



Research article

Phase transitions and scaling in an interacting elephant random walk with random amplitudes

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Abstract: We investigated a modified two-elephant random walk in which one walker follows the classical dynamics, while the other recalls past steps of its partner and multiplies them by independent random amplitudes, introducing heterogeneous reinforcement effects. Using martingale techniques, we established a law of large numbers (LLN), a central limit theorem (CLT), and a law of the iterated logarithm (LIL) for the position of the first elephant. A phase transition driven by an effective memory parameter was identified. In both the diffusive and critical regimes, Gaussian fluctuations arise, with a change of scaling at criticality due to logarithmic corrections. In the super-diffusive regime, the walk converges to a non-Gaussian limit at a polynomial rate. Numerical simulations support the theoretical results. Unlike previous interacting elephant walks, we uncovered a non-Gaussian limit in the super-diffusive regime and explicit logarithmic corrections at criticality, highlighting the profound impact of random amplitudes on memory-driven systems.

Keywords: elephant random walk; non-Markovian processes; anomalous diffusion; interacting random walks; central limit theorem; asymptotic normality; martingale methods; law of large numbers

Mathematics Subject Classification: 60G50, 60F05, 60G42, 60K35, 82C41

1. Introduction

Non-Markovian stochastic processes—where the future evolution depends on the entire past of the system—play a central role in probability theory and in modeling complex phenomena. Among them, the elephant random walk (ERW), introduced by Schütz and Trimper [1], has become a paradigmatic model for processes with complete memory. In its simplest form, at each step, the walker randomly recalls one of its own past steps and then, with a fixed probability, either repeats or inverts that step; the increments are traditionally restricted to ± 1 . Despite its simplicity, this memory mechanism generates rich asymptotic behavior, including a phase transition between diffusive and super-diffusive regimes.

The ERW has also inspired numerous generalizations. Memory-based random walks arise naturally in statistical mechanics to describe anomalous transport, reinforced diffusion, and systems with long-range temporal correlations (see, e.g., [2]). The two-elephant framework can be seen as a minimal interacting system where agents exchange memories, mimicking coupled particles or information flow in complex systems. The elephant random walk has been extended in several directions. In particular, Dedecker et al. [3] investigated an elephant random walk with random step sizes and established rates of convergence in the central limit theorem.

A significant extension of the classical ERW is the elephant random walk with random amplitudes, where increments are no longer restricted to ± 1 but are drawn from a more general distribution. This variant allows the step sizes to be random, introducing an additional source of variability and capturing situations where the impact or weight of past actions fluctuates. Such models have been studied, for example, by Dedecker, Fan, Hu, and Merlevede [3], who investigated rates of convergence in the central limit theorem for ERWs with random amplitudes. Their work shows how the moment properties of the step distribution influence scaling limits and fluctuation behavior, thereby enriching the universality classes observed in memory-based random processes.

Recently, Aguech and Shuo [4] extended this framework by introducing a model of two interacting elephants. In their system, two walkers $S^{(1)}$ and $S^{(2)}$ evolve by recalling not their own past steps but rather the past steps of the other agent. The dynamics are governed by memory parameters p_1 and p_2 : At each step, elephant i chooses uniformly at random a past step of the other elephant and then, with probability p_i , repeats it, or with probability $1 - p_i$, inverts it. This interaction captures scenarios of mutual learning or reciprocal influence, where the history of one agent directly shapes the behavior of the other. A key contribution of [4] is a detailed asymptotic analysis of this coupled system. The authors proved that, under mild conditions, the two-elephant walk is recurrent, satisfies a law of large numbers with zero speed, and obeys a central limit theorem with an explicit covariance structure. Moreover, they established a law of the iterated logarithm that precisely characterizes the almost-sure fluctuations. Their results highlight how the interaction parameters p_1 and p_2 modulate the long-term correlation structure and scaling limits, providing a rigorous probabilistic foundation for modeling bidirectional memory-based interaction.

Building on this interactive framework, the present work introduces a novel extension: the two-elephant model with random step amplitudes. We modify the dynamics of the first elephant so that, whenever it recalls a past step of the second elephant, it multiplies that step by a random amplitude L_{n+1} (with mean m_L and variance σ_L^2) before applying the sign rule determined by p_1 . This mechanism reflects realistic situations where the intensity of influence, the learning rate, or the impact of copied information varies stochastically over time. For instance, it can model the spread of rumors in social networks where certain relays amplify messages with variable strength, or reinforcement learning contexts where rewards are associated with random weights.

The random-amplitude interaction captures several realistic phenomena:

- **Social networks:** When a user retweets a message, the visibility (e.g., number of likes or shares) varies randomly; our amplitude L_n models this variable amplification.
- **Collective animal behavior:** In bird flocks or fish schools, an individual following a neighbor's move may be influenced by random factors like distance or visibility—captured by random amplitudes.
- **Multi-agent reinforcement learning:** An agent copying another's action may receive a

stochastically scaled reward, which fits our amplitude mechanism.

These examples illustrate that the model is not purely abstract but has clear potential applications.

Unlike previous interacting elephant walks [4], where the influence of one agent on the other is purely deterministic in magnitude (only the sign is random), our random-amplitude mechanism introduces heterogeneous reinforcement—a feature known to qualitatively change the long-time behavior in memory-driven systems [5]. Indeed, as we show below, the random amplitudes do not merely renormalize constants; they give rise to a new effective parameter $\gamma = \alpha_3(1 - \beta)$ that controls a phase transition, and in the super-diffusive regime, the limiting distribution becomes non-Gaussian—a phenomenon absent in the standard two-elephant model [4]. Thus, our extension is not a parametric exercise but a minimal model that uncovers richer scaling universality classes.

Such interacting memory-driven walkers may represent coupled active particles [6], agents with adaptive learning rates [7], or information propagation where influence strength fluctuates randomly [8]. From a statistical physics perspective, this framework can model systems with heterogeneous reinforcement [5], where past interactions are not only remembered but also amplified with stochastic intensity. This framework naturally arises in anomalous transport phenomena [1, 2], collective learning dynamics, and socio-physical systems where agents exchange information with variable reliability or impact. Introducing random amplitudes therefore provides a minimal yet flexible mechanism to capture fluctuating memory effects and non-uniform influence in interacting non-Markovian systems.

We provide a comprehensive asymptotic analysis of this new model. By decomposing the process into martingale components and exploiting recursive tree representations, we derive a law of large numbers, a central limit theorem, and a law of the iterated logarithm for the position $S_n^{(1)}$ of the first elephant. Our results reveal that, while the qualitative asymptotic behaviors (recurrence, Gaussian fluctuations in the diffusive regime) are preserved, the scaling constants and limiting variances explicitly depend on the moments of the amplitude L_1 . Furthermore, we identify a phase transition governed by an effective parameter $\gamma = \alpha_3(1 - \beta)$, where $\alpha_3 = 2p_3 - 1$ and β is the probability of cross-recall. In the critical case $\gamma = 1/2$, logarithmic corrections appear in the scaling, whereas for $\gamma > 1/2$, the process converges in distribution to a non-Gaussian random variable at a super-diffusive rate n^γ .

Why random amplitudes? In many real-world systems, the strength of a remembered event is not constant. For example, when information spreads in a social network, a user may retweet a message with or without amplification (likes, shares). In collective animal behavior, the influence of a neighbor's move may depend on distance or on random environmental factors. Our model captures this variability by multiplying each recalled step by an (i.i.d.) amplitude L_n . As we demonstrate, this variability fundamentally alters the scaling limits compared to the constant-amplitude case studied in [4]. In particular, the second moment of L_1 (not just its mean) appears explicitly in the asymptotic variance and in the critical threshold, highlighting that the distribution of amplitudes matters beyond first-order moments.

In summary, our main contributions are:

- A complete asymptotic analysis (law of large numbers, central limit theorem, and law of the iterated logarithm) for an interacting two-elephant random walk with random step amplitudes.
- The identification of a phase transition governed by the effective parameter $\gamma = \alpha_3(1 - \beta)$, separating diffusive, critical, and super-diffusive regimes.

