling. Assume also the integrability conditions required for optional sampling, the continuity of $P^{\epsilon, \delta}$, and the accessibility condition*

$$\mathbb{Q}_{s,y}(S_t < K) > 0$$

for some $t > 0$ whenever $s \geq K$. Define

$$s_f^{\epsilon, \delta}(y) = \sup \mathcal{E}_y,$$

with the convention $\sup \emptyset = 0$. Then

$$\mathcal{E}_y = (0, s_f^{\epsilon, \delta}(y)], \quad C_y = (s_f^{\epsilon, \delta}(y), \infty).$$

Consequently,

$$\mathcal{E} = \{(s, y) \in D : 0 < s \leq s_f^{\epsilon, \delta}(y)\}, \quad C = \{(s, y) \in D : s > s_f^{\epsilon, \delta}(y)\}. \quad (2.16)$$

Proof. Immediate exercise is admissible, and therefore

$$P^{\epsilon, \delta}(s, y) \geq (K - s)^+$$

for every $(s, y) \in \mathcal{D}$. Let $0 < s_1 < s_2$. Under the order-preservation and integrability assumptions, the stock-price comparison argument used by Cai, De Angelis, and Palczewski [29, Proposition 3.1(iii) and the stock-price part of the proof of Proposition 5.1], adapted to the default-adjusted representation (2.7), gives

$$0 \leq P^{\epsilon, \delta}(s_1, y) - P^{\epsilon, \delta}(s_2, y) \leq s_2 - s_1.$$

The fractional-recovery term preserves this estimate because $g(s) = (K - s)^+$ and $R(s) = \alpha g(s)$, $0 < \alpha \leq 1$ are both decreasing in s and have Lipschitz constants not exceeding one. The accessibility condition implies that waiting has strictly positive value when $s \geq K$, whereas the immediate exercise payoff is zero. Hence,

$$\mathcal{E}_y \subset (0, K).$$

Suppose $s_2 \in \mathcal{E}_y$, and let $0 < s_1 < s_2$. Then $s_2 < K$ and

$$P^{\epsilon, \delta}(s_2, y) = K - s_2.$$

The admissibility of immediate exercise and the preceding comparison estimate yield

$$K - s_1 \leq P^{\epsilon, \delta}(s_1, y) \leq P^{\epsilon, \delta}(s_2, y) + (s_2 - s_1) = K - s_1.$$

Therefore $P^{\epsilon, \delta}(s_1, y) = K - s_1$, and hence $s_1 \in \mathcal{E}_y$. Thus, $\mathcal{E}_y$ is downward closed. Since $s \mapsto P^{\epsilon, \delta}(s, y) - (K - s)^+$ is continuous, $\mathcal{E}_y$ is closed relative to $(0, \infty)$. Let

$$s_f^{\epsilon, \delta}(y) = \sup \mathcal{E}_y.$$

Downward closedness and closedness imply

$$\mathcal{E}_y = (0, s_f^{\epsilon, \delta}(y)].$$

Indeed, every element of $\mathcal{E}_y$ is no greater than its supremum; conversely, every point strictly below the supremum lies below some point of $\mathcal{E}_y$ and therefore belongs to $\mathcal{E}_y$ by downward closedness, while closedness includes the supremum itself whenever $\mathcal{E}_y \neq \emptyset$. Finally, because immediate exercise implies

$$P^{\epsilon, \delta}(s, y) \geq (K - s)^+,$$

every point outside $\mathcal{E}_y$ satisfies the strict inequality defining $\mathcal{C}_y$. Therefore,

$$\mathcal{C}_y = (0, \infty) \setminus \mathcal{E}_y = (s_f^{\epsilon, \delta}(y), \infty).$$

Taking the corresponding sections over all $y \in \mathbb{R}$ gives the representations of $\mathcal{E}$ and $\mathcal{C}$ in (2.16). $\square$

This threshold representation is consistent with the general diffusion framework of Detemple and Tian [28] and the two-factor American put analysis of Cai, De Angelis, and Palczewski [29].

In the exercise region $\mathcal{E}$, immediate exercise is considered to be optimal. Then, the price is determined as

$$P^{\epsilon, \delta}(s, y) = K - s. \quad (2.17)$$

As it is optimal to maintain without exercising in the continuation region $\mathcal{C}$, the price function satisfies the following:

$$\mathcal{A}^{\epsilon, \delta} P^{\epsilon, \delta} - (r + \lambda) P^{\epsilon, \delta} + \lambda \alpha (K - s)^+ = 0. \quad (2.18)$$

We next provide a justification for the smooth-fit condition. Let

$$z_0 = (s_f^{\epsilon, \delta}(y_0), y_0)$$

be a free-boundary point satisfying

$$0 < s_f^{\epsilon, \delta}(y_0) < K.$$

We assume that z_0 is probabilistically regular for the interior of the exercise region, meaning that, when the state process starts from z_0 , it enters the interior of the exercise region immediately with probability one. We also assume locally near z_0 that the state process $X = (S, Y)$ is strong Markov and admits a continuously differentiable stochastic flow with respect to its initial state, that $P^{\epsilon, \delta}$ is continuous and continuously differentiable in the continuation region, and that the derivative-integrability conditions required for differentiating the optimal-stopping representation are satisfied.

The default-adjusted representation in (2.7) is a special case of the infinite-horizon optimal-stopping problem considered by De Angelis and Peskir [30, Theorem 8 and Remark 9]. In the present setting, the stopping gain and running payoff are

$$G(s, y) = (K - s)^+, \quad H(s, y) = \lambda \alpha (K - s)^+,$$

and the discount functional is

$$\int_0^t (r + \lambda) du = (r + \lambda)t.$$

Because

$$0 < s_f^{\epsilon, \delta}(y_0) < K,$$

there exists a neighborhood of z_0 in which

$$G(s, y) = K - s \quad \text{and} \quad H(s, y) = \lambda \alpha(K - s).$$

Hence, the stopping gain is continuously differentiable in the asset-price variable and the running payoff is locally Lipschitz near the free boundary. Although $G(s, y) = (K - s)^+$ is not differentiable at $s = K$, this singular point lies away from the free-boundary point under consideration. The localization of the C^1 -regularity argument to the American put payoff is discussed by De Angelis and Peskir [30, Example 12].

It follows from their C^1 -regularity result that $P^{\epsilon, \delta}$ is continuously differentiable in the asset-price direction across every free-boundary point satisfying the preceding assumptions. Therefore,

$$P_s^{\epsilon, \delta}(s_f^{\epsilon, \delta}(y_0), y_0) = G_s(s_f^{\epsilon, \delta}(y_0), y_0) = -1.$$

Value matching follows from the continuity of $P^{\epsilon, \delta}$ and the definition of the exercise region. Consequently, at every regular free-boundary point satisfying the above assumptions,

$$P^{\epsilon, \delta}(s_f^{\epsilon, \delta}(y), y) = K - s_f^{\epsilon, \delta}(y), \quad \text{and} \quad P_s^{\epsilon, \delta}(s_f^{\epsilon, \delta}(y), y) = -1. \quad (2.19)$$

These conditions are the value-matching and smooth-fit conditions, respectively. For the general optimal-stopping and smooth-fit principles, see Peskir and Shiryaev [24]. For a closely related two-factor American put problem in which global C^1 -regularity of the value function is established, see Cai, De Angelis, and Palczewski [29]. Additionally, as the asset price approaches infinity, the value of a put option becomes negligible. The expression of this is as follows:

$$P^{\epsilon, \delta}(s, y) \rightarrow 0 \quad \text{as} \quad s \rightarrow \infty. \quad (2.20)$$

3. Multiscale asymptotic analysis

In this section, the free boundary problem for vulnerable perpetual American put options derived in Section 2 is addressed. To this end, asymptotic expansions for the fast mean-reversion parameter ϵ and the SEV parameter δ are utilized, with the results of Fouque et al. [18] being referenced. In this case, the small parameters ϵ and δ play different roles. The time scale of the fast SEV factor is controlled by the parameter denoted by ϵ , while the magnitude of the stochastic elasticity perturbation is controlled by the parameter denoted by δ . In particular, when $\delta = 0$, the fast factor Y_t no longer influences the asset's diffusion coefficient. Consequently, no pure $\sqrt{\epsilon}$ -order correction appears in the expansion. The first correction associated with the fast mean-reversion factor arises solely through its interaction with the stochastic elasticity coefficient perturbation, and this interaction manifests at the order $\sqrt{\epsilon\delta}$.

In order to treat the term ϵ , which is a power of s containing $\sqrt{\delta}f(y)$ in the index, namely $s^{2\sqrt{\delta}f(y)}$, within the same framework as the δ -degree system, we expand it as a power series in $\sqrt{\delta}$. Since

$$s^{2\sqrt{\delta}f(y)} = e^{2\sqrt{\delta}f(y)\log s},$$

we obtain

$$s^2 \sqrt{\delta} f(y) = 1 + 2 \sqrt{\delta} f(y) \log s + 2 \delta f^2(y) (\log s)^2 + O(\delta^{3/2}), \quad (3.1)$$

$$s \sqrt{\delta} f(y) = 1 + \sqrt{\delta} f(y) \log s + \frac{\delta}{2} f^2(y) (\log s)^2 + O(\delta^{3/2}). \quad (3.2)$$

Substituting the aforementioned expansion into the pricing equation and rearranging the terms according to the order of ϵ and δ , the pricing equation in the continuation domain C can be rewritten as follows.

$$\left[\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_{10} + \sqrt{\delta} \mathcal{L}_{11} + \delta \mathcal{L}_{12}) + (\mathcal{L}_{20} + \sqrt{\delta} \mathcal{L}_{21} + \delta \mathcal{L}_{22}) - \lambda \right] P^{\epsilon, \delta} + \lambda \alpha (K - s)^+ = 0. \quad (3.3)$$

Here,

$$\begin{aligned} \mathcal{L}_0 &= v^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}, \\ \mathcal{L}_{10} &= v \sqrt{2} \left(\rho \sigma s \frac{\partial^2}{\partial s \partial y} - \Lambda \frac{\partial}{\partial y} \right), & \mathcal{L}_{20} &= \frac{1}{2} \sigma^2 s^2 \frac{\partial^2}{\partial s^2} + r s \frac{\partial}{\partial s} - r, \\ \mathcal{L}_{11} &= \rho v \sqrt{2} \sigma s f(y) \log s \frac{\partial^2}{\partial s \partial y}, & \mathcal{L}_{21} &= \sigma^2 s^2 f(y) \log s \frac{\partial^2}{\partial s^2}, \\ \mathcal{L}_{12} &= \frac{\rho v \sigma}{\sqrt{2}} s f^2(y) (\log s)^2 \frac{\partial^2}{\partial s \partial y}, & \mathcal{L}_{22} &= \sigma^2 s^2 f^2(y) (\log s)^2 \frac{\partial^2}{\partial s^2}. \end{aligned} \quad (3.4)$$

$\mathcal{L}_0$ is the infinitesimal generator of the fast mean-reverting elasticity factor y , and $\mathcal{L}_{20}$ is the Black–Scholes operator with volatility σ . The operators $\mathcal{L}_{10}$, $\mathcal{L}_{11}$, and $\mathcal{L}_{12}$ contain the s – y mixed derivative generated by correlation, whereas $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$, already defined in Eq (3.4), are the first- and second-order elasticity perturbations of the asset diffusion operator, respectively.

Now, we expand the price and free boundary as follows:

$$P^{\epsilon, \delta}(s, y) = P_0(s) + \sqrt{\delta} P_{1,0}(s) + \sqrt{\delta \epsilon} P_{1,1}(s) + \delta P_{2,0}(s) + \cdots, \quad (3.5)$$

and

$$s_f^{\epsilon, \delta} = s_0 + \sqrt{\delta} s_{1,0} + \sqrt{\delta \epsilon} s_{1,1} + \delta s_{2,0} + \cdots. \quad (3.6)$$

Now, to apply the multi-scale asymptotic method of Fouque et al. [18], we substitute the expansions (3.5) and (3.6) into (3.3) and organize the terms by the order of ϵ and δ as follows.

$$\begin{aligned} 0 &= \frac{1}{\epsilon} \mathcal{L}_0 P_0 + \frac{\sqrt{\delta}}{\epsilon} \mathcal{L}_0 P_{1,0} + \frac{\delta}{\epsilon} \mathcal{L}_0 P_{2,0} \\ &\quad + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_{10} P_0 \\ &\quad + \frac{\sqrt{\delta}}{\sqrt{\epsilon}} (\mathcal{L}_0 P_{1,1} + \mathcal{L}_{10} P_{1,0} + \mathcal{L}_{11} P_0) \\ &\quad + \frac{\delta}{\sqrt{\epsilon}} (\mathcal{L}_{10} P_{2,0} + \mathcal{L}_{11} P_{1,0} + \mathcal{L}_{12} P_0) \\ &\quad + ((\mathcal{L}_{20} - \lambda) P_0 + \lambda \alpha (K - s)^+) \\ &\quad + \sqrt{\delta} ((\mathcal{L}_{20} - \lambda) P_{1,0} + \mathcal{L}_{21} P_0 + \mathcal{L}_{10} P_{1,1}) \\ &\quad + \delta ((\mathcal{L}_{20} - \lambda) P_{2,0} + \mathcal{L}_{21} P_{1,0} + \mathcal{L}_{22} P_0 + \mathcal{L}_{11} P_{1,1}) + \cdots. \end{aligned} \quad (3.7)$$

If we compare the terms $O(1/\epsilon)$, $O(\sqrt{\delta}/\epsilon)$, and $O(\delta/\epsilon)$ individually, we find that

$$\mathcal{L}_0 P_0 = 0, \quad \mathcal{L}_0 P_{1,0} = 0, \quad \mathcal{L}_0 P_{2,0} = 0. \quad (3.8)$$

Here, $\mathcal{L}_0$ is the infinitesimal generator of the fast mean-reverting Ornstein–Uhlenbeck process. This process is ergodic, and the null space of $\mathcal{L}_0$ consists of functions that do not depend on y . Therefore,

$$P_0(s, y) = P_0(s), \quad P_{1,0}(s, y) = P_{1,0}(s), \quad P_{2,0}(s, y) = P_{2,0}(s). \quad (3.9)$$

In other words, the main price terms in the asymptotic expansion all depend only on the original asset price s and not on the variable y .

