



Research article

Sharpness results for a class of analytic functions defined via a generalized Balloon-shaped domain

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Abstract: In geometric function theory, the estimation of initial coefficients for analytic functions is a fundamental topic of considerable interest to researchers. To gain a more profound understanding of these estimations, it becomes essential to identify the values that satisfy the state of equality (the sharpness result). This state represents one of the most difficult challenges in this field. For this reason, many studies that do not achieve the sharpness remain open for further investigations by other researchers. In this paper, we present an extreme value function that enables the acquisition of sharpness for a class of symmetric functions with real order. Furthermore, we verify the accuracy of the representation used to determine the initial coefficients. We also derive the sharp estimates by selecting appropriate values for some parameters affecting the extreme value functions. Finally, the sharp bounds of the initial coefficients for the inverse of analytic functions and Hankel determinants are derived.

Keywords: analytic functions; inverse analytic functions; Hankel determinants

Mathematics Subject Classification: 30C45, 30C50

1. Introduction

To provide a clearer framework for the results investigated in this paper, we first recall several fundamental concepts from geometric function theory. In this regard, we denote by $\mathcal{A}$ the class of normalized analytic functions and by $\mathcal{S}$ the class of univalent functions. These classes are defined in

the open unit disk $\mathbb{E} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ through the following set representation:

$$\mathcal{A} = \left\{ \zeta \in \mathbb{E} : h(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k \right\} \quad (1.1)$$

and $\mathcal{S}$ represents the set of univalent functions defined in $\mathbb{E}$.

Researchers studied many subclasses of univalent functions associated with different types of image fields. Among the most prominent and important of these are the starlike and convex functions, usually denoted as $\mathcal{S}^*$ and $\mathcal{C}$, respectively.

The class $\mathcal{S}^*$ is given by

$$\Re\left(\frac{\zeta h'(\zeta)}{h(\zeta)}\right) > 0, \quad \zeta \in \mathbb{E},$$

while convex functions are given by the following form

$$\Re\left(1 + \frac{\zeta h''(\zeta)}{h'(\zeta)}\right) > 0, \quad \zeta \in \mathbb{E}.$$

For any analytic functions $h_1(\zeta), h_2(\zeta) \in \mathcal{A}$, it is said that $h_1(\zeta)$ is subordinated to $h_2(\zeta)$, and we write $h_1(\zeta) < h_2(\zeta)$ if the Schwarz function ν exists in $\mathbb{E}$ such that $\nu(0) = 0$ and $|\nu(\zeta)| < 1$ and $h_1(\zeta) = h_2(\nu(\zeta))$. Furthermore, if $h_2(\zeta)$ is a univalent function, then this relation is equivalent to the following formula:

$$h_1(\zeta) < h_2(\zeta) \iff h_1(0) = h_2(0) \text{ with } h_1(\mathbb{E}) \subset h_2(\mathbb{E}).$$

Analytical functions are typically divided into diverse classes depending on the geometric properties of their images on the disk $\mathbb{E}$. This division allows for a deeper understanding of their analytic structure and functional behavior. In this context, Ma and Minda [4] considered an important generalization of the starlike function family $\mathcal{S}^*$ in 1992, defining an extended subclass denoted as $\mathcal{S}^*(\Phi)$. This family involves a function $h \in \mathcal{A}$ that satisfy the appropriate subsequent condition

$$\frac{\zeta h'(\zeta)}{h(\zeta)} < \Phi(\zeta),$$

where Φ is a univalent function satisfying $\Phi(0) = 1$, $\Phi'(0) > 0$, and $\Re e(\Phi(\zeta)) > 0$. Moreover, the image domain $\Phi(\mathbb{E})$ is starlike concerning to the point $\Phi(0) = 1$ and symmetric about the real axis. In their study, the authors investigated several geometric properties of this class, including distortion, growth, and covering theorems. The family $\mathcal{S}^*(\Phi)$ serves as a unifying framework that encompasses many well-known subfamilies of starlike functions through suitable choices of the function Φ , as formulated in Table 1.

Table 1. Various families depend on the choice of behavior for the function $\Phi(\zeta)$.

Family $\mathcal{S}^*(\Phi)$	Authors (Year)	Reference
$\mathcal{S}^*(1 + \frac{4}{3}\zeta + \frac{2}{3}\zeta^2)$	Sharma et al. (2016)	[5]
$\mathcal{S}^*(\varpi + (1 - \varpi)e^\zeta), \varpi \in [0, 1)$	Khatter et al. (2017)	[6]
$\mathcal{S}^*(\zeta + \sqrt{1 + \zeta^2})$	Raina et al. (2018)	[7]
$\mathcal{S}^*(1 + \sqrt{2}\zeta + \frac{\zeta^2}{2})$	Yunus et al. (2018)	[8]
$\mathcal{S}^*(\frac{2}{1+e^{-\zeta}})$	Goel & Kumar (2019)	[10]
$\mathcal{S}^*(\text{sech}(\zeta))$	Al-Shbeil et al. (2022)	[11]
$\mathcal{S}^*(1 + \sinh^{-1}(\zeta))$	Kumar & Arora (2022)	[9]
$\mathcal{S}^*(\frac{2\sqrt{1+\zeta}}{1+e^{-\zeta}})$	Ahmad et al. (2023)	[12]
$\mathcal{S}^*(2 - \sqrt[3]{\frac{2-\zeta^2}{2(1+\arcsin(\frac{2\zeta}{3}))}})$	Tayyah et al. (2025)	[25]

For the function h , Pommerenke [17] derived Hankel determinants of order s , denoted as $\mathcal{H}_{s,k}$, as follows:

$$\mathcal{H}_{s,k}(h) := \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+s-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+s-1} & a_{k+s} & \cdots & a_{k+2s-2} \end{vmatrix}, \quad (s, k \geq 1).$$

Familiar special determinants include the following:

$$\mathcal{H}_{2,1}(h) = a_3 - a_2^2, \quad (1.2)$$

$$\mathcal{H}_{2,2}(h) = a_2a_4 - a_3^2, \quad (1.3)$$

$$\mathcal{H}_{3,1}(h) = a_3(a_2a_4 - a_3^2) + a_4(a_2a_3 - a_4) + a_5(a_3 - a_2^2). \quad (1.4)$$

Nowadays, the topic of determining the sharp limits of Hankel determinants at specific values of both k and s has received increasing attention due to its theoretical and practical importance in the study of various subclasses of analytic functions. Several investigators have contributed to exploring and validating these limits across a wide range of such classes. Khan et al. [21] investigated a Hankel determinants and derived distinctive results and criteria for functions involving tangent function. In particular, robust estimates of the upper limits of 2nd- and 3rd-order Hankel determinants for several classes of standard analytic functions have been presented in references [22, 24, 27].

Ahmad et al. [12] investigated analytic functions associated with a balloon-shaped domain and obtained several coefficient estimates and bounds for Hankel determinants, while Tayyah et al. [15] introduced families of Ma-Minda-type analytic functions associated with a kidney-shaped domain and explored a number of their geometric and coefficient properties. In a related direction, Riaz et al. [20] studied starlike and convex functions associated with a three-leaf-shaped domain and established bounds for the second- and third-order Hankel determinants. These contributions illustrate the growing interest in connecting subclasses of analytic functions with prescribed geometric domains. Motivated by these developments, the present work extends this direction to a parameter-dependent generalized balloon-shaped domain and introduces a unified symmetric starlike-convex framework in which the parameters ϖ and σ control the geometric and analytic behavior of the class, respectively.

Based on the concept of Ma-Minda, Khattar [6] derived two functions dependent on the selection of the functions $\Phi(\zeta) = \varpi + (1 - \varpi)e^\zeta$ and $\Phi(\zeta) = \varpi + (1 - \varpi)(\zeta + \sqrt{1 + \zeta^2})$.

Motivated by that, we define the following function:

$$\Phi_\varpi(\zeta) = \varpi + (1 - \varpi) \frac{2\sqrt{1 + \zeta}}{1 + \exp(-\zeta)} = 1 + (1 - \varpi)\zeta + \frac{1}{8}(1 - \varpi)\zeta^2 - \frac{1}{24}(1 - \varpi)\zeta^3 - \frac{11}{384}(1 - \varpi)\zeta^4 + \dots, \quad (\varpi \in [0, 1)).$$

Figure 1 below shows the geometric behavior of $\Phi_\varpi(\mathbb{E})$ with different values of ϖ .