- For $\gamma > 1/2$, the first demonstration of a non-Gaussian limit in an interacting elephant random walk (Theorem 2 (3-iii)).
- Explicit logarithmic scaling laws in the critical regime $\gamma = 1/2$: $\sqrt{n \log n}$ for the CLT and

$$\sqrt{2n \log n \log \log(\log n)}$$

for the LIL.

The paper is organized as follows. Section 2 gives a precise definition of the model and the underlying assumptions. Section 3 presents the main asymptotic results (Theorems 3.1, 3.3 and 3.5), covering the law of large numbers, the law of the iterated logarithm, and the central limit theorem for both diffusive and critical/super-diffusive regimes. Section 4 illustrates these theoretical findings with numerical simulations, highlighting the influence of parameters and the convergence to limiting distributions. Section 5 contains the detailed proofs, based on martingale decompositions, stochastic approximation techniques, and careful control of interaction terms. A brief conclusion and perspectives for future work are provided in Section 6.

2. Definition of the model

At each integer time $n \geq 0$, with probability β , the first elephant (position $S_n^{(1)}$) copies a randomly chosen past step of the second elephant (position $S_n^{(2)}$), multiplies it by a random amplitude L_{n+1} , and then either repeats it (with probability p_1) or inverts it (with probability $1 - p_1$). With probability $1 - \beta$, it copies a randomly chosen past step of its own, and then either repeats it (with probability p_3) or inverts it (with probability $1 - p_3$). The second elephant evolves independently: With probability r , it takes a step $+1$ or -1 ; otherwise, it copies its own past step with sign determined by p_2 . The following formal definition uses the same notation as in [4] for consistency, but we have added a summary in Table 1 for quick reference.

Table 1. Summary of main notations.

Symbol	Meaning
p_1	Probability that the first elephant repeats a copied step.
p_2	Probability that the second elephant repeats its own past step.
p_3	Probability that the first elephant repeats its own past step.
$\alpha_i = 2p_i - 1$	Memory parameter for elephants (for $i = 1, 2, 3$).
β	Probability that the first elephant copies the second elephant.
L_n	Random amplitude (i.i.d., with mean m_L and variance σ_L^2).
$\gamma = \alpha_3(1 - \beta)$	Effective memory parameter controlling the phase transition.
Σ^2	Asymptotic variance (given in Theorem 3.5).
$\mathcal{F}_n$	Filtration $\sigma\{X_k^{(1)}, X_k^{(2)} : 1 \leq k \leq n\}$.

Let $(p_1, p_2, p_3, \beta) \in [0, 1]^4$ and define $\alpha_i := 2p_i - 1$ for $i = 1, 3$. Consider independent sequences:

- $(\xi_n^{(i)})_{n \geq 2}$ i.i.d., Bernoulli with $\mathbb{P}(\xi_n^{(i)} = 1) = p_i$;
- $(\eta_n)_{n \geq 2}$ i.i.d., Bernoulli with $\mathbb{P}(\eta_n = 1) = \beta$;
- $(u_n^{(i)})_{n \geq 1}$ and $(v_n)_{n \geq 1}$ are uniformly distributed on $\{1, \dots, n\}$, independent of all else;
- $(L_n)_{n \geq 1}$ i.i.d., random amplitudes with mean m_L and variance σ_L^2 .

Assume that all the processes are independent. Define the step processes and positions:

$$S_0^{(i)} = 0, \quad S_n^{(i)} = \sum_{k=1}^n X_k^{(i)}, \quad i = 1, 2.$$

The update rules are:

$$\begin{aligned} X_1^{(2)} &= \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}, \\ X_1^{(1)} &= \begin{cases} L_1 & \text{with probability } 1/2 \\ -L_1 & \text{with probability } 1/2 \end{cases}, \\ X_{n+1}^{(2)} &= (2\xi_{n+1}^{(2)} - 1)X_{u_n^{(1)}}^{(2)}, \\ X_{n+1}^{(1)} &= \eta_{n+1}(2\xi_{n+1}^{(1)} - 1)L_{n+1}X_{u_n^{(2)}}^{(2)} + (1 - \eta_{n+1})(2\xi_{n+1}^{(3)} - 1)X_{v_n}^{(1)}. \end{aligned}$$

Let

$$\mathcal{F}_n = \sigma(X_1^{(1)}, \dots, X_n^{(1)}, X_1^{(2)}, \dots, X_n^{(2)})$$

denote the natural filtration (the history of both walkers up to time n).

2.1. Physical and socio-physical interpretations

To make the model less abstract, we provide explicit interpretations for the parameters and random variables:

- $\eta_n \in \{0, 1\}$ indicates whether at step n the first elephant copies the second elephant ($\eta_n = 1$) or itself ($\eta_n = 0$). In social network terms, this is the probability β that an agent adopts information from an external source rather than from its own memory.
- $\xi_n^{(1)}$ (resp., $\xi_n^{(3)}$) determines whether the copied step is repeated ($\xi = 1$) or inverted ($\xi = 0$). In opinion dynamics, this corresponds to agreeing or disagreeing with the recalled opinion.
- L_n is a random amplitude scaling the copied step. In animal behavior, this could represent the varying intensity of a movement (e.g., jump length, vocal volume). In finance, it could model the size of a trade influenced by a peer.
- p_1 and p_3 control the propensity to repeat vs. invert. For $p_i > 1/2$, the walker tends to confirm the recalled step; for $p_i < 1/2$, it tends to oppose it.
- The second elephant's parameter p_2 : When $p_2 = 1/2$, the second elephant performs a simple symmetric random walk, representing an unbiased environmental signal.

Thus, despite its mathematical appearance, the model captures essential features of interacting memory-driven systems with heterogeneous influence strengths.

3. Main results

We now state the asymptotic behavior of $S^{(1)}$. Precisely, we study some statistical properties of the random walk $S^{(1)}$ such as the law of large numbers, the law of the iterated logarithm, and the central limit theorem.

Theorem 3.1. Let $(S_n^{(1)}, S_n^{(2)})_{n \geq 0}$ be the random walk with dimension 2, we assume that $\beta = 1$ and $p_2 = 1/2$. Then we have:

(1) The law of large numbers is

$$\frac{S_n^{(1)}}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} 0.$$

(2) The law of the iterated logarithm is expressed as follows

$$\left| 1 - |\alpha_1| m_L \sqrt{2} \right| \leq \limsup_{n \rightarrow \infty} \frac{S_n^{(1)}}{\sqrt{2n \log \log n}} \leq 1 + |\alpha_1| m_L \sqrt{2}.$$

(3) The central limit theorem is

$$Z_n := \frac{S_n^{(1)}}{\sqrt{n(\sigma_L^2 + (2\alpha_1^2 + 1)m_L^2)}} \xrightarrow[n \rightarrow +\infty]{D} \mathcal{N}(0, 1).$$

Comment 3.2. Theorem 3.1 describes the asymptotic behavior of the first coordinate $S_n^{(1)}$ of a two-dimensional random walk in the critical regime $\beta = 1$ and $p_2 = 1/2$, highlighting a combination of classical and non-standard features. The law of large numbers shows that the process is centered, as $S_n^{(1)}/n$ converges in probability to zero, indicating the absence of drift. The central limit theorem confirms Gaussian fluctuations, but with an adjusted variance σ_L^2 , which reflects the influence of the underlying dependence structure or interaction encoded by α_1 and the moments of L_1 . More interestingly, the law of the iterated logarithm is not expressed as an exact limit but rather as bounds, suggesting that the fluctuations of $S_n^{(1)}$ deviate from the classical i.i.d. setting and are affected by additional structural constraints.

Theorem 3.3. Consider the two-dimensional random walk $(S_n^{(1)}, S_n^{(2)})_{n \geq 0}$ defined in Section 2, and assume $\beta \in (0, 1)$. Then the following law of large numbers holds:

$$\left(\frac{S_n^{(1)}}{n}, \frac{S_n^{(2)}}{n} \right) \xrightarrow[n \rightarrow +\infty]{a.s.} (0, 0).$$

Recall, in the case where $\alpha_2 > 1/2$, it is known, from [9], that the almost sure limit of $n^{-\alpha_2} S_n^{(2)}$ is a non-degenerate random variable called L .

