



*Research article***A fully decoupled scheme for the simplified Ericksen–Leslie system with variable density****Jiaoyan Hou¹, Zhaowei Wang² and Danxia Wang^{2,*}**¹ Shanxi College of Technology, Shuozhou, 036000, Shanxi, China² School of Mathematics, Taiyuan University of Technology, Taiyuan, 030024, Shanxi, China*** Correspondence:** Email: 2621259544@qq.com.

Abstract: In this work, we develop a fully decoupled sphere-constraint-preserving scheme for the simplified Ericksen–Leslie system with variable density. To begin, the original system is reformulated so that the constraint condition $|\mathbf{b}| = 1$ can be maintained in the discrete numerical scheme. Next, by employing the Gauge–Uzawa method together with the scalar auxiliary variable (SAV) approach, we construct a first-order time discretization scheme that is fully decoupled and unconditionally energy stable. Combining this with the finite element method (FEM) yields a first-order fully discrete scheme along with its implementation details. A rigorous error analysis is carried out for the temporal semi-discretization. Subsequently, the convergence of the proposed numerical method is established through a rigorous error analysis. Finally, several numerical experiments are carried out to demonstrate the effectiveness of the scheme.

Keywords: Ericksen–Leslie; variable density; Gauge–Uzawa method; SAV method**Mathematics Subject Classification:** 35Q35, 65M60

1. Introduction

This work is concerned with full decoupling strategies and error analysis for numerical approximations to the 2D variable-density Ericksen–Leslie system [1, 2]. The governing equations are presented below

$$\varrho_t + \nabla \cdot (\varrho \mathbf{w}) = 0, \quad \text{in } \Omega_T, \quad (1.1a)$$

$$\varrho(\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w}) + \nabla p - \mu \Delta \mathbf{w} + \lambda \nabla \cdot ((\nabla \mathbf{b})^T \nabla \mathbf{b}) = 0, \quad \text{in } \Omega_T, \quad (1.1b)$$

$$\mathbf{b}_t + (\mathbf{w} \cdot \nabla) \mathbf{b} - \gamma (\Delta \mathbf{b} + |\nabla \mathbf{b}|^2 \mathbf{b}) = 0, \quad \text{in } \Omega_T, \quad (1.1c)$$

$$\nabla \cdot \mathbf{w} = 0, \quad \text{in } \Omega_T, \quad (1.1d)$$

$$|\mathbf{b}| = 1, \quad \text{in } \Omega_T. \quad (1.1e)$$

Here, $\Omega_T = \Omega \times (0, T]$, where $\Omega \subset \mathbb{R}^2$ is a bounded open domain with smooth boundary $\partial\Omega$, and $T > 0$ represents the terminal time. The positive constant $\gamma > 0$ stands for the relaxation time, $\mu > 0$ is associated with the fluid viscosity, and $\lambda > 0$ denotes the elastic coefficient. $\mathbf{b} : \Omega_T \rightarrow \mathbb{R}^2$ describes the molecular orientation, $\mathbf{w} : \Omega_T \rightarrow \mathbb{R}^2$ denotes the fluid velocity field, and $p : \Omega_T \rightarrow \mathbb{R}$ is the fluid pressure. In addition, $(\nabla \mathbf{b})^T$ refers to the transpose of $\nabla \mathbf{b} = (\partial_j \mathbf{b}_i)_{i,j}$, while $|\mathbf{b}|$ corresponds to the Euclidean norm of $\mathbf{b}$ in $\mathbb{R}^2$.

We further present the initial and boundary conditions associated with system (1.1):

$$\begin{cases} \mathbf{w}(x, 0) = \mathbf{w}_0(x), & \mathbf{w}(x, t)|_{\Gamma} = \mathbf{g}, \\ \varrho(x, 0) = \varrho_0(x), & \varrho(x, t)|_{\Gamma_{in}} = a(x, t), \\ \mathbf{b}(x, 0) = \mathbf{b}_0(x), & \partial_n \mathbf{b}(x, t) = \mathbf{0}, \end{cases}$$

where $\Gamma = \partial\Omega$ denotes the boundary of Ω . Γ_{in} stands for the inflow boundary, defined by $\Gamma_{in} = \{x \in \Gamma : \mathbf{g} \cdot \mathbf{n} < 0\}$. For convenience, as adopted in [3], we consider an impermeable boundary, i.e., $\Gamma_{in} = \emptyset$ and $\mathbf{g} = \mathbf{0}$. ϱ_0 denotes the initial density, which is uniformly positive. There exist two positive constants $\varrho_{\min}^0 > 0$ and $\varrho_{\max}^0 > 0$ such that

