



Research article

New bounds for the eigenvalue of the Schur product of an M -matrix and inverse M -matrix

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Abstract: M -matrices are an important class of special matrices for numerical modeling in biology, physics, and other disciplines. Estimation of the minimum eigenvalue of the Schur product of M -matrices is a significant research topic in the field of matrix analysis. Based on the Gerschgorin disk theorem and the Brauer oval theorem, this paper established new lower bound estimates for the minimum eigenvalue of the Schur product of an M -matrix and an inverse M -matrix. The proposed bounds only rely on the entries of the matrices, with compact expressions and high computational efficiency. A series of numerical examples illustrate that the bounds derived in this paper are tighter than several estimation formulas from existing literature, serving as a valuable supplement to the theory of spectral bound estimation for Schur products of M -matrices.

Keywords: M -matrix; Schur product; minimum eigenvalue; lower bound

Mathematics Subject Classification: 15A15, 15A42

1. Introduction

The set of all real (complex) matrices is written as $R^{m \times n}$ ($C^{m \times n}$). Let $G \in Z_n = \{G = (g_{ij}) \in R^{n \times n} \mid g_{ij} \leq 0; i, j \in N, i \neq j\}$, where N denote the index set $\{1, 2, \dots, n\}$. If a matrix $G \in Z_n$ can be written as $G = \alpha I - P$ with nonnegative matrix P and $\alpha > \rho(P)$, then G is called a nonsingular M -matrix, and the set of all nonsingular M -matrices is denoted by M_n . If $\alpha = \rho(P)$, G is referred to as a singular M -matrix. We define

$$\tau(\mathbf{G}) = \min\{\operatorname{Re}(\lambda) : \lambda \in \sigma(\mathbf{G})\},$$

where $\tau(\mathbf{G})$ stands for the minimum real part of eigenvalues of matrix $\mathbf{G}$ [1].

If there exists a permutation matrix $\mathbf{P}$ satisfying

$$\mathbf{P}\mathbf{G}\mathbf{P}^T = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{pmatrix}$$

for some $1 \leq k \leq n-1$, where $\mathbf{B}_{11}, \mathbf{B}_{22}$ are square matrices of order k and $n-k$, then $\mathbf{G}$ is reducible. Otherwise, $\mathbf{G}$ is irreducible. Equivalently, $\mathbf{G}$ is irreducible if and only if its associated directed graph is strongly connected [2].

A nonnegative matrix $\mathbf{G} = (g_{ij}) \in R^{n \times n}$ is called a doubly stochastic matrix if it satisfies both unit column sum and unit row sum $\sum_{i=1}^n g_{ij} = 1, j \in \mathbf{N}, \sum_{j=1}^n g_{ij} = 1, i \in \mathbf{N}$ [2]. For any invertible matrix $\mathbf{G}$ with $\mathbf{G}^{-1}$ being doubly stochastic, one obtains the following equalities:

$$\mathbf{G}\mathbf{e} = \mathbf{e}, \mathbf{G}^T\mathbf{e} = \mathbf{e}, \mathbf{e} = [1, 1, \dots, 1]^T.$$

Let $\mathbf{G} = (g_{ij}) \in R^{n \times n}$. If $|g_{ii}| > \sum_{j \neq i} |g_{ij}|$ holds for every row, then $\mathbf{G}$ is called a strictly row diagonally dominant matrix. If the transpose $\mathbf{G}^T$ is strictly row diagonally dominant, then $\mathbf{G}$ is also called a strictly column diagonally dominant matrix.

The Schur product (Hadamard product) of matrices $\mathbf{G} = (g_{ij})$ and $\mathbf{H} = (h_{ij})$ is denoted by $\mathbf{G} \circ \mathbf{H} \equiv (g_{ij}h_{ij})$, which means that the corresponding entries of $\mathbf{G}$ and $\mathbf{H}$ are multiplied element-wise. If $\mathbf{G}, \mathbf{H} \in M_n$, then $\mathbf{G} \circ \mathbf{H}^{-1} \in M_n$ [1].

The Schur product is not merely a theoretical matrix operation; it has been widely utilized in econometrics [3–5], biosignal processing [6], image processing [7–10], and financial time-series analysis [11–13].

Over recent years, eigenvalue localization [14–16], spectral perturbation [17–19], and spectral radius estimation for the Schur product of matrices [20–24] have drawn substantial attention from mathematicians across the globe. This field has witnessed intensive ongoing research and a steady stream of new theoretical results.

In particular, estimating the lower bound of the minimum eigenvalue for the Schur product of an $\mathbf{M}$ -matrix and an inverse $\mathbf{M}$ -matrix is a core problem of remarkable research value in matrix spectral theory. Recently, a number of papers have carried out in-depth investigations into this issue and obtained several relevant lower bound estimation results [25–33].