Remark 3.4. The case where $\beta = 1$ was previously studied in [4]. In the present paper, we restrict our attention to the non-resonant regime: $\gamma \neq \alpha_2$. The resonant case, $\gamma = \alpha_2$, is particularly delicate because the two characteristic memory parameters of the system coincide. This parameter configuration is expected to generate critical phenomena, including modified scaling laws and logarithmic corrections that do not arise away from resonance. As a consequence, the techniques developed for the non-resonant regime cannot be applied directly and must be supplemented by a refined analysis. Since this regime represents a key component of the phase diagram of the model, a comprehensive investigation will be carried out in a separate paper.

The asymptotic behavior of $S_n^{(1)}$ in the non-resonant regime $\gamma \neq \alpha_2$ is given by the following theorem.

Theorem 3.5. Consider the two-dimensional random walk $(S_n^{(1)}, S_n^{(2)})_{n \geq 0}$ defined in Section 2, and assume $\beta \in (0, 1)$ and $\alpha_3(1 - \beta) \neq \alpha_2$. Define

$$\gamma = \alpha_3(1 - \beta), \quad \lambda = \frac{\beta\alpha_1 m_L}{\gamma - \alpha_2},$$

and let Σ^2 be given by

$$\Sigma^2 = \begin{cases} \lambda^2, & \text{if } \gamma < 1/2, \alpha_2 = 1/2, \\ \frac{\sigma_L^2 + m_L^2 + \lambda^2}{1 - 2\gamma} + \frac{\lambda^2}{1 - 2\alpha_2} - \frac{2\lambda^2}{1 - \alpha_2 - \gamma}, & \text{if } \gamma < 1/2, \alpha_2 < 1/2, \\ \sigma_L^2 + m_L^2 + \lambda^2, & \text{if } \alpha_2 < 1/2, \gamma = 1/2. \end{cases} \quad (3.1)$$

Then the following asymptotic results hold.

(1) **Case where** $\alpha_2 > 1/2$.

(1-i) If $\gamma < \alpha_2$, then

$$\frac{S_n^{(1)}}{n^{\alpha_2}} \xrightarrow[n \rightarrow +\infty]{a.s.} -\lambda L.$$

(1-ii) If $\gamma > \alpha_2$, then

$$\frac{S_n^{(1)}}{n^\gamma} \xrightarrow[n \rightarrow +\infty]{a.s.} W^{(x)}.$$

(2) **Case where** $\alpha_2 = 1/2$.

(2-i) If $\gamma < 1/2$, then

$$\frac{S_n^{(1)}}{\sqrt{n \log n}} \xrightarrow[n \rightarrow +\infty]{D} \mathcal{N}(0, \Sigma^2)$$

and

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(1)}|}{\sqrt{2n \log n \log \log n}} \leq |\lambda|.$$

(2-ii) If $\gamma > 1/2$, then

$$\frac{S_n^{(1)}}{n^\gamma} \xrightarrow[n \rightarrow +\infty]{a.s.} W^{(x)}.$$

(3) **Case where** $\alpha_2 < 1/2$.

(3-i) If $\gamma < 1/2$ and $\gamma \neq \alpha_2$, then

$$\frac{S_n^{(1)}}{\sqrt{n}} \xrightarrow[n \rightarrow +\infty]{D} \mathcal{N}(0, \Sigma^2)$$

and

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(1)}|}{\sqrt{2n \log \log n}} \leq \sqrt{C(\gamma)} + \frac{|\lambda|}{\sqrt{1 - 2\alpha_2}}.$$

(3-ii) If $\gamma = 1/2$, then

$$\frac{S_n^{(1)}}{\sqrt{n \log n}} \xrightarrow[n \rightarrow +\infty]{D} \mathcal{N}(0, \Sigma^2)$$

and

$$\limsup_{n \rightarrow \infty} \frac{|S_n^{(1)}|}{\sqrt{2n \log n \log \log n}} \leq \sqrt{C(\gamma)}.$$

(3-iii) If $\gamma > 1/2$, then

$$\frac{S_n^{(1)}}{n^\gamma} \xrightarrow[n \rightarrow +\infty]{a.s.} W^{(x)}.$$

In the statements above, L is the almost-sure limit of $n^{-\alpha_2} S_n^{(2)}$ (see [9]) and $W^{(x)}$ is a non-degenerate finite random variable defined in Eq (5.14).

Terminological note. Following the established convention in the elephant random walk literature (see, e.g., [1, 4]), we use the term “phase transition” to describe a sharp change in the asymptotic behavior (scaling exponents and/or the nature of the limiting distribution) as the parameter γ crosses the critical value $1/2$. This does not refer to a thermodynamic phase transition but to a qualitative change in the stochastic dynamics.

Comment 3.6. Theorem 3.5 provides a detailed description of the asymptotic behavior of the first coordinate $S_n^{(1)}$ in the regime $\beta \in (0, 1)$. The dynamics are governed by the effective parameter $\gamma = \alpha_3(1 - \beta)$ and the memory parameter α_2 of the second elephant.

The finer asymptotic behavior splits into three main regimes based on γ relative to $1/2$, but with additional nuances depending on α_2 :

• **Diffusive-like regime** ($\gamma < 1/2$):

- (1) If $\alpha_2 < 1/2$ and $\gamma \neq \alpha_2$, then $S_n^{(1)}/\sqrt{n}$ converges to a Gaussian distribution with variance Σ^2 and a classical law of the iterated logarithm (LIL) holds with

$$\sqrt{2n \log \log n}.$$

- (2) If $\alpha_2 = 1/2$, the normalization becomes $\sqrt{n \log n}$ for the CLT, and the LIL involves

$$\sqrt{2n \log n \log \log n}.$$

- **Critical regime** ($\gamma = 1/2$): For $\alpha_2 < 1/2$, the CLT uses $\sqrt{n \log n}$ and the LIL is the same iterated logarithmic factor as above.
- **Super-diffusive regime** ($\gamma > 1/2$): The process exhibits almost-sure convergence (and L^2 convergence) to a non-degenerate, non-Gaussian random variable after scaling by $n^{\max(\gamma, \alpha_2)}$. This marks a sharp departure from Gaussian fluctuations.

In all regimes, the random amplitudes L_n influence the scale of fluctuations (e.g., Σ^2 in the diffusive case) and the nature of the limit in the super-diffusive regime (non-Gaussian). However, the phase transition boundary is determined solely by the memory and cross-interaction parameters (γ and α_2), while the amplitudes modify the quantitative and distributional properties within each regime.

4. Simulations

For the numerical illustration, we consider the two-dimensional random walk defined in Section 2 with parameters $p_1 = 0.7$, $p_2 = 0.5$, and $p_3 = 0.6$. The random amplitudes $(L_n)_{n \geq 1}$ are assumed to be independent and identically distributed according to an exponential distribution with rate 1, that is, $L_n \sim \text{Exp}(1)$, and the sample size is fixed at $n = 10^4$. To illustrate Theorem 3.1, we simulate the first component $S_n^{(1)}$ under the assumptions $\beta = 1$ and $p_2 = 1/2$, and plot the normalized trajectory $S_n^{(1)}/n$. To illustrate Theorem 3.3, we simulate the second component $S_n^{(2)}$ with $\beta = 0.75$ and plot the

normalized trajectory $S_n^{(2)}/n$. Figure 1 shows that both normalized trajectories are displayed on the same graph, where the horizontal dashed line corresponds to the limiting value 0 predicted by the law of large numbers.

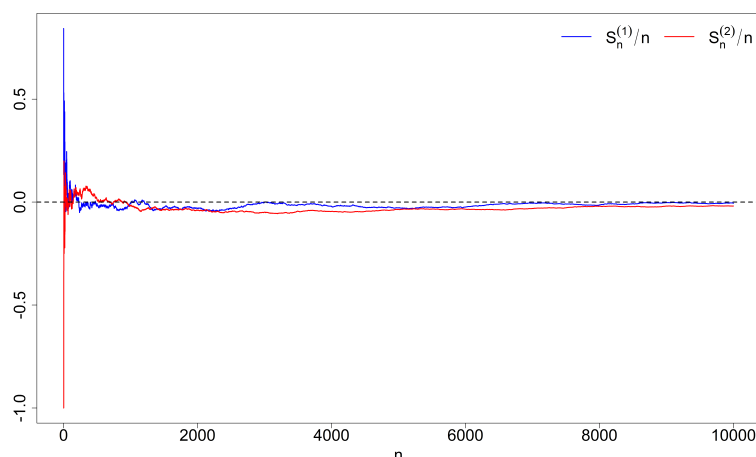


Figure 1. Law of large numbers (LLN) of $\frac{S_n^{(1)}}{n}$ and $\frac{S_n^{(2)}}{n}$.