$$\varrho_{\min}^0 \leq \varrho_0(x) \leq \varrho_{\max}^0 \quad \text{in } \Omega, \quad (1.2)$$

which ensures that the system is non-vacuous over the domain Ω . The initial values $\mathbf{b}_0$ and $\mathbf{w}_0$ satisfy the compatibility conditions $|\mathbf{b}_0| = 1$ and $\nabla \cdot \mathbf{w}_0 = 0$.

As a macroscopic phenomenological description, system (1.1) governs the hydrodynamic behavior of nematic liquid crystals [4–6]. The continuity equation is given by (1.1a), and the conservation of angular momentum by (1.1c). Together with (1.1e), these equations describe a convective harmonic heat-map flow taking values on the unit sphere. Here, $\mathbf{b}$ serves merely to characterize the molecular orientation of nematic liquid crystals, rather than acting as a thermodynamic state variable. Equations (1.1b) and (1.1d) are Navier–Stokes systems accounting for linear momentum conservation. Elastic stress exerted by liquid-crystal molecules on the fluid is incorporated via the term $\lambda \nabla \cdot ((\nabla \mathbf{b})^T \nabla \mathbf{b})$, whereas the advection term $(\mathbf{w} \cdot \nabla) \mathbf{b}$ represents the transport of molecular orientation by the fluid flow.

In recent years, numerical approximations of liquid-crystal flows have attracted considerable research attention. This is mainly due to the fact that liquid-crystal systems are strongly constrained by geometric boundary conditions, which cannot be easily studied from experimental observations. Therefore, numerical simulations can provide a clear understanding of the behaviour of liquid crystals and their underlying physical properties. For example, the design of practical devices can be improved through numerical simulations.

There are some existing numerical studies for the Ericksen–Leslie model. In [7], Liu et al. first studied a simplified system using the Ginzburg–Landau approximation to numerically solve the Ericksen–Leslie model. In [8], Lin et al. proposed a nonlinear scheme for computing the Ginzburg–Landau model. This scheme differs only slightly from the standard Ericksen–Leslie system, where the stretching term appears in the $\mathbf{b}$ -system. To improve Ericksen–Leslie fluid models for nematic liquid crystals, Zhao et al. [9] designed first- and second-order coupled numerical schemes that maintain stability. In [10], Bao et al. introduced a simple time-discrete scheme for the 2D simplified Ericksen–Leslie system that maintains sphere constraints nodewise while obeying a discrete energy law. In [11], Cabrales et al. presented a time-split finite element format to approximate the solution of the Ericksen–Leslie equations that control the flow of nematic liquid crystals. In [12], Guillén-González et al. studied

the approximation of the Ericksen–Leslie nematic liquid crystal model using the Ginzburg–Landau penalty problem based on a fully discrete hybrid format of continuous finite elements in space and linear semi-implicit first-order integrals in time. Recently, Zhang et al. [13] developed a second-order numerical scheme featuring full decoupling, linearization, and unconditional energy stability for the numerical approximation of the penalized Ericksen–Leslie equations.

However, we note that most of the above studies are based on simplified versions of the Ericksen–Leslie system. In fact, the simplified version of the Ericksen–Leslie system ignores the Leslie stresses, but still retains the core of the mathematical structure and some of the fundamental difficulties of the Ericksen–Leslie system. It turns out that the mathematical techniques developed for the study of simplified systems are applicable to the general Ericksen–Leslie system. As a result, many mathematicians have devoted themselves to the study of the simplified Ericksen–Leslie system rather than the original system [14–16].