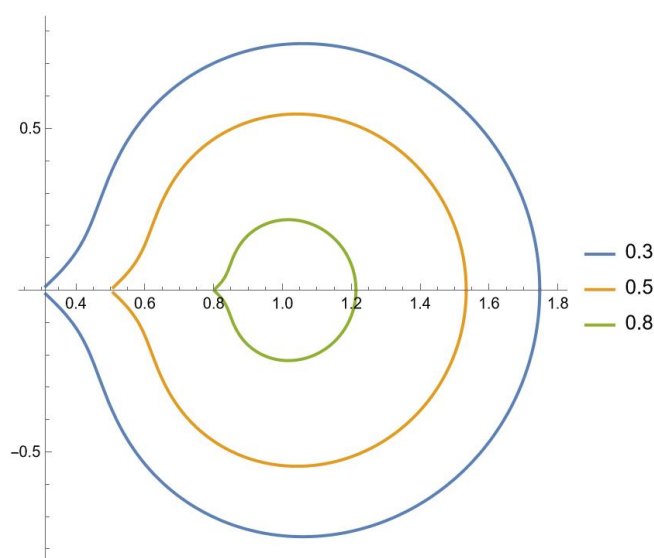


Figure 1. Geometric behavior of $\Phi_\varpi(\mathbb{E})$.

Theorem 1.1. For $\varpi \in [0, 1)$, the function

$$\Phi_\varpi(\zeta) = \varpi + (1 - \varpi) \frac{2\sqrt{1 + \zeta}}{1 + e^{-\zeta}}$$

is univalent in $\mathbb{E}$.

Proof. Consider

$$\mathcal{F}(\zeta) = \frac{2\sqrt{1 + \zeta}}{1 + e^{-\zeta}} = \frac{e^{\zeta/2} \sqrt{1 + \zeta}}{\cosh(\zeta/2)}.$$

Since $\mathcal{F}(\zeta) \neq 0$ in $\mathbb{E}$, we may define

$$\mathcal{G}(\zeta) := \log \mathcal{F}(\zeta).$$

Then,

$$\mathcal{G}'(\zeta) = \frac{1}{2(1 + \zeta)} + \frac{1}{1 + e^\zeta}.$$

For $|\zeta| < 1$, we have

$$\Re\left(\frac{1}{2(1 + \zeta)}\right) > \frac{1}{4}.$$

Moreover, writing $\zeta = x + iy$, we obtain

$$\Re\left(\frac{1}{1+e^\zeta}\right) = \frac{1+e^x \cos y}{|1+e^\zeta|^2} > 0,$$

since $|y| < 1$ and hence $\cos y > 0$. Therefore,

$$\Re \mathcal{G}'(\zeta) > \frac{1}{4} > 0.$$

Thus, by the Noshiro–Warschawski criterion, $\mathcal{G}$ is univalent in $\mathbb{E}$.

Furthermore,

$$\mathcal{G}(\zeta) = \frac{\zeta}{2} + \frac{1}{2} \log(1 + \zeta) - \log \cosh\left(\frac{\zeta}{2}\right).$$

Since

$$\left|\Im \frac{\zeta}{2}\right| < \frac{1}{2}, \quad |\arg(1 + \zeta)| < \frac{\pi}{2}$$

and

$$\left|\arg \cosh\left(\frac{\zeta}{2}\right)\right| < \frac{1}{2},$$

it follows that

$$|\Im \mathcal{G}(\zeta)| < \frac{1}{2} + \frac{\pi}{4} + \frac{1}{2} = 1 + \frac{\pi}{4}.$$

Hence, for any $\zeta_1, \zeta_2 \in \mathbb{E}$,

$$|\Im \mathcal{G}(\zeta_1) - \Im \mathcal{G}(\zeta_2)| < 2 + \frac{\pi}{2} < 2\pi.$$

Consequently, if

$$e^{\mathcal{G}(\zeta_1)} = e^{\mathcal{G}(\zeta_2)},$$

then

$$\mathcal{G}(\zeta_1) - \mathcal{G}(\zeta_2) = 2k\pi i$$

for some $k \in \mathbb{Z}$. The preceding inequality implies $k = 0$. Since $\mathcal{G}$ is univalent, we obtain $\zeta_1 = \zeta_2$. Hence, $\mathcal{F} = e^{\mathcal{G}}$ is univalent in $\mathbb{E}$.

Finally, since $1 - \varpi > 0$ for $0 \leq \varpi < 1$,

$$\Phi_\varpi(\zeta) = \varpi + (1 - \varpi)\mathcal{F}(\zeta)$$

is also univalent in $\mathbb{E}$. □

We consider now the class of analytic functions as follows:

Definition 1.1. For $\sigma \in [0, 1]$, then $h \in \mathcal{A}$ is within the symmetric class $\mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$ if the following holds:

$$\frac{2\zeta h'(\zeta) + 2\sigma\zeta^2 h''(\zeta)}{(1 - \sigma)(h(\zeta) - h(-\zeta)) + \sigma\zeta(h'(\zeta) + h'(-\zeta))} < \Phi_\varpi(\zeta), \quad (\zeta \in \mathbb{E}), \quad (1.5)$$

where $\Phi_\varpi(\zeta) = \varpi + (1 - \varpi)\frac{2\sqrt{1+\zeta}}{1+\exp(-\zeta)}$, ($\varpi \in [0, 1)$).

Example 1.1. The class $\mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$ is not empty; it involves a range of analytic functions within its structure. Let

$$h_2(\zeta) = \zeta + \frac{1}{2}\zeta^2 + \frac{1}{3}\zeta^3, \quad |\zeta| \leq 1.$$

By substituting the function $h_2(\zeta)$ into the class (1.5), we obtain

$$\begin{aligned} \Sigma_{\sigma,1}^\varpi(\zeta) &= \frac{2\zeta h_2'(\zeta) + 2\sigma\zeta^2 h_2''(\zeta)}{(1-\sigma)(h_2(\zeta) - h_2(-\zeta)) + \sigma\zeta(h_2'(\zeta) + h_2'(-\zeta))} \\ &= \frac{2(1^2(2\sigma+1)\zeta^2 + 1(\sigma+1)\zeta + 1)}{21^2\sigma\zeta^2 + \zeta(1-1\sigma) + 2}. \end{aligned}$$

1. If $\sigma = 0$, then

$$\Sigma_{0,1}^\varpi(\zeta) = \frac{2(1^2(2\sigma+1)\zeta^2 + 1(\sigma+1)\zeta + 1)}{21^2\sigma\zeta^2 + \zeta(1-1\sigma) + 2} = \frac{2(1^2\zeta^2 + 1\zeta + 1)}{1\zeta + 2} < \Phi_{0.3}(\zeta), \quad |\zeta| \leq 0.57,$$

see Figure 2(a).

2. $\sigma = 0.5$, then

$$\Sigma_{0.5,1}^\varpi(\zeta) = \frac{2(1^2(2\sigma+1)\zeta^2 + 1(\sigma+1)\zeta + 1)}{21^2\sigma\zeta^2 + \zeta(1-1\sigma) + 2} = \frac{2(21^2\zeta^2 + 1.51\zeta + 1)}{1^2\zeta^2 + 0.51\zeta + 2} < \Phi_{0.3}(\zeta), \quad |\zeta| \leq 0.45,$$

see Figure 2(b).

3. $\sigma = 1$, then

$$\Sigma_{1,1}^\varpi(\zeta) = \frac{2(1^2(2\sigma+1)\zeta^2 + 1(\sigma+1)\zeta + 1)}{21^2\sigma\zeta^2 + \zeta(1-1\sigma) + 2} = \frac{2(31^2\zeta^2 + 21\zeta + 1)}{21^2\zeta^2 + 2} < \Phi_{0.3}(\zeta), \quad |\zeta| \leq 0.31,$$

see Figure 2(c).

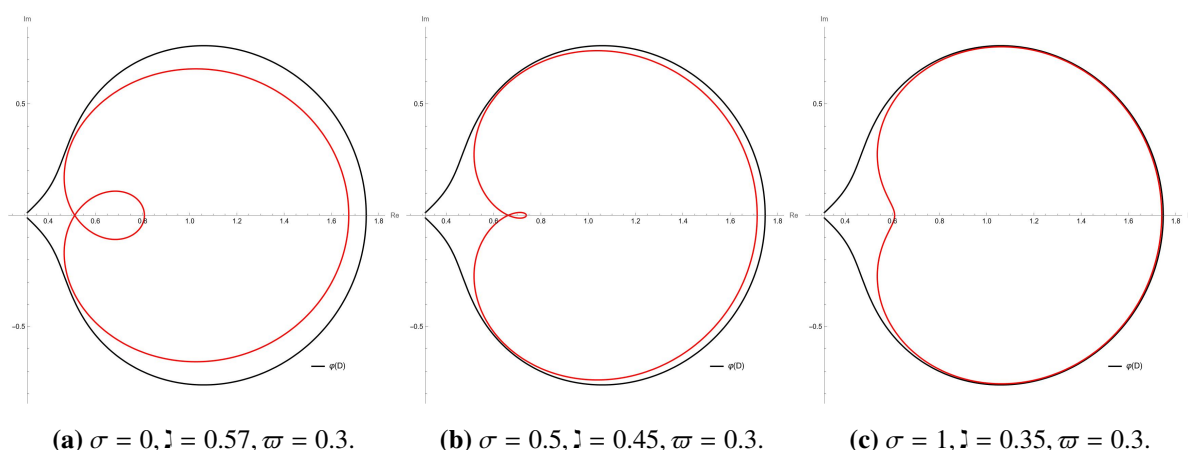


Figure 2. Representation of $\Sigma_{\sigma,1}^\varpi(\zeta) \subset \Phi_\varpi(\mathbb{E})$ with different parameters.