Figure 2 illustrates the three fluctuation regimes described in Theorem 3.5.

(i) The first two panels illustrate the case $\alpha_2 > 1/2$. For $(\alpha_2, \gamma) = (0.7, 0.3)$, where $\gamma < \alpha_2$, the trajectory of $S_n^{(1)}/n^{\alpha_2}$ stabilizes, in agreement with the convergence

$$\frac{S_n^{(1)}}{n^{\alpha_2}} \xrightarrow{\mathbb{P}} -\lambda L.$$

For $(\alpha_2, \gamma) = (0.6, 0.855)$, where $\gamma > \alpha_2$, the normalized process $S_n^{(1)}/n^\gamma$ approaches a non-degenerate random limit, as predicted by Case (1-ii).

(ii) The top-right panel corresponds to $\alpha_2 = 1/2$ with $\gamma = 0.3$. The empirical density of the normalized process agrees closely with the theoretical Gaussian density $\mathcal{N}(0, \Sigma^2)$, confirming the central limit theorem stated in Case (2-i). The bottom-left panel illustrates the supercritical case $\gamma = 0.72 > 1/2$, where the normalized process $S_n^{(1)}/n^\gamma$ is shown for $\alpha_2 = 0.5$ (red). The trajectory stabilizes to a finite random limit, consistently with the convergence in probability predicted by Case (2-ii).

(iii) The bottom-middle and bottom-right panels correspond to $\alpha_2 < 1/2$. For $(\alpha_2, \gamma) = (0.3, 0.375)$ and $(0.4, 0.5)$, respectively, the empirical densities are in good agreement with the limiting Gaussian distributions predicted by Cases (3-i) and (3-ii). The bottom-left panel illustrates the supercritical case $\gamma = 0.72 > 1/2$, where the normalized process $S_n^{(1)}/n^\gamma$ is shown for $\alpha_2 = 0.3$ (blue). The trajectory stabilizes to a finite random limit, consistent with the convergence in probability predicted by Case (3-iii).

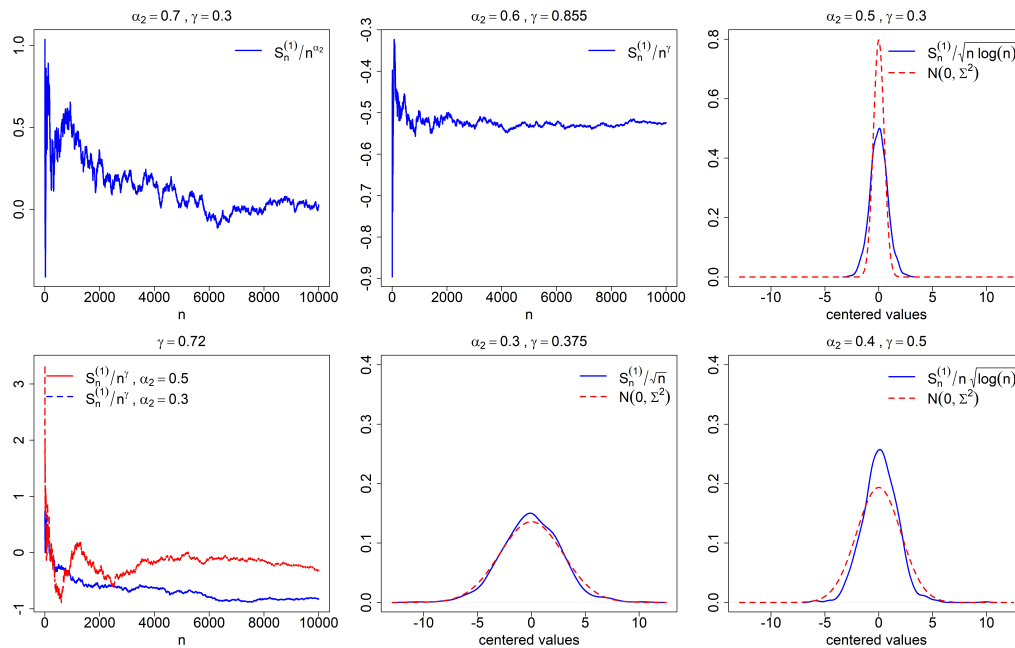


Figure 2. Numerical illustration of the asymptotic regimes in Theorem 3.5.

In all numerical experiments, we considered $M = 1,000$ independent Monte Carlo replications, each consisting of a trajectory of length $N = 10,000$. The amplitude sequence $(L_n)_{n \geq 1}$ was generated 10,000 times according to L_n following an exponential distribution with parameter 1, independently for each n . Table 2 shows that the remaining parameters were selected to produce the three fluctuation regimes described in Theorem 3.5.

Table 2. Parameter values used in the simulations of Figure 2.

Case	p_1	p_2	p_3	β	α_1	α_2	α_3	γ	Σ^2
(1-i)	0.70	0.85	0.70	0.25	0.40	0.70	0.40	0.30	
(1-ii)	0.70	0.80	0.95	0.05	0.40	0.60	0.90	0.855	
(2-i)	0.70	0.75	0.70	0.25	0.40	0.50	0.40	0.30	0.25
(2-iii)	0.70	0.75	0.95	0.20	0.40	0.50	0.90	0.72	
(3-i)	0.70	0.65	0.75	0.25	0.40	0.30	0.50	0.375	8.62
(3-ii)	0.70	0.70	0.90	0.375	0.40	0.40	0.80	0.50	4.25
(3-iii)	0.70	0.65	0.95	0.20	0.40	0.30	0.90	0.72	

5. Proofs

5.1. Proof of Theorem 3.1

Conditionally on $\mathcal{F}_n$, $u_n^{(2)}$ is uniform on $\{1, \dots, n\}$ and independent of $(L_{n+1}, \xi_{n+1}^{(1)})$. Therefore, we define the following conditional moments:

$$\mathbb{E}[X_{n+1}^{(1)} \mid \mathcal{F}_n] = \alpha_1 m_L \frac{S_n^{(2)}}{n}, \quad (5.1)$$

while

$$\mathbb{E}\left[(X_{n+1}^{(1)})^2 \mid \mathcal{F}_n\right] = \mathbb{E}[L_1^2] = m_L^2 + \sigma_L^2. \quad (5.2)$$

Thus,

$$\text{Var}\left[X_{n+1}^{(1)} \mid \mathcal{F}_n\right] = \sigma_L^2 + m_L^2 - \left(\alpha_1 m_L \frac{S_n^{(2)}}{n}\right)^2.$$

Define the martingale M_n of the walk by:

$$M_n = S_n^{(1)} - \sum_{k=1}^{n-1} \mathbb{E}\left[X_{k+1}^{(1)} \mid \mathcal{F}_k\right] = S_n^{(1)} - \alpha_1 m_L \sum_{k=1}^{n-1} \frac{S_k^{(2)}}{k}. \quad (5.3)$$

The increment of the martingale $(M_n, \mathcal{F}_n)$ is

$$\Delta M_{n+1} = X_{n+1}^{(1)} - \mathbb{E}\left[X_{n+1}^{(1)} \mid \mathcal{F}_n\right].$$

By (5.2), the conditional variance satisfies

$$\mathbb{E}\left[(\Delta M_{n+1})^2 \mid \mathcal{F}_n\right] = (\sigma_L^2 + m_L^2) - \left(\alpha_1 m_L \frac{S_n^{(2)}}{n}\right)^2.$$

Summing, the predictable quadratic variation is

$$\langle M \rangle_n = \sum_{k=1}^n \mathbb{E}\left[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}\right] = (\sigma_L^2 + m_L^2)n - (\alpha_1 m_L)^2 \sum_{k=1}^{n-1} \left(\frac{S_k^{(2)}}{k}\right)^2.$$