Applying (3.9) to the $O(\sqrt{\delta}/\epsilon)$ terms, we obtain $\mathcal{L}_0 P_{1,1} = 0$, and therefore,

$$P_{1,1}(s, y) = P_{1,1}(s). \quad (3.10)$$

We now apply the Fredholm solvability condition to the equations of each order to derive the ordinary differential equations (ODEs) satisfied by P_0 , $P_{1,0}$, $P_{1,1}$, and $P_{2,0}$. To this end, we define $\langle \cdot \rangle$ as the average with respect to the invariant distribution of the fast-mean reverting factor Y_t . That is, for some function g ,

$$\langle g \rangle = \int_{\mathbb{R}} g(y) \Phi(y) dy,$$

where $\Phi(y)$ is the invariant density function of Y by

$$\Phi(y) = \frac{1}{\sqrt{2\pi\nu}} e^{-(y-m)^2/2\nu^2}.$$

Applying the Fredholm solvability condition (centering condition; cf. [31]) to the equation obtained from the $O(1)$ -order term, we have

$$\langle (\mathcal{L}_{20} - \lambda) P_0 + \lambda \alpha (K - s)^+ \rangle = 0. \quad (3.11)$$

However, since the leading-order term P_0 is independent of y and $\mathcal{L}_{20}$ acts only on the s -variable, the average operator can be eliminated. Therefore, the leading-order term P_0 satisfies

$$(\mathcal{L}_{20} - \lambda) P_0 + \lambda \alpha (K - s)^+ = 0. \quad (3.12)$$

The free-boundary problem for P_0 is then as follows:

$$\begin{cases} \mathcal{M} P_0 = -\lambda \alpha (K - s)^+, & s > s_0, \\ P_0(s_0) = K - s_0, \\ \left. \frac{dP_0}{ds} \right|_{s=s_0} = -1, \end{cases} \quad (3.13)$$

where

$$\mathcal{M} = \frac{1}{2} \sigma^2 s^2 \frac{d^2}{ds^2} + rs \frac{d}{ds} - (r + \lambda). \quad (3.14)$$

We shall now derive the ODEs satisfied by $P_{1,0}$, $P_{1,1}$, and $P_{2,0}$, respectively. First, we apply the Fredholm solvability condition for the Poisson equation with respect to $\mathcal{L}_0$ to the term of order $O(\sqrt{\delta})$, which gives

$$(\mathcal{L}_{20} - \lambda)P_{1,0} = -\langle \mathcal{L}_{21} \rangle P_0 = -\sigma^2 \langle f \rangle s^2 \log s \frac{d^2 P_0}{ds^2}. \quad (3.15)$$

Furthermore, expanding the value-matching and smooth-pasting conditions at $s = s_f^{\epsilon, \delta}$ yields

$$P_{1,0}(s_0) = 0, \quad s_{1,0} = -\frac{\frac{d}{ds}P_{1,0}(s_0)}{\frac{d^2}{ds^2}P_0(s_0)}. \quad (3.16)$$

Next, we derive the equation for the mixing correction term $P_{1,1}$ of order $O(\sqrt{\delta\epsilon})$. This term arises from the interaction between the fast mean-reversion factor and the stochastic elasticity perturbation. To represent this, we define ϕ as the solution to the following Poisson equation:

$$\mathcal{L}_0 \phi = f - \langle f \rangle, \quad \text{where } \langle \phi \rangle = 0. \quad (3.17)$$

Applying the centering condition to an equation of order $O(\sqrt{\delta\epsilon})$, following the method of Fouque et al. [18], we obtain

$$(\mathcal{L}_{20} - \lambda)P_{1,1} = \langle \mathcal{L}_{10} \mathcal{L}_0^{-1} (\mathcal{L}_{21} - \langle \mathcal{L}_{21} \rangle) \rangle P_0. \quad (3.18)$$

More specifically, this can be written as

$$\mathcal{M}P_{1,1} = V_\rho s \frac{d}{ds} \left(s^2 \log s \frac{d^2 P_0}{ds^2} \right) + V_\Lambda s^2 \log s \frac{d^2 P_0}{ds^2}, \quad (3.19)$$

where

$$V_\rho = -\sqrt{2}\nu\rho\sigma\langle\phi'\rangle, \quad V_\Lambda = -\sqrt{2}\nu\langle\Lambda\phi'\rangle. \quad (3.20)$$

Furthermore, expanding the boundary conditions gives

$$P_{1,1}(s_0) = 0, \quad s_{1,1} = -\frac{\frac{d}{ds}P_{1,1}(s_0)}{\frac{d^2}{ds^2}P_0(s_0)}. \quad (3.21)$$

Finally, by applying the solvability condition to the term of order $O(\delta)$ in the same manner, we obtain

$$(\mathcal{L}_{20} - \lambda)P_{2,0} = -\langle \mathcal{L}_{21} \rangle P_{1,0} - \langle \mathcal{L}_{22} \rangle P_0. \quad (3.22)$$

Then,

$$\mathcal{M}P_{2,0} = -\sigma^2 \langle f \rangle s^2 \log s \frac{d^2 P_{1,0}}{ds^2} - \sigma^2 \langle f^2 \rangle s^2 (\log s)^2 \frac{d^2 P_0}{ds^2}. \quad (3.23)$$

Furthermore, the following holds from the value-matching and smooth-pasting conditions:

$$P_{2,0}(s_0) = -s_{1,0} \frac{d}{ds} P_{1,0}(s_0) - \frac{1}{2} s_{1,0}^2 \frac{d^2}{ds^2} P_0(s_0), \quad (3.24)$$

$$s_{2,0} = -\frac{\frac{d}{ds}P_{2,0}(s_0) + s_{1,0} \frac{d^2}{ds^2}P_{1,0}(s_0) + \frac{1}{2} s_{1,0}^2 \frac{d^3}{ds^3}P_0(s_0)}{\frac{d^2}{ds^2}P_0(s_0)}. \quad (3.25)$$

Proposition 3.1. Suppose that the leading-order price $P_0 = P_{0,0}$ is the leading-order solution of the vulnerable perpetual American put option and satisfies the free-boundary problem (3.13). The corresponding characteristic equation is

$$\frac{1}{2}\sigma^2\eta(\eta-1) + r\eta - (r+\lambda) = 0. \quad (3.26)$$

Let $\eta_1 > 1$ and $\eta_2 < 0$ denote the two roots. Then the leading-order price takes the form

$$P_0(s) = \begin{cases} K - s, & 0 < s \leq s_0, \\ As^{\eta_2} + Bs^{\eta_1} + a_0 - \alpha s, & s_0 < s < K, \\ Cs^{\eta_2}, & s \geq K, \end{cases} \quad (3.27)$$

where the constants are given by

$$\begin{aligned} a_0 &= \frac{\lambda\alpha K}{r+\lambda}, \\ A &= s_0^{-\eta_2} \frac{\eta_1(K-a_0) - (\eta_1-1)(1-\alpha)s_0}{\eta_1-\eta_2}, \\ B &= -s_0^{-\eta_1} \frac{\eta_2(K-a_0) + (1-\eta_2)(1-\alpha)s_0}{\eta_1-\eta_2}, \\ C &= A + BK^{\eta_1-\eta_2} + (a_0 - \alpha K)K^{-\eta_2}. \end{aligned} \quad (3.28)$$

Also, the leading-order free boundary s_0 is determined by

$$-\eta_2(K-a_0) + (\eta_2-1)(1-\alpha)s_0 = [\alpha K(1-\eta_2) + \eta_2 a_0] \left(\frac{s_0}{K}\right)^{\eta_1}. \quad (3.29)$$

Proof. This proof follows the leading-order free-boundary argument for vulnerable perpetual American put options presented by Ha and Yoon [32]. In the SEV setting, the leading-order equation is derived from the Black–Scholes equation, with the effective volatility in [32] replaced by a constant σ and a constant default intensity λ .

Substituting the expression $P(s) = s^\eta$ into the homogeneous equation $\mathcal{M}P_0 = 0$ results in the characteristic equation (3.26). The two roots of this equation are defined as $\eta_1 > 1$ and $\eta_2 < 0$. First, within the interval $s_0 < s < K$, the source term is $-\lambda\alpha(K-s)$, and the particular solution is

$$a_0 - \alpha s,$$

where $a_0 = \lambda\alpha K/(r+\lambda)$. It can be demonstrated that the following equation is true: $\mathcal{M}(a_0 - \alpha s) = -\lambda\alpha(K-s)$. The general solution is therefore as follows.

$$P_0(s) = As^{\eta_2} + Bs^{\eta_1} + a_0 - \alpha s, \quad s_0 < s < K.$$

Subsequently, when $s \geq K$, the source term becomes zero. As $s \rightarrow \infty$, the value must converge to 0; therefore, the increasing term s^{η_1} is ruled out, and thus

$$P_0(s) = Cs^{\eta_2}, \quad s \geq K.$$

Within the exercise region, a fundamental calculation reveals that when $0 < s \leq s_0$, $P_0(s) = K - s$.

We must now determine the constants and boundary conditions. The value-matching at $s = s_0$ and the smooth-pasting conditions yield the following:

$$\begin{aligned} As_0^{\eta_2} + Bs_0^{\eta_1} + a_0 - \alpha s_0 &= K - s_0, \\ \eta_2 As_0^{\eta_2-1} + \eta_1 Bs_0^{\eta_1-1} - \alpha &= -1. \end{aligned}$$

The solution to these two linear equations results in the coefficients A and B that are presented in Eq (3.28). The expression for C can be obtained from the continuity condition $P_0(K^-) = P_0(K^+)$. It is evident that, upon applying the differential continuity condition $\frac{d}{ds}P_0(K^-) = \frac{d}{ds}P_0(K^+)$ and substituting the expressions for A , B , and C , the free boundary equation is obtained. This completes the proof. $\square$

The leading-order price P_0 is representative of the price of a vulnerable perpetual American put option under the averaged Black–Scholes benchmark. The default intensity λ has been shown to increase the effective discount rate. The recovery rate α is reflected via the source term $\lambda R(S_t) = \lambda \alpha(K - s_t)$. Consequently, P_0 reflects the average impact of the counterparty's default risk and partial recovery but does not yet incorporate stochastic elasticity corrections.

Next, the first-order elasticity correction $P_{1,0}$ is generated by the averaged operator $\mathcal{L}_{21}$. Because the payoff changes its analytic form at $s = K$, the price correction is represented separately on the two sides of the strike. However, crossing K does not itself trigger exercise or a cash flow. The option value and its delta are therefore matched continuously at the strike, as in the piecewise formulation of American put prices by Zhu and Zhang [33]. Applying these matching conditions order by order to the asymptotic price expansion, and noting that P_0 already satisfies them, gives

$$P_{1,0}(K^-) = P_{1,0}(K^+), \quad \frac{d}{ds}P_{1,0}(K^-) = \frac{d}{ds}P_{1,0}(K^+).$$

These two conditions determine the integration constants in the piecewise solution for $P_{1,0}$.

Proposition 3.2. *Suppose that the first-order elasticity correction $P_{1,0}$ satisfies the following free-boundary problem:*

$$\begin{cases} \mathcal{M}P_{1,0} = -\sigma^2 \langle f \rangle s^2 \log s \frac{d^2 P_0}{ds^2}, \\ P_{1,0}(s_0) = 0, \\ P_{1,0}(K^-) = P_{1,0}(K^+), \\ \left. \frac{dP_{1,0}}{ds} \right|_{s=K^-} = \left. \frac{dP_{1,0}}{ds} \right|_{s=K^+}, \end{cases} \quad (3.30)$$

where the operator $\mathcal{M}$ is defined as (3.14). Define $\chi(\eta)$ by

$$\chi(\eta) := \frac{1}{2}\sigma^2\eta(\eta-1) + r\eta - (r+\lambda), \quad (3.31)$$

and let $\chi'(\eta)$ denote its derivative with respect to η . Then $P_{1,0}$ has the following piecewise representation:

$$P_{1,0}(s) = \begin{cases} 0, & 0 < s \leq s_0, \\ s^{\eta_2} \left(a_{1,0}^- (\log s)^2 + b_{1,0}^- \log s + c_{1,0}^- \right) + s^{\eta_1} \left(a_{1,0}^+ (\log s)^2 + b_{1,0}^+ \log s + c_{1,0}^+ \right), & s_0 < s < K, \\ s^{\eta_2} \left(\widehat{a}_{1,0} (\log s)^2 + \widehat{b}_{1,0} \log s + \widehat{c}_{1,0} \right), & s \geq K, \end{cases} \quad (3.32)$$

where the coefficients of the logarithmic terms are given by

$$\begin{aligned} a_{1,0}^- &= -\frac{q_1 A \eta_2 (\eta_2 - 1)}{2\chi'(\eta_2)}, & a_{1,0}^+ &= -\frac{q_1 B \eta_1 (\eta_1 - 1)}{2\chi'(\eta_1)}, & \widehat{a}_{1,0} &= -\frac{q_1 C \eta_2 (\eta_2 - 1)}{2\chi'(\eta_2)}, \\ b_{1,0}^- &= \frac{q_1 A \eta_2 (\eta_2 - 1) \sigma^2}{2(\chi'(\eta_2))^2}, & b_{1,0}^+ &= \frac{q_1 B \eta_1 (\eta_1 - 1) \sigma^2}{2(\chi'(\eta_1))^2}, & \widehat{b}_{1,0} &= \frac{q_1 C \eta_2 (\eta_2 - 1) \sigma^2}{2(\chi'(\eta_2))^2}, \\ c_{1,0}^- &= -U_{1,0}^-(\log s_0) - s_0^{\eta_1 - \eta_2} (c_{1,0}^+ + U_{1,0}^+(\log s_0)), & c_{1,0}^+ &= \frac{R_2 - \eta_2 R_1}{K^{\eta_1 - \eta_2} (\eta_1 - \eta_2)}, \\ \widehat{c}_{1,0} &= c_{1,0}^- + K^{\eta_1 - \eta_2} c_{1,0}^+ - R_1. \end{aligned} \quad (3.33)$$

Here, the remaining constants are given by

$$\begin{aligned} q_1 &= \sigma^2 \langle f \rangle, \\ R_1 &= \widehat{U}_{1,0}(\log K) - U_{1,0}^-(\log K) - K^{\eta_1 - \eta_2} U_{1,0}^+(\log K), \\ R_2 &= \eta_2 \left(\widehat{U}_{1,0}(\log K) - U_{1,0}^-(\log K) \right) + \widehat{U}'_{1,0}(\log K) - (U_{1,0}^-)'(\log K) - K^{\eta_1 - \eta_2} \left(\eta_1 U_{1,0}^+(\log K) + (U_{1,0}^+)'(\log K) \right), \end{aligned} \quad (3.34)$$

where η_1 and η_2 are two roots of the characteristic equation (3.26) and $U_{1,0}^-(x) = a_{1,0}^- x^2 + b_{1,0}^- x$, $U_{1,0}^+(x) = a_{1,0}^+ x^2 + b_{1,0}^+ x$, $\widehat{U}_{1,0}(x) = \widehat{a}_{1,0} x^2 + \widehat{b}_{1,0} x$.

Furthermore, the first correction to the free boundary is determined by

$$s_{1,0} = -\frac{\frac{d}{ds} P_{1,0}(s_0)}{\frac{d^2}{ds^2} P_0(s_0)}. \quad (3.35)$$

Proof. See Appendix A. □

Next, the mixed correction term $P_{1,1}$ arises from the interaction between the fast mean-reverting factor and the stochastic elasticity structure.