The introduction of the class $\mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$ stems from the desire to construct a unified framework that combines symmetric-starlike and symmetric-convex behavior within a single analytical structure.

The operator σ allows for a smooth transition between these two classes, such that the symmetric starlike character becomes apparent as it approaches zero, while the symmetric convex behavior becomes more pronounced as it approaches one. Simultaneously, the operator ϖ plays a crucial role in shaping a directed geometric distortion within the image domain, enriching the analytical properties of the class and giving it greater depth. This interplay between the geometric and analytical dimensions lends the class a dynamic quality that transcends traditional models.

More specifically, the present construction extends the balloon-shaped framework considered by Khan et al. [3], where a symmetric starlike class was associated with a fixed balloon-shaped mapping. In the present setting, the introduction of the parameter-dependent function Φ_{ϖ} produces a broader family of generalized balloon-shaped domains. Notably, the original balloon mapping studied in [3] is recovered when $\varpi = 0$. The additional parameter σ further enriches this framework by linking the symmetric starlike and symmetric convex settings within the same class. The integral representation is also developed in this generalized setting, leading to the family of functions $\mathbb{T}_s$, $s \in \mathbb{N}$, whose particular choices provide the extremal functions associated with the corresponding sharp coefficient estimates. Thus, the novelty of the present approach lies not only in generalizing the underlying balloon-shaped geometry but also in providing a two-parameter analytical framework through which the effects of ϖ and σ on coefficient bounds and extremal behavior can be explicitly investigated.

This construction also represents an advanced generalization of previous classes while maintaining strict precision in the parameters and Henkel determinants.

The main objectives of this research are summarized as follows. In the first part, we introduce a new concept based on a balloon-like behavior domain. We also establish a symmetric starlike-convex class dependently related to the given domain. The integral representation of the extreme function is derived, which then satisfies the boundary of the studied results. In the second part, Hankel determinants of different orders are derived and verified. Finally, we deduce the fundamental coefficients and Hankel determinants of the inverse analytic function with some special sharpness cases.

2. Extreme function for $\mathcal{S}_{\sigma}(\zeta; \Phi_{\varpi}(\zeta))$

Theorem 2.1. *The class $\mathcal{S}_{\sigma}(\zeta; \Phi_{\varpi}(\zeta))$ contains $h \in \mathcal{A}$ if and only if it can be represented by the following integral form:*

$$h(\zeta) = \zeta^{-\frac{1-\sigma}{\sigma}} \left(\int_0^{\zeta} \frac{1}{\sigma} y^{\frac{1-\sigma}{\sigma}-1} \left(\int_0^y q(t) \exp \left(\frac{1}{2} \int_0^t \frac{q(u) + q(-u) - 2}{u} du \right) dt \right) dy \right), \quad (2.1)$$

where

$$q(\zeta) < \Phi_{\varpi}(\zeta).$$

Proof. Define $k(\zeta) = (1 - \sigma)h(\zeta) + \sigma\zeta[h(\zeta)]'$. Then,

$$\begin{aligned} k(\zeta) - k(-\zeta) &= (1 - \sigma)h(\zeta) + \sigma\zeta[h(\zeta)]' - \{(1 - \sigma)h(-\zeta) + \sigma\zeta[h(-\zeta)]'\} \\ &= (1 - \sigma)h(\zeta) + \sigma\zeta h'(\zeta) - \{(1 - \sigma)h(-\zeta) - \sigma\zeta h'(-\zeta)\} \\ &= (1 - \sigma)(h(\zeta) - h(-\zeta)) + \sigma\zeta(h'(\zeta) + h'(-\zeta)). \end{aligned}$$

Hence,

$$\frac{2\zeta k'(\zeta)}{k(\zeta) - k(-\zeta)} = \frac{2\zeta \frac{d}{dz} [(1 - \sigma)h(\zeta) + \sigma\zeta h'(-\zeta)]}{(1 - \sigma)(h(\zeta) - h(-\zeta)) + \sigma\zeta(h'(\zeta) + h'(-\zeta))}$$

$$\begin{aligned}
&= \frac{2\zeta [(1-\sigma)h'(\zeta) + \sigma\zeta h''(\zeta) + \sigma h'(\zeta)]}{(1-\sigma)(h(\zeta) - h(-\zeta)) + \sigma\zeta(h'(\zeta) + h(-\zeta))} \\
&= \frac{2\zeta h'(\zeta) + 2\sigma\zeta^2 h''(\zeta)}{(1-\sigma)(h(\zeta) - h(-\zeta)) + \sigma\zeta(h'(\zeta) + h(-\zeta))}.
\end{aligned}$$

Put $\frac{2\zeta k'(\zeta)}{k(\zeta) - k(-\zeta)} = q(\zeta)$. Since $\frac{h(\zeta) - h(-\zeta)}{2}$ is symmetric-starlike and by straightforward calculations from [3, Theorem 2.1], we find that

$$k(\zeta) = \int_0^\zeta q(t) \exp\left(\frac{1}{2} \int_0^t \frac{q(u) + q(-u) - 2}{u} du\right) dt. \quad (2.2)$$

So,

$$\sigma\zeta h'(\zeta) + (1-\sigma)h(\zeta) = \int_0^\zeta q(t) \exp\left(\frac{1}{2} \int_0^t \frac{q(u) + q(-u) - 2}{u} du\right) dt.$$

Hence,

$$h'(\zeta) + \frac{1-\sigma}{\sigma\zeta} h(\zeta) = \frac{1}{\sigma\zeta} \int_0^\zeta q(t) \exp\left(\frac{1}{2} \int_0^t \frac{q(u) + q(-u) - 2}{u} du\right) dt. \quad (2.3)$$

It is observed that equation (2.3) represents a linear differential with respect to the function h , and its integral coefficient is $\zeta^{\frac{1-\sigma}{\sigma}}$. By the fact $h(0) = h'(0) - 1 = 0$, the constant of integration can be determined, and the solution to the equation is given in the following form:

$$h(\zeta) = \zeta^{-\frac{1-\sigma}{\sigma}} \left(\int_0^\zeta \frac{1}{\sigma} y^{\frac{1-\sigma}{\sigma}-1} \left(\int_0^y q(t) \exp\left(\frac{1}{2} \int_0^t \frac{q(u) + q(-u) - 2}{u} du\right) dt \right) dy \right).$$

□

Example 2.1. We take into account the analytical functions

$$q_1(\zeta) = 1 + \frac{2\zeta \exp(\zeta)}{9}, \quad q_2(\zeta) = \sqrt{1 + \sin(\sin(\zeta))}, \quad q_3(\zeta) = 1 + 1/3\zeta - 1/9\zeta^3.$$

Clearly, we can investigate that $q_j(0) = 1$ and $q_j(\mathbb{E}) \subset \Phi_{0.3}(\mathbb{E})$, ($j = 1, 2, 3$). Since $\Phi_{0.3}(\mathbb{E})$, then $q_j < \Phi_{0.3}$ (see Figure 3). Therefore, the theorem states that the functions recalled involve to the family $\mathcal{S}_\sigma(\zeta; \Phi_{0.3}(\zeta))$.