By [9], $S_n^{(2)}/n \rightarrow 0$, almost surely, and then

$$\langle M \rangle_n = \sum_{k=1}^n \mathbb{E}\left[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}\right] = (\sigma_L^2 + m_L^2)n + o(n).$$

Using [9] applied to the walk $S_n^{(2)}$ and [10, Theorem 1.3.15] applied to the martingale $(M_n)_n$ and Eq (5.3), we conclude assertion (1) of Theorem 3.1:

$$\frac{S_n^{(1)}}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} 0.$$

In order to prove assertion (2) of Theorem 3.1, we use Stout's martingale (LIL) to prove the results concerning the LIL. Since $\langle M \rangle_n / ((\sigma_L^2 + m_L^2)n) \rightarrow 1$ and the Lindeberg condition holds, we obtain, by [11, Theorem 4.8],

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{M_n}{\sqrt{2(\sigma_L^2 + m_L^2)n \log \log n}} &= 1, \\ \liminf_{n \rightarrow \infty} \frac{M_n}{\sqrt{2(\sigma_L^2 + m_L^2)n \log \log n}} &= -1, \quad \text{a.s.} \end{aligned} \quad (5.4)$$

To transfer this result to $S_n^{(1)}$, observe by [4, Eq (9)] that, almost surely,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{4n \log \log n}} \sum_{i=1}^{n-1} \frac{S_i^{(2)}}{i} &= -1, \\ \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{4n \log \log n}} \sum_{i=1}^{n-1} \frac{S_i^{(2)}}{i} &= 1, \quad \text{a.s.} \end{aligned} \quad (5.5)$$

Combining with (5.4), we obtain almost surely

$$\left| 1 - |\alpha_1| m_L \sqrt{2} \right| \leq \limsup_{n \rightarrow \infty} \frac{S_n^{(1)}}{\sqrt{2n \log \log n}} \leq 1 + |\alpha_1| m_L \sqrt{2}.$$

Finally, to prove assertion (3) of Theorem 3.1, define, for each $n \geq k \geq 1$, that

$$d_k(n) := \# \{j \in (k, n] : u_j^{(2)} = k\}$$

with the convention that $d_n(n) = 0$. For $n \geq 1$, by a slight abuse of notation, in distribution, we have

$$S_n^{(1)} = X_1^{(1)} + \sum_{k=1}^n \left(\sum_{j=1}^{d_k(n)} (2\xi_{j,k}^{(1)} - 1) L_{j,k} \right) X_k^{(2)}, \quad (5.6)$$

where, by convention, $\sum_{j=1}^{d_k(n)} a_j = 0$ if $d_k(n) = 0$ and for all fixed j , $(\xi_{j,k}^{(1)})_k$ and $(L_{j,k})_k$ are two i.i.d. sequences having, respectively, the same distribution as $\xi_1^{(1)}$ and L_1 . Using the following notation:

$$\tilde{S}_n^{(k)} = \sum_{j=1}^n (2\xi_{j,k}^{(1)} - 1) L_{j,k}.$$

Equation (5.6) can be expressed as

$$S_n^{(1)} = X_1^{(1)} + \sum_{k=1}^n \tilde{S}_{d_k(n)}^{(k)} X_k^{(2)}.$$

Since, for all k , $\tilde{S}_{d_k(n)}^{(k)}$ and $X_k^{(2)}$ are independent, we have

$$\begin{aligned} \mathbb{E} \tilde{S}_n^{(k)} &= n \alpha_1 m_L, \\ \mathbb{E} \left(\tilde{S}_n^{(k)} \right)^2 &= n(\sigma_L^2 + m_L^2) + n(n-1) \alpha_1^2 m_L^2, \\ \text{Var} \tilde{S}_n^{(k)} &= n \sigma_L^2 + n m_L^2 (1 - \alpha_1^2). \end{aligned}$$

We recall the following central limit theorem for martingale triangular arrays, see, e.g., [11].

Proposition 5.1. *Suppose that for each $n \geq 1$, we have random variables $X_{n,1}, \dots, X_{n,n}$ on a probability space $(\Omega, \mathcal{F}, \text{Pr})$, with sub σ -fields $\mathcal{F}_{n,0} \subset \mathcal{F}_{n,1} \subset \dots \subset \mathcal{F}_{n,n}$ of $\mathcal{F}$ such that $X_{n,k}$ is $\mathcal{F}_{n,k}$ -measurable and*

$\mathbb{E}(X_{n,k} \mid \mathcal{F}_{n,k-1}) = 0$ a.s. for $k = 1, 2, \dots, n$. Such an array is called a martingale triangular array. For all $n \geq 1$ and for all $0 \leq k \leq n$, define $S_{n,k} = X_{n,1} + \dots + X_{n,k}$. If

$$\sum_{k=1}^n \mathbb{E}[X_{n,k}^2 \mid \mathcal{F}_{n,k-1}] \longrightarrow 1 \quad \text{in probability as } n \rightarrow \infty, \quad (5.7)$$

and for any $\varepsilon > 0$,

$$\sum_{k=1}^n \mathbb{E}[X_{n,k}^2 \mathbf{1}_{\{|X_{n,k}| \geq \varepsilon\}} \mid \mathcal{F}_{n,k-1}] \longrightarrow 0 \quad \text{in probability as } n \rightarrow \infty, \quad (5.8)$$

then $S_{n,n}$ converges in law as $n \rightarrow \infty$ to $\mathcal{N}(0, 1)$.

We use the following result, where the proof is similar to [4, Proposition 2.2],

Proposition 5.2. As $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n (\tilde{S}_{d_k(n)}^{(k)})^2}{n(1 + 2\alpha_1^2 m_L^2)} = 1 \quad \text{in probability.}$$

Assertion (3) of Theorem 3.1 follows by applying the same arguments as in the proof of [4].

5.2. Proof of Theorem 3.3

Recall that the first elephant evolves according to

$$X_{n+1}^{(1)} = \eta_{n+1}(2\xi_{n+1}^{(1)} - 1)L_{n+1}X_{u_n^{(2)}}^{(2)} + (1 - \eta_{n+1})(2\xi_{n+1}^{(3)} - 1)X_{v_n}^{(1)}, \quad n \geq 0,$$

and

$$\mathcal{F}_n = \sigma(X_1^{(1)}, \dots, X_n^{(1)}, X_1^{(2)}, \dots, X_n^{(2)})$$

denotes the natural filtration.

Conditionally on $\mathcal{F}_n$, all random variables at time $n + 1$ are independent. Hence,

$$\begin{aligned} \mathbb{E}[X_{n+1}^{(1)} \mid \mathcal{F}_n] &= \mathbb{E}[\eta_{n+1}] \mathbb{E}[2\xi_{n+1}^{(1)} - 1] \mathbb{E}[L_{n+1}] \mathbb{E}[X_{u_n^{(2)}}^{(2)} \mid \mathcal{F}_n] + \mathbb{E}[1 - \eta_{n+1}] \mathbb{E}[2\xi_{n+1}^{(3)} - 1] \mathbb{E}[X_{v_n}^{(1)} \mid \mathcal{F}_n] \\ &= \beta \alpha_1 m_L \left(\frac{1}{n} \sum_{k=1}^n X_k^{(2)} \right) + (1 - \beta) \alpha_3 \left(\frac{1}{n} \sum_{k=1}^n X_k^{(1)} \right) \\ &= \beta \alpha_1 m_L \frac{S_n^{(2)}}{n} + (1 - \beta) \alpha_3 \frac{S_n^{(1)}}{n}. \end{aligned}$$

Similarly, we have

$$\mathbb{E}[(X_{n+1}^{(1)})^2 \mid \mathcal{F}_n] = \beta(\sigma_L^2 + m_L^2) + \frac{(1 - \beta)}{n} \sum_{k=1}^n (X_k^{(1)})^2.$$

The martingale component is defined by

$$M_n := \sum_{k=1}^n (X_k^{(1)} - \mathbb{E}[X_k^{(1)} \mid \mathcal{F}_{k-1}]).$$

The process $(M_n, \mathcal{F}_n)_{n \geq 1}$ is a square-integrable martingale with increments

$$\Delta M_{n+1} = X_{n+1}^{(1)} - \mathbb{E}[X_{n+1}^{(1)} \mid \mathcal{F}_n].$$

This decomposition separates the purely stochastic fluctuations encoded in the martingale M_n from the cumulative memory effects contained in the compensator C_n , and constitutes the starting point for the proofs of the law of large numbers, the central limit theorem, and the law of the iterated logarithm for $S_n^{(1)}$.