Proposition 3.3. Suppose that the mixed correction $P_{1,1}$ satisfies the following free-boundary problem:

$$\begin{cases} \mathcal{M}P_{1,1} = V_\rho s \frac{d}{ds} \left(s^2 \log s \frac{d^2 P_0}{ds^2} \right) + V_\Lambda s^2 \log s \frac{d^2 P_0}{ds^2}, \\ P_{1,1}(s_0) = 0, \\ P_{1,1}(K^-) = P_{1,1}(K^+), \\ \left. \frac{dP_{1,1}}{ds} \right|_{s=K^-} = \left. \frac{dP_{1,1}}{ds} \right|_{s=K^+}, \end{cases} \quad (3.36)$$

where the operator $\mathcal{M}$ is defined as (3.14) and

$$V_\rho = -\sqrt{2} \nu \rho \sigma \langle \phi' \rangle, \quad V_\Lambda = -\sqrt{2} \nu \langle \Lambda \phi' \rangle. \quad (3.37)$$

Then $P_{1,1}$ captures the effects of the correlation ρ between the asset price and the elasticity factor, as well as the market price of elasticity risk Λ . Its solution again takes a power-logarithmic form:

$$P_{1,1}(s) = \begin{cases} 0, & 0 < s \leq s_0, \\ s^{\eta_2} \left(a_{1,1}^- (\log s)^2 + b_{1,1}^- \log s + c_{1,1}^- \right) + s^{\eta_1} \left(a_{1,1}^+ (\log s)^2 + b_{1,1}^+ \log s + c_{1,1}^+ \right), & s_0 < s < K, \\ s^{\eta_2} \left(\widehat{a}_{1,1} (\log s)^2 + \widehat{b}_{1,1} \log s + \widehat{c}_{1,1} \right), & s \geq K, \end{cases} \quad (3.38)$$

where the coefficients of the logarithmic terms are given by

$$\begin{aligned} a_{1,1}^- &= -\frac{H_{\eta_2}(A)L_{\eta_2}}{2\chi'(\eta_2)}, & a_{1,1}^+ &= -\frac{H_{\eta_1}(B)L_{\eta_1}}{2\chi'(\eta_1)}, & \widehat{a}_{1,1} &= -\frac{H_{\eta_2}(C)L_{\eta_2}}{2\chi'(\eta_2)}, \\ b_{1,1}^- &= \frac{-H_{\eta_2}(A)M_{\eta_2} - \sigma^2 a_{1,1}^-}{\chi'(\eta_2)}, & b_{1,1}^+ &= \frac{-H_{\eta_1}(B)M_{\eta_1} - \sigma^2 a_{1,1}^+}{\chi'(\eta_1)}, & \widehat{b}_{1,1} &= \frac{-H_{\eta_2}(C)M_{\eta_2} - \sigma^2 \widehat{a}_{1,1}}{\chi'(\eta_2)}, \\ c_{1,1}^- &= -U_{1,1}^-(\log s_0) - s_0^{\eta_1 - \eta_2} (c_{1,1}^+ + U_{1,1}^+(\log s_0)), & c_{1,1}^+ &= \frac{R_4 - \eta_2 R_3}{K^{\eta_1 - \eta_2}(\eta_1 - \eta_2)}, \\ \widehat{c}_{1,1} &= c_{1,1}^- + K^{\eta_1 - \eta_2} c_{1,1}^+ - R_3. \end{aligned} \quad (3.39)$$

Here, the remaining constants are given by

$$\begin{aligned} H_\eta(D) &= D\eta(\eta - 1), & L_\eta &= V_\rho\eta + V_\Lambda, & M_\eta &= V_\rho, \\ R_3 &= \widehat{U}_{1,1}(\log K) - U_{1,1}^-(\log K) - K^{\eta_1 - \eta_2} U_{1,1}^+(\log K), \\ R_4 &= \eta_2 \left(\widehat{U}_{1,1}(\log K) - U_{1,1}^-(\log K) \right) + \widehat{U}_{1,1}'(\log K) - (U_{1,1}^-)'(\log K) - K^{\eta_1 - \eta_2} \left(\eta_1 U_{1,1}^+(\log K) + (U_{1,1}^+)'(\log K) \right), \end{aligned} \quad (3.40)$$

where η_1 and η_2 are two roots of the characteristic equation (3.26), $U_{1,1}^-(x) = a_{1,1}^- x^2 + b_{1,1}^- x$, $U_{1,1}^+(x) = a_{1,1}^+ x^2 + b_{1,1}^+ x$, $\widehat{U}_{1,1}(x) = \widehat{a}_{1,1} x^2 + \widehat{b}_{1,1} x$, and $\chi(\eta)$ is given by (3.31).

Furthermore, the corresponding free-boundary correction is

$$s_{1,1} = -\frac{\frac{d}{ds}P_{1,1}(s_0)}{\frac{d^2}{ds^2}P_0(s_0)}. \quad (3.41)$$

Proof. Refer to Appendix B. □

Finally, the second-order elasticity correction $P_{2,0}$ is generated by the action of $\mathcal{L}_{22}$ and the interaction between $\mathcal{L}_{21}$ and $P_{1,0}$.

Proposition 3.4. Suppose that the second-order elasticity correction $P_{2,0}$ satisfies the following free boundary problem:

$$\begin{cases} \mathcal{M}P_{2,0} = -\sigma^2 \langle f^2 \rangle s^2 (\log s)^2 \frac{d^2 P_0}{ds^2} - \sigma^2 \langle f \rangle s^2 \log s \frac{d^2 P_{1,0}}{ds^2}, \\ P_{2,0}(s_0) = -s_{1,0} \frac{dP_{1,0}}{ds} \Big|_{s=s_0} - \frac{1}{2} s_{1,0}^2 \frac{d^2 P_0}{ds^2} \Big|_{s=s_0}, \\ P_{2,0}(K^-) = P_{2,0}(K^+), \\ \frac{dP_{2,0}}{ds} \Big|_{s=K^-} = \frac{dP_{2,0}}{ds} \Big|_{s=K^+}, \end{cases} \quad (3.42)$$

where the operator $\mathcal{M}$ is defined as (3.14). Then $P_{2,0}$ admits the following piecewise form:

$$P_{2,0}(s) = \begin{cases} 0, & 0 < s \leq s_0, \\ \sum_{\eta \in \{\eta_1, \eta_2\}} s^\eta \left(a_{2,0}^\eta (\log s)^4 + b_{2,0}^\eta (\log s)^3 + c_{2,0}^\eta (\log s)^2 + d_{2,0}^\eta \log s + e_{2,0}^\eta \right), & s_0 < s < K, \\ s^{\eta_2} \left(\widehat{a}_{2,0} (\log s)^4 + \widehat{b}_{2,0} (\log s)^3 + \widehat{c}_{2,0} (\log s)^2 + \widehat{d}_{2,0} \log s + \widehat{e}_{2,0} \right), & s \geq K, \end{cases} \quad (3.43)$$

where the coefficients of the logarithmic terms are given by

$$\begin{aligned}
 a_{2,0}^{(\eta)} &= \frac{g_3^{(\eta)}}{4\chi'(\eta)}, \quad \widehat{a}_{2,0} = \frac{\widehat{g}_3}{4\chi'(\eta_2)}, \\
 b_{2,0}^{(\eta)} &= \frac{g_2^{(\eta)} - 6\sigma^2 a_{2,0}^{(\eta)}}{3\chi'(\eta)}, \quad \widehat{b}_{2,0} = \frac{\widehat{g}_2 - 6\sigma^2 \widehat{a}_{2,0}}{3\chi'(\eta_2)}, \\
 c_{2,0}^{(\eta)} &= \frac{g_1^{(\eta)} - 3\sigma^2 b_{2,0}^{(\eta)}}{2\chi'(\eta)}, \quad \widehat{c}_{2,0} = \frac{\widehat{g}_1 - 3\sigma^2 \widehat{b}_{2,0}}{2\chi'(\eta_2)}, \\
 d_{2,0}^{(\eta)} &= -\frac{\sigma^2 c_{2,0}^{(\eta)}}{\chi'(\eta)}, \quad \widehat{d}_{2,0} = -\frac{\sigma^2 \widehat{c}_{2,0}}{\chi'(\eta_2)}, \\
 e_{2,0}^{(\eta_1)} &= \frac{R_6 - \eta_2 R_5}{K^{\eta_1 - \eta_2} (\eta_1 - \eta_2)}, \quad e_{2,0}^{(\eta_2)} = s_0^{-\eta_2} \Theta_0 - G_-(\log s_0) - s_0^{\eta_1 - \eta_2} (G_+(\log s_0) + e_{2,0}^{(\eta_1)}), \\
 \widehat{e}_{2,0} &= e_{2,0}^{(\eta_2)} + K^{\eta_1 - \eta_2} e_{2,0}^{(\eta_1)} - R_5.
 \end{aligned} \tag{3.44}$$

Here, the remaining constants are given by

$$\begin{aligned}
 q_1 &:= \sigma^2 \langle f \rangle, \quad q_2 := \sigma^2 \langle f^2 \rangle, \\
 \mathcal{A}_\eta &= \eta(\eta - 1), \quad \mathcal{B}_\eta = 2\eta - 1, \\
 D_{\eta_2} &:= A, \quad D_{\eta_1} := B, \\
 a_{1,0}^{(\eta_2)} &:= a_{1,0}^-, \quad b_{1,0}^{(\eta_2)} := b_{1,0}^-, \quad c_{1,0}^{(\eta_2)} := c_{1,0}^-, \\
 a_{1,0}^{(\eta_1)} &:= a_{1,0}^+, \quad b_{1,0}^{(\eta_1)} := b_{1,0}^+, \quad c_{1,0}^{(\eta_1)} := c_{1,0}^+, \\
 R_5 &= \widehat{G}(\log K) - G_-(\log K) - K^{\eta_1 - \eta_2} G_+(\log K), \\
 R_6 &= \eta_2 (\widehat{G}(\log K) - G_-(\log K)) + \widehat{G}'(\log K) - G'_-(\log K) - K^{\eta_1 - \eta_2} (\eta_1 G_+(\log K) + G'_+(\log K)),
 \end{aligned} \tag{3.45}$$

where A, B are given by Eq (3.28), η_1, η_2 are two roots of the characteristic equation (3.26), and $\chi(\eta)$ is described by (3.31). The source coefficients are defined by

$$\begin{aligned}
 g_3^{(\eta)} &= -q_1 \mathcal{A}_\eta a_{1,0}^{(\eta)}, \\
 g_2^{(\eta)} &= -q_2 D_\eta \mathcal{A}_\eta - q_1 [\mathcal{A}_\eta b_{1,0}^{(\eta)} + 2\mathcal{B}_\eta a_{1,0}^{(\eta)}], \\
 g_1^{(\eta)} &= -q_1 [\mathcal{A}_\eta c_{1,0}^{(\eta)} + \mathcal{B}_\eta b_{1,0}^{(\eta)} + 2a_{1,0}^{(\eta)}], \\
 \widehat{g}_3 &= -q_1 \mathcal{A}_{\eta_2} \widehat{a}_{1,0}, \\
 \widehat{g}_2 &= -q_2 C \mathcal{A}_{\eta_2} - q_1 [\mathcal{A}_{\eta_2} \widehat{b}_{1,0} + 2\mathcal{B}_{\eta_2} \widehat{a}_{1,0}], \\
 \widehat{g}_1 &= -q_1 [\mathcal{A}_{\eta_2} \widehat{c}_{1,0} + \mathcal{B}_{\eta_2} \widehat{b}_{1,0} + 2\widehat{a}_{1,0}].
 \end{aligned} \tag{3.46}$$

Moreover,

$$\Theta_0 := -s_{1,0} \frac{d}{ds} P_{1,0}(s_0) - \frac{1}{2} s_{1,0}^2 \frac{d^2}{ds^2} P_0(s_0), \tag{3.47}$$

and

$$\begin{aligned}
 G_-(x) &:= a_{2,0}^{(\eta_2)} x^4 + b_{2,0}^{(\eta_2)} x^3 + c_{2,0}^{(\eta_2)} x^2 + d_{2,0}^{(\eta_2)} x, \\
 G_+(x) &:= a_{2,0}^{(\eta_1)} x^4 + b_{2,0}^{(\eta_1)} x^3 + c_{2,0}^{(\eta_1)} x^2 + d_{2,0}^{(\eta_1)} x, \\
 \widehat{G}(x) &:= \widehat{a}_{2,0} x^4 + \widehat{b}_{2,0} x^3 + \widehat{c}_{2,0} x^2 + \widehat{d}_{2,0} x.
 \end{aligned} \tag{3.48}$$

Furthermore, the corresponding free-boundary correction is determined by

$$s_{2,0} = -\frac{\frac{1}{2}s_{1,0}^2 \frac{d^3}{ds^3} P_0(s_0) + s_{1,0} \frac{d^2}{ds^2} P_{1,0}(s_0) + \frac{d}{ds} P_{2,0}(s_0)}{\frac{d^2}{ds^2} P_0(s_0)}. \quad (3.49)$$

Proof. See Appendix C. □

Collecting all terms, the asymptotic approximation for the vulnerable perpetual American put option price is given by

$$P^{\text{app}}(s) = P_0(s) + \sqrt{\delta} P_{1,0}(s) + \delta P_{2,0}(s) + \sqrt{\delta \epsilon} P_{1,1}(s). \quad (3.50)$$

The corresponding approximation for the free boundary is

$$s_f^{\text{app}} = s_0 + \sqrt{\delta} s_{1,0} + \delta s_{2,0} + \sqrt{\delta \epsilon} s_{1,1}. \quad (3.51)$$

Therefore, under the asymptotic regime

$$\epsilon \ll \delta \ll \sqrt{\epsilon}, \quad (3.52)$$

the exact price and free boundary satisfy

$$P^{\epsilon, \delta}(s, y) = P^{\text{app}}(s) + O(\delta^{3/2} + \epsilon + \delta \sqrt{\epsilon}), \quad (3.53)$$

$$s_f^{\epsilon, \delta}(y) = s_f^{\text{app}} + O(\delta^{3/2} + \epsilon + \delta \sqrt{\epsilon}). \quad (3.54)$$

This result shows that the leading-order term P_0 incorporates the average effect of counterparty default risk and fractional recovery, while the higher-order corrections $P_{1,0}$, $P_{1,1}$, and $P_{2,0}$ capture the effects of stochastic elasticity, fast mean reversion, correlation structure, and the market price of elasticity risk. In particular, the recovery rate α enters the pricing equation through the inhomogeneous source term

$$\lambda \alpha (K - s)^+$$

and consequently affects all higher-order correction terms through the structure of the leading-order price.

4. Numerical results

This section presents numerical experiments designed to evaluate both the pricing accuracy and the economic implications of the asymptotic approximation (3.50) derived in Section 3. We first assess the accuracy of the proposed approximations by comparing the leading-order benchmark price P_0 and the corrected asymptotic price P^{app} with Monte Carlo benchmark values. This comparison examines the convergence behavior of the asymptotic approximations with respect to the small parameters ϵ and δ , as well as the stability of the Monte Carlo benchmark prices as the number of simulation paths increases. We then investigate the effects of counterparty credit-risk parameters and SEV-specific parameters on option prices and the optimal exercise boundary. In particular, we analyze the roles of the default intensity, recovery rate, stochastic elasticity corrections, asset–elasticity correlation, and the elasticity risk premium through price comparisons, correction-term decompositions, free-boundary analyses, and sensitivity analyses.

Before proceeding to the numerical results, we briefly comment on the economic meaning and choice of the SEV group parameters $\langle f \rangle$, $\langle f^2 \rangle$, $\langle \phi' \rangle$, ν , ρ , and Λ , following the closely related analysis of Yoon and Kim [25].

Since, following Lee, Yang, and Kim [26], f satisfies only the general bound $-\infty < c_1 \leq f(y) \leq c_2 < \infty$ without a fixed sign, whether the SEV effect reinforces or opposes the leverage effect depends on the sign of $\langle f \rangle$: $\langle f \rangle < 0$ implies that local volatility decrease as the asset price rises, consistent with the leverage effect, while $\langle f \rangle > 0$ acts in the opposite direction. The non-negative second moment $\langle f^2 \rangle$ governs the curvature of $P_{2,0}$, and ν , the standard deviation of the invariant distribution of Y_t , controls the amplitude of the fluctuations feeding into both.

The correlation ρ and the market price of elasticity risk Λ jointly enter the mixed correction $P_{1,1}$ through $\langle \phi' \rangle$, the average derivative of the solution to $L_0 \phi = f - \langle f \rangle$. Following [25], the direction in which ρ and Λ shift the optimal exercise boundary depends on the sign of $\langle \phi' \rangle$; since $\langle \phi' \rangle < 0$ here, an increase in Λ or a decrease in ρ strengthens the incentive for early exercise, consistent with $\partial s_f^{\text{app}} / \partial \Lambda < 0$ and $\partial s_f^{\text{app}} / \partial \rho > 0$ in Figure 6(e)–(f). Economically, ρ captures the co-movement between shocks to the asset price and to Y_t , a channel distinct from the leverage effect embedded in $\langle f \rangle$, while Λ reflects the risk-neutral compensation for bearing elasticity risk, analogous to a volatility risk premium.

Regarding parameter values, the group parameters of European-style options can be estimated from the implied volatility smile following Fouque et al. [17, 18], and, since the asymptotic expansion is linear in these parameters, the resulting estimates apply in principle to American-style options under the same SEV dynamics. However, the vulnerable American-style contract considered here is not liquidly traded, so direct calibration is currently infeasible; we therefore adopt parameter values consistent with the closely related studies [20, 25], leaving direct calibration to future work once market data become available.