The authors of [25] established the following classical conclusion

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \frac{1}{n},$$

and put forward a conjecture that

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \frac{2}{n}.$$

Suppose $\mathbf{G} = (g_{ij}) \in M_n$ and $\mathbf{G}^{-1} = (\beta_{ij})$. In [26,27], the authors presented the result below

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \min_i \left\{ \frac{g_{ii} - s_i \sum_{j \neq i} |g_{ji}|}{1 + \sum_{j \neq i} s_{ji}} \right\},$$

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \min_{j \neq i} \frac{1}{2} \left\{ g_{ii} \beta_{ii} + g_{jj} \beta_{jj} - \left[(g_{ii} \beta_{ii} - g_{jj} \beta_{jj})^2 + 4 \left(v_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right) \left(v_j \sum_{k \neq j} |g_{kj}| \beta_{jj} \right) \right]^{\frac{1}{2}} \right\},$$

where

$$s_i = \max_{j \neq i} \{s_{ij}\}, \quad s_{ij} = \frac{|g_{ij}| + \sum_{k \neq j, i} |g_{ik}| d_k}{|g_{ii}|}, \quad d_i = \frac{R_i}{|g_{ii}|}, \quad R_i = \sum_{j \neq i} |g_{ij}|,$$

$$v_{ij} = \frac{|g_{ij}| + \sum_{k \neq j, i} |g_{ik}| s_{kj}}{|g_{ii}|}, \quad v_i = \max_{j \neq i} \{v_{ij}\}, \quad i, j, k \in \mathbf{N}, \quad j \neq i.$$

Subsequently, the authors in [28–30] put forward the following conclusion

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \min_i \left\{ \frac{g_{ii} - m_i \sum_{j \neq i} |g_{ji}|}{1 + \sum_{j \neq i} m_{ji}} \right\},$$

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \min_i \left\{ \frac{g_{ii} - u_i \sum_{j \neq i} |g_{ji}|}{1 + \sum_{j \neq i} u_{ji}} \right\},$$

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \min_i \left\{ \frac{g_{ii} - w_i \sum_{j \neq i} |g_{ji}|}{1 + \sum_{j \neq i} w_{ji}} \right\},$$

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \min_{j \neq i} \frac{1}{2} \left\{ g_{ii} \beta_{ii} + g_{jj} \beta_{jj} - \left[(g_{ii} \beta_{ii} - g_{jj} \beta_{jj})^2 + 4 \left(w_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right) \left(w_j \sum_{k \neq j} |g_{kj}| \beta_{jj} \right) \right]^{\frac{1}{2}} \right\},$$

where

$$m_i = \max_{j \neq i} \{m_{ij}\}, \quad m_{ij} = \frac{|g_{ij}| + \sum_{k \neq j, i} |g_{ik}| r_j}{|g_{ii}|}, \quad r_i = \max_{j \neq i} \left(\frac{|g_{ji}|}{|g_{jj}| - \sum_{k \neq j, i} |g_{jk}|} \right),$$

$$u_{ij} = \frac{|g_{ij}| + \sum_{k \neq j, i} |g_{ik}| m_{kj}}{|g_{ii}|}, \quad u_i = \max_{j \neq i} \{u_{ij}\}, \quad h_i = \max_{j \neq i} \left\{ \frac{|g_{ji}|}{|g_{jj}| m_{ji} - \sum_{k \neq j, i} |g_{jk}| m_{ki}} \right\},$$

$$w_{ij} = \frac{|g_{ij}| + \sum_{k \neq j, i} |g_{ik}| m_{kj} h_j}{|g_{ii}|}, \quad w_i = \max_{j \neq i} \{w_{ij}\}, \quad i, j, k \in \mathbf{N}, \quad j \neq i.$$

This paper further investigates the problem of lower bounds for the minimum eigenvalue of the Schur product of an M -matrix and an inverse M -matrix, and establishes estimation formulas with higher accuracy. The validity of the new bounds is verified by theoretical proofs and numerical examples, which provides a reliable reference for relevant research on matrix theory.

2. Main results

First, some lemmas that we use in the proof of our conclusions are given.

Lemma 1. [1] If G is an M -matrix, then there exists a positive diagonal matrix D such that $D^{-1}GD$ is a row strictly diagonally dominant M -matrix.

Lemma 2. [2] Let $G = (g_{ij}) \in C^{n \times n}$ and $x_1, x_2, \dots, x_n$ are positive real numbers. All eigenvalues of G are contained in the following region

$$\bigcup_{i=1}^n \left\{ z \in C : |z - g_{ii}| \leq x_i \sum_{j \neq i} \frac{1}{x_j} |g_{ji}| \right\}.$$

Lemma 3. [2] Suppose $G = (g_{ij}) \in C^{n \times n}$ and $x_1, x_2, \dots, x_n$ are positive real numbers. Every eigenvalue of G lies within the domain given below

$$\bigcup_{\substack{i, j=1 \\ i \neq j}}^n \left\{ z \in C : |z - g_{ii}| |z - g_{jj}| \leq \left(x_i \sum_{k \neq i} \frac{1}{x_k} |g_{ki}| \right) \left(x_j \sum_{k \neq j} \frac{1}{x_k} |g_{kj}| \right) \right\}.$$

Lemma 4. [34] If $G = (g_{ij}) \in R^{n \times n}$ is a strictly diagonally dominant matrix of rows, then the inverse matrix $G^{-1} = (\beta_{ij})$ satisfies

$$|\beta_{ji}| \leq \frac{\sum_{k \neq j} |g_{jk}|}{|g_{jj}|} |\beta_{ii}|, \quad j \neq i, \quad i, j \in \mathbf{N}.$$