Proof of Theorem 3.3. We first establish a strong law of large numbers for the martingale part M_n .

Lemma 5.3. *For the martingale (M_n) defined in (5.3), we have*

$$\frac{M_n}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} 0.$$

Proof. From the dynamics, we have

$$\begin{aligned} \mathbb{E}[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}] &\leq \mathbb{E}[(X_k^{(1)})^2 \mid \mathcal{F}_{k-1}] \\ &= \beta(\sigma_L^2 + m_L^2) + \frac{1-\beta}{k-1} \sum_{l=1}^{k-1} (X_l^{(1)})^2. \end{aligned}$$

Since $\mathbb{E}[(X_l^{(1)})^2] \leq \sigma_L^2 + m_L^2$, we obtain $\mathbb{E}[(\Delta M_k)^2] \leq \sigma_L^2 + m_L^2$ for all $k \geq 2$. Consequently,

$$\mathbb{E} \left(\sum_{k=2}^{\infty} \frac{\Delta M_k}{k} \right)^2 = \sum_{k=2}^{\infty} \frac{1}{k^2} \mathbb{E}[(\Delta M_k)^2] \leq (\sigma_L^2 + m_L^2) \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

Hence

$$\sum_{k=2}^{\infty} \frac{\Delta M_k}{k}$$

converges almost surely and, by Kronecker's lemma, we deduce $M_n/n \rightarrow 0$ a.s. □

Now we prove that $S_n^{(1)}/n \rightarrow 0$ a.s. From the martingale decomposition (5.3), we have

$$T_n = A_n + \frac{\gamma}{n} \sum_{k=1}^{n-1} T_k,$$

where

$$\begin{aligned} T_n &= \frac{S_n^{(1)}}{n}, \quad \gamma = (1-\beta)\alpha_3, \\ A_n &= \frac{M_n}{n} + \beta\alpha_1 m_L \frac{1}{n} \sum_{k=1}^{n-1} \frac{S_k^{(2)}}{k}. \end{aligned}$$

By Lemma 5.3, $M_n/n \rightarrow 0$ a.s. For the second elephant, it is known (see [4] or [9]) that $S_k^{(2)}/k \rightarrow 0$ a.s., and therefore its Cesàro mean also tends to 0 a.s. Thus $A_n \rightarrow 0$ a.s.

Let $a = \limsup_{n \rightarrow \infty} |T_n|$. Since $A_n \rightarrow 0$, for any $\varepsilon > 0$, there exists N such that $|A_n| \leq \varepsilon$ for all $n \geq N$. Then for $n \geq N$,

$$|T_n| \leq \varepsilon + \frac{|\gamma|}{n} \sum_{k=1}^{n-1} |T_k|.$$

Taking the limit superior as $n \rightarrow \infty$ and using

$$\frac{1}{n} \sum_{k=1}^{n-1} |T_k| \leq a,$$

we obtain $a \leq |\gamma|a$. Because $|\gamma| = |(1 - \beta)\alpha_3| < 1$ (as $0 < \beta < 1$ and $|\alpha_3| \leq 1$), we conclude that $a = 0$. Hence $T_n \rightarrow 0$ a.s., which completes the proof. $\square$

5.3. Proof of Theorem 3.5

We need the following lemma:

Lemma 5.4 (Convergence of the empirical second moment). *Under the assumptions of Theorem 3.5, we have*

$$\frac{1}{n} \sum_{k=1}^n (X_k^{(1)})^2 \xrightarrow{a.s.} \sigma_L^2 + m_L^2.$$

Proof. From the dynamics,

$$\mathbb{E}[(X_{k+1}^{(1)})^2 \mid \mathcal{F}_k] = \beta(\sigma_L^2 + m_L^2) + (1 - \beta) \frac{1}{k} \sum_{i=1}^k (X_i^{(1)})^2.$$

Set

$$Y_k = \frac{1}{k} \sum_{i=1}^k (X_i^{(1)})^2.$$

Then

$$\mathbb{E}[Y_{k+1} \mid \mathcal{F}_k] = Y_k + \frac{\beta}{k+1} (\sigma_L^2 + m_L^2 - Y_k).$$

Let $W_k = Y_k - (\sigma_L^2 + m_L^2)$. Then

$$\mathbb{E}[W_{k+1} \mid \mathcal{F}_k] = \left(1 - \frac{\beta}{k+1}\right) W_k.$$

Since $\beta \in (0, 1)$, a standard super-martingale argument (or the Robbins-Siegmund lemma) implies $W_k \rightarrow 0$ almost surely. Hence $Y_k \rightarrow \sigma_L^2 + m_L^2$ a.s. $\square$

Proof of Theorem 3.5. Recall the definitions

$$\gamma = \alpha_3(1 - \beta), \quad \gamma \neq \alpha_2, \quad \lambda = \frac{\beta\alpha_1 m_L}{\gamma - \alpha_2},$$

and set

$$x_n = \frac{S_n^{(1)} + \lambda S_n^{(2)}}{n},$$

$$y_n = \frac{S_n^{(2)}}{n}.$$

Define the martingale increments

$$\begin{aligned}\varepsilon_{n+1}^{(x)} &= \frac{X_{n+1}^{(1)} + \lambda X_{n+1}^{(2)} - \gamma x_n}{1 - \gamma}, \\ \varepsilon_{n+1}^{(y)} &= \frac{X_{n+1}^{(2)} - \alpha_2 y_n}{1 - \alpha_2}.\end{aligned}$$

For $\eta \notin \{-1, 1\}$ and $n \geq 1$, set

$$\gamma_n(\pm\eta) := \frac{1 \pm \eta}{n+1}, \quad \beta_n(\pm\eta) := \prod_{k=1}^{n-1} (1 - \gamma_k(\pm\eta)) = \frac{\Gamma(n \mp \eta)}{\Gamma(1 \mp \eta) \Gamma(n+1)}, \quad (5.9)$$

with $\beta_1(\pm\eta) := 1$. Notice that

$$\lim_{n \rightarrow \infty} \beta_n(\pm\eta) n^{1 \pm \eta} = \frac{1}{\Gamma(1 \mp \eta)}. \quad (5.10)$$

Consequently,

$$\begin{aligned}x_n &= \beta_n(-\gamma) \left(x_1 + \sum_{j=1}^{n-1} \frac{\gamma_j(-\gamma)}{\beta_{j+1}(-\gamma)} \varepsilon_{j+1}^{(x)} \right), \\ y_n &= \beta_n(-\alpha_2) \left(y_1 + \sum_{j=1}^{n-1} \frac{\gamma_j(-\alpha_2)}{\beta_{j+1}(-\alpha_2)} \varepsilon_{j+1}^{(y)} \right).\end{aligned}$$

Moreover,

$$S_n^{(1)} = n(x_n - \lambda y_n) = n\beta_n(-\gamma)(x_1 + M_n^{(x)}) - \lambda S_n^{(2)},$$

where we define two martingales

$$\begin{aligned}M_n^{(x)} &= \sum_{j=1}^{n-1} \frac{\gamma_j(-\gamma)}{\beta_{j+1}(-\gamma)} \varepsilon_{j+1}^{(x)}, \\ M_n^{(y)} &= \sum_{j=1}^{n-1} \frac{\gamma_j(-\alpha_2)}{\beta_{j+1}(-\alpha_2)} \varepsilon_{j+1}^{(y)},\end{aligned}$$

with $M_1^{(x)} = M_1^{(y)} = 0$. Their quadratic variations satisfy

$$\langle M^{(x)} \rangle_n = \sum_{j=1}^{n-1} \left(\frac{\gamma_j(-\gamma)}{\beta_{j+1}(-\gamma)} \right)^2 \mathbb{E}[(\varepsilon_{j+1}^{(x)})^2 \mid \mathcal{F}_j] \approx C(\gamma) \sum_{j=1}^{n-1} j^{-2\gamma}, \quad (5.11)$$

$$\langle M^{(y)} \rangle_n = \sum_{j=1}^{n-1} \left(\frac{\gamma_j(-\alpha_2)}{\beta_{j+1}(-\alpha_2)} \right)^2 \mathbb{E}[(\varepsilon_{j+1}^{(y)})^2 \mid \mathcal{F}_j] \approx C(\alpha_2) \sum_{j=1}^{n-1} j^{-2\alpha_2}, \quad (5.12)$$

where

$$C(\gamma) = \Gamma^2(1 + \gamma) (\lambda^2 + \sigma_L^2 + m_L^2), \quad C(\alpha_2) = \Gamma^2(1 + \alpha_2). \quad (5.13)$$