4.1. Numerical accuracy of the asymptotic approximation

In this subsection, we evaluate the numerical accuracy of the proposed asymptotic approximation by comparing the leading-order price P_0 , the CEV price

$$P_{\text{CEV}}(s) := P_0(s) + \sqrt{\delta} P_{1,0}(s) + \delta P_{2,0}(s),$$

and the corrected asymptotic approximation P^{app} with the Monte Carlo benchmark price P_{MC} . Here, P_{CEV} denotes the part of the asymptotic approximation that excludes the fast-factor mixed correction $\sqrt{\epsilon \delta} P_{1,1}$; we refer to this as the constant-elasticity correction.

Since the target contract is perpetual, the benchmark price is obtained by approximating the original problem over a sufficiently long finite time horizon $[0, T_{\text{max}}]$. In the numerical implementation, we set $T_{\text{max}} = 100$ and $N_t = 10,000$, yielding a uniform time step $\Delta t = T_{\text{max}}/N_t = 0.01$. The resulting benchmark price is then used to evaluate the accuracy of P_0 and P^{app} .

The benchmark price is computed using the Longstaff–Schwartz least-squares Monte Carlo (LSMC) method [34], adapted to the defaultable payoff structure. For each simulated path, the counterparty default time is generated consistently with the Cox specification with constant default intensity λ . At each exercise date, among paths that have not defaulted and are in the money, the immediate exercise payoff $(K - S_t)^+$ is compared with the continuation value estimated by cross-sectional regression. The exercise decision is then determined according to this comparison. With $X = S_t/K$, the continuation-

value regression uses the following seven-term raw polynomial and interaction design vector:

$$\mathbf{b}(X, Y_t) = (1, X, X^2, X^3, Y_t, Y_t^2, XY_t).$$

Let τ denote the exercise time selected by the LSMC algorithm, with $\tau = T_{\max}$ if no earlier exercise occurs, and let τ_d denote the simulated default time. The discounted payoff on each path is given by

$$\Pi = e^{-r\tau}(K - S_\tau)^+ \mathbf{1}_{\{\tau < \tau_d\}} + e^{-r\tau_d} \alpha(K - S_{\tau_d})^+ \mathbf{1}_{\{\tau_d \leq \tau\}}.$$

Thus, if exercise occurs before default, the option holder receives the standard American put payoff, whereas if default occurs before exercise, the holder receives only the fractional recovery payoff. If neither early exercise nor default occurs before $T_{\max}$, the payoff reduces to the terminal payoff $(K - S_{T_{\max}})^+$ at the final date of the finite-horizon approximation. The Monte Carlo benchmark price is computed as the sample average of the discounted pathwise payoffs,

$$P_{\text{MC}} = \frac{1}{N} \sum_{i=1}^N \Pi_i,$$

where N denotes the number of simulated paths.

To quantify the pricing accuracy, we define the relative errors as

$$\text{RE}_1(\%) = 100 \times \frac{|P_0 - P_{\text{MC}}|}{P_{\text{MC}}}, \quad \text{RE}_2(\%) = 100 \times \frac{|P_{\text{CEV}} - P_{\text{MC}}|}{P_{\text{MC}}}, \quad \text{RE}_3(\%) = 100 \times \frac{|P^{\text{app}} - P_{\text{MC}}|}{P_{\text{MC}}}.$$

Here, RE_1 measures the relative error of the leading-order price P_0 , RE_2 measures the relative error of the CEV price P_{CEV} , which includes the constant elasticity correction terms, whereas RE_3 measures that of the corrected asymptotic approximation P^{app} , which includes the stochastic elasticity correction terms. We also report the standard error and the 95% confidence interval of the Monte Carlo estimate. The standard error is computed as

$$\text{SE} = \frac{\widehat{\text{sd}}(\Pi)}{\sqrt{N}},$$

where $\widehat{\text{sd}}(\Pi)$ denotes the sample standard deviation of the discounted pathwise payoffs. The corresponding 95% confidence interval is given by $P_{\text{MC}} \pm 1.96 \text{ SE}$.

Table 1 reports the numerical results obtained by progressively decreasing the small parameters (ϵ, δ) while fixing the number of Monte Carlo simulation paths at $N = 20,000$. Since the leading-order price P_0 is obtained from the averaged leading-order problem, it is independent of ϵ and δ and therefore remains unchanged across all parameter combinations. The CEV approximate price P_{CEV} is the part of the overall asymptotic approximation that excludes the mixed correction term $\sqrt{\epsilon\delta}P_{1,1}(s)$, which arises from the fast Ornstein–Uhlenbeck factor, and depends solely on δ . By contrast, the asymptotic approximation P^{app} depends on ϵ and δ through the higher-order correction terms $\sqrt{\delta}P_{1,0}$, $\sqrt{\epsilon\delta}P_{1,1}$, and $\delta P_{2,0}$. The numerical results demonstrate that the proposed asymptotic approximation provides a substantial improvement in accuracy over the leading-order solution and CEV solution. For example, when $(\epsilon, \delta) = (10^{-4}, 10^{-3})$, the relative error is reduced from 11.127043% for P_0 to 0.552414% for P^{app} . Also, as the relative error of P_{CEV} is found to be 1.269951% for the same set of parameters, the relative error decreases in the order P_0 , P_{CEV} , and P^{app} . Moreover, as ϵ and δ decrease, P^{app}

converges toward the Monte Carlo benchmark price, with the relative error decreasing to 0.027985% at $(\epsilon, \delta) = (10^{-7}, 10^{-6})$. In all cases, the asymptotic approximation falls within the 95% confidence interval of the Monte Carlo benchmark estimate, thereby providing strong numerical support for the consistency and reliability of the proposed asymptotic approximation.

Table 1. $s = 90, K = 100, r = 0.02, \sigma = 0.1, \lambda = 0.1, \alpha = 0.1, \langle f \rangle = 1.2840, \langle f^2 \rangle = 2.7183, \langle \phi' \rangle = -1.2840, \nu = \frac{1}{\sqrt{2}}, \rho = -0.2, \Lambda = 0$, and simulation number = 20,000.

ϵ	δ	ϵ/δ	$\delta/\sqrt{\epsilon}$	P_0	P_{CEV}	P^{app}	P_{MC}	CI (95%)	SE	$ P_0 - P_{MC} $	$ P_{CEV} - P_{MC} $	$ P^{app} - P_{MC} $	RE ₁ (%)	RE ₂ (%)	RE ₃ (%)
10^{-4}	10^{-3}	0.1	0.1000	10.570550	11.742953	11.828297	11.894001	[11.779714, 12.008289]	0.058310	1.323451	0.151048	0.065704	11.127043	1.269951	0.552414
10^{-5}	10^{-4}	0.1	0.0316	10.570550	10.863656	10.872191	10.894115	[10.768583, 11.019647]	0.064047	0.323565	0.030459	0.021924	2.970086	0.279587	0.201242
10^{-6}	10^{-5}	0.1	0.0100	10.570550	10.655475	10.656328	10.660188	[10.545014, 10.775362]	0.058762	0.089638	0.004713	0.003860	0.840865	0.044209	0.036207
10^{-7}	10^{-6}	0.1	0.0031	10.570550	10.596629	10.596715	10.599681	[10.496573, 10.702789]	0.052606	0.029131	0.003052	0.002966	0.274832	0.028796	0.027985

Table 2 reports the numerical results obtained by varying the number of Monte Carlo simulation paths while fixing the small parameters at $(\epsilon, \delta) = (10^{-5}, 10^{-4})$. As the number of simulation paths increases from 10,000 to 100,000, the standard error decreases from 0.045859 to 0.020901, and the corresponding 95% confidence interval becomes progressively narrower. These results indicate that the Monte Carlo benchmark price converges steadily as the number of simulation paths increases. At the same time, the absolute error between the asymptotic approximation P^{app} and the Monte Carlo benchmark price decreases from 0.081921 to 0.011738, while the relative error is reduced substantially from 0.747856% to 0.107847%. In addition, the absolute and relative errors of P_{CEV} are smaller than those of P_0 but larger than those of P^{app} . This demonstrates that the regularised CEV correction term significantly reduces the error in the leading-order price, while the mixed correction term arising from the fast Ornstein–Uhlenbeck factor further improves the agreement with the Monte Carlo price. This demonstrates that the proposed asymptotic approximation converges rapidly toward the Monte Carlo benchmark as the benchmark estimate becomes more stable. By contrast, the leading-order price P_0 maintains a persistent discrepancy from the Monte Carlo benchmark. This is because P_0 incorporates only the averaged effects of counterparty default risk and fractional recovery, whereas P^{app} additionally captures the effects of stochastic elasticity, fast mean reversion, asset–elasticity correlation, and the elasticity risk premium.

Table 2. $s = 90, K = 100, r = 0.02, \sigma = 0.1, \lambda = 0.1, \alpha = 0.1, \epsilon = 0.00001, \delta = 0.00001, \langle f \rangle = 1.2840, \langle f^2 \rangle = 2.7183, \langle \phi' \rangle = -1.2840, \nu = \frac{1}{\sqrt{2}}, \rho = -0.2$, and $\Lambda = 0$.

simulation #	P_0	P_{CEV}	P^{app}	P_{MC}	CI (95%)	SE	$ P_0 - P_{MC} $	$ P_{CEV} - P_{MC} $	$ P^{app} - P_{MC} $	RE ₁ (%)	RE ₂ (%)	RE ₃ (%)
10,000	10.570550	10.863656	10.872191	10.954112	[10.864228, 11.043996]	0.045859	0.383562	0.090456	0.081921	3.501534	0.825770	0.747856
20,000	10.570550	10.863656	10.872191	10.912858	[10.824107, 11.001609]	0.045281	0.342308	0.049202	0.040667	3.136739	0.450861	0.372651
50,000	10.570550	10.863656	10.872191	10.902190	[10.844380, 10.959999]	0.029494	0.331640	0.038534	0.029999	3.041954	0.353448	0.275161
100,000	10.570550	10.863656	10.872191	10.883929	[10.842963, 10.924895]	0.020901	0.313379	0.020273	0.011738	2.879283	0.186267	0.107848

Taken together, Tables 1 and 2 demonstrate the practical accuracy of the proposed asymptotic approximation over the finite parameter configurations considered and the numerical stability of the Monte Carlo benchmark.

It should be noted that the parameter pairs in Table 1 are designed to assess the finite-parameter robustness of the approximation as both ϵ and δ decrease, rather than to provide a direct numerical verification of the distinguished asymptotic limit in (3.52). In particular, since $\epsilon/\delta = 0.1$ is held fixed across the parameter pairs in Table 1, these experiments do not constitute a sequence along which both $\epsilon/\delta \rightarrow 0$ and $\delta/\sqrt{\epsilon} \rightarrow 0$. To complement this finite-parameter robustness analysis, we therefore conduct an additional numerical experiment using a sequence of parameter pairs constructed

to approach systematically the distinguished regime $\epsilon \ll \delta \ll \sqrt{\epsilon}$ in (3.52). The resulting comparison, reported in Table 3, provides a more direct assessment of the asymptotic approximation under progressively stronger scale separation.

Table 3. $s = 90, K = 100, r = 0.02, \sigma = 0.1, \lambda = 0.1, \alpha = 0.1, \langle f \rangle = 1.2840, \langle f^2 \rangle = 2.7183, \langle \phi' \rangle = -1.2840, \nu = \frac{1}{\sqrt{2}}, \rho = -0.2, \Lambda = 0$, and simulation number = 20,000.

ϵ	δ	ϵ/δ	$\delta/\sqrt{\epsilon}$	P_0	P_{CEV}	P^{app}	P_{MC}	CI (95%)	SE	$ P_0 - P_{MC} $	$ P_{CEV} - P_{MC} $	$ P^{app} - P_{MC} $	RE ₁ (%)	RE ₂ (%)	RE ₃ (%)
1.00×10^{-4}	1.00×10^{-3}	0.100	0.100	10.570550	11.742953	11.828297	11.894001	[11.779714, 12.008289]	0.058310	1.323451	0.151048	0.065704	11.127043	1.269951	0.552414
6.25×10^{-6}	1.25×10^{-4}	0.050	0.050	10.570550	10.902985	10.910528	10.954112	[10.831289, 11.076935]	0.062665	0.383562	0.051127	0.043584	3.501532	0.466734	0.397876
1.60×10^{-7}	8.00×10^{-6}	0.020	0.020	10.570550	10.646168	10.646473	10.662636	[10.536469, 10.766821]	0.058763	0.092086	0.016468	0.016163	0.863636	0.154445	0.151588
1.00×10^{-8}	1.00×10^{-6}	0.010	0.010	10.570550	10.596629	10.596655	10.601250	[10.498142, 10.704358]	0.052606	0.030700	0.004621	0.004595	0.289584	0.043586	0.043344
1.00×10^{-8}	1.00×10^{-5}	0.001	0.100	10.570550	10.655473	10.655558	10.671017	[10.555843, 10.786191]	0.058762	0.100467	0.015544	0.015459	0.941494	0.145666	0.144867

Table 3 complements Table 1 by examining the approximation along a parameter sequence explicitly constructed to approach the distinguished asymptotic regime in (3.52). Whereas Table 1 keeps $\epsilon/\delta = 0.1$ fixed while decreasing the absolute magnitudes of ϵ and δ , Table 3 sets

$$\epsilon = c^4, \quad \delta = c^3,$$

so that

$$\frac{\epsilon}{\delta} = \frac{\delta}{\sqrt{\epsilon}} = c.$$

Hence, as c decreases, both scale-separation ratios approach zero simultaneously, providing a systematic numerical path toward $\epsilon \ll \delta \ll \sqrt{\epsilon}$. We consider $c = 0.10, 0.05, 0.02$, and 0.01 . In addition, we include the pair $(\epsilon, \delta) = (10^{-8}, 10^{-5})$ as separate benchmark. For all cases, P^{app} is compared with the full-model Monte Carlo estimate P_{MC} using the same simulation procedure and error measures as in Table 1. Taken together, Tables 1 and 3 provide complementary evidence: Table 1 evaluates finite-parameter numerical robustness along the original fixed-ratio sequence, whereas Table 3 assesses the approximation under systematically stronger scale separation consistent with the theoretical asymptotic hierarchy.

4.2. Credit-risk and SEV effects on option prices and the optimal exercise boundary

Figure 2 illustrates the effects of the recovery rate α and the default intensity λ on the asymptotic option price P^{app} . Figures 2(a) and 2(b) display the option price under varying values of α and λ , respectively, whereas Figures 2(c) and 2(d) depict the corresponding deviations from the baseline parameter values.

Figure 2(a) shows that the option price increases as the recovery rate α increases. This behavior reflects the fact that, in the event of counterparty default, the option holder recovers the fractional payoff $\alpha(K - s)^+$, so that a higher recovery rate mitigates losses arising from default. In contrast, Figure 2(b) shows that the option price decreases as the default intensity λ increases. A higher default intensity increases the likelihood that the counterparty defaults before the option is exercised, and when $\alpha < 1$, the holder recovers only a fraction of the contractual payoff. Consequently, the continuation value of the option is reduced.

Figures 2(c) and 2(d) provide a clearer illustration of these credit-risk effects. The price differences in Figure 2(c) are positive, indicating that an increase in the recovery rate enhances the value of the vulnerable option. In contrast, the price differences in Figure 2(d) are negative, indicating that a

higher default intensity reduces the option value by increasing the likelihood that, before exercise, the holder receives only the fractional recovery payoff rather than the full contractual payoff. Both effects are most pronounced in the continuation region rather than in the deep in-the-money or far out-of-the-money regions. This observation suggests that counterparty credit risk primarily affects the continuation premium, rather than the intrinsic value of the option, by altering the value of delaying exercise.