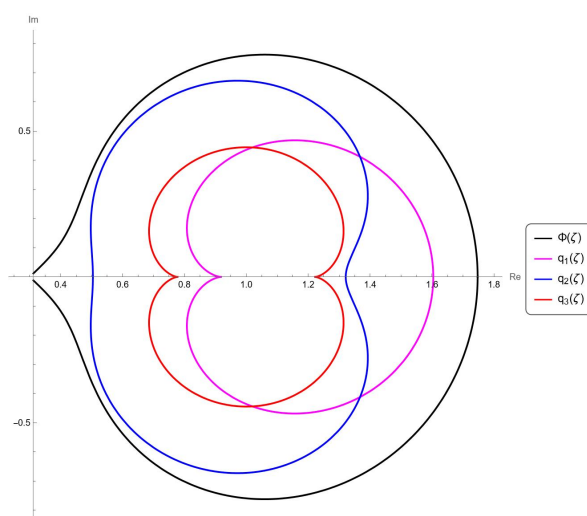


Figure 3. Plots boundary of $\Phi_{0.3}(\mathbb{E})$ and $q_j(\mathbb{E})$ ($j = 1, 2, 3$).

$$\begin{aligned}
h_1(\zeta) &:= \zeta + \frac{\zeta^2}{9(\sigma+1)} + \frac{\zeta^3}{9(2+\sigma)} + \frac{11\zeta^4}{324(3\sigma+1)} + \frac{5\zeta^5}{324(4\sigma+1)} + \cdots, \\
h_2(\zeta) &:= \zeta + \frac{\zeta^2}{4(\sigma+1)} - \frac{\zeta^3}{16(\sigma+2)} - \frac{13\zeta^4}{384(3\sigma+1)} + \frac{5\zeta^5}{384(4\sigma+1)} + \cdots, \\
h_3(\zeta) &:= \zeta - \frac{\zeta^2}{6(\sigma+1)} - \frac{\zeta^4}{36(3\sigma+1)} + \cdots.
\end{aligned}$$

Example 2.2. The class $\mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$ is a non-empty class, containing an unlimited number of functions given by the following formula:

$$\mathbb{T}_s(\zeta) = \zeta^{-\frac{1-\sigma}{\sigma}} \left(\int_0^\zeta \frac{1}{\sigma} y^{\frac{1-\sigma}{\sigma}-1} \left(\int_0^y q_s(t) \exp\left(\frac{1}{2} \int_0^t \frac{q_s(u) + q_s(-u) - 2}{u} du\right) dt \right) dy \right), \quad (2.4)$$

where $q_s(\zeta) = \varpi + (1 - \varpi) \frac{2\sqrt{1+\zeta^s}}{1+\exp(-\zeta^s)}$, ($s \in \mathbb{N}$).

The extremal functions $\mathbb{T}_s$ are obtained by setting $q_s(\zeta) = \Phi_\varpi(\zeta^s)$, where the appropriate choice of s yields equality in the corresponding sharp coefficient bound.

Particularly, we deduce for $s = 1, 2, 3, 4$ that

$$\mathbb{T}_1(\zeta) := \zeta + \frac{(1-\varpi)\zeta^2}{2(\sigma+1)} + \frac{(1-\varpi)\zeta^3}{16(2\sigma+1)} - \frac{(1-\varpi)(3\varpi-1)\zeta^4}{192(3\sigma+1)} - \frac{(1-\varpi)(3\varpi+8)\zeta^5}{1536(4\sigma+1)} + \cdots, \quad (2.5)$$

$$\mathbb{T}_2(\zeta) := \zeta + \frac{(1-\varpi)\zeta^3}{2(2\sigma+1)} - \frac{(1-\varpi)(4\varpi-5)\zeta^5}{32(4\sigma+1)} + \frac{(1-\varpi)(3\varpi(4\varpi-11)+17)\zeta^7}{576(6\sigma+1)} + \cdots, \quad (2.6)$$

$$\mathbb{T}_3(\zeta) := \zeta + \frac{(1-\varpi)\zeta^4}{4(3\sigma+1)} + \frac{(1-\varpi)\zeta^7}{48(6\sigma+1)} - \frac{(1-\varpi)(\varpi+1)\zeta^{10}}{480(9\sigma+1)} + \cdots, \quad (2.7)$$

$$\mathbb{T}_4(\zeta) := \zeta - \frac{(1-\varpi)\zeta^5}{4(4\sigma+1)} - \frac{(1-\varpi)(2\varpi-3)\zeta^9}{64(8\sigma+1)} + \frac{(1-\varpi)(6\varpi^2-21\varpi+7)\zeta^{13}}{2304(12\sigma+1)} + \cdots. \quad (2.8)$$

Remark 2.1. Based on the definition of the extreme function representation in Theorem 2.1 by Khan et al. [3], it becomes clear that the integral representation of odd analytic functions is equivalent to the extreme function representation of the class $h'(\zeta) < \Phi(\zeta)$. To illustrate this, let us take the odd analytic function $q(\zeta) = 1 + 1/3\zeta - 1/9\zeta^3$, and by substituting into the integral representation in [3], we obtain $q(u) + q(-u) - 2 = 0$. Therefore,

$$h(\zeta) = \int_0^\zeta q(t) dt.$$

Remark 2.2. The integral representation establishes a direct connection between the analytic structure of the class and the generalized balloon-shaped geometry, where ϖ controls the deformation of the associated domain. Meanwhile, the parameters σ and s determine the analytic structure and the corresponding members of the extremal family $\mathbb{T}_s$.

We now begin by revisiting several key lemmas that serve as fundamental instruments in deriving our main results. Carefully chosen for their significance and suitability, these lemmas form the backbone of our analysis and provide the necessary groundwork for the study. Each one offers a distinct contribution, collectively supporting the progression toward our principal conclusions.

3. Main lemmas

To start, let $\mathcal{P}$ denote the Carathéodory class, consisting of analytic functions $\kappa(\zeta)$ whose real part is positive. These functions are normalized and expressed through the following series expansion.

$$\kappa(\zeta) = 1 + \sum_{k=1}^{\infty} d_k \zeta^k. \quad (3.1)$$

Lemma 3.1. [16] *For the Carathéodory function (3.1), we have*

$$|d_k| \leq 2, \quad k \geq 1, \quad (3.2)$$

$$|d_{k+i} - \nu d_k d_i| < 2, \quad 0 < \nu \leq 1, \quad (3.3)$$

$$|d_2 - \alpha d_1^2| \leq 2 \max\{1, |2\alpha - 1|\}, \quad \alpha \in \mathbb{C}. \quad (3.4)$$

Lemma 3.2. [2] *For the function κ , the following inequality holds:*

$$|\Lambda_1 d_1^3 - \Lambda_2 d_1 d_2 + \Lambda_3 d_3| \leq 2|\Lambda_1| + 2|\Lambda_2 - 2\Lambda_1| + 2|\Lambda_1 - \Lambda_2 + \Lambda_3|.$$

Lemma 3.3. [17] *For the Carathéodory function (3.1), if $\Lambda_2 \in [0, 1]$ holds $\Lambda_2(2\Lambda_2 - 1) \leq \Lambda_1 \leq \Lambda_2$, then*

$$|\Lambda_1 d_1^3 - 2\Lambda_2 d_1 d_3 + d_3| \leq 2.$$

Lemma 3.4. [14] *We assume that $\gamma_1, \gamma_2, \gamma_3$, and γ_4 are satisfied the conditions $\gamma_1, \gamma_4 \in (0, 1)$, and*

$$\begin{aligned} & 8\gamma_4(1 - \gamma_4) \left[(\gamma_1\gamma_2 - 2\gamma_3)^2 + (\gamma_1(\gamma_4 + \gamma_1) - \gamma_2)^2 \right] + \gamma_1(1 - \gamma_1)(\gamma_2 - 2\gamma_4\gamma_1)^2 \\ & \leq 4\gamma_4\gamma_1^2(1 - \gamma_1)^2(1 - \gamma_4). \end{aligned}$$

Then, for the Carathéodory function (3.1), we have

$$\left| \gamma_3 d_1^4 + \gamma_4 d_2^2 + 2\gamma_1 d_1 d_3 - \frac{3}{2} \gamma_2 d_1^2 d_2 - d_4 \right| \leq 2.$$

Lemma 3.5. [18, 19] *Assume that $\kappa \in \mathcal{P}$ is represented as in (3.1). If ω and ϱ are elements of $\partial\mathbb{E}$, then*

$$2d_2 = d_1^2 + \omega(4 - d_1^2), \quad (3.5)$$

$$4d_3 = d_1^3 + 2\omega(4 - d_1^2)d_1 - \omega^2(4 - d_1^2)d_1 + 2\varrho(1 - |\omega|^2)(4 - d_1^2). \quad (3.6)$$

4. Sharp bounds for the class $\mathcal{S}_\sigma(\zeta; \Phi_\omega(\zeta))$

In this section, we develop a series of important results that lead to a deeper understanding of its structural properties. In particular, we determine the sharp bounds for the first four coefficients and derive the sharp form of the Fekete-Szegő inequality. These findings not only highlight the inherent features of the class but also provide essential reference points for future investigations and comparative studies in this field.

Theorem 4.1. If $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then the bounds of initial coefficients are hold as follows:

$$|a_2| \leq \frac{1 - \varpi}{2(\sigma + 1)}, \quad (4.1)$$

$$|a_3| \leq \frac{1 - \varpi}{2(2\sigma + 1)}, \quad (4.2)$$

$$|a_4| \leq \frac{1 - \varpi}{4(3\sigma + 1)}, \quad (4.3)$$

$$|a_5| \leq \frac{1 - \varpi}{4(4\sigma + 1)}. \quad (4.4)$$

The equality in constraints is achieved through functions $\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_4$, and $\mathbb{T}_4$ in (2.5)–(2.8), respectively.