Lemma 5. Let $G = (g_{ij}) \in R^{n \times n}$ be a strictly row diagonally dominant matrix. For all $i, j \in \mathbf{N}$ with $j \neq i$, the following conclusions hold:

$$(1) \quad 0 \leq \eta_i \leq 1, \quad 0 \leq \gamma_i \leq 1,$$

$$(2) \ 0 \leq \rho_{ji} \leq g_{ji} \leq w_{ji} < 1,$$

where

$$\eta_i = \max_{j \neq i} \left\{ \frac{|g_{ji}|}{|g_{jj}|w_{ji} - \sum_{k \neq j,i} |g_{jk}|w_{ki}} \right\}, \quad g_{ij} = \frac{|g_{ij}| + \sum_{k \neq j,i} |g_{ik}|w_{kj}\eta_j}{|g_{ii}|},$$

$$\gamma_i = \max_{j \neq i} \left\{ \frac{|g_{ji}|}{|g_{jj}|g_{ji} - \sum_{k \neq j,i} |g_{jk}|g_{ki}} \right\}, \quad \rho_{ij} = \frac{|g_{ij}| + \sum_{k \neq j,i} |g_{ik}|g_{kj}\gamma_j}{|g_{ii}|},$$

$$\rho_i = \max_{j \neq i} \{\rho_{ij}\}, \quad i, j, k \in \mathbf{N}, \quad j \neq i.$$

Proof. Since $w_{ji} \leq m_{ji}h_i$ [30]. According to the definition of w_{ji} , we have

$$|g_{ji}| = |g_{jj}|w_{ji} - \sum_{k \neq j,i} |g_{jk}|m_{ki}h_i.$$

It follows that

$$\frac{|g_{ji}|}{|g_{jj}|w_{ji} - \sum_{k \neq j,i} |g_{jk}|w_{ki}} = \frac{|g_{ji}|w_{ji} - \sum_{k \neq j,i} |g_{jk}|m_{ki}h_i}{|g_{jj}|w_{ji} - \sum_{k \neq j,i} |g_{jk}|w_{ki}} \leq 1,$$

which yields $0 \leq \eta_i \leq 1$. From the definition of η_i , we have

$$\eta_i \geq \frac{|g_{ji}|}{|g_{jj}|w_{ji} - \sum_{k \neq j,i} |g_{jk}|w_{ki}}.$$

Further,

$$w_{ji}\eta_i \geq \frac{|g_{ji}| + \sum_{k \neq j,i} |g_{jk}|w_{ki}\eta_i}{|g_{jj}|} = g_{ji}.$$

Combined with $0 \leq \eta_i \leq 1$, we arrive at $0 \leq g_{ji} \leq w_{ji}$.

Hence,

$$\frac{|g_{ji}|}{|g_{jj}|g_{ji} - \sum_{k \neq j,i} |g_{jk}|g_{ki}} = \frac{|g_{ji}|g_{ji} - \sum_{k \neq j,i} |g_{jk}|w_{ki}\eta_i}{|g_{jj}|g_{ji} - \sum_{k \neq j,i} |g_{jk}|g_{ki}} \leq 1.$$

Furthermore, by the definition of γ_i , we have $0 \leq \gamma_i \leq 1$.

Since

$$\gamma_i \geq \frac{|g_{ji}|}{|g_{jj}|g_{ji} - \sum_{k \neq j,i} |g_{jk}|g_{ki}},$$

we have

$$g_{ji}\gamma_i \geq \frac{|g_{ji}| + \sum_{k \neq j,i} |g_{jk}|g_{ki}\gamma_i}{|g_{jj}|} = \rho_{ji}.$$

By $0 \leq \gamma_i \leq 1$, we have $0 \leq \rho_{ji} \leq g_{ji}$. Hence, $0 \leq \rho_{ji} \leq g_{ji} \leq w_{ji} < 1$.

From the inequalities $0 \leq \rho_{ji} \leq w_{ji} < 1$, we see that the quantity ρ_{ji} obtained by our iterative refinement is tighter than the starting iterate w_{ji} [30]. This improvement forms the theoretical basis for deriving sharper lower bounds for $\tau(\mathbf{G} \circ \mathbf{G}^{-1})$ in our subsequent corollaries.

Lemma 6. If $\mathbf{G} = (g_{ij}) \in R^{n \times n}$ is a strictly diagonally dominant matrix of rows, then the inverse matrix $\mathbf{G}^{-1} = (\beta_{ij})$ satisfies

$$\beta_{ji} \leq \frac{|g_{ji}| + \sum_{k \neq j,i} |g_{jk}|g_{ki}\gamma_i}{|g_{jj}|} \beta_{ii} \leq \rho_j \beta_{ii}, \quad j \neq i, \quad i, j \in \mathbb{N}.$$

Proof. Define

$$\begin{aligned} r_i(\varepsilon) &= \max_{j \neq i} \left\{ \frac{|a_{ji}| + \varepsilon}{|a_{jj}| - \sum_{k \neq j,i} |a_{jk}|} \right\}, \quad m_{ji}(\varepsilon) = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|r_i(\varepsilon) + \varepsilon}{a_{jj}}, \\ h_i(\varepsilon) &= \max_{j \neq i} \left\{ \frac{|a_{ji}| + \varepsilon}{|a_{jj}|m_{ji}(\varepsilon) - \sum_{k \neq j,i} |a_{jk}|m_{ki}(\varepsilon)} \right\}, \quad w_{ji}(\varepsilon) = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|m_{ki}(\varepsilon)h_i(\varepsilon) + \varepsilon}{|a_{jj}|}, \\ \eta_i(\varepsilon) &= \max_{j \neq i} \left\{ \frac{|g_{ji}| + \varepsilon}{|g_{jj}|\theta_{ji}(\varepsilon) - \sum_{k \neq j,i} |g_{jk}|\theta_{ki}(\varepsilon)} \right\}, \quad g_{ji}(\varepsilon) = \frac{|g_{ji}| + \sum_{k \neq j,i} |g_{jk}|g_{ki}(\varepsilon)\eta_i(\varepsilon) + \varepsilon}{|g_{jj}|}, \\ \gamma_i(\varepsilon) &= \max_{j \neq i} \left\{ \frac{|g_{ji}| + \varepsilon}{|g_{jj}|g_{ji}(\varepsilon) - \sum_{k \neq j,i} |g_{jk}|g_{ki}(\varepsilon)} \right\}. \end{aligned}$$