To derive numerical approximations for the system (1.1), we reformulate the Ericksen–Leslie system [17] as given below

$$\varrho_t + \nabla \cdot (\varrho \mathbf{w}) = 0, \quad \text{in } \Omega_T, \quad (1.3a)$$

$$\varrho (\mathbf{w}_t + \mathbf{w} \cdot \nabla \mathbf{w}) + \nabla \hat{p} - \mu \Delta \mathbf{w} + \lambda \Delta \eta \nabla \eta = 0, \quad \text{in } \Omega_T, \quad (1.3b)$$

$$\eta_t + \mathbf{w} \cdot \nabla \eta - \gamma \Delta \eta = 0, \quad \text{in } \Omega_T, \quad (1.3c)$$

$$\nabla \cdot \mathbf{w} = 0, \quad \text{in } \Omega_T, \quad (1.3d)$$

with boundary and initial conditions

$$\begin{cases} \varrho(x, 0) = \varrho_0(x), & \mathbf{w}(x, 0) = \mathbf{w}_0(x), & \eta(x, 0) = \eta_0(x), \\ \mathbf{w}(x, t) \cdot \mathbf{n}|_{\Gamma} = 0, & \nabla \eta(x, t) \cdot \mathbf{n}|_{\Gamma} = 0, \end{cases}$$

where $\hat{p} := p + \frac{\lambda}{2} |\nabla \eta|^2$ denotes the modified pressure and $\eta \in [0, 2\pi)$.

System (1.3) was first derived by Wang et al, who combined the equivalent formulations proposed in [10] with the variable-density Navier–Stokes equations. In [17], Wang et al. adopted a sphere-constraint-preserving method to address the associated numerical difficulties. Specifically, they parameterized $\mathbf{b}(x, t) = (d_1, d_2)^T$ via

$$\mathbf{b}(x, t) = (\cos \eta(x, t), \sin \eta(x, t))^T,$$

where $\eta(x, t) = \arg \mathbf{b}(x, t)$ with $\arg$ being the argument from the polar coordinates. Based on this method, they rewrote the system (1.1) in dimension two and obtained an equivalent system (1.3) of functions $\varrho, \mathbf{w}, \eta, \hat{p}$. They then proved the unconditional stability of the proposed scheme and gave an error estimate. Compared to the original Ericksen–Leslie system (1.1), the system (1.3) is free from the strong nonlinear coupling due to the presence of $\lambda \nabla \cdot ((\nabla \mathbf{b})' \nabla \mathbf{b})$ [15].

Developing an efficient, stable, and fully decoupled numerical scheme for the system (1.3) remains a challenging task. This is because the model inherits not only the numerical difficulties of constant-density incompressible flows but also additional complexities arising from the variable-density continuity equation [18]. Specifically, the main numerical challenges can be summarized as follows: (i) preserving the positivity of density during fluid evolution, which is strictly required by the continuity equation; (ii) the strong velocity–pressure coupling induced by the incompressibility

constraint, which complicates the pressure update, especially under significant density variations; (iii) the intrinsic coupling between the velocity field $\mathbf{w}$ and the orientation variable η . Accordingly, this paper mainly focuses on resolving the aforementioned technical obstacles.

A universal framework for developing linear, decoupled, and unconditionally energy-stable numerical schemes is offered by the scalar auxiliary variable (SAV) method, originally designed for gradient flow models [19]. The existing literature [20] has well documented its successful extension to the Navier–Stokes equations and magnetohydrodynamic systems. Inspired by the above research progress, this work further extends the SAV method to the variable-density Ericksen–Leslie system.

Standard pressure correction schemes [21, 22] cannot decouple velocity and pressure in variable-density Navier–Stokes equations. Guermond and Salgado [23] introduced a fractional time-stepping method that solves one pressure Poisson equation per step, reducing cost and ensuring stability, though its error analysis remains open. Meanwhile, the Gauge–Uzawa method [24] approximates variable density as constant under the Boussinesq approximation [25], which limits its applicability to small density variations. For larger density variations, Jae–Hong et al. [26] proposed two unconditionally stable Gauge–Uzawa schemes—one convective, one conserved—each requiring only a single projection step. Several studies [24, 25, 27, 28] have demonstrated the advantages of the Gauge–Uzawa method over both the pressure-correction [29] and original Gauge methods [30]. Leveraging these benefits, we extend the Gauge–Uzawa method to the variable-density Ericksen–Leslie system.

It is worth emphasizing the distinction between the current work and the existing literature. Compared with the sphere-constraint-preserving scheme in [17], our algorithm achieves full decoupling by integrating the Gauge–Uzawa method with the SAV technique, thereby eliminating the coupled velocity–pressure–density system and significantly reducing computational costs. Moreover, unlike the fully decoupled schemes developed in [13] for the penalized Ericksen–Leslie model, our approach strictly preserves the unit-sphere constraint $|\mathbf{b}| = 1$ pointwise. While the Gauge–Uzawa and SAV methods are established tools, their combination for the variable-density Ericksen–Leslie system is novel and faces unique challenges, such as maintaining density positivity and deriving uniform error estimates in the presence of a variable coefficient σ . Our work provides the first fully decoupled, unconditionally stable, and rigorously error-controlled scheme for this specific physical model.