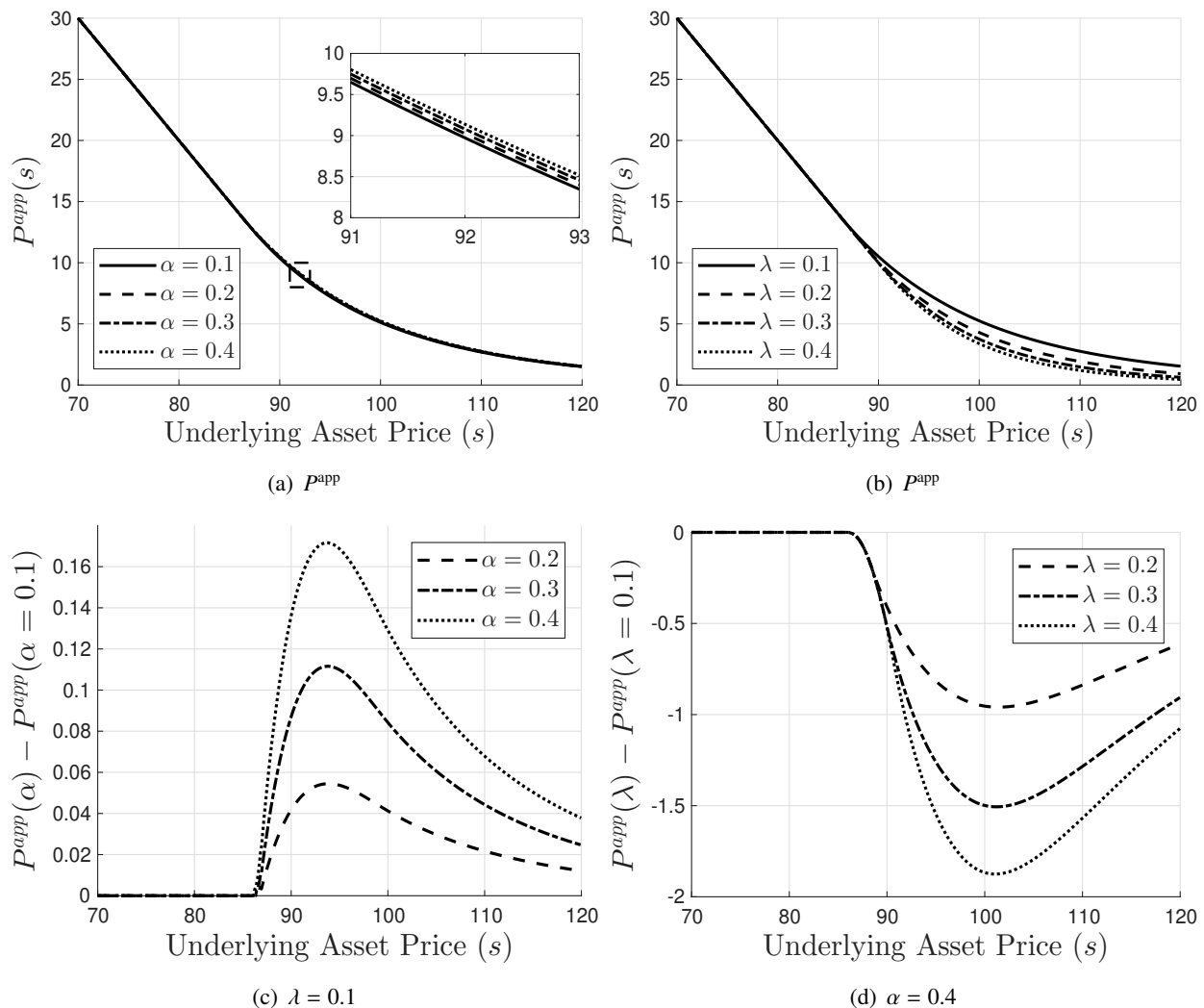


Figure 2. Credit-risk effects on the asymptotic option price P^{app} . Figure (a) and (c) show the effect of the recovery rate α with $\lambda = 0.1$, whereas figures (b) and (d) show the effect of the default intensity λ with $\alpha = 0.4$. The parameters selected are $K = 100$, $r = 0.02$, $\sigma = 0.1$, $\epsilon = 0.0001$, $\delta = 0.001$, $\langle f \rangle = -0.1284$, $\langle f^2 \rangle = 0.272$, $\langle \phi' \rangle = -0.098277$, $\nu = \frac{1}{\sqrt{2}}$, $\rho = -0.2$, and $\Lambda = 0$.

Figure 3 presents a decomposition of the scaled SEV price corrections under different recovery rates α . Whereas Figure 2 illustrates the direct impact of counterparty credit risk on the option price, Figure 3 examines how the recovery structure affects the SEV correction terms after removing the leading-order price P_0 . Figure 3(a) displays the total correction, $P^{\text{app}} - P_0 = \sqrt{\delta}P_{1,0} + \delta P_{2,0} + \sqrt{\epsilon}\delta P_{1,1}$,

whereas Figures 3(b)–3(d) separately display the first-order averaged elasticity correction, the second-order elasticity correction, and the mixed correction associated with the fast mean-reverting factor, respectively.

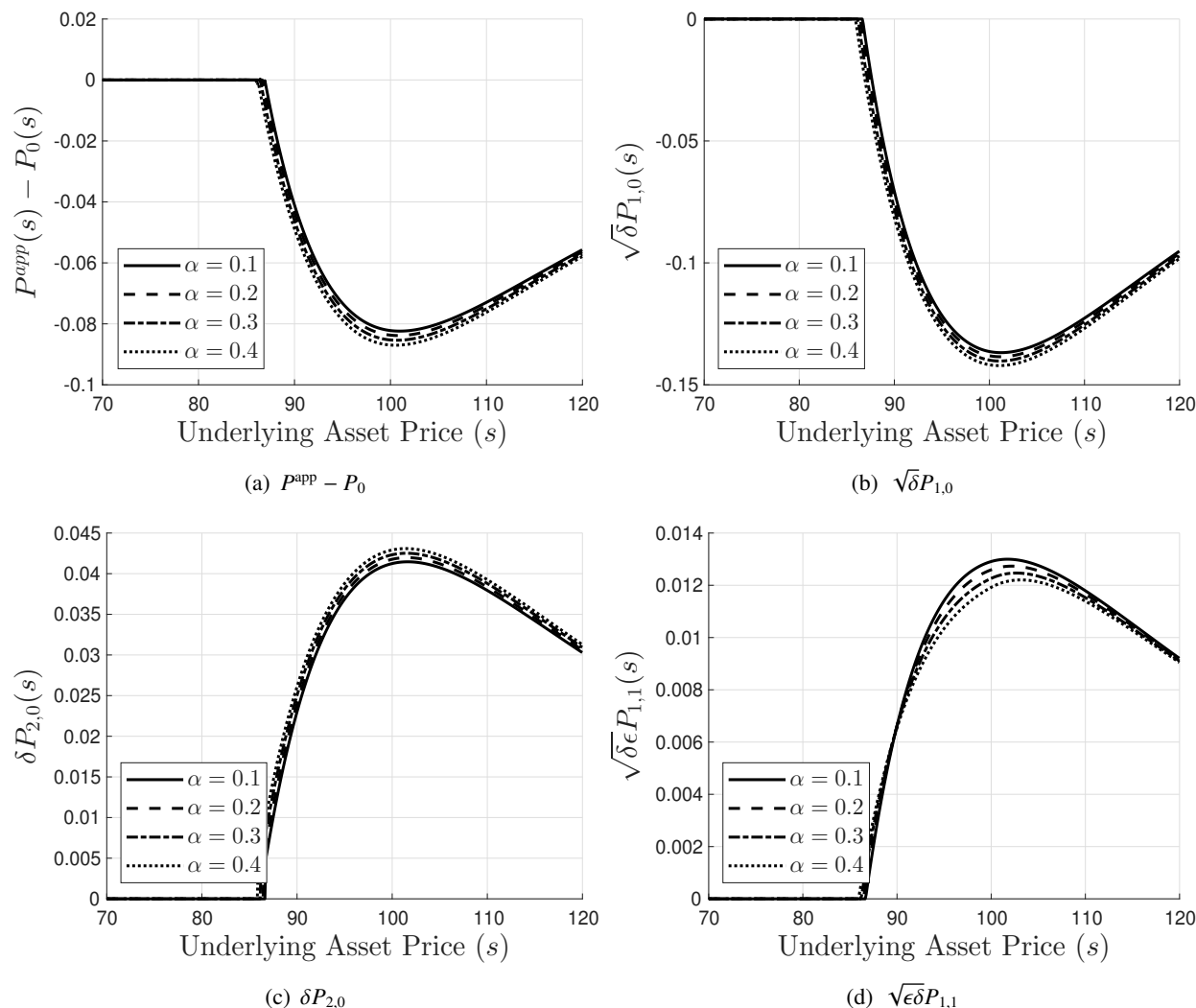


Figure 3. Scaled SEV price-correction decomposition under different recovery rates α . The figures display the total SEV correction $P^{\text{app}} - P_0$, the first-order averaged elasticity correction $\sqrt{\delta}P_{1,0}$, the second-order elasticity correction $\delta P_{2,0}$, and the mixed fast-factor correction $\sqrt{\epsilon\delta}P_{1,1}$, respectively. The parameters taken are $K = 100$, $r = 0.02$, $\sigma = 0.1$, $\epsilon = 0.0001$, $\delta = 0.001$, $\langle f \rangle = -0.1284$, $\langle f^2 \rangle = 0.272$, $\langle \phi' \rangle = -0.098277$, $\nu = \frac{1}{\sqrt{2}}$, $\rho = -0.2$, and $\Lambda = 0$.

Figure 3(a) shows that the total SEV correction, $P^{\text{app}} - P_0$, is negative over most of the continuation region. This indicates that, under the selected parameter configuration, the SEV effect reduces the value of the vulnerable put option relative to the leading-order price P_0 , which already incorporates counterparty default risk and fractional recovery. The correction is close to zero in the exercise region, where the option value is determined primarily by its intrinsic value, $K - s$. By contrast, the correction becomes much more pronounced in the continuation region, where the option value is driven by the prospect of future price movements and the optimal exercise opportunity.

Figure 3(b) further reveals that the dominant contribution to the total correction arises from the first-order term $\sqrt{\delta}P_{1,0}$, which represents the averaged stochastic elasticity effect through $\langle f \rangle$. Since this term is negative throughout the continuation region, it largely determines the negative sign of the total correction observed in Figure 3(a). Economically, this suggests that stochastic elasticity modifies the local diffusion structure of the underlying asset and, under the present parameter setting, reduces the continuation premium of the vulnerable put option. Figure 3(c) shows that the second-order correction $\delta P_{2,0}$ is positive and has a larger magnitude than the mixed correction term. Figure 3(d) displays the mixed correction term $\sqrt{\epsilon\delta}P_{1,1}$, which captures the interaction among the fast mean-reverting elasticity factor, the asset–elasticity correlation, and the elasticity risk premium. Although this correction is positive, its magnitude is relatively small, indicating that the effects of fast mean reversion and correlation partially offset the dominant negative first-order correction. Consequently, the higher-order SEV corrections partially compensate for the dominant negative contribution of $\sqrt{\delta}P_{1,0}$.

The fact that the curves corresponding to different recovery rates α lie very close to one another in Figure 3 should not be interpreted as indicating that the recovery rate has little influence on the option value. Since Figure 2 presents the residual SEV corrections after removing the leading-order price P_0 , the recovery rate first affects the leading-order price and the associated free-boundary structure, with its influence subsequently propagating indirectly to the higher-order correction terms $P_{1,0}$, $P_{2,0}$, and $P_{1,1}$. Consequently, Figure 3 demonstrates that the recovery structure and the SEV effect interact primarily through the continuation value, implying that the impact of stochastic elasticity is more pronounced in the continuation premium than in the intrinsic value of the option.

Figure 4 presents a decomposition of the scaled SEV price corrections under different default intensities λ . Whereas Figure 3 examines how the recovery structure influences the SEV correction terms, Figure 4 illustrates how the correction components vary with the default intensity. Figure 4(a) displays the total SEV correction, $P^{\text{app}} - P_0 = \sqrt{\delta}P_{1,0} + \delta P_{2,0} + \sqrt{\epsilon\delta}P_{1,1}$, whereas Figures 4(b)–4(d) separately display the first-order averaged elasticity correction $\sqrt{\delta}P_{1,0}$, the second-order elasticity correction $\delta P_{2,0}$, and the mixed correction associated with the fast mean-reverting factor $\sqrt{\epsilon\delta}P_{1,1}$, respectively.

Figure 4(a) shows that the total SEV correction, $P^{\text{app}} - P_0$, is negative over most of the continuation region. This indicates that, under the present parameter configuration, stochastic elasticity reduces the value of the vulnerable put option relative to the leading-order price P_0 , which already incorporates counterparty default risk and fractional recovery. Moreover, as the default intensity λ increases, the magnitude of the total correction decreases, while the region over which the correction is significant shifts toward higher asset prices. This behavior reflects the fact that a higher default intensity weakens the holder's incentive to postpone exercise, thereby reducing the continuation region over which the SEV effect can influence the option value.

Figure 4(b) further reveals that the dominant contribution to the total SEV correction arises from the first-order term $\sqrt{\delta}P_{1,0}$. This term represents the averaged stochastic elasticity effect through $\langle f \rangle$ and remains negative for all values of λ . Consequently, the total correction shown in Figure 4(a) is likewise negative throughout the continuation region. In contrast, the second-order correction $\delta P_{2,0}$ in Figure 4(c) and the mixed correction term $\sqrt{\epsilon\delta}P_{1,1}$ in Figure 4(d) are both positive, thereby partially offsetting the dominant negative first-order correction. Moreover, since the magnitude of $\delta P_{2,0}$ exceeds that of $\sqrt{\epsilon\delta}P_{1,1}$, the second-order elasticity correction plays a more significant compensating role in the overall SEV correction.