Lemma 5.5. For $\gamma \neq \alpha_2$, the following convergences in probability hold:

$$\begin{aligned}(1 - \gamma)^2 \mathbb{E} \left[(\varepsilon_{n+1}^{(x)})^2 \mid \mathcal{F}_n \right] &\xrightarrow{\mathbb{P}} \lambda^2 + \sigma_L^2 + m_L^2, \\ (1 - \alpha_2)^2 \mathbb{E} \left[(\varepsilon_{n+1}^{(y)})^2 \mid \mathcal{F}_n \right] &\xrightarrow{\mathbb{P}} 1, \\ (1 - \alpha_2)(1 - \gamma) \mathbb{E} \left[\varepsilon_{n+1}^{(x)} \varepsilon_{n+1}^{(y)} \mid \mathcal{F}_n \right] &\xrightarrow{\mathbb{P}} \lambda.\end{aligned}$$

Proof of Lemma 5.5. Expanding

$$(1 - \gamma)^2 (\varepsilon_{n+1}^{(x)})^2$$

gives

$$(1 - \gamma)^2 (\varepsilon_{n+1}^{(x)})^2 = (X_{n+1}^{(1)})^2 + \lambda^2 (X_{n+1}^{(2)})^2 + \gamma^2 x_n^2 + 2\lambda X_{n+1}^{(1)} X_{n+1}^{(2)} - 2\gamma X_{n+1}^{(1)} x_n - 2\lambda\gamma X_{n+1}^{(2)} x_n.$$

From Lemma 5.4,

$$\mathbb{E} \left[(X_{n+1}^{(1)})^2 \mid \mathcal{F}_n \right] \xrightarrow[n \rightarrow +\infty]{a.s.} \sigma_L^2 + m_L^2,$$

and the remaining terms converge to 0 in probability. Hence the first claim follows. Next,

$$(1 - \alpha_2)^2 \mathbb{E} \left[(\varepsilon_{n+1}^{(y)})^2 \mid \mathcal{F}_n \right] = 1 + \alpha_2^2 y_n^2 - 2\alpha_2 X_{n+1}^{(2)} y_n \xrightarrow[n \rightarrow +\infty]{a.s.} 1.$$

Finally, expanding the product yields

$$(1 - \alpha_2)(1 - \gamma) \varepsilon_{n+1}^{(x)} \varepsilon_{n+1}^{(y)} = X_{n+1}^{(1)} X_{n+1}^{(2)} - \alpha_2 y_n X_{n+1}^{(1)} + \lambda - \lambda \alpha_2 y_n X_{n+1}^{(2)} - \gamma x_n X_{n+1}^{(2)} + \gamma \alpha_2 y_n x_n,$$

and taking the conditional expectation gives the convergence to λ . $\square$

Now we treat the different regimes.

(1-i) **Case where $\alpha_2 > 1/2$ and $\gamma < \alpha_2$:** From [9], we have

$$\frac{S_n^{(2)}}{n^{\alpha_2}} \xrightarrow[n \rightarrow +\infty]{a.s.} L.$$

It immediately follows that

$$\frac{S_n^{(1)}}{n^{\alpha_2}} \xrightarrow[n \rightarrow +\infty]{a.s.} -\lambda L.$$

(1-ii) **Case where $\alpha_2 > 1/2$ and $\gamma > \alpha_2$:** This case can be proved using the same arguments as in Case (1-i).

(2-i) **Case where $\alpha_2 = 1/2$ and $\gamma < 1/2$.** Define

$$X_{n,k} = \frac{\sqrt{n+1}}{\log(n+1)} \left(\beta_{n+1}(-\gamma) \frac{\gamma_k(-\gamma)}{\beta_{k+1}(-\gamma)} \varepsilon_{k+1}^{(x)} - \lambda \beta_{n+1}(-\alpha_2) \frac{\gamma_k(-\alpha_2)}{\beta_{k+1}(-\alpha_2)} \varepsilon_{k+1}^{(y)} \right).$$

Then

$$\mathbb{E} \left[X_{n,k}^2 \mid \mathcal{F}_{n,k-1} \right] \approx \frac{1}{\log(n+1)} \left[(\sigma_L^2 + m_L^2 + \lambda^2) n^{2\gamma-1} k^{-2\gamma} + \lambda^2 k^{-1} - 2\lambda^2 n^{\gamma-1/2} k^{-\gamma-1/2} \right],$$

so that

$$\sum_{k=1}^n \mathbb{E} [X_{n,k}^2 \mid \mathcal{F}_{n,k-1}] \xrightarrow[n \rightarrow +\infty]{a.s.} \lambda^2 > 0.$$

Applying [9, Theorem 3.5] and [12, Theorem 1.3], we obtain

$$\limsup \frac{M_n^{(x)}}{\sqrt{2n^{1-2\gamma} \log \log n}} = -\liminf \frac{M_n^{(x)}}{\sqrt{2n^{1-2\gamma} \log \log n}} = +1$$

and

$$\limsup \frac{S_n^{(2)}}{\sqrt{2n \log n \log \log n}} = -\liminf \frac{S_n^{(2)}}{\sqrt{2n \log n \log \log n}} = +1.$$

The law of the iterated logarithm then follows.

(2-ii) **Case where $\alpha_2 = 1/2$ and $\gamma > 1/2$.** Denote by $M_\infty^{(x)}$ the almost-sure limit of $M_n^{(x)}$. Then, using the same arguments as before, we have

$$\frac{S_n^{(1)}}{n^\gamma} \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{x_1 + M_\infty^{(x)}}{\Gamma(1 + \gamma)} =: W^{(x)}. \quad (5.14)$$

(3-i) **Case where $\alpha_2 < 1/2$, $\gamma < 1/2$, and $\gamma \neq \alpha_2$.** Define

$$X_{n,k} = \sqrt{n+1} \left(\beta_{n+1}(-\gamma) \frac{\gamma_k(-\gamma)}{\beta_{k+1}(-\gamma)} \varepsilon_{k+1}^{(x)} - \lambda \beta_{n+1}(-\alpha_2) \frac{\gamma_k(-\alpha_2)}{\beta_{k+1}(-\alpha_2)} \varepsilon_{k+1}^{(y)} \right).$$

Then

$$\begin{aligned} \frac{X_{n,k}^2}{n+1} &= \beta_{n+1}^2(-\gamma) \frac{\gamma_k^2(-\gamma)}{\beta_{k+1}^2(-\gamma)} (\varepsilon_{k+1}^{(x)})^2 \\ &\quad + \lambda^2 \beta_{n+1}^2(-\alpha_2) \frac{\gamma_k^2(-\alpha_2)}{\beta_{k+1}^2(-\alpha_2)} (\varepsilon_{k+1}^{(y)})^2 \\ &\quad - 2\lambda \beta_{n+1}(-\gamma) \beta_{n+1}(-\alpha_2) \frac{\gamma_k(-\gamma) \gamma_k(-\alpha_2)}{\beta_{k+1}(-\gamma) \beta_{k+1}(-\alpha_2)} \varepsilon_{k+1}^{(x)} \varepsilon_{k+1}^{(y)}. \end{aligned}$$