This paper pursues two main goals. The first involves designing a fully decoupled stable scheme for the variable-density simplified Ericksen–Leslie system. Pressure is handled implicitly via the Gauge–Uzawa method to separate velocity from pressure, while the SAV method takes care of the remaining coupled terms. A finite-element-based decoupling procedure is spelled out in detail to meet this goal. The second goal concerns error estimation for the proposed scheme. As a prerequisite, we prove that the density approximation from the semi-discrete formulation is bounded in L^∞ and stays positive. In summary, besides the advantage of full decoupling, the proposed scheme exhibits the following features: (i) the sphere constraint $|\mathbf{b}| = 1$ is preserved at the discrete level; (ii) no initial pressure value is required; (iii) no artificial boundary conditions for the pressure are needed; (iv) convergence is proved under minimal smoothness assumptions. The authors are not aware of any existing work that offers both a fully decoupled numerical scheme and its error analysis for the variable-density Ericksen–Leslie system. This study presents the first such effort.

Very recently, Wang et al. [31] developed first-order fully decoupled methods for the *penalty* Ericksen–Leslie system with variable density using the Gauge–Uzawa method, where the unit-sphere constraint $|\mathbf{b}| = 1$ is relaxed via a Ginzburg–Landau penalty term. In contrast, the present work

treats the *original* (unpenalized) system, preserves the pointwise sphere constraint exactly through the angular parametrization $\mathbf{b} = (\cos \eta, \sin \eta)^T$, and provides, for the first time, a rigorous error analysis for such a fully decoupled scheme applied directly to the variable-density Ericksen–Leslie system without penalty.

The SAV methodology has also been extended to various contexts, including structure-preserving spectral methods for Hamiltonian systems [32], constraint-preserving schemes for gradient flows [33], and efficient constant-coefficient Multiple SAV approaches for vesicle simulations [34]; coupled numerical methods have also been explored for fractional partial differential equations (PDEs) [35]. These developments further motivate the present study, which combines SAV and Gauge–Uzawa techniques to achieve full decoupling while maintaining both energy stability and sphere constraint preservation.

The remainder of this paper is structured as follows. Section 2 introduces the notation, inequalities, and lemmas used throughout the study. In Section 3, a first-order time discretization scheme is presented, and its unconditional stability is proved. Additionally, a first-order finite element scheme and its implementation are described. Section 4 provides rigorous error estimates for the proposed method. Numerical results demonstrating the accuracy and efficiency of the scheme are reported in Section 5. Finally, a brief conclusion is offered in the last section.

2. Preliminaries

For the convenience of the reader, we first summarize in Table 1 the key symbols used throughout this paper.

Table 1. Notation of key variables used in this paper.

Symbol	Definition
σ	square root of density, $\sigma = \sqrt{\rho}$
ρ	fluid density
$\mathbf{w}$	fluid velocity field
$\widehat{\mathbf{w}}$	intermediate velocity in Gauge–Uzawa correction
ϕ	pressure correction variable in Gauge–Uzawa method
η	scalar orientation angle, $\mathbf{b} = (\cos \eta, \sin \eta)^T$
$\hat{p}$	modified pressure
p	original physical pressure, $p = \hat{p} - \frac{\lambda}{2} \nabla \eta ^2$
Q	scalar auxiliary variable for energy decoupling
s	auxiliary variable in Gauge–Uzawa

In this section, to support the error analysis in the next section, we present the relevant spaces, notation, and lemmas. The constant C appears throughout and may take different values in different contexts. Standard Sobolev spaces $L^2(\Omega)$, $H^k(\Omega)$ and $H_0^k(\Omega)$ are employed, with the norm in $H^k(\Omega)$, denoted by $\|\cdot\|_k$. Boldface is used for vector functions and vector spaces.