Given that $\mathbf{G}$ is strictly diagonally dominant by rows, one can always pick a sufficiently small $\varepsilon > 0$ fulfilling

$$0 < g_{ji}(\varepsilon) \leq w_{ji}(\varepsilon) \leq m_{ji}(\varepsilon) \leq r_i(\varepsilon) < 1, \quad 0 < h_i(\varepsilon) < 1, \quad 0 < \eta_i(\varepsilon) < 1, \quad 0 < \gamma_i(\varepsilon) < 1.$$

Denote the diagonal matrix

$$W_i(\varepsilon) = \text{diag}(\mathcal{G}_{1i}(\varepsilon)\gamma_i(\varepsilon), \dots, \mathcal{G}_{i-1,i}(\varepsilon)\gamma_i(\varepsilon), 1, \mathcal{G}_{i+1,i}(\varepsilon)\gamma_i(\varepsilon), \dots, \mathcal{G}_{ni}(\varepsilon)\gamma_i(\varepsilon)), \quad i \in \mathbf{N}.$$

In fact, for $j \neq i$, from the definition of $\gamma_i(\varepsilon)$, we have

$$\gamma_i(\varepsilon) > \frac{|g_{ji}|}{|g_{jj}|\mathcal{G}_{ji}(\varepsilon) - \sum_{k \neq j,i} |g_{jk}|\mathcal{G}_{ki}(\varepsilon)},$$

It then follows that

$$|g_{ji}| + \sum_{k \neq j,i} |g_{jk}|\mathcal{G}_{ki}(\varepsilon)\gamma_i(\varepsilon) < |g_{jj}|\mathcal{G}_{ji}(\varepsilon)\gamma_i(\varepsilon). \quad (1)$$

For $j = i$, we have

$$\sum_{k \neq j,i} |g_{ik}|\mathcal{G}_{ki}(\varepsilon)\gamma_i(\varepsilon) < \sum_{k \neq i} |g_{ik}| < |g_{ii}|. \quad (2)$$

Combining relations (1) and (2), the matrix $\mathbf{G}W_i(\varepsilon)$ is strictly row diagonally dominant and also an $\mathbf{M}$ -matrix. By Lemma 4, we obtain

$$\frac{\beta_{ji}}{\mathcal{G}_{ji}(\varepsilon)\gamma_i(\varepsilon)} \leq \frac{|g_{ji}| + \sum_{k \neq j,i} |g_{jk}|\mathcal{G}_{ki}(\varepsilon)\gamma_i(\varepsilon)}{|g_{jj}|\mathcal{G}_{ji}(\varepsilon)\gamma_i(\varepsilon)} \beta_{ii},$$

which simplifies to

$$\beta_{ji} \leq \frac{|g_{ji}| + \sum_{k \neq j,i} |g_{jk}|\mathcal{G}_{ki}(\varepsilon)\gamma_i(\varepsilon)}{|g_{jj}|} \beta_{ii},$$

Letting $\varepsilon \rightarrow 0$, one deduces that

$$\beta_{ji} \leq \frac{|g_{ji}| + \sum_{k \neq j,i} |g_{jk}|\mathcal{G}_{ki}\gamma_i}{|g_{jj}|} \beta_{ii} \leq \rho_j \beta_{ii}, \quad j \neq i, \quad i, j \in \mathbf{N}.$$

Remark 1. Since $\rho_{ji} \leq w_{ji} \leq u_{ji} \leq m_{ji} \leq s_{ji}$, $j \neq i, i, j \in \mathbf{N}$. This indicates that Lemma 6 provides an improved bound compared with the relevant conclusions established in the literature [26,28–30].

Lemma 7. If $\mathbf{G} \in M_n$ and $\mathbf{G}^{-1} = (\beta_{ij})$ is a doubly stochastic matrix, then

$$\beta_{ii} \geq \frac{1}{1 + \sum_{j \neq i} \rho_{ji}}, \quad i \in \mathbf{N}.$$

Proof. Since $\mathbf{G}$ is an $\mathbf{M}$ -matrix and $\mathbf{G}^{-1} = (\beta_{ij})$ is a doubly stochastic matrix, we have

$$g_{ii} = \sum_{k \neq i} |g_{ik}| + 1 = \sum_{k \neq i} |g_{ki}| + 1, \quad \beta_{ii} + \sum_{j \neq i} \beta_{ji} = 1, \quad i \in \mathbf{N}.$$