Proof. Let $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$. Then, using the Schwarz function $w(\zeta)$, we attain

$$A_\sigma(\zeta) = \frac{2\zeta h'(\zeta) + 2\sigma \zeta^2 h''(\zeta)}{(1 - \sigma)(h(\zeta) - h(-\zeta)) + \sigma \zeta(h'(\zeta) + h'(-\zeta))} = \Phi_\varpi(w(\zeta)). \quad (4.5)$$

Let us assume that the function $\kappa \in \mathcal{P}$ and is given by the following form:

$$\kappa(\zeta) = \frac{1 + w(\zeta)}{1 - w(\zeta)} = 1 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \dots. \quad (4.6)$$

Thus,

$$\begin{aligned} \phi_\varpi(w(\zeta)) &= \varpi + (1 - \varpi) \frac{2\sqrt{1 + w(\zeta)}}{1 + \exp(-w(\zeta))} = 1 + \frac{1 - \varpi}{2}d_1\zeta \\ &+ \frac{1 - \varpi}{32}(16d_2 - 7d_1^2)\zeta^2 + \frac{1 - \varpi}{192}(17d_1^3 - 84d_2d_1 + 96d_3)\zeta^3 \\ &- \frac{1 - \varpi}{6144}(203d_1^4 - 1632d_2d_1^2 + 2688d_3d_1 + 1344d_2^2 - 3072d_4)\zeta^4 + \dots \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} A_\sigma(\zeta) &= 1 + 2(\sigma + 1)a_2\zeta + 2(2\sigma + 1)a_3\zeta^2 + (4a_4(3\sigma + 1) - 2a_2a_3(2\sigma^2 + 3\sigma + 1))\zeta^3 \\ &+ (4a_5(4\sigma + 1) - 2a_3^2(2\sigma + 1)^2)\zeta^4 + \dots. \end{aligned} \quad (4.8)$$

Comparing the expressions in (4.7) and (4.8) shows that

$$a_2 = \frac{1 - \varpi}{4(\sigma + 1)}d_1, \quad (4.9)$$

$$a_3 = \frac{1 - \varpi}{64(2\sigma + 1)}(16d_2 - 7d_1^2), \quad (4.10)$$

$$a_4 = \frac{(1 - \varpi)\left(\frac{1}{192}(21\varpi + 13)d_1^3 - \frac{1}{8}(2\varpi + 5)d_1d_2 + d_3\right)}{8(3\sigma + 1)}, \quad (4.11)$$

$$a_5 = -\frac{1 - \varpi}{8(4\sigma + 1)}\left(\frac{7(21\varpi + 8)d_1^4 - 96(7\varpi + 10)d_2d_1^2 + 192(4\varpi + 3)d_2^2 + 2688d_3d_1 - 3072d_4}{3072}\right). \quad (4.12)$$

Substituting (3.2) into (4.9), we obtain

$$|a_2| \leq \frac{1 - \varpi}{2(\sigma + 1)}.$$

To determine the maximum value of a_3 , we substitute (3.3) into (4.10) and obtain the following:

$$|a_3| = \frac{1 - \varpi}{4(2\sigma + 1)} \left| d_2 - \frac{7}{16} d_1^2 \right| \leq \frac{1 - \varpi}{2(2\sigma + 1)}.$$

To find the limit a_4 , we compare Lemma 3.3 and the Eq (4.11), which leads to the following result:

$$|a_4| = \frac{1 - \varpi}{8(3\sigma + 1)} \left| \left(\frac{(21\varpi + 13)}{192} d_1^3 - 2 \frac{(2\varpi + 5)}{16} d_1 d_2 + d_3 \right) \right|.$$

Considering $\Lambda_1 = \frac{(21\varpi + 13)}{192}$ and $\Lambda_2 = \frac{(2\varpi + 5)}{16}$, we readily have $\Lambda_2 \in [0, 1]$ and $\Lambda_2(2\Lambda_2 - 1) \leq \Lambda_1 \leq \Lambda_2$. Then,

$$|a_4| \leq \frac{2(1 - \varpi)}{8(3\sigma + 1)} = \frac{1 - \varpi}{4(3\sigma + 1)}.$$

Now, the upper bound of the fifth coefficient $|a_5|$ is as follows:

$$|a_5| = \frac{1 - \varpi}{8(4\sigma + 1)} \left| \frac{7(21\varpi + 8)}{3072} d_1^4 + \frac{(4\varpi + 3)}{16} d_2^2 + 2 \frac{7}{16} d_3 d_1 - \frac{3}{2} \frac{(7\varpi + 10)}{48} d_2 d_1^2 - d_4 \right|.$$

It is obvious that the value of the coefficients by Lemma 3.4 are

$$\gamma_1 = \frac{7}{16}, \quad \gamma_2 = \frac{7\varpi + 10}{48}, \quad \gamma_3 = \frac{7(21\varpi + 8)}{3072}, \quad \gamma_4 = \frac{4\varpi + 3}{16}.$$

Since $0 \leq \varpi < 1$, we have

$$0 < \gamma_1 = \frac{7}{16} < 1, \quad 0 < \gamma_4 = \frac{4\varpi + 3}{16} < \frac{7}{16} < 1.$$

Let

$$F(\varpi) = 8\gamma_4(1 - \gamma_4) \left[(\gamma_1\gamma_2 - 2\gamma_3)^2 + (\gamma_1(\gamma_4 + \gamma_1) - \gamma_2)^2 \right] \\ + \gamma_1(1 - \gamma_1)(\gamma_2 - 2\gamma_4\gamma_1)^2$$

and

$$G(\varpi) = 4\gamma_4\gamma_1^2(1 - \gamma_1)^2(1 - \gamma_4).$$

A direct calculation gives

$$G(\varpi) - F(\varpi) = \frac{P(\varpi)}{75497472},$$

where

$$P(\varpi) = 2084640(1 - \varpi)^4 + 11391800\varpi(1 - \varpi)^3 \\ + 21259937\varpi^2(1 - \varpi)^2 + 16150642\varpi^3(1 - \varpi) + 4286457\varpi^4.$$

Since $0 \leq \varpi < 1$, each term of $P(\varpi)$ is nonnegative, and hence

$$G(\varpi) - F(\varpi) > 0,$$

as shown in Figure 4.

Therefore,

$$F(\varpi) \leq G(\varpi).$$

Therefore, Lemma 3.4 gives

$$\left| \frac{7(21\varpi + 8)}{3072}d_1^4 + \frac{(4\varpi + 3)}{16}d_2^2 + \frac{7}{8}d_3d_1 - \frac{3}{2} \frac{(7\varpi + 10)}{48}d_2d_1^2 - d_4 \right| \leq 2$$

and

$$|a_5| \leq \frac{1 - \varpi}{4(4\sigma + 1)}.$$

This includes the sharp proofs. □

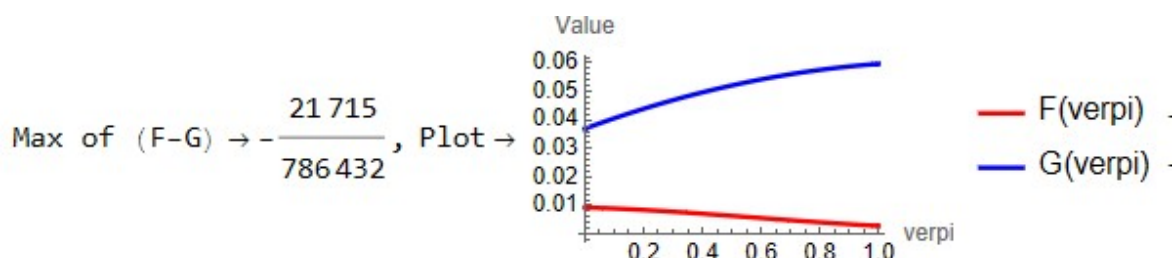


Figure 4. This figure shows the relation between $F(\varpi)$ and $G(\varpi)$.

Theorem 4.2. If $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then the bounds of Fekete-Szegő coefficients are hold as follows:

$$|a_3 - \alpha a_2^2| \leq \frac{1 - \varpi}{2(2\sigma + 1)} \max \left\{ 1, \left| -\frac{\alpha(\varpi - 1)(2\sigma + 1)}{2(\sigma + 1)^2} - \frac{1}{8} \right| \right\}. \quad (4.13)$$

Proof. Since $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$\begin{aligned} |a_3 - \alpha a_2^2| &= \left| \frac{(1 - \varpi)d_2}{4(2\sigma + 1)} - \frac{(1 - \varpi)(7(\sigma + 1)^2 + 4\alpha(1 - \varpi)(2\sigma + 1))}{64(\sigma + 1)^2(2\sigma + 1)}d_1^2 \right| \\ &= \frac{(1 - \varpi)}{4(2\sigma + 1)} \left| d_2 - \frac{7(\sigma + 1)^2 + 4\alpha(1 - \varpi)(2\sigma + 1)}{16(\sigma + 1)^2}d_1^2 \right|. \end{aligned}$$

With the application of (3.4), the required finding is directly obtained. □

Theorem 4.3. If $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$ with $\alpha = 1$, then

$$|a_3 - a_2^2| \leq \frac{1 - \varpi}{2(2\sigma + 1)}. \quad (4.14)$$

The equality is attained by (2.6).