The following spaces are introduced

$$\begin{aligned}\mathbf{H}^k(\Omega) &= (H^k(\Omega))^2, \quad \mathbf{H}_0^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u}|_{\partial\Omega} = 0\}, \\ \mathbf{H}_n^1(\Omega) &= \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0\}, \\ \mathbf{V}(\Omega) &= \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0\}, \\ \mathbf{V}_n(\Omega) &= \{\mathbf{u} \in \mathbf{H}_n^1(\Omega) : \nabla \cdot \mathbf{u} = 0\}, \\ \mathbf{H}(\Omega) &= \{\mathbf{u} \in (L^2(\Omega))^2 : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0\}, \\ L_0^2(\Omega) &= \left\{ q \in L^2 : \int_{\Omega} q dx = 0 \right\}.\end{aligned}$$

Let us recall several important inequalities that will be used later [36, 37]:

$$\{\|\nabla \cdot \mathbf{u}\| \leq c\|\nabla \mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega)\}, \quad (2.1)$$

$$\|\mathbf{u}\|_{L^4} \leq c\|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_1^{1/2}, \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega). \quad (2.2)$$

$$\|\mathbf{u}\|_{L^\infty} \leq c\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|_2^{1/2}, \quad \forall \mathbf{u} \in \mathbf{H}^2(\Omega). \quad (2.3)$$

By applying the Poincaré inequality, we obtain

$$\|\mathbf{u}\| \leq c\|\nabla \mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega). \quad (2.4)$$

We next define the trilinear form

$$b(\mathbf{v}, \mathbf{u}, \mathbf{x}) = \int_{\Omega} (\mathbf{v} \cdot \nabla) \mathbf{u} \cdot \mathbf{x} dx.$$

Based on the above-defined characteristics, we obtain

$$b(\mathbf{v}, \mathbf{u}, \mathbf{z}) = -b(\mathbf{v}, \mathbf{z}, \mathbf{u}), \quad \forall \mathbf{v} \in \mathbf{H}(\Omega), \quad \mathbf{u}, \mathbf{z} \in \mathbf{H}_0^1(\Omega), \quad (2.5)$$

and it further satisfies a key property

$$b(\mathbf{v}, \mathbf{u}, \mathbf{u}) = 0, \quad \forall \mathbf{v} \in \mathbf{H}(\Omega), \quad \mathbf{u} \in \mathbf{H}_0^1(\Omega). \quad (2.6)$$

Moreover, applying several standard inequalities [38–40] yields

$$b(\mathbf{v}, \mathbf{u}, \mathbf{z}) \leq \begin{cases} c\|\mathbf{v}\|_1\|\mathbf{u}\|_1\|\mathbf{z}\|_1, & \forall \mathbf{v}, \mathbf{u} \in \mathbf{H}, \mathbf{z} \in \mathbf{H}_0^1(\Omega), \\ c\|\mathbf{v}\|_2\|\mathbf{u}\|_1\|\mathbf{z}\|_1, & \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}, \mathbf{u} \in \mathbf{H}, \mathbf{z} \in \mathbf{H}_0^1(\Omega), \\ c\|\mathbf{v}\|_2\|\mathbf{u}\|_1\|\mathbf{z}\|, & \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}, \mathbf{u} \in \mathbf{H}, \mathbf{z} \in \mathbf{H}_0^1(\Omega), \\ c\|\mathbf{v}\|_1\|\mathbf{u}\|_2\|\mathbf{z}\|, & \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}, \mathbf{v} \in \mathbf{H}, \mathbf{z} \in \mathbf{H}_0^1(\Omega), \\ c\|\mathbf{v}\|_1\|\mathbf{u}\|_2\|\mathbf{z}\|_1, & \forall \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}, \mathbf{v} \in \mathbf{H}, \mathbf{z} \in \mathbf{H}_0^1(\Omega), \\ c\|\mathbf{v}\|_1^{1/2}\|\mathbf{v}\|^{1/2}\|\mathbf{u}\|_1^{1/2}\|\mathbf{u}\|^{1/2}\|\mathbf{z}\|_1, & \forall \mathbf{v}, \mathbf{u} \in \mathbf{H}, \mathbf{z} \in \mathbf{H}_0^1(\Omega). \end{cases} \quad (2.7)$$

We now introduce the projection operator $P_{\mathbf{H}}$ from $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$ [41]. We get

$$\|P_{\mathbf{H}}\mathbf{u}\|_1 \leq C\|\mathbf{u}\|_1, \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega). \quad (2.8)$$

Below, we provide an important lemma.