We thus deduce that $\mathbf{G}$ is strictly diagonally dominant by rows. By Lemma 6, we have

$$\begin{aligned} 1 &= \beta_{ii} + \sum_{j \neq i} \beta_{ji} \leq \beta_{ii} + \sum_{j \neq i} \frac{|g_{ji}| + \sum_{k \neq j, i} |g_{jk}| \varrho_{ki} \gamma_i}{|g_{jj}|} \beta_{ii} \\ &= \left(1 + \sum_{j \neq i} \frac{|g_{ji}| + \sum_{k \neq j, i} |g_{jk}| \varrho_{ki} \gamma_i}{|g_{jj}|} \right) \beta_{ii} \\ &= \left(1 + \sum_{j \neq i} \rho_{ji} \right) \beta_{ii}. \end{aligned}$$

Thus, we obtain

$$\beta_{ii} \geq \frac{1}{1 + \sum_{j \neq i} \rho_{ji}}, \quad i \in \mathbf{N}.$$

Theorem 1. Let $\mathbf{G} = (g_{ij})$, $\mathbf{H} = (h_{ij}) \in M_n$, and $\mathbf{G}^{-1}$ for the inverse matrix of $\mathbf{G}$. Then

$$\tau(\mathbf{H} \circ \mathbf{G}^{-1}) \geq \min_{1 \leq i \leq n} \left\{ \frac{h_{ii} - \rho_i \sum_{j \neq i} |h_{ji}|}{g_{ii}} \right\}, \quad i, j \in \mathbf{N}.$$

Proof. From Lemma 1 and the identity

$$\tau(\mathbf{H} \circ \mathbf{G}^{-1}) = \tau(\mathbf{X}^{-1}(\mathbf{H} \circ \mathbf{G}^{-1})\mathbf{X}) = \tau(\mathbf{H} \circ (\mathbf{X}^{-1}\mathbf{G}\mathbf{X})),$$

we may assume without loss of generality that $\mathbf{G}$ is a strictly diagonally dominant $\mathbf{M}$ -matrix.

(1) Both $\mathbf{G}$ and $\mathbf{H}$ are irreducible. Suppose $\mathbf{G}^{-1} = (\beta_{ij})$. For any $i \in \mathbf{N}$, we have $0 < \rho_i < 1$.

Thus, based on Lemma 2 and the inequality $0 \leq \tau(\mathbf{H} \circ \mathbf{G}^{-1}) \leq h_{ii}\beta_{ii}$, there exists $i \in \mathbf{N}$ such that

$$|\tau(\mathbf{H} \circ \mathbf{G}^{-1}) - h_{ii}\beta_{ii}| \leq \rho_i \sum_{j \neq i} \frac{1}{\rho_j} |h_{ji}\beta_{ji}|.$$

It follows that

$$\begin{aligned} \tau(\mathbf{H} \circ \mathbf{G}^{-1}) &\geq h_{ii}\beta_{ii} - \rho_i \sum_{j \neq i} \frac{1}{\rho_j} |h_{ji}\beta_{ji}| \\ &\geq h_{ii}\beta_{ii} - \rho_i \sum_{j \neq i} \frac{1}{\rho_j} |h_{ji}| n_j \beta_{ii} \\ &= \left(h_{ii} - \rho_i \sum_{j \neq i} |h_{ji}| \right) \beta_{ii}. \end{aligned}$$

Since $\mathbf{G}\mathbf{G}^{-1} = \mathbf{I}$, we have

$$1 = \sum_{j=1}^n g_{ij} \beta_{ji} = g_{ii} \beta_{ii} - \sum_{j \neq i} |g_{ij}| \beta_{ji}.$$

This immediately yields $g_{ii} \beta_{ii} \geq 1$, or equivalently, $\beta_{ii} \geq \frac{1}{g_{ii}}$, $i \in \mathbf{N}$. Consequently

$$\tau(\mathbf{H} \circ \mathbf{G}^{-1}) \geq \frac{h_{ii} - \rho_i \sum_{j \neq i} |h_{ji}|}{g_{ii}} \geq \min_{1 \leq i \leq n} \left\{ \frac{h_{ii} - \rho_i \sum_{j \neq i} |h_{ji}|}{g_{ii}} \right\}.$$

(2) At least one of $\mathbf{G}$ and $\mathbf{H}$ is reducible. We construct a permutation matrix $\mathbf{M} = (m_{ij})$ such that $m_{12} = m_{23} = \cdots = m_{n1} = 1$ and all other entries of $\mathbf{M}$ equal zero. For any sufficiently small positive real number ε , the matrices $\mathbf{G} - \varepsilon \mathbf{M}$ and $\mathbf{H} - \varepsilon \mathbf{M}$ are both irreducible M -matrices. We replace $\mathbf{G}$ and $\mathbf{H}$ with $\mathbf{G} - \varepsilon \mathbf{M}$ and $\mathbf{H} - \varepsilon \mathbf{M}$, respectively. The inequality holds as we take the limit $\varepsilon \rightarrow 0$.

The following Corollaries are derived from Theorem 1.