Proof. Since $\alpha = 1$ and $0 \leq \varpi < 1, 0 \leq \sigma \leq 1$, we have $0 < 1 - \varpi \leq 1$ and $\frac{1+2\sigma}{(1+\sigma)^2} \leq 1$, since

$$(1 + \sigma)^2 - (1 + 2\sigma) = \sigma^2 \geq 0.$$

Therefore,

$$\left| -\frac{1}{8} - \frac{(\varpi - 1)(1 + 2\sigma)}{2(1 + \sigma)^2} \right| = \left| -\frac{1}{8} + \frac{(1 - \varpi)(1 + 2\sigma)}{2(1 + \sigma)^2} \right| \leq \frac{1}{8} + \frac{(1 - \varpi)(1 + 2\sigma)}{2(1 + \sigma)^2} \leq \frac{1}{8} + \frac{1}{2} = \frac{5}{8} < 1.$$

Hence,

$$\left| -\frac{1}{8} - \frac{(\varpi - 1)(1 + 2\sigma)}{2(1 + \sigma)^2} \right| < 1.$$

Therefore, the outcome is verified. $\square$

Theorem 4.4. If $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$\mathcal{Z} = |a_2a_3 - a_4| \leq \frac{1 - \varpi}{4(3\sigma + 1)}. \quad (4.15)$$

The equality is attained by (2.7).

Proof. We attain from (4.9)–(4.11) that

$$|a_2a_3 - a_4| = \frac{1 - \varpi}{8(3\sigma + 1)} \left| \Lambda_1 d_1^3 - 2\Lambda_2 d_2 d_1 + d_3 \right|,$$

where

$$\Lambda_1 = \frac{21\varpi(\sigma(2\sigma - 3) - 1) + \sigma(26\sigma + 165) + 55}{192(\sigma + 1)(2\sigma + 1)}, \quad \Lambda_2 = \frac{\varpi(4\sigma^2 - 6\sigma - 2) + 10\sigma^2 + 27\sigma + 9}{16(\sigma + 1)(2\sigma + 1)}.$$

Since $\Lambda_2 \in [0, 1]$ and $\Lambda_2(2\Lambda_2 - 1) \leq \Lambda_1 \leq \Lambda_2$. Employing Lemma 3.3 yields to the required finding. $\square$

Theorem 4.5. If $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$|a_2a_4 - a_3^2| \leq \frac{(1 - \varpi)^2}{4(2\sigma + 1)^2}, \quad (0 \leq \varpi < 0.611). \quad (4.16)$$

The equality is attained by (2.6).

Proof. Based on the provisions of (4.9)–(4.11), the following results are obtained:

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{(1 - \varpi)^2}{4(2\sigma + 1)^2} \left| \frac{(42\varpi(2\sigma + 1)^2 - 337\sigma^2 - 484\sigma - 121)}{3072(\sigma + 1)(3\sigma + 1)} d_1^4 \right. \\ &\quad \left. - \frac{(2\varpi(2\sigma + 1)^2 - 22\sigma^2 - 36\sigma - 9)}{64(\sigma + 1)(3\sigma + 1)} d_2 d_1^2 - \frac{1}{4} d_2^2 + \frac{(2\sigma + 1)^2}{8(\sigma + 1)(3\sigma + 1)} d_3 d_1 \right| \\ &= \frac{(1 - \varpi)^2}{4(2\sigma + 1)^2} \left| \frac{(42\varpi(2\sigma + 1)^2 - 337\sigma^2 - 484\sigma - 121)}{3072(\sigma + 1)(3\sigma + 1)} d_1^4 \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{(2\varpi(2\sigma+1)^2 - 22\sigma^2 - 36\sigma - 9)}{128(\sigma+1)(3\sigma+1)}(d_1^4 + \omega(4-d_1^2)d_1^2) \\
& + \frac{(2\sigma+1)^2}{32(\sigma+1)(3\sigma+1)}(d_1^4 + 2\omega(4-d_1^2)d_1^2 - \omega^2(4-d_1^2)d_1^2 + 2\varrho(1-|\omega|^2)(4-d_1^2)d_1) \\
& - \frac{1}{16}(d_1^4 + 2\omega d_1^2(4-d_1^2) + \omega^2(4-d_1^2)^2) \Big|.
\end{aligned}$$

Considering $d_1 = d$ and $|\omega| = \varepsilon$ within the statement of Lemma 3.5, we obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \frac{(1-\varpi)^2}{4(2\sigma+1)^2} \Big| - \frac{6\varpi(2\sigma+1)^2 + \sigma^2 + 4\sigma + 1}{3072(\sigma+1)(3\sigma+1)} d^4 \\
&+ \frac{(2\sigma+1)^2}{16(\sigma+1)(3\sigma+1)} \varrho(1-|\omega|^2)(4-d^2)d - \frac{1}{16}\omega^2(4-d^2)^2 \\
&+ \frac{-2\varpi(2\sigma+1)^2 + 6\sigma^2 + 4\sigma + 1}{128(\sigma+1)(3\sigma+1)} \omega(4-d^2)d^2 \\
&- \frac{(2\sigma+1)^2}{32(\sigma+1)(3\sigma+1)} \omega^2(4-d^2)d^2 \Big|
\end{aligned}$$

with the absolute value property, and thus

$$\begin{aligned}
|a_2a_4 - a_3^2| &= \frac{(1-\varpi)^2}{4(2\sigma+1)^2} \Big[\frac{6\varpi(2\sigma+1)^2 + \sigma^2 + 4\sigma + 1}{3072(\sigma+1)(3\sigma+1)} d^4 \\
&+ \frac{(2\sigma+1)^2}{16(\sigma+1)(3\sigma+1)} (1-\varepsilon^2)(4-d^2)d + \frac{1}{16}\varepsilon^2(4-d^2)^2 \\
&+ \frac{-2\varpi(2\sigma+1)^2 + 6\sigma^2 + 4\sigma + 1}{128(\sigma+1)(3\sigma+1)} \varepsilon(4-d^2)d^2 \\
&+ \frac{(2\sigma+1)^2}{32(\sigma+1)(3\sigma+1)} \varepsilon^2(4-d^2)d^2 \Big] = T(\varepsilon, d).
\end{aligned}$$

Restricting the variable such that $0 \leq \varpi < 0.611$, ensures that the coefficient of $\varepsilon(4-d^2)d^2$ is positive over the entire studied domain.

To determine the highest values, the derivative of the function $T(\varepsilon, d)$ must be used as follows:

$$\begin{aligned}
\frac{\partial T(\varepsilon, d)}{\partial \varepsilon} &= \frac{(1-\varpi)^2}{4(2\sigma+1)^2} \Big[\frac{-2\varpi(2\sigma+1)^2 + 6\sigma^2 + 4\sigma + 1}{128(\sigma+1)(3\sigma+1)} (4-d^2)d^2 \\
&+ \frac{(2\sigma+1)^2\varepsilon}{16(\sigma+1)(3\sigma+1)} (4-d^2)d^2 - \frac{(2\sigma+1)^2\varepsilon}{8(\sigma+1)(3\sigma+1)} (4-d^2)d + \frac{1}{8}(4-d^2)^2\varepsilon \Big].
\end{aligned}$$

Figure 5 accurately shows the critical value of the function $\partial T/\partial \varepsilon$, and this verification was performed when $\varepsilon \in [0, 1]$. The function $T(\varepsilon, d)$ takes its highest value at $\varepsilon = 1$. Hence,

$$\begin{aligned}
T(d) = T(1, d) &= \frac{(1-\varpi)^2}{4(2\sigma+1)^2} \Big[\frac{6\varpi(2\sigma+1)^2 + \sigma^2 + 4\sigma + 1}{3072(\sigma+1)(3\sigma+1)} d^4 + \frac{1}{16}(4-d^2)^2 \\
&+ \frac{-2\varpi(2\sigma+1)^2 + 6\sigma^2 + 4\sigma + 1}{128(\sigma+1)(3\sigma+1)} (4-d^2)d^2 + \frac{(2\sigma+1)^2}{32(\sigma+1)(3\sigma+1)} (4-d^2)d^2 \Big].
\end{aligned}$$

We now seek to find the Max value of $T(d)$, and accordingly,

$$T'(d) = \frac{(1-\varpi)^2}{4(2\sigma+1)^2} \left[\frac{d^3 (54\varpi(2\sigma+1)^2 + 49\sigma^2 + 292\sigma + 73)}{768(\sigma+1)(3\sigma+1)} - \frac{d(2\varpi(2\sigma+1)^2 + 26\sigma^2 + 44\sigma + 11)}{16(\sigma+1)(3\sigma+1)} \right].$$

The roots we obtain by making $T'(d) = 0$ are

$$\begin{aligned} r_1 &= 0, \\ r_2 &= \frac{4\sqrt{3}\sqrt{8\varpi\sigma^2 + 8\varpi\sigma + 2\varpi + 26\sigma^2 + 44\sigma + 11}}{\sqrt{216\varpi\sigma^2 + 216\varpi\sigma + 54\varpi + 49\sigma^2 + 292\sigma + 73}}, \\ r_3 &= -\frac{4\sqrt{3}\sqrt{8\varpi\sigma^2 + 8\varpi\sigma + 2\varpi + 26\sigma^2 + 44\sigma + 11}}{\sqrt{216\varpi\sigma^2 + 216\varpi\sigma + 54\varpi + 49\sigma^2 + 292\sigma + 73}}, \end{aligned}$$

where $r_1, r_2 \in [0, 2]$ and $r_3 \notin [0, 2]$.