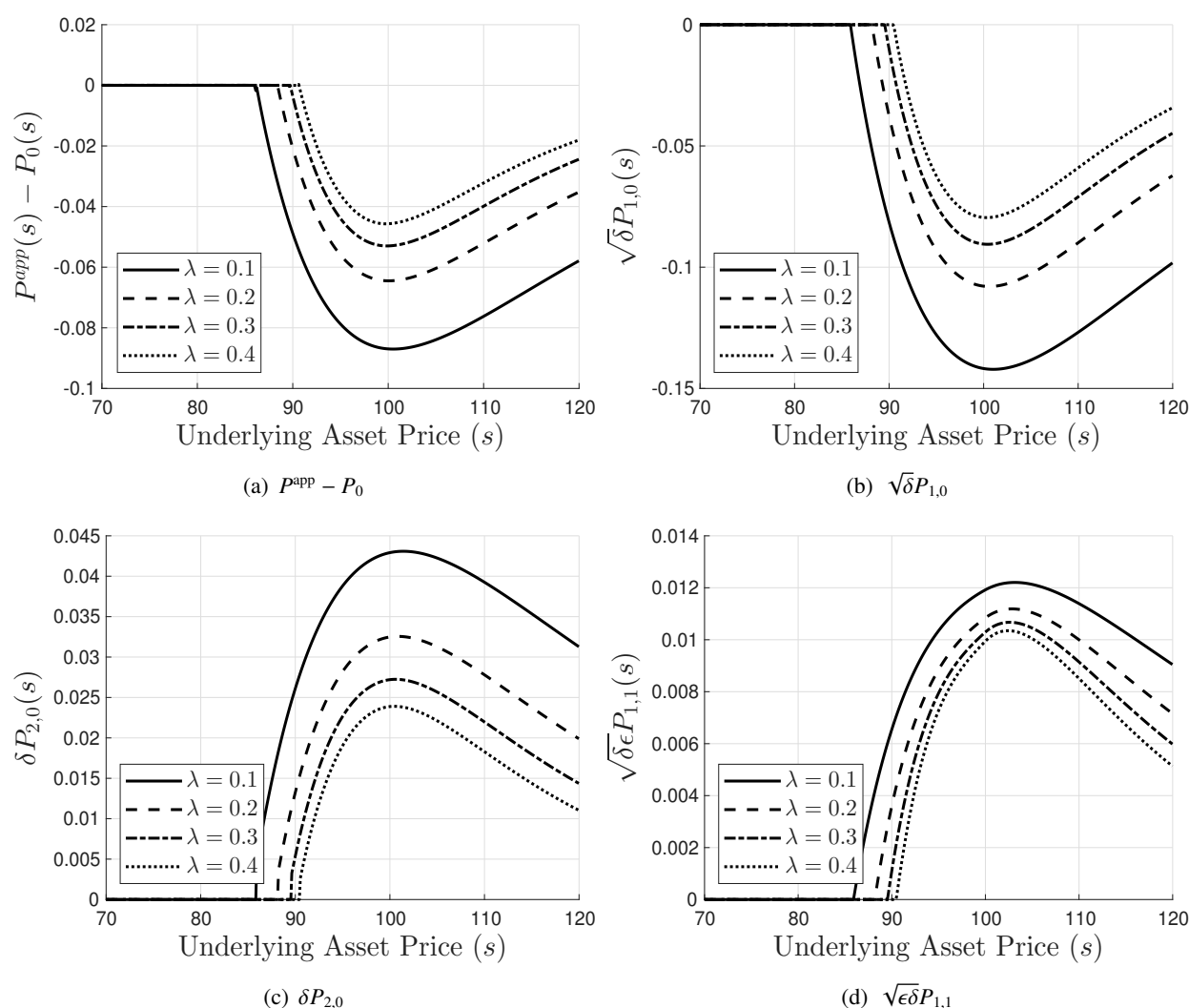


Figure 4. Scaled SEV price-correction decomposition under different default intensities λ . The figures display the total SEV correction $P^{\text{app}} - P_0$, the first-order averaged elasticity correction $\sqrt{\delta}P_{1,0}$, the second-order elasticity correction $\delta P_{2,0}$, and the mixed fast-factor correction $\sqrt{\epsilon\delta}P_{1,1}$, respectively. The recovery rate is fixed at $\alpha = 0.4$. The parameters chosen are $K = 100, r = 0.02, \sigma = 0.1, \epsilon = 0.0001, \delta = 0.001, \langle f \rangle = -0.1284, \langle f^2 \rangle = 0.272, \langle \phi' \rangle = -0.098277, \nu = \frac{1}{\sqrt{2}}, \rho = -0.2$ and $\Lambda = 0$.

Compared with the recovery-rate effects shown in Figure 3, Figure 4 demonstrates that variations in the default intensity λ have a more pronounced impact on the SEV price corrections. This is because the default intensity directly affects both the probability of counterparty default and the default-adjusted discount rate, $r + \lambda$. Consequently, changes in λ modify not only the level of the leading-order price P_0 , but also its curvature, the continuation region, and the associated free-boundary structure, thereby inducing more substantial changes in the SEV correction terms $P_{1,0}$, $P_{2,0}$, and $P_{1,1}$. Overall, Figure 4 demonstrates that the effects of default intensity and stochastic elasticity are not independent. As λ increases, counterparty credit risk becomes increasingly dominant, reducing the continuation premium and, in turn, weakening the SEV price corrections generated by stochastic elasticity.

Figure 5 presents a decomposition of the scaled SEV free-boundary corrections under different recovery rates α as the default intensity λ varies. Figure 5(a) displays the total boundary correction, $s_f^{\text{app}} - s_0 = \sqrt{\delta}s_{1,0} + \delta s_{2,0} + \sqrt{\epsilon\delta}s_{1,1}$, whereas Figures 5(b)–5(d) separately display the first-order averaged elasticity correction $\sqrt{\delta}s_{1,0}$, the second-order elasticity correction $\delta s_{2,0}$, and the mixed correction associated with the fast mean-reverting factor $\sqrt{\epsilon\delta}s_{1,1}$, respectively. Here, s_0 denotes the leading-order free boundary incorporating counterparty default risk and fractional recovery, while s_f^{app} denotes the SEV-corrected approximate free boundary.

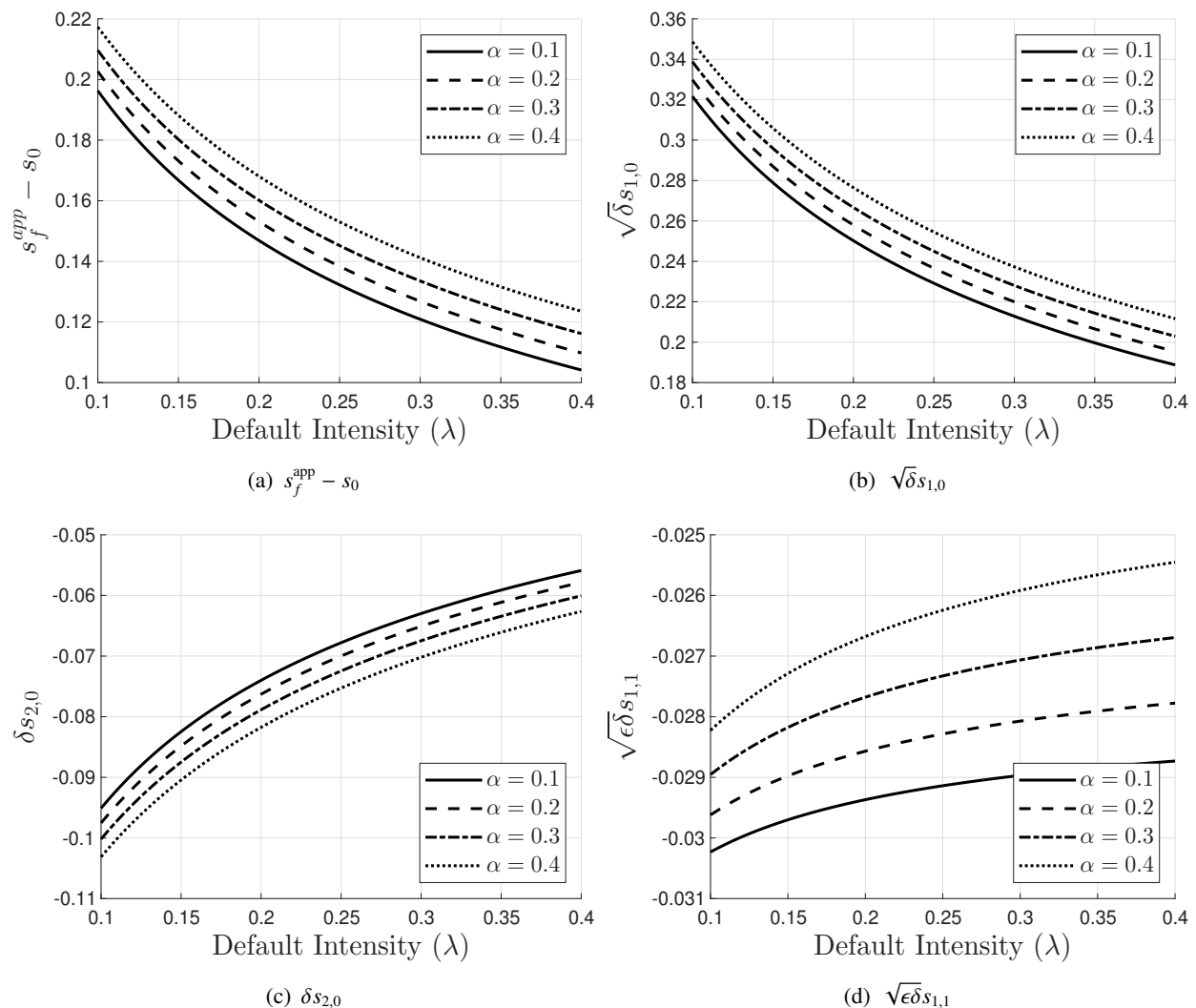


Figure 5. Scaled SEV free-boundary correction decomposition under different recovery rates α . The figures display the total boundary correction $s_f^{\text{app}} - s_0$, the first-order averaged elasticity correction $\sqrt{\delta}s_{1,0}$, the second-order elasticity correction $\delta s_{2,0}$, and the mixed fast-factor correction $\sqrt{\epsilon\delta}s_{1,1}$, respectively, as functions of the default intensity λ . The parameters selected are $K = 100, r = 0.02, \sigma = 0.1, \epsilon = 0.0001, \delta = 0.001, \langle f \rangle = -0.1284, \langle f^2 \rangle = 0.272, \langle \phi' \rangle = -0.098277, \nu = \frac{1}{\sqrt{2}}, \rho = -0.2$, and $\Lambda = 0$.

Figure 5(a) shows that the total boundary correction, $s_f^{\text{app}} - s_0$, remains positive throughout the

parameter range considered. This indicates that the SEV effect shifts the optimal exercise boundary upward relative to the leading-order boundary s_0 . Since the exercise region of an American put option is characterized by $s \leq s_f$, an upward shift in the free boundary implies that stochastic elasticity increases the holder's incentive to exercise early even at higher underlying asset prices. Figure 5(b) shows that the upward shift of the optimal exercise boundary is primarily driven by the positive first-order averaged elasticity correction $\sqrt{\delta}s_{1,0}$. This term represents the averaged stochastic elasticity effect through $\langle f \rangle$ and remains positive for all values of λ and α . By contrast, the second-order correction $\delta s_{2,0}$ shown in Figure 5(c) and the mixed correction $\sqrt{\epsilon}\delta s_{1,1}$ shown in Figure 5(d) are both negative, thereby partially offsetting the dominant positive first-order correction. Moreover, since the magnitude of $\delta s_{2,0}$ exceeds that of $\sqrt{\epsilon}\delta s_{1,1}$, the second-order elasticity correction plays the more significant compensating role in the overall free-boundary correction.

As illustrated in Figure 5, both the total boundary correction $s_f^{\text{app}} - s_0$ and the dominant positive first-order averaged elasticity correction $\sqrt{\delta}s_{1,0}$ decrease as the default intensity λ increases. This reflects the fact that a higher default intensity reduces the continuation value and shortens the period during which the option holder remains exposed to future price movements and SEV effects. By contrast, the negative second-order correction $\delta s_{2,0}$ and the negative mixed correction $\sqrt{\epsilon}\delta s_{1,1}$ move closer to zero as λ increases. Although these correction terms increase in value, their magnitudes decrease, implying that their offsetting effects on the positive first-order correction become weaker. Nevertheless, the overall boundary correction still decreases because the reduction in the dominant first-order correction outweighs the weakening of the higher-order offsetting effects. In contrast, as the recovery rate α increases, the positive boundary correction becomes stronger because higher recovery mitigates default losses and preserves the continuation value.

The key economic implication of Figure 5 is that the SEV effect is more pronounced in the optimal exercise boundary than in the option price. Since the free boundary is the threshold at which the continuation value equals the immediate exercise value, even small changes in the local slope of the pricing function can produce substantial shifts in the exercise boundary. Indeed, the boundary corrections satisfy $s_{1,0} = -\frac{P'_{1,0}(s_0)}{P''_0(s_0)}$ and $s_{1,1} = -\frac{P'_{1,1}(s_0)}{P''_0(s_0)}$, showing that they are determined by the slope of the price corrections and the curvature of the leading-order price at the free boundary. Consequently, even when the SEV correction appears relatively small in the option price itself, it can be amplified through the smooth-fit condition, leading to a more pronounced effect on the optimal exercise boundary. Overall, Figure 5 demonstrates that the SEV corrections shift the optimal exercise boundary upward relative to the leading-order free boundary s_0 , with this adjustment being driven primarily by the first-order averaged elasticity correction.

Figure 6 extends the free-boundary corrections presented in Figure 5 to the full (λ, α) parameter space through a boundary-correction map. The panels display the (λ, α) -dependence of the total boundary correction $s_f^{\text{app}} - s_0$, the first-order averaged elasticity correction $\sqrt{\delta}s_{1,0}$, the second-order correction $\delta s_{2,0}$, and the mixed correction $\sqrt{\epsilon}\delta s_{1,1}$. Both the total boundary correction and the dominant positive first-order correction attain their largest values in the region of low default intensity and high recovery rate. This reflects the fact that a low default intensity and a high recovery rate preserve the continuation value, thereby allowing stochastic elasticity to exert a stronger influence on the optimal exercise boundary. Conversely, as the default intensity increases, the stronger default-adjusted discounting suppresses the SEV free-boundary corrections. Although the negative mixed

and second-order corrections partially offset the positive first-order correction, the total boundary correction remains positive throughout the parameter range considered, implying that the SEV effect ultimately shifts the optimal exercise boundary upward relative to the leading-order boundary s_0 .

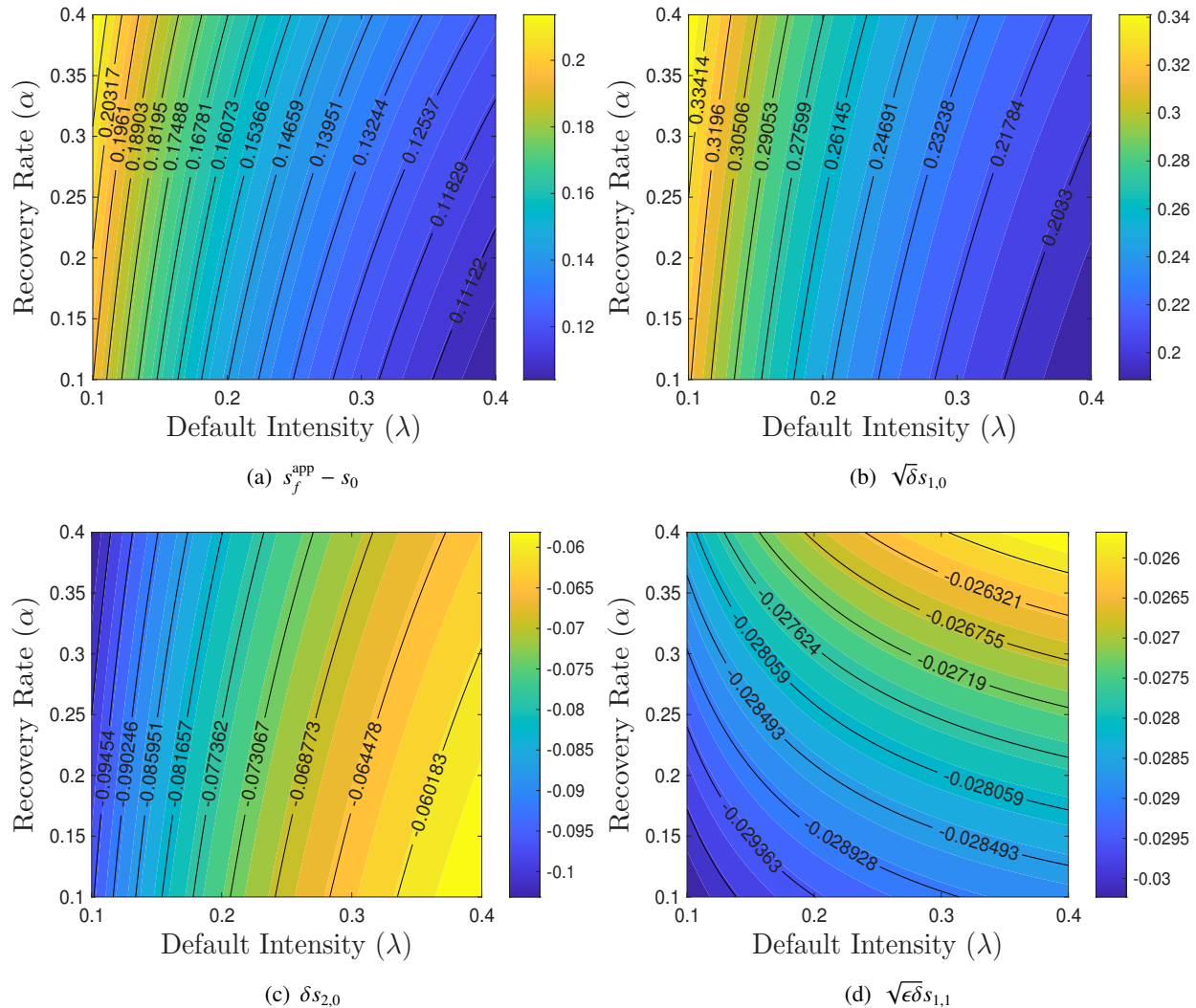


Figure 6. Joint credit-risk map of scaled SEV free-boundary corrections over (λ, α) . The figures display the total boundary correction $s_f^{\text{app}} - s_0$, the first-order averaged elasticity correction $\sqrt{\delta}s_{1,0}$, the second-order elasticity correction $\delta s_{2,0}$, and the mixed fast-factor correction $\sqrt{\epsilon\delta}s_{1,1}$, respectively. The parameters chosen are $K = 100, r = 0.02, \sigma = 0.1, \lambda \in [0.1, 0.4], \alpha \in [0.1, 0.4], \epsilon = 0.0001, \delta = 0.001, \langle f \rangle = -0.1284, \langle f^2 \rangle = 0.272, \langle \phi \rangle = -0.098277, \nu = \frac{1}{\sqrt{2}}, \rho = -0.2$, and $\Lambda = 0$.