Corollary 1. Let $\mathbf{G} = (g_{ij})$, $\mathbf{H} = (h_{ij}) \in M_n$, and $\mathbf{G}^{-1}$ be a double stochastic matrix. Then

$$\tau(\mathbf{H} \circ \mathbf{G}^{-1}) \geq \min_{1 \leq i \leq n} \left\{ \frac{h_{ii} - \rho_i \sum_{j \neq i} |h_{ji}|}{1 + \sum_{j \neq i} \rho_{ji}} \right\}, \quad i, j \in \mathbf{N}.$$

Corollary 2. Let $\mathbf{G} = (g_{ij}) \in M_n$ and $\mathbf{G}^{-1}$ be a double stochastic matrix. Then

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \min_{1 \leq i \leq n} \left\{ \frac{g_{ii} - \rho_i \sum_{j \neq i} |g_{ji}|}{1 + \sum_{j \neq i} \rho_{ji}} \right\}, \quad i, j \in \mathbf{N}.$$

Theorem 2. If $\mathbf{G} = (g_{ij})$, $\mathbf{H} = (h_{ij}) \in M_n$, and $\mathbf{G}^{-1} = (\beta_{ij})$, then

$$\tau(\mathbf{H} \circ \mathbf{G}^{-1}) \geq \min_{j \neq i} \frac{1}{2} \left\{ h_{ii} \beta_{ii} + h_{jj} \beta_{jj} - \left[(h_{ii} \beta_{ii} - h_{jj} \beta_{jj})^2 + 4(\rho_i \sum_{k \neq i} |h_{ki}| \beta_{ii})(\rho_j \sum_{k \neq j} |h_{kj}| \beta_{jj}) \right]^{\frac{1}{2}} \right\}.$$

Proof. It is trivial that the conclusion holds when $n = 1$. We shall verify the case $n \geq 2$ below.

From Lemma 1 and the identity

$$\tau(\mathbf{H} \circ \mathbf{G}^{-1}) = \tau(\mathbf{X}^{-1}(\mathbf{H} \circ \mathbf{G}^{-1})\mathbf{X}) = \tau(\mathbf{H} \circ (\mathbf{X}^{-1}\mathbf{G}\mathbf{X})),$$

we may assume without loss of generality that $\mathbf{G}$ is a strictly diagonally dominant M -matrix, and set $\tau(\mathbf{H} \circ \mathbf{G}^{-1}) = \lambda$.

(1) If both $\mathbf{G}$ and $\mathbf{H}$ are irreducible, since λ is an eigenvalue of $\mathbf{H} \circ \mathbf{G}^{-1}$, we have $0 < \lambda < h_{ii} \beta_{ii}$. From Lemma 3, there exist $i, j \in \mathbf{N}$ with $j \neq i$ such that

$$\begin{aligned}
& |\lambda - h_{ii}\beta_{ii}| |\lambda - h_{jj}\beta_{jj}| \\
& \leq \left(\rho_i \sum_{k \neq i} \frac{1}{\rho_k} |h_{ki}\beta_{ki}| \right) \left(\rho_j \sum_{k \neq j} \frac{1}{\rho_k} |h_{kj}\beta_{kj}| \right) \\
& \leq \left(\rho_i \sum_{k \neq i} \frac{1}{\rho_k} |h_{ki}| \rho_k \beta_{ii} \right) \cdot \left(\rho_j \sum_{k \neq j} \frac{1}{\rho_k} |h_{kj}| \rho_k \beta_{jj} \right) \\
& = \left(\rho_i \sum_{k \neq i} |h_{ki}| \beta_{ii} \right) \cdot \left(\rho_j \sum_{k \neq j} |h_{kj}| \beta_{jj} \right).
\end{aligned}$$

Then,

$$\lambda \geq \frac{1}{2} \left\{ h_{ii}\beta_{ii} + h_{jj}\beta_{jj} - \left[(h_{ii}\beta_{ii} - h_{jj}\beta_{jj})^2 + 4(\rho_i \sum_{k \neq i} |h_{ki}| \beta_{ii})(\rho_j \sum_{k \neq j} |h_{kj}| \beta_{jj}) \right]^{\frac{1}{2}} \right\}.$$

Therefore,

$$\begin{aligned}
\tau(\mathbf{H} \circ \mathbf{G}^{-1}) & \geq \frac{1}{2} \left\{ h_{ii}\beta_{ii} + h_{jj}\beta_{jj} - \left[(h_{ii}\beta_{ii} - h_{jj}\beta_{jj})^2 + 4(\rho_i \sum_{k \neq i} |h_{ki}| \beta_{ii})(\rho_j \sum_{k \neq j} |h_{kj}| \beta_{jj}) \right]^{\frac{1}{2}} \right\} \\
& \geq \min_{j \neq i} \frac{1}{2} \left\{ h_{ii}\beta_{ii} + h_{jj}\beta_{jj} - \left[(h_{ii}\beta_{ii} - h_{jj}\beta_{jj})^2 + 4(\rho_i \sum_{k \neq i} |h_{ki}| \beta_{ii})(\rho_j \sum_{k \neq j} |h_{kj}| \beta_{jj}) \right]^{\frac{1}{2}} \right\}.
\end{aligned}$$

(2) At least one of $\mathbf{G}$ and $\mathbf{H}$ is reducible. We construct a permutation matrix $\mathbf{M} = (m_{ij})$ such that $m_{12} = m_{23} = \dots = m_{n1} = 1$ and all other entries of $\mathbf{M}$ equal zero. For any sufficiently small positive real number ε , the matrices $\mathbf{G} - \varepsilon\mathbf{M}$ and $\mathbf{H} - \varepsilon\mathbf{M}$ are both irreducible M -matrices. We replace $\mathbf{G}$ and $\mathbf{H}$ with $\mathbf{G} - \varepsilon\mathbf{M}$ and $\mathbf{H} - \varepsilon\mathbf{M}$, respectively. The inequality holds as we take the limit $\varepsilon \rightarrow 0$.