We observe that $T''(r_1) < 0$ (as in Figure 6(a), indicating that the function reaches a maximum value at r_1 . Conversely, since $T''(r_2) > 0$ (as shown in Figure 6(b), the function takes a minimum value at r_2 . Thus,

$$|a_2a_4 - a_3^2| \leq T(1, 0) = \frac{(1-\varpi)^2}{4(2\sigma+1)^2}.$$

The argument is thus complete, confirming the truth of the finding.

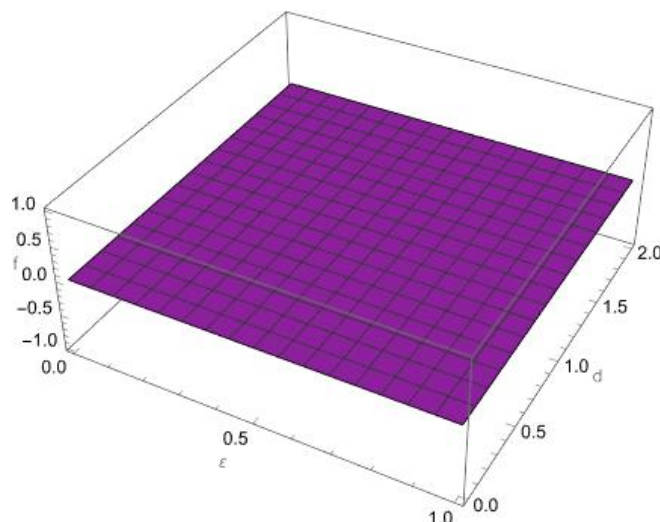


Figure 5. Graphing the derivative of function $T(\varepsilon, d)$.

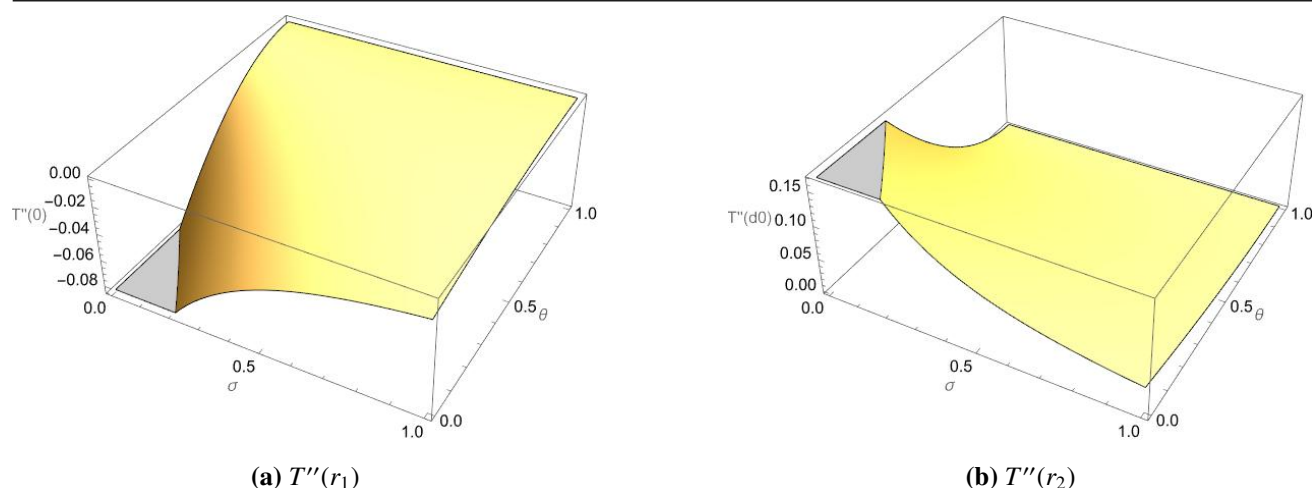


Figure 6. Plot $T''(r_1)$ and $T''(r_2)$.

□

Theorem 4.6. For $0 \leq \varpi < 0.611$ and $0 \leq \sigma \leq 1$. If $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$|\mathcal{H}_{3,1}(h)| \leq \frac{(1 - \varpi)^2}{16(3\sigma + 1)^2} \mathbb{L},$$

where

$$\mathbb{L} = \left| \frac{2\varpi(3\sigma + 1)^2(4\sigma + 1) + 104\sigma^4 + 104\sigma^3 + 44\sigma^2 + 10\sigma + 1}{(2\sigma + 1)^3(4\sigma + 1)} \right| \quad (4.17)$$

and $1 \leq \mathbb{L} \leq 2.67$.

Proof. The desired result can be obtained by relying on the constraints of (4.1)–(4.4), (4.14)–(4.16) through the following calculations to arrive at the expected formula

$$\begin{aligned} |\mathcal{H}_{3,1}(h)| &\leq |a_3|(a_2a_4 - a_3^2) + |a_4|(a_2a_3 - a_4) + |a_5|(a_3 - a_2^2) \\ &= \frac{(1 - \varpi)^2}{16(3\sigma + 1)^2} \left| \frac{2\varpi(3\sigma + 1)^2(4\sigma + 1) + 104\sigma^4 + 104\sigma^3 + 44\sigma^2 + 10\sigma + 1}{(2\sigma + 1)^3(4\sigma + 1)} \right| = \mathbb{L}. \end{aligned}$$

The smallest and largest values that the modulus $\mathbb{L}$ can reach are $1 \leq \mathbb{L} \leq 2.67$. □

4.1. Numerical evaluation

The proposed theoretical framework defines the bounds of $|a_k|$ and Hankel determinants based on ϖ and σ . Justifying the choice of these values is essential for understanding behavior of the domain and its application. This part give an explanation of the coefficients value used in the numerical evaluation.

Different values were adopted for the two parameters ϖ and σ , see Table 2.

Table 2. Upper limits by selected values for ϖ and σ .

(ϖ, σ)	$ a_2 $	$ a_3 $	$ a_4 $	$ a_5 $	$\mathcal{H}_2(1)$	$\mathcal{L}$	$\mathcal{H}_2(2)$
$(0, 0)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
$(0, 0.25)$	0.400	0.333	0.142	0.125	0.333	0.142	0.111
$(0.25, 0)$	0.375	0.375	0.187	0.187	0.375	0.187	0.140
$(0.25, 0.5)$	0.250	0.187	0.075	0.062	0.187	0.075	0.035
$(0.5, 0.25)$	0.200	0.167	0.071	0.062	0.166	0.071	0.027
$(0.5, 0.5)$	0.166	0.125	0.050	0.041	0.125	0.050	0.015
$(0.5, 0.75)$	0.142	0.100	0.038	0.031	0.100	0.038	0.010
$(0.75, 0.5)$	0.083	0.062	0.025	0.021	0.062	0.025	—
$(0.75, 1)$	0.062	0.041	0.015	0.012	0.041	0.015	—
$(0, 1)$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{16}$	$\frac{1}{20}$	$\frac{1}{6}$	$\frac{1}{16}$	$\frac{1}{36}$

5. Sharp bounds for inverse of the function h

This section examines the coefficient bounds of initial coefficients and Hankel determinants for the inverse of univalent functions, establishing sharp estimates for the initial coefficients.