Although Figures 5 and 6 present the same free-boundary correction terms, they serve complementary purposes. Figure 5 provides one-dimensional cross-sectional views with respect to the default intensity λ for several fixed values of the recovery rate α , making it convenient to compare the sign, magnitude, and monotonicity of each correction component. By contrast, Figure 6 illustrates the joint effects of λ and α over the entire parameter space. It confirms that the patterns observed in

Figure 5 are not confined to particular recovery rates but remain consistent throughout the parameter region considered. Furthermore, the orientation and spacing of the contour lines indicate that the free-boundary corrections are more sensitive to changes in the default intensity than to changes in the recovery rate, suggesting that the default intensity exerts a more dominant influence on the SEV free-boundary corrections.

Figure 7 illustrates the sensitivities of the asymptotic option price P^{app} and the optimal exercise boundary s_f^{app} with respect to the SEV-specific parameters, namely the elasticity risk premium Λ and the asset–elasticity correlation coefficient ρ . The elasticity risk premium Λ represents the market price of elasticity risk and modifies the risk-neutral dynamics of the fast mean-reverting factor Y_t , whereas ρ governs the instantaneous dependence between the underlying asset price and the elasticity factor. Since both parameters enter the asymptotic approximation through the mixed correction terms $P_{1,1}$ and $s_{1,1}$, Figure 7 highlights SEV-specific sensitivities that cannot be captured by the standard Black–Scholes or CEV models. Under the parameter configuration considered, an increase in Λ raises the option price while lowering the optimal exercise boundary, whereas an increase in ρ reduces the option price while shifting the optimal exercise boundary upward.

As the default intensity λ increases, the probability of counterparty default prior to exercise becomes higher, thereby shortening the period during which the option remains exposed to future price movements and stochastic elasticity effects. Consequently, the sensitivities of both the option price and the optimal exercise boundary with respect to Λ and ρ become weaker overall. By contrast, a higher recovery rate α mitigates default losses and preserves the continuation value. Its impact, however, differs across the two SEV-specific parameters: The sensitivity with respect to Λ becomes stronger as α increases, whereas the magnitude of the sensitivity with respect to ρ decreases. This suggests that the recovery structure does not amplify the SEV effects uniformly but instead modifies the direction and magnitude of each risk channel differently.

Figure 7 further shows that the SEV sensitivities are more pronounced in the optimal exercise boundary than in the option price. This is because the free boundary is the threshold at which the continuation value equals the immediate exercise value, making it highly sensitive to local changes in the slope and curvature of the pricing function through the smooth-fit condition. In particular, $s_{1,1} = -\frac{P'_{1,1}(s_0)}{P''_0(s_0)}$, so that even a relatively small contribution of $P_{1,1}$ to the option price can produce a substantial shift in the optimal exercise boundary through changes in the local slope of the pricing function near s_0 .

Overall, Figure 7 demonstrates that the effects of counterparty credit-risk parameters and SEV-specific parameters are not independent. A higher default intensity λ reduces both the exposure to and the sensitivity with respect to SEV effects, whereas a higher recovery rate α mitigates default losses and modulates the influence of the individual SEV risk channels. Consequently, variations in the elasticity risk premium Λ and the asset–elasticity correlation ρ may have a relatively limited effect on the option price, but they exert a much more pronounced influence on the optimal exercise boundary, thereby having a substantial impact on the holder's optimal early-exercise decision.

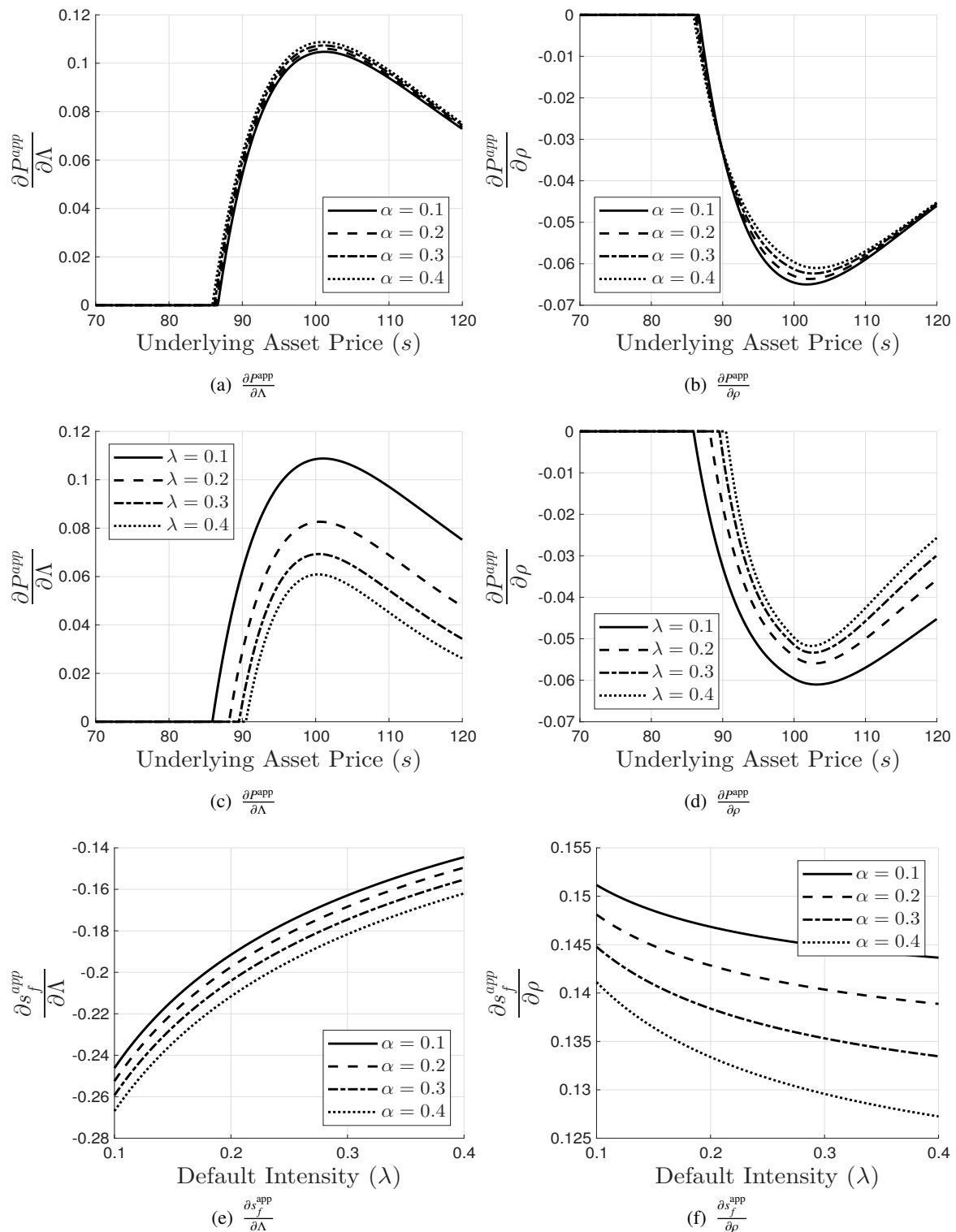


Figure 7. SEV-specific sensitivities with respect to the elasticity risk premium Λ and the asset–elasticity correlation ρ . The figures display the sensitivities of the asymptotic option price and the optimal exercise boundary with respect to Λ and ρ , under different credit-risk parameters λ and α . The parameters taken are $K = 100$, $r = 0.02$, $\sigma = 0.1$, $\epsilon = 0.0001$, $\delta = 0.001$, $\langle f \rangle = -0.1284$, $\langle f^2 \rangle = 0.272$, $\langle \phi \rangle = -0.098277$, and $\nu = \frac{1}{\sqrt{2}}$.

5. Discussion and extensions

The preceding sections focus on vulnerable perpetual American put options. This section provides two preliminary extensions that clarify the broader scope of the framework. First, we formulate the corresponding vulnerable perpetual American call problem and discuss the limitations of a direct call–put symmetry argument. Second, we describe the finite-maturity formulation and highlight the distinction between a stationary perpetual problem and a time-dependent finite-horizon problem. The results below are intended as formulations and directions for further analysis; no additional asymptotic solution is claimed in this section.

5.1. Extension to vulnerable perpetual American call options

Although the main analysis concerns vulnerable perpetual American put options, a corresponding call problem can be formulated within the same default-adjusted optimal-stopping framework. Under an analogous recovery-of-payoff convention, define

$$g_C(s) = (s - K)^+, \quad R_C(s) = \alpha(s - K)^+. \quad (5.1)$$

Here, g_C is the exercise payoff and R_C is the fractional recovery payoff received if counterparty default occurs before exercise. This recovery specification is an additional modeling convention for the call case and is not implied by the put formulation.

Let $\mathcal{T}_{\mathbb{F}}$ denote the set of admissible $\mathbb{F}$ -stopping times on the infinite horizon. Assuming the integrability and transversality conditions required for the call value to be finite, the pre-default value of the vulnerable perpetual American call is formally defined by

$$C^{\epsilon, \delta}(s, y) = \sup_{\tau \in \mathcal{T}_{\mathbb{F}}} \mathbb{E}_{s, y}^{\mathbb{Q}} \left[e^{-(r+\lambda)\tau} g_C(S_{\tau}) + \int_0^{\tau} e^{-(r+\lambda)u} \lambda R_C(S_u) du \right]. \quad (5.2)$$

The same dynamic-programming argument used for the put problem then leads to the formal variational inequality

$$\max \left\{ A^{\epsilon, \delta} C^{\epsilon, \delta} - (r + \lambda) C^{\epsilon, \delta} + \lambda \alpha (s - K)^+, (s - K)^+ - C^{\epsilon, \delta} \right\} = 0 \quad \text{on } D. \quad (5.3)$$

Equations (5.1)–(5.3) provide a formal correspondence between the put and call formulations, but they do not establish an exact call–put symmetry within the same SEV model. In a one-dimensional time-homogeneous local-volatility setting, Alfonsi and Jourdain [35] show that perpetual American call–put duality generally requires simultaneous transformations of the spot and strike, the interest and dividend rates, and the local-volatility function. In the present two-factor setting, a rigorous duality would additionally have to account for the stochastic elasticity factor, the asset–elasticity correlation structure, and the default-recovery term.

Accordingly, the existence, finiteness, and structure of the vulnerable call exercise boundary are not asserted here. A complete derivation of the call price, the associated free-boundary problem, the multiscale asymptotic corrections, and their numerical implementation is left for future work.

5.2. Extension to finite-maturity vulnerable American options

The perpetual formulation may also be extended to a vulnerable American option with a fixed contractual maturity $T > 0$. Related finite-maturity vulnerable American option problems are studied by Klein and Yang [19], although under a different default specification, while general finite-horizon optimal-stopping and free-boundary methods are presented by Peskir and Shiryaev [24]. For concreteness, we state the extension for the put payoff considered in the present paper; replacing the put payoffs below by the call payoffs in (5.1) gives the corresponding finite-maturity call formulation.

Let $t \in [0, T]$, and let $\mathcal{T}_{t,T}^{\mathbb{F}}$ denote the set of admissible $\mathbb{F}$ -stopping times taking values in $[t, T]$. Conditional on survival up to time t , the holder selects $\tau \in \mathcal{T}_{t,T}^{\mathbb{F}}$. The contract termination time is

$$\theta = \tau \wedge \tau_d \wedge T. \quad (5.4)$$

Because $\tau \leq T$, this is equivalently $\theta = \tau \wedge \tau_d$; writing T explicitly displays the three competing termination events. If $\tau < T$ and $\tau < \tau_d$, the option is exercised early. If $\tau_d \leq \tau$, counterparty default occurs first, and the recovery payoff is received. If $\tau = T < \tau_d$, the terminal payoff is paid at maturity.

Set

$$g_P(s) = (K - s)^+, \quad R_P(s) = \alpha(K - s)^+,$$

and write

$$\mathbb{E}_{t,s,y}^{\mathbb{Q}}[\cdot] = \mathbb{E}^{\mathbb{Q}}[\cdot | S_t = s, Y_t = y, \tau_d > t].$$

The finite-maturity pre-default put value is

$$P_T^{\epsilon,\delta}(t, s, y) = \sup_{\tau \in \mathcal{T}_{t,T}^{\mathbb{F}}} \mathbb{E}_{t,s,y}^{\mathbb{Q}} \left[e^{-r(\tau-t)} g_P(S_\tau) \mathbf{1}_{\{\tau < \tau_d\}} + e^{-r(\tau_d-t)} R_P(S_{\tau_d}) \mathbf{1}_{\{\tau_d \leq \tau\}} \right]. \quad (5.5)$$

When $\tau = T < \tau_d$, the first term in (5.5) is the terminal payoff $e^{-r(T-t)} g_P(S_T)$.

Under the constant-intensity Cox specification, the residual default time, conditional on survival up to t , is exponentially distributed with the same intensity λ . Following the reduced-form framework of Lando [8] and Bielecki and Rutkowski [10], Eq (5.5) therefore admits the equivalent default-adjusted representation

$$P_T^{\epsilon,\delta}(t, s, y) = \sup_{\tau \in \mathcal{T}_{t,T}^{\mathbb{F}}} \mathbb{E}_{t,s,y}^{\mathbb{Q}} \left[e^{-(r+\lambda)(\tau-t)} g_P(S_\tau) + \int_t^\tau e^{-(r+\lambda)(u-t)} \lambda R_P(S_u) du \right]. \quad (5.6)$$

Applying the dynamic-programming principle, the finite-maturity value formally satisfies the parabolic variational inequality

$$\max \left\{ \frac{\partial P_T^{\epsilon,\delta}}{\partial t} + A^{\epsilon,\delta} P_T^{\epsilon,\delta} - (r + \lambda) P_T^{\epsilon,\delta} + \lambda \alpha (K - s)^+, (K - s)^+ - P_T^{\epsilon,\delta} \right\} = 0 \quad (5.7)$$

on $[0, T] \times D$, together with the terminal condition

$$P_T^{\epsilon,\delta}(T, s, y) = (K - s)^+. \quad (5.8)$$

The fixed date T breaks the stationarity of the perpetual problem. Consequently, $P_T^{\epsilon,\delta}$ depends explicitly on calendar time (or, equivalently, on the remaining time to maturity). Under the

corresponding monotonicity and regularity conditions, when the stopping region admits a one-sided threshold representation, the finite-maturity exercise boundary may be written as $s_{f,T}^{\epsilon,\delta}(t, y)$, which generally depends on both t and y . This contrasts with the stationary perpetual boundary $s_f^{\epsilon,\delta}(y)$ considered in Section 2.