Corollary 3. Let $\mathbf{G} = (g_{ij}) \in M_n$ and $\mathbf{G}^{-1} = (\beta_{ij})$ be a double stochastic matrix. Then,

$$\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq \min_{j \neq i} \frac{1}{2} \left\{ g_{ii}\beta_{ii} + g_{jj}\beta_{jj} - \left[(g_{ii}\beta_{ii} - g_{jj}\beta_{jj})^2 + 4 \left(\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right) \left(\rho_j \sum_{k \neq j} |g_{kj}| \beta_{jj} \right) \right]^{\frac{1}{2}} \right\}.$$

Remark 2. In view of $\rho_{ji} \leq w_{ji}$, it follows that $\rho_i \leq w_i$. Hence, Corollary 2 and Corollary 3 improve the results of [30].

Theorem 3. Let $\mathbf{G} = (g_{ij}) \in M_n$ and $\mathbf{G}^{-1} = (\beta_{ij})$ be a double stochastic matrix. Then,

$$\min_{j \neq i} \frac{1}{2} \left\{ g_{ii}\beta_{ii} + g_{jj}\beta_{jj} - \left[(g_{ii}\beta_{ii} - g_{jj}\beta_{jj})^2 + 4(\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii})(\rho_j \sum_{k \neq j} |g_{kj}| \beta_{jj}) \right]^{\frac{1}{2}} \right\}$$

$$\geq \min_{1 \leq i \leq n} \left\{ \frac{g_{ii} - \rho_i \sum_{j \neq i} |g_{ji}|}{1 + \sum_{j \neq i} \rho_{ji}} \right\}.$$

Proof. Without loss of generality, for $i \neq j$, we may suppose

$$g_{ii}\beta_{ii} - \rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \leq g_{jj}\beta_{jj} - \rho_j \sum_{k \neq j} |g_{kj}| \beta_{jj}.$$

Then,

$$\rho_j \sum_{k \neq j} |g_{kj}| \beta_{jj} \leq g_{jj}\beta_{jj} - g_{ii}\beta_{ii} + \rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii}.$$

It then follows that

$$\begin{aligned} & (g_{ii}\beta_{ii} - g_{jj}\beta_{jj})^2 + 4 \left(\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right) \left(\rho_j \sum_{k \neq j} |g_{kj}| \beta_{jj} \right) \\ & \leq (g_{ii}\beta_{ii} - g_{jj}\beta_{jj})^2 + 4 \left(\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right) \cdot \left(g_{jj}\beta_{jj} - g_{ii}\beta_{ii} + \rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right) \\ & = (g_{ii}\beta_{ii} - g_{jj}\beta_{jj})^2 + 4 \left(\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right) \cdot (g_{jj}\beta_{jj} - g_{ii}\beta_{ii}) + \left(2\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right)^2 \\ & = \left(g_{jj}\beta_{jj} - g_{ii}\beta_{ii} + 2\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right)^2. \end{aligned}$$

Accordingly,

$$\begin{aligned} & \frac{1}{2} \left\{ g_{ii}\beta_{ii} + g_{jj}\beta_{jj} - \left[(g_{ii}\beta_{ii} - g_{jj}\beta_{jj})^2 + 4(\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii})(\rho_j \sum_{k \neq j} |g_{kj}| \beta_{jj}) \right]^{\frac{1}{2}} \right\} \\ & \geq \frac{1}{2} \left\{ g_{ii}\beta_{ii} + g_{jj}\beta_{jj} - \left[\left(g_{jj}\beta_{jj} - g_{ii}\beta_{ii} + 2\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right)^2 \right]^{\frac{1}{2}} \right\} \\ & = \frac{1}{2} \left\{ g_{ii}\beta_{ii} + g_{jj}\beta_{jj} - g_{jj}\beta_{jj} + g_{ii}\beta_{ii} - 2\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \right\} \\ & = g_{ii}\beta_{ii} - \rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii} \\ & \geq \frac{g_{ii} - \rho_i \sum_{k \neq i} |g_{ki}|}{1 + \sum_{k \neq i} \rho_{ki}}. \end{aligned}$$

Thus, we draw the following conclusion:

$$\min_{j \neq i} \frac{1}{2} \left\{ g_{ii}\beta_{ii} + g_{jj}\beta_{jj} - \left[(g_{ii}\beta_{ii} - g_{jj}\beta_{jj})^2 + 4(\rho_i \sum_{k \neq i} |g_{ki}| \beta_{ii})(\rho_j \sum_{k \neq j} |g_{kj}| \beta_{jj}) \right]^{\frac{1}{2}} \right\}$$

$$\geq \min_{1 \leq i \leq n} \left\{ \frac{g_{ii} - \rho_i \sum_{j \neq i} |g_{ji}|}{1 + \sum_{j \neq i} \rho_{ji}} \right\}.$$

3. Numerical examples

To further illustrate the superiority of our novel bounds over several known results, three numerical examples are presented in this section.

Example 1. As the first illustration, we take the matrix adopted in [26] for comparison.

$$\mathbf{G} = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}.$$

Table 1 presents the comparison between the estimation results in [25–30] and Corollary 2 and Corollary 3 proposed in this paper.

Table 1 The lower bound of $\tau(\mathbf{G} \circ \mathbf{G}^{-1})$.