Lemma 2.1. (Discrete Gronwall inequality) [42] Fix $T > 0$, and suppose $\{a^m\}_{m=1}^M$, $\{b^m\}_{m=1}^M$, and $\{c^m\}_{m=1}^{M-1}$ are non-negative sequences such that $\tau \sum_{m=1}^{M-1} c^m \leq C_1$, where C_1 is independent of τ and M , and $M\tau = T$. Suppose that, for all $\tau > 0$,

$$a^M + \tau \sum_{m=1}^M b^m \leq C_2 + \tau \sum_{m=1}^{M-1} a^m c^m, \quad (2.9)$$

where $C_2 > 0$ is a constant independent of τ and M . Then, for all $\tau > 0$,

$$a^M + \tau \sum_{m=1}^M b^m \leq C_2 \exp \left(\tau \sum_{m=1}^{M-1} c^m \right) \leq C_2 \exp(C_1). \quad (2.10)$$

3. Numerical scheme and its unconditional energy stability

Before presenting the time discretization, we specify the boundary conditions used throughout. Throughout this work, we adopt the no-penetration condition $\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0$ and the Neumann condition $\nabla\eta \cdot \mathbf{n}|_{\partial\Omega} = 0$. In the Gauge–Uzawa scheme, the intermediate velocity $\widehat{\mathbf{w}}^{n+1}$ satisfies $\widehat{\mathbf{w}}^{n+1}|_{\partial\Omega} = 0$ (see (3.7c)), while the pressure correction ϕ^{n+1} satisfies $\partial_n \phi^{n+1}|_{\partial\Omega} = 0$ (see (3.7g)); hence the corrected velocity $\mathbf{w}^{n+1}$ given by (3.7h) satisfies $\mathbf{w}^{n+1} \cdot \mathbf{n}|_{\partial\Omega} = 0$. These settings are consistent with the original system (1.1) under the impermeable boundary assumption $\mathbf{g} = 0$, and are used in all numerical experiments.

3.1. Equivalent system reformulation

From (1.3a) and (1.3d), we have

$$\begin{aligned} \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{w}) &= \frac{\partial(\sigma)^2}{\partial t} + \nabla \cdot (\sigma^2 \mathbf{w}) \\ &= 2\sigma \left(\frac{\partial \sigma}{\partial t} + \nabla \sigma \cdot \mathbf{w} \right) = 2\sigma \left(\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{w}) \right) = 0, \end{aligned}$$

and

$$\frac{\partial(\varrho \mathbf{w})}{\partial t} = \mathbf{w} \frac{\partial \varrho}{\partial t} + \varrho \frac{\partial \mathbf{w}}{\partial t} = \varrho \frac{\partial \mathbf{w}}{\partial t} - \mathbf{w} \nabla \cdot (\varrho \mathbf{w}).$$

Thus, it holds that

$$\varrho \frac{\partial \mathbf{w}}{\partial t} = \frac{\partial(\varrho \mathbf{w})}{\partial t} + \mathbf{w} \nabla \cdot (\varrho \mathbf{w}). \quad (3.1)$$

The right-hand side term of (3.1) satisfies, for its first component,

$$\begin{aligned} \frac{\partial(\varrho \mathbf{w})}{\partial t} &= \frac{\partial(\sigma \sigma \mathbf{w})}{\partial t} = \sigma \frac{\partial(\sigma \mathbf{w})}{\partial t} + \sigma \mathbf{w} \frac{\partial \sigma}{\partial t} = \sigma \frac{\partial(\sigma \mathbf{w})}{\partial t} + \frac{\mathbf{w}}{2} \frac{\partial \varrho}{\partial t} \\ &= \sigma \frac{\partial(\sigma \mathbf{w})}{\partial t} - \frac{\mathbf{w}}{2} \nabla \cdot (\varrho \mathbf{w}), \end{aligned}$$

which, together with (3.1), yields

$$\varrho \frac{\partial \mathbf{w}}{\partial t} = \sigma \frac{\partial(\sigma \mathbf{w})}{\partial t} + \frac{\mathbf{w}}{2} \nabla \cdot (\varrho \mathbf{w}).$$

The system (1.3a)–(1.3b) can then be recast into the following equivalent form:

$$\begin{aligned}\sigma_t + \mathbf{w} \cdot \nabla \sigma &= 0, \\ \sigma(\sigma \mathbf{w})_t - \mu \Delta \mathbf{w} + \varrho(\mathbf{w} \cdot \nabla) \mathbf{w} + \frac{\mathbf{w}}{2} \nabla \cdot (\varrho \mathbf{w}) + \lambda \Delta \eta \nabla \eta + \nabla \hat{p} &= 0, \\ \nabla \cdot \mathbf{w} &= 0.\end{aligned}$$