The finite-horizon formulation above is preliminary. A rigorous study of the time-dependent free boundary, a multiscale asymptotic expansion for $P_T^{\epsilon,\delta}$, and the corresponding numerical implementation are left for future work.

6. Conclusions

This paper develops a pricing framework for vulnerable perpetual American put options under an SEV model with constant-intensity Cox default and fractional recovery. The default-adjusted optimal-stopping problem yields a variational inequality whose exercise and continuation regions are separated by a stationary boundary; smooth fit holds under the stated regularity conditions. A multiscale expansion provides explicit approximations for price and boundary: The leading term is a default-adjusted Black–Scholes benchmark, while higher-order terms capture stochastic elasticity, fast mean reversion, asset–elasticity correlation, and the elasticity risk premium.

Least-squares Monte Carlo comparisons show that the corrected approximation substantially improves on the leading-order solution and closely matches the full-model benchmark across the tested parameter sets. Higher recovery rates increase values by mitigating default losses, whereas higher default intensities reduce them by lowering continuation value. SEV corrections are concentrated in the continuation region and affect the exercise boundary more strongly than price. Thus, modest price corrections can imply material changes in optimal exercise decisions. Credit-risk parameters and SEV channels interact through the continuation premium and free-boundary structure.

Section 5 further formulates vulnerable perpetual American calls through a default-adjusted variational inequality and finite-maturity vulnerable American options through a time-dependent obstacle problem. Complete analyses of these extensions, together with state-dependent default intensities, stochastic interest rates, and jump risk, remain for future research.

Author contributions

Ji-Hun Yoon: Formal analysis, writing – review & editing, writing - original draft, methodology; Sangmin Park: Software, formal analysis; Mijin Ha: Software, formal analysis, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors would like to state that, in the course of writing this paper, ChatGPT were used solely for grammar checking and organizing references.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix

A. Proof of Proposition 3.2

Substituting the expansion (3.5) into (3.3) and collecting the terms of order $\sqrt{\delta}$ give, after averaging with respect to the invariant distribution of Y_t ,

$$\mathcal{M}P_{1,0} = -\sigma^2 \langle f \rangle s^2 \log s \frac{d^2}{ds^2} P_0''(s),$$

which is (3.30). Since the leading-order exercise payoff is already matched by P_0 , the correction satisfies $P_{1,0}(s_0) = 0$. The conditions at $s = K$ follow from the continuity of the price and its first derivative with respect to s across the kink of the payoff.

Let $\chi(\eta) = \frac{1}{2}\sigma^2\eta(\eta - 1) + r\eta - (r + \lambda)$. For any polynomial $U(x)$ with $x = \log s$, a direct calculation gives

$$\mathcal{M}(s^\eta U(\log s)) = s^\eta \left[\chi(\eta)U + \chi'(\eta)U' + \frac{1}{2}\sigma^2 U'' \right].$$

Because $\chi(\eta_i) = 0$ for $i = 1, 2$, the source terms generated by As^{η_2} , Bs^{η_1} , and Cs^{η_2} are inverted by quadratic polynomials in $\log s$. This yields the logarithmic representation (3.32), and comparison of the coefficients of $\log s$ and the constant terms gives $a_{1,0}^\pm$, $b_{1,0}^\pm$, $\widehat{a}_{1,0}$, and $\widehat{b}_{1,0}$.

The remaining constants are determined by the three linear matching conditions

$$P_{1,0}(s_0) = 0, \quad P_{1,0}(K^-) = P_{1,0}(K^+), \quad \frac{d}{ds}P_{1,0}(K^-) = \frac{d}{ds}P_{1,0}(K^+),$$

which gives the displayed formulas for $c_{1,0}^-$, $c_{1,0}^+$, and $\widehat{c}_{1,0}$. Finally, expanding the smooth-pasting condition at (3.6) gives

$$\frac{d^2}{ds^2}P_0(s_0)s_{1,0} + \frac{d}{ds}P_{1,0}(s_0) = 0.$$

Provided that $\frac{d^2}{ds^2}P_0(s_0) \neq 0$, we obtain $s_{1,0} = -\frac{\frac{d}{ds}P_{1,0}(s_0)}{\frac{d^2}{ds^2}P_0(s_0)}$. This completes the proof of Proposition 3.2.

B. Proof of Proposition 3.3

Collecting the terms of order $\sqrt{\delta\epsilon}$ in (3.3) and applying the Fredholm solvability condition for the fast generator $\mathcal{L}_0$ gives the averaged equation (3.36). The constants V_ρ and V_Λ arise respectively from the correlation term and the market price of elasticity risk through the Poisson equation associated with $\mathcal{L}_0$.

For a mode Ds^η of P_0 , we have

$$s^2 \log s \frac{d^2}{ds^2}(Ds^\eta) = H_\eta(D)s^\eta \log s,$$

and therefore

$$V_\rho s \frac{d}{ds} \left(s^2 \log s \frac{d^2}{ds^2}(Ds^\eta) \right) + V_\Lambda s^2 \log s \frac{d^2}{ds^2}(Ds^\eta) = H_\eta(D)s^\eta (L_\eta \log s + M_\eta).$$

Using again

$$\mathcal{M}(s^\eta U(\log s)) = s^\eta \left[\chi(\eta)U + \chi'(\eta)U' + \frac{1}{2}\sigma^2 U'' \right],$$

with $\chi(\eta_i) = 0$, the inverse of each resonant source term is a quadratic polynomial in $\log s$. Matching the coefficients of $\log s$ and the constant term gives the displayed coefficients $a_{1,1}^\pm$, $b_{1,1}^\pm$, $\widehat{a}_{1,1}$, and $\widehat{b}_{1,1}$.

The constants $c_{1,1}^-$, $c_{1,1}^+$, and $\widehat{c}_{1,1}$ are then obtained from

$$P_{1,1}(s_0) = 0, \quad P_{1,1}(K^-) = P_{1,1}(K^+), \quad \frac{d}{ds}P_{1,1}(K^-) = \frac{d}{ds}P_{1,1}(K^+),$$

which is the same linear matching system used for $P_{1,0}$. Finally, the smooth-pasting condition at (3.6) gives

$$\frac{d^2}{ds^2}P_0(s_0)s_{1,1} + \frac{d}{ds}P_{1,1}(s_0) = 0,$$

provided that $\frac{d^2}{ds^2}P_0(s_0) \neq 0$ and hence $s_{1,1} = -\frac{\frac{d}{ds}P_{1,1}(s_0)}{\frac{d^2}{ds^2}P_0(s_0)}$. This completes the proof of Proposition 3.3.

C. Proof of Proposition 3.4

We provide the details because the source term contains both the direct $\mathcal{L}_{22}$ contribution and the interaction between $\mathcal{L}_{21}$ and $P_{1,0}$.

Set

$$q_1 := \sigma^2 \langle f \rangle, \quad q_2 := \sigma^2 \langle f^2 \rangle, \quad x := \log s,$$

and, for each characteristic mode $\eta \in \{\eta_1, \eta_2\}$,

$$D_{\eta_2} := A, \quad D_{\eta_1} := B, \quad \mathcal{A}_\eta := \eta(\eta - 1), \quad \mathcal{B}_\eta := 2\eta - 1,$$

where A and B are given by (3.28). On $s_0 < s < K$, $P_{1,0}^{(\eta)}$ can be written as follows.

$$P_{1,0}^{(\eta)}(s) = s^\eta \left(a_{1,0}^{(\eta)} x^2 + b_{1,0}^{(\eta)} x + c_{1,0}^{(\eta)} \right),$$

where the coefficients with $\eta = \eta_2$ are $a_{1,0}^-$, $b_{1,0}^-$, and $c_{1,0}^-$ and those with $\eta = \eta_1$ are $a_{1,0}^+$, $b_{1,0}^+$, and $c_{1,0}^+$. For $F_\eta(z) = a_{1,0}^{(\eta)}z^2 + b_{1,0}^{(\eta)}z + c_{1,0}^{(\eta)}$, direct differentiation gives

$$s^2 \frac{d^2}{ds^2}(s^\eta F_\eta(x)) = s^\eta (\mathcal{A}_\eta F_\eta(x) + \mathcal{B}_\eta F'_\eta(x) + F''_\eta(x)).$$

Therefore the source term in (3.42) corresponding to $D_\eta s^\eta$ can be written as

$$G_\eta(x)s^\eta,$$

where

$$G_\eta(x) = g_3^{(\eta)}x^3 + g_2^{(\eta)}x^2 + g_1^{(\eta)}x$$

with

$$\begin{aligned} g_3^{(\eta)} &= -q_1 \mathcal{A}_\eta a_{1,0}^{(\eta)}, \\ g_2^{(\eta)} &= -q_2 D_\eta \mathcal{A}_\eta - q_1 (\mathcal{A}_\eta b_{1,0}^{(\eta)} + 2\mathcal{B}_\eta a_{1,0}^{(\eta)}), \\ g_1^{(\eta)} &= -q_1 (\mathcal{A}_\eta c_{1,0}^{(\eta)} + \mathcal{B}_\eta b_{1,0}^{(\eta)} + 2a_{1,0}^{(\eta)}). \end{aligned}$$

Now take the ansatz

$$P_{2,0}^{(\eta)}(s) = s^\eta (a_{2,0}^{(\eta)}x^4 + b_{2,0}^{(\eta)}x^3 + c_{2,0}^{(\eta)}x^2 + d_{2,0}^{(\eta)}x + e_{2,0}^{(\eta)}).$$

If $\chi(\eta) = \frac{1}{2}\sigma^2\eta(\eta-1) + r\eta - (r+\lambda)$, then $\chi(\eta_i) = 0$ for $i = 1, 2$, and

$$\mathcal{M}(s^\eta F(x)) = s^\eta \left(\chi'(\eta)F'(x) + \frac{1}{2}\sigma^2 F''(x) \right).$$

Matching the coefficients of x^3 , x^2 , x , and the constant term yields

$$a_{2,0}^{(\eta)} = \frac{g_3^{(\eta)}}{4\chi'(\eta)}, \quad b_{2,0}^{(\eta)} = \frac{g_2^{(\eta)} - 6\sigma^2 a_{2,0}^{(\eta)}}{3\chi'(\eta)}, \quad c_{2,0}^{(\eta)} = \frac{g_1^{(\eta)} - 3\sigma^2 b_{2,0}^{(\eta)}}{2\chi'(\eta)}, \quad d_{2,0}^{(\eta)} = -\frac{\sigma^2 c_{2,0}^{(\eta)}}{\chi'(\eta)}.$$

This proves the polynomial-logarithmic form on $s_0 < s < K$.

In the case of $s \geq K$, only the decreasing term s^{η_2} remains. The same calculation, with D_{η_2} replaced by C and the coefficients of $P_{1,0}$ replaced by $\widehat{a}_{1,0}$, $\widehat{b}_{1,0}$, and $\widehat{c}_{1,0}$, gives

$$\widehat{G}(x) = \widehat{g}_3 x^3 + \widehat{g}_2 x^2 + \widehat{g}_1 x,$$

where

$$\begin{aligned} \widehat{g}_3 &= -q_1 \mathcal{A}_{\eta_2} \widehat{a}_{1,0}, \\ \widehat{g}_2 &= -q_2 C \mathcal{A}_{\eta_2} - q_1 [\mathcal{A}_{\eta_2} \widehat{b}_{1,0} + 2\mathcal{B}_{\eta_2} \widehat{a}_{1,0}], \\ \widehat{g}_1 &= -q_1 [\mathcal{A}_{\eta_2} \widehat{c}_{1,0} + \mathcal{B}_{\eta_2} \widehat{b}_{1,0} + 2\widehat{a}_{1,0}]. \end{aligned}$$

Thus

$$\widehat{a}_{2,0} = \frac{\widehat{g}_3}{4\chi'(\eta_2)}, \quad \widehat{b}_{2,0} = \frac{\widehat{g}_2 - 6\sigma^2\widehat{a}_{2,0}}{3\chi'(\eta_2)}, \quad \widehat{c}_{2,0} = \frac{\widehat{g}_1 - 3\sigma^2\widehat{b}_{2,0}}{2\chi'(\eta_2)}, \quad \widehat{d}_{2,0} = -\frac{\sigma^2\widehat{c}_{2,0}}{\chi'(\eta_2)}.$$

It remains to determine the constants $e_{2,0}^{(\eta_2)}$, $e_{2,0}^{(\eta_1)}$, and $\widehat{e}_{2,0}$. Define

$$\begin{aligned} G_-(x) &:= a_{2,0}^{(\eta_2)}x^4 + b_{2,0}^{(\eta_2)}x^3 + c_{2,0}^{(\eta_2)}x^2 + d_{2,0}^{(\eta_2)}x, \\ G_+(x) &:= a_{2,0}^{(\eta_1)}x^4 + b_{2,0}^{(\eta_1)}x^3 + c_{2,0}^{(\eta_1)}x^2 + d_{2,0}^{(\eta_1)}x, \\ \widehat{G}(x) &:= \widehat{a}_{2,0}x^4 + \widehat{b}_{2,0}x^3 + \widehat{c}_{2,0}x^2 + \widehat{d}_{2,0}x. \end{aligned}$$

The second-order value-matching expansion at $s = s_0$ gives

$$P_{2,0}(s_0) = -s_{1,0}\frac{d}{ds}P_{1,0}(s_0) - \frac{1}{2}s_{1,0}^2\frac{d^2}{ds^2}P_0(s_0) =: \Theta_0.$$

The continuity of $P_{2,0}$ and $\frac{dP_{2,0}}{ds}$ at $s = K$ gives the two linear relations encoded by

$$\begin{aligned} R_5 &= \widehat{G}(\log K) - G_-(\log K) - K^{\eta_1-\eta_2}G_+(\log K), \\ R_6 &= \eta_2 \left(\widehat{G}(\log K) - G_-(\log K) \right) + \widehat{G}'(\log K) - G'_-(\log K) - K^{\eta_1-\eta_2}(\eta_1 G_+(\log K) + G'_+(\log K)). \end{aligned}$$

Solving this linear system yields

$$\begin{aligned} e_{2,0}^{(\eta_1)} &= \frac{R_6 - \eta_2 R_5}{K^{\eta_1-\eta_2}(\eta_1 - \eta_2)}, \\ e_{2,0}^{(\eta_2)} &= s_0^{-\eta_2}\Theta_0 - G_-(x_0) - s_0^{\eta_1-\eta_2} \left(G_+(x_0) + e_{2,0}^{(\eta_1)} \right), \end{aligned}$$

and

$$\widehat{e}_{2,0} = e_{2,0}^{(\eta_2)} + K^{\eta_1-\eta_2}e_{2,0}^{(\eta_1)} - R_5.$$

These are precisely the constants stated in Proposition 3.4.

Finally, expanding the smooth-pasting condition at (3.6) to order δ gives

$$\frac{d^2}{ds^2}P_0(s_0)s_{2,0} + \frac{1}{2}s_{1,0}^2\frac{d^3}{ds^3}P_0(s_0) + s_{1,0}\frac{d^2}{ds^2}P_{1,0}(s_0) + \frac{d}{ds}P_{2,0}(s_0) = 0.$$

Hence

$$s_{2,0} = -\frac{\frac{1}{2}s_{1,0}^2\frac{d^3}{ds^3}P_0(s_0) + s_{1,0}\frac{d^2}{ds^2}P_{1,0}(s_0) + \frac{d}{ds}P_{2,0}(s_0)}{\frac{d^2}{ds^2}P_0(s_0)},$$

which completes the proof of Proposition 3.4.



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