Method	$\tau(\mathbf{G} \circ \mathbf{G}^{-1})$
Conjecture of [25]	0.5000
Theorem 3.1 of [26]	0.6624
Theorem 3.2 of [27]	0.8457
Theorem 3.2 of [28]	0.8000
Theorem 3.1 of [29]	0.8250
Corollary 2 of [30]	0.8320
Corollary 3 of [30]	0.8880
Corollary 2	0.8363
Corollary 3	0.8929

Example 2. Now we present the second example, where we consider the following 10×10 matrix taken from Example 1 of [33].

$$\mathbf{G} = \begin{pmatrix} 20 & -1 & -2 & -3 & -4 & -1 & -1 & -3 & -2 & -2 \\ -1 & 18 & -3 & -1 & -1 & -4 & -2 & -1 & -3 & -1 \\ -2 & -1 & 10 & -1 & -1 & -1 & 0 & -1 & -1 & -1 \\ -3 & -1 & 0 & 16 & -4 & -2 & -1 & -1 & -1 & -2 \\ -1 & -3 & 0 & -2 & 15 & -1 & -1 & -1 & -2 & -3 \\ -3 & -2 & -1 & -1 & -1 & 12 & -2 & 0 & -1 & 0 \\ -1 & -3 & -1 & -1 & 0 & -1 & 9 & 0 & -1 & 0 \\ -3 & -1 & -1 & -4 & -1 & 0 & 0 & 12 & 0 & -1 \\ -2 & -4 & -1 & -1 & -1 & 0 & -1 & -3 & 14 & 0 \\ -3 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -2 & 11 \end{pmatrix}.$$

Based on $\mathbf{G} \in Z_n$ and $\mathbf{G}\mathbf{e} = \mathbf{e}$, $\mathbf{G}^T\mathbf{e} = \mathbf{e}$, it is easy to see that $\mathbf{G}$ is a nonsingular $\mathbf{M}$ -matrix, and $\mathbf{G}^{-1}$ is doubly stochastic. Table 2 illustrates the comparison between the results derived in this paper and those presented in [25–30].

Table 2. The lower bound of $\tau(\mathbf{G} \circ \mathbf{G}^{-1})$.

Method	$\tau(\mathbf{G} \circ \mathbf{G}^{-1})$
Conjecture of [25]	0.2000
Theorem 3.1 of [26]	0.2519
Theorem 3.2 of [27]	0.4732
Theorem 3.2 of [28]	0.4125
Theorem 3.1 of [29]	0.4471
Corollary 2 of [30]	0.5062
Corollary 3 of [30]	0.6064
Corollary 2	0.6130
Corollary 3	0.6668

Remark 3. The numerical results listed in the above two tables demonstrate that the lower bounds derived from Corollary 2 and Corollary 3 are sharper than those provided in some existing literature. Moreover, the result obtained via Corollary 3 in this paper outperforms that from Corollary 2.

Example 3. Consider the following circulant nonsingular $\mathbf{M}$ -matrix taken from Example 3 of [33].

Let $\mathbf{G} = (g_{ij})$, where $g_{11} = g_{22} = \cdots g_{n,n} = 2$, $g_{12} = g_{23} = \cdots = g_{n-1,n} = g_{n,1} = -1$, and $g_{ij} = 0$ elsewhere.

It can be readily verified that the circulant matrix $\mathbf{G}$ constructed above is a nonsingular $\mathbf{M}$ -matrix, and its inverse $\mathbf{G}^{-1}$ is a doubly stochastic matrix. For the case $n=10$, we compute the lower bounds $\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq 0.7164$ and $\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq 0.7507$ by virtue of Corollary 2 and Corollary 3, respectively, while the exact value is $\tau(\mathbf{G} \circ \mathbf{G}^{-1}) = 0.7507$. When $n=100$, the corresponding bounds from the two corollaries are $\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq 0.3057$ and $\tau(\mathbf{G} \circ \mathbf{G}^{-1}) \geq 0.7500$, where the exact value satisfies $\tau(\mathbf{G} \circ \mathbf{G}^{-1}) = 0.7500$.

The iterative scheme in [33] can also attain the exact lower bound for this example, yet it requires more iteration steps. By contrast, our proposed iterative formulas yield the exact bound with fewer iterations, which demonstrates the superiority of our method in terms of convergence performance and computational efficiency.

4. Conclusions

$\mathbf{M}$ -matrices are widely employed in matrix theory and engineering problems such as convergence analysis of numerical iterations, system stability judgment, and eigenvalue estimation of Schur products due to their favorable spectral properties. The Schur product, also known as the Hadamard product of matrices, refers to the operation of multiplying the corresponding entries of two matrices, which finds extensive applications in mathematics, engineering, data science, and other disciplines.

This paper mainly investigates the lower bound estimation for the minimum eigenvalue of the

Schur product of an M -matrix and the inverse of another M -matrix and derives novel and effective new inequalities. Theoretical proofs and numerical examples demonstrate that the proposed lower bounds can well approximate the true minimum eigenvalue, featuring superior estimation accuracy and universality and improving several existing results. The relevant conclusions further enrich the theoretical research on Schur products of M -matrices and provide theoretical references for related studies in matrix theory.

Author contributions

Xiaofeng Zhang: Conceptualization, Data curation, Writing-original draft, Methodology; Fubin Chen: Formal analysis, Writing-review & editing, Investigation, Data curation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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