The Taylor expansion of the inverse of an analytic function h^{-1} is defined by

$$h(h^{-1}(w)) = w \quad (|w| < r_0(h), \quad r_0(h) \geq \frac{1}{4})$$

and can be written as

$$h^{-1}(w) = w + \ell_2 w^2 + \ell_3 w^3 + \ell_4 w^4 + \dots, \quad (5.1)$$

where the coefficients satisfy

$$\ell_2 = -a_2, \quad \ell_3 = 2a_2^2 - a_3, \quad \ell_4 = -(5a_2^3 - 5a_2a_3 + a_4).$$

Recently, coefficient problems for inverse functions have received considerable attention in geometric function theory, particularly in estimating the initial inverse coefficients and their associated Hankel determinants; see, for example, [1, 13, 26].

Theorem 5.1. *For the univalent function $h \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$ has an inverse of h^{-1} , we have*

$$|\ell_2| \leq \frac{1 - \varpi}{2(\sigma + 1)}, \quad (5.2)$$

$$|\ell_3| \leq \frac{1 - \varpi}{2(2\sigma + 1)}, \quad (5.3)$$

$$|\ell_4| \leq \frac{1 - \varpi}{4(3\sigma + 1)}, \quad \left(\frac{5}{8} \leq \varpi < 1\right). \quad (5.4)$$

The equality is achieved by $\mathbb{T}_1, \mathbb{T}_2$, and $\mathbb{T}_3$ in (2.5)–(2.7), respectively.

Proof. Since h is univalent within $\mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$\ell_2 = -a_2 = -\frac{1 - \varpi}{4(\sigma + 1)}d_1. \quad (5.5)$$

It attains from (3.2) that

$$|\ell_2| \leq \frac{1 - \varpi}{2(\sigma + 1)}.$$

By (3.3) in Lemma 3.1, it follows that

$$|\ell_3| = \frac{1 - \varpi}{4(2\sigma + 1)} \left| d_2 - \frac{(7\sigma^2 - 8\varpi(2\sigma + 1) + 30\sigma + 15)}{16(\sigma + 1)^2} d_1^2 \right|.$$

Since $0 < \frac{7\sigma^2 - 8\varpi(2\sigma + 1) + 30\sigma + 15}{16(\sigma + 1)^2} \leq 1$ for all ϖ and σ , then

$$|\ell_3| \leq \frac{1 - \varpi}{2(2\sigma + 1)}.$$

Now, applying (4.9)–(4.11), and (5.1), we proceed to determine an upper bound for ℓ_4 as follows:

$$|\ell_4| = \frac{1 - \varpi}{8(1 + 3\sigma)} \left| \Lambda_1 d_1^3 - 2\Lambda_2 d_1 d_2 + d_3 \right|,$$

where

$$\begin{aligned} \Lambda_1 := & (120\varpi^2(6\sigma^2 + 5\sigma + 1) + 3\varpi(14\sigma^4 - 161\sigma^3 - 907\sigma^2 - 715\sigma - 143) + 26\sigma^4 \\ & + 721\sigma^3 + 2307\sigma^2 + 1715\sigma + 343) / (192(\sigma + 1)^3(2\sigma + 1)) \end{aligned}$$

and

$$\Lambda_2 := \frac{2\varpi(2\sigma^2 - 27\sigma - 9) + 5(2\sigma^2 + 15\sigma + 5)}{16(\sigma + 1)(2\sigma + 1)}.$$

To apply Lemma 3.2, assume that

$$0 \leq \sigma \leq 1 \quad \text{and} \quad \frac{5}{8} \leq \varpi < 1,$$

we first have

$$0 \leq \Lambda_2 \leq 1, \quad \Lambda_2 - \Lambda_1 \geq 0 \text{ and } \Lambda_1 - \Lambda_2(2\Lambda_2 - 1) \geq 0.$$

Consequently,

$$\Lambda_2(2\Lambda_2 - 1) \leq \Lambda_1 \leq \Lambda_2, \quad 0 \leq \Lambda_2 \leq 1.$$

Thus, all the hypotheses of Lemma 3.2 are satisfied, and hence

$$\left| \Lambda_1 d_1^3 - 2\Lambda_2 d_1 d_2 + d_3 \right| \leq 2.$$

Therefore, the required upper bound for ℓ_4 follows.

Thus, the claims are proven. $\square$

Theorem 5.2. *If $h^{-1} \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then*

$$\left| \ell_3 - \alpha \ell_2^2 \right| \leq \frac{1 - \varpi}{2(2\sigma + 1)} \max \left\{ 1, \left| \frac{4\alpha(\varpi - 1)(2\sigma + 1) - 8\varpi(2\sigma + 1) - \sigma^2 + 14\sigma + 7}{8(\sigma + 1)^2} \right| \right\}. \quad (5.6)$$

The equality is attained by (2.6).

Proof. Since $h^{-1} \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$|\ell_3 - \alpha \ell_2^2| = \frac{1 - \varpi}{4(2\sigma + 1)} \left| d_2 - \frac{4\alpha(\varpi - 1)(2\sigma + 1) - 8\varpi(2\sigma + 1) + 7\sigma^2 + 30\sigma + 15}{16(\sigma + 1)^2} d_1^2 \right|.$$

With the application of (3.4), the required finding is directly obtained. $\square$

Corollary 5.1. If $h^{-1} \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$ with $\alpha = 1$, then

$$|\ell_3 - \ell_2^2| \leq \frac{1 - \varpi}{2(2\sigma + 1)} \max \left\{ 1, \left| \frac{\varpi(8\sigma + 4) + \sigma^2 - 6\sigma - 3}{8(\sigma + 1)^2} \right| \right\}. \quad (5.7)$$

The equality is attained by (2.6).

Theorem 5.3. If $h^{-1} \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$|\ell_2 \ell_3 - \ell_4| \leq \frac{1 - \varpi}{4(3\sigma + 1)}, \quad \left(\frac{5}{8} \leq \varpi < 1 \right).$$

The equality is attained by (2.7).

Proof. Since $h^{-1} \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$\begin{aligned} |\ell_2 \ell_3 - \ell_4| &= \frac{(1 - \varpi)}{8(1 + 3\sigma)} \left| \Lambda_1 d_1^3 - 2\Lambda_2 d_1 d_2 + d_3 \right| \\ &\leq \frac{(1 - \varpi)}{8(1 + 3\sigma)} (2) = \frac{(1 - \varpi)}{4(1 + 3\sigma)}, \quad \left(\frac{5}{8} \leq \varpi < 1 \right), \end{aligned}$$

where

$$\begin{aligned} \Lambda_1 &:= (120\varpi^2(6\sigma^2 + 5\sigma + 1) - 3\varpi(14\sigma^4 + 1027\sigma^3 + 2057\sigma^2 + 1233\sigma + 221) + 982\sigma^4 \\ &\quad + 5699\sigma^3 + 8001\sigma^2 + 4105\sigma + 677) / (192(\sigma + 1)^3(2\sigma + 1)) \end{aligned}$$

and

$$\Lambda_2 = \frac{(2\varpi(2\sigma^2 + 27\sigma + 9) + 10\sigma^2 - 33\sigma - 11)}{16(\sigma + 1)(2\sigma + 1)}.$$

By applying Lemma 3.3, we can attain the required outcome. $\square$

Theorem 5.4. If $h^{-1} \in \mathcal{S}_\sigma(\zeta; \Phi_\varpi(\zeta))$, then

$$|\ell_2 \ell_4 - \ell_3^2| \leq \frac{(1 - \varpi)^2}{4(2\sigma + 1)}.$$

The equality is attained by (2.6).

Proof. Following an approach analogous to that used in establishing the upper bound in Theorem 4.5, the desired result is verified. $\square$

6. Concluding results

By choosing the values of $\varpi = \sigma = 0$, it is easy to access the initial coefficients and Hankel determinants for analytic and inverse of analytic functions in [3, Theorems 4.1–4.5 and 6.1–6.3], respectively.

7. Conclusions

In this work, we have presented a new generalization of the balloon domain, providing a unified geometric framework for studying classes of symmetric analytic functions. Based on this generalization, we derive a generalized formula for the extreme function, yielding the sharpness estimate for a class of symmetric-starlike functions.

Furthermore, we investigated the sharpness findings of the initial coefficients for a unified class of symmetric starlike and convex functions associated with the proposed domain. We also calculated sharp limits for Henkel determinants of different orders, revealing the structural properties of this class.

Moreover, we extended these results to the inverse of univalent functions, verifying the sharpness and consistency of the extracted estimates for both the initial coefficients and Henkel determinants.

Author contributions

Sarem H. Hadi: Conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing—original draft preparation, writing—review and editing, visualization, supervision, project administration, funding acquisition; Adel Salim Tayyah: Conceptualization, software, formal analysis, investigation, resources, writing—original draft preparation, writing—review and editing, supervision, project administration, funding acquisition; Alina Alb Lupas: Conceptualization, software, formal analysis, investigation, visualization, supervision, project administration, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are pleased to express their gratitude to the journal's editors and reviewers.

Funding

The publication of this paper was supported by the University of Oradea, Romania.

Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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