Using (5.10), we have

$$\beta_{k+1}(-\gamma) \sim \frac{k^{\gamma-1}}{\Gamma(1 + \gamma)}, \quad \beta_{k+1}(-\alpha_2) \sim \frac{k^{\alpha_2-1}}{\Gamma(1 + \alpha_2)}, \quad \gamma_k(-\gamma) = \frac{1 - \gamma}{k + 1}.$$

Hence

$$\mathbb{E} [X_{n,k}^2 \mid \mathcal{F}_{n,k-1}] \approx (\sigma_L^2 + m_L^2 + \lambda^2) n^{2\gamma-1} k^{-2\gamma} + \lambda^2 n^{2\alpha_2-1} k^{-2\alpha_2} - 2\lambda^2 n^{\gamma+\alpha_2-1} k^{-\alpha_2-\gamma}.$$

Therefore, by (5.11) and (5.12),

$$\sum_{k=1}^n \mathbb{E} [X_{n,k}^2 \mid \mathcal{F}_{n,k-1}] \xrightarrow[n \rightarrow +\infty]{a.s.} \frac{\sigma_L^2 + m_L^2 + \lambda^2}{1 - 2\gamma} + \frac{\lambda^2}{1 - 2\alpha_2} - \frac{2\lambda^2}{1 - \alpha_2 - \gamma} > 0,$$

so condition (5.7) holds. The law of the iterated logarithm then follows by the same argument.

(3-ii) **Case where $\alpha_2 < 1/2$ and $\gamma = 1/2$.** Define

$$X_{n,k} = \frac{\sqrt{n+1}}{\log(n+1)} \left(\beta_{n+1}(-\gamma) \frac{\gamma_k(-\gamma)}{\beta_{k+1}(-\gamma)} \varepsilon_{k+1}^{(x)} - \lambda \beta_{n+1}(-\alpha_2) \frac{\gamma_k(-\alpha_2)}{\beta_{k+1}(-\alpha_2)} \varepsilon_{k+1}^{(y)} \right).$$

Then

$$\mathbb{E} \left[X_{n,k}^2 \mid \mathcal{F}_{n,k-1} \right] \approx \frac{1}{\log(n+1)} \left[(\sigma_L^2 + m_L^2 + \lambda^2) k^{-1} + \lambda^2 n^{2\alpha_2-1} k^{-2\alpha_2} - 2\lambda^2 n^{\alpha_2-1/2} k^{-\alpha_2-1/2} \right],$$

and consequently

$$\sum_{k=1}^n \mathbb{E} \left[X_{n,k}^2 \mid \mathcal{F}_{n,k-1} \right] \xrightarrow[n \rightarrow +\infty]{a.s.} \sigma_L^2 + m_L^2 + \lambda^2 > 0,$$

so condition (5.7) is satisfied. The law of the iterated logarithm is proved similarly.

(3-iii) **Case where $\alpha_2 < 1/2$ and $\gamma > 1/2$.** Recall that

$$S_n^{(1)} = n(x_n - \lambda y_n) = n \left[\beta_n(-\gamma)(x_1 + M_n^{(x)}) - \lambda \beta_n(-\alpha_2)(y_1 + M_n^{(y)}) \right].$$

In this regime,

$$\langle M^{(x)} \rangle_\infty < \infty, \quad \langle M^{(y)} \rangle_n \approx \frac{\Gamma^2(1 + \alpha_2)}{1 - 2\alpha_2} n^{1-2\alpha_2}.$$

By [12],

$$M_n^{(x)} \xrightarrow[n \rightarrow +\infty]{a.s.} M_\infty^{(x)}$$

(a finite random variable) and

$$\limsup_n \frac{M_n^{(y)}}{\sqrt{2\langle M^{(y)} \rangle_n \log \log \langle M^{(y)} \rangle_n}} = -\liminf_n \frac{M_n^{(y)}}{\sqrt{2\langle M^{(y)} \rangle_n \log \log \langle M^{(y)} \rangle_n}} = 1.$$

Setting

$$a = \frac{\Gamma^2(1 + \alpha_2)}{1 - 2\alpha_2},$$

we obtain

$$\limsup_n \frac{M_n^{(y)}}{\sqrt{n^{1-2\alpha_2} \log \log n}} = -\liminf_n \frac{M_n^{(y)}}{\sqrt{n^{1-2\alpha_2} \log \log n}} = \frac{\sqrt{2}\Gamma(1 + \alpha_2)}{\sqrt{1 - 2\alpha_2}}.$$

After simplification, we deduce that

$$\lim_{n \rightarrow \infty} \frac{S_n^{(1)}}{n^\gamma} = \frac{x_1 + M_\infty^{(x)}}{\Gamma(1 + \gamma)} = W^{(x)}.$$

Finally, for cases (2-i), (2-ii), (3-i), and (3-ii), since $|X_n^{(2)}| = 1$ for all n and $\mathbb{E}[(X_n^{(1)})^2] \leq m_L^2 + \sigma_L^2$, we have, for any $\varepsilon > 0$,

$$\sum_{k=1}^n \mathbb{E} [X_{n,k}^2 \mathbf{1}_{\{|X_{n,k}| > \varepsilon\}} \mid \mathcal{F}_{n,k-1}] \xrightarrow[n \rightarrow +\infty]{a.s.} 0,$$

so condition (5.8) is satisfied in each of these cases. $\square$

6. Conclusions and future outlook

In this work, we have studied an interacting two-elephant random walk with random step amplitudes. This extension provides a minimal yet flexible mechanism for modeling heterogeneous reinforcement in memory-driven systems, a feature that is natural in many physical and socio-physical contexts where the intensity of retrieved information varies stochastically over time.

This extension enriches the classical two-elephant framework by allowing fluctuations in the intensity of recalled steps. Crucially, we show that this seemingly minor modification leads to a non-trivial phase transition with three distinct regimes (diffusive, critical, super-diffusive), and—unlike the model in [4]—produces a non-Gaussian limit when $\gamma > 1/2$. Such behavior cannot be captured by simply adjusting parameters in existing models, confirming that random amplitudes are a physically and mathematically essential ingredient.

We have established the law of large numbers, Gaussian and non-Gaussian scaling limits, and the law of the iterated logarithm for the position of the first elephant. The analysis reveals a sharp phase transition controlled by the effective memory parameter $\gamma = \alpha_3(1 - \beta)$. In the diffusive regime ($\gamma < 1/2$), the fluctuations are Gaussian with renormalised variances that explicitly depend on the moments of the random amplitudes. At the critical threshold ($\gamma = 1/2$), the scaling acquires logarithmic corrections, while in the super-diffusive regime ($\gamma > 1/2$), the process converges almost surely to a non-Gaussian limit at a polynomial rate, reflecting strong memory persistence.

These results go beyond a mere extension of the model in [4] by revealing that random amplitudes can fundamentally change the universality class of the interacting system, pushing it from Gaussian to non-Gaussian fluctuations in the super-diffusive regime.

From a statistical mechanics viewpoint, our model provides a minimal setting to study anomalous transport induced by interacting memory effects. More generally, the results demonstrate that random amplitudes qualitatively modify the long-term dynamics and generate new scaling behaviors beyond those of standard elephant random walks.

Several extensions deserve further investigation. Natural directions include considering correlated or heavy-tailed amplitudes, analyzing the joint scaling properties of the two interacting walkers, and extending the framework to many-agent systems or network-based interactions. Such developments may shed further light on collective memory effects and non-equilibrium transport phenomena in complex systems.

Our analysis groups the memory parameters into an effective constant γ , which is sufficient because the conditional expectation is linear. However, more complex models—for instance, with state-dependent or correlated random amplitudes, or with non-linear reinforcement rules—could exhibit richer dynamics that our effective parameter cannot capture. Investigating such extensions is a promising direction for future research.

As noted by one reviewer, the martingale decomposition $S_n^{(1)} = M_n + C_n$ (see (5.9)) explicitly reveals the role of the memory parameter γ in the drift. Such representations are not only mathematically convenient but also physically interpretable, as they separate predictable drift from random fluctuations. We believe this approach may prove useful in other physical models of non-Markovian dynamics.

Author contributions

Mohamed Abdelkader: Conceptualization, methodology, software, formal analysis, writing—original draft; Rafik Aguech: Conceptualization, software, formal analysis, writing—original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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