Leveraging the “zero-energy contribution”, we introduce a nonlocal variable $Q(t)$ to construct a specific ODE system as follows

$$\begin{cases} Q_t = (\varrho \mathbf{w} \cdot \nabla \mathbf{w}, \mathbf{w}) + \frac{1}{2}(\nabla \cdot (\varrho \mathbf{w}) \mathbf{w}, \mathbf{w}) + \lambda(\Delta \eta \nabla \eta, \mathbf{w}) - \lambda(\mathbf{w} \cdot \nabla \eta, \Delta \eta), \\ Q(t_0) = 1. \end{cases} \quad (3.2)$$

Clearly, the exact solution to the above equation is the constant function $Q(t) = 1$, as it is equivalent to

$$\begin{cases} Q_t = 0, \\ Q(t = 0) = 1. \end{cases} \quad (3.3)$$

Then, we rewrite the system (1.3) as

$$\sigma_t + \mathbf{w} \cdot \nabla \sigma = 0, \quad (3.4a)$$

$$\sigma(\sigma \mathbf{w})_t - \mu \Delta \mathbf{w} + \nabla \hat{p} + Q(\varrho \mathbf{w} \cdot \nabla \mathbf{w} + \frac{1}{2} \nabla \cdot (\varrho \mathbf{w}) \mathbf{w} + \lambda \Delta \eta \nabla \eta) = 0, \quad (3.4b)$$

$$\eta_t + Q \mathbf{w} \cdot \nabla \eta - \gamma \Delta \eta = 0, \quad (3.4c)$$

$$\nabla \cdot \mathbf{w} = 0, \quad (3.4d)$$

$$Q_t = (\varrho \mathbf{w} \cdot \nabla \mathbf{w}, \mathbf{w}) + \frac{1}{2}(\nabla \cdot (\varrho \mathbf{w}) \mathbf{w}, \mathbf{w}) + \lambda(\Delta \eta \nabla \eta, \mathbf{w}) - \lambda(\mathbf{w} \cdot \nabla \eta, \Delta \eta), \quad (3.4e)$$

with initial and boundary conditions as below

$$\begin{cases} \sigma(x, 0) = \sigma_0(x) = \sqrt{\varrho_0(x)}, & \mathbf{w}(x, 0) = \mathbf{w}_0(x), & \eta(x, 0) = \eta_0(x), & Q(0) = 1, \\ \mathbf{w}(x, t) \cdot \mathbf{n}|_{\Gamma} = 0, & \nabla \eta(x, t) \cdot \mathbf{n}|_{\Gamma} = 0, \end{cases} \quad (3.5)$$

where $\sigma = \sqrt{\varrho}$.

Thus, one can readily verify that system (3.4) obeys the following modified energy dissipation law

$$\frac{1}{2} \frac{d}{dt} \left(\|\sigma \mathbf{w}\|^2 + \lambda \|\nabla \eta\|^2 + |Q|^2 - 1 \right) = -\mu \|\nabla \mathbf{w}\|^2 - \gamma \lambda \|\Delta \eta\|^2. \quad (3.6)$$

3.2. Semi-discrete scheme of first order in time

This subsection presents a first-order Gauge–Uzawa time discretization scheme and demonstrates its unconditional stability.

Let $\tau \geq 0$ be the time step, and let $t_n = n\tau$, $0 \leq n \leq N \leq T/\tau - 1$. Let $\sigma^n, \mathbf{w}^n, \eta^n, p^n$ be numerical approximations in time discretization scheme of $\sigma(t_n), \mathbf{w}(t_n), \eta(t_n), p(t_n)$, respectively.

Algorithm 1. Given $\sigma^0 = \sigma_0, \mathbf{w}^0 = \mathbf{w}_0, \eta^0 = \eta_0$, and $s^0 = 0$, $(\sigma^{n+1}, \mathbf{w}^{n+1}, \eta^{n+1}, \hat{p}^{n+1})$ are calculated by the following:

$$\frac{\sigma^{n+1} - \sigma^n}{\tau} + \nabla \sigma^{n+1} \cdot \mathbf{w}^n = 0, \quad (3.7a)$$

$$\sigma^{n+1} \frac{\sigma^{n+1} \widehat{\mathbf{w}}^{n+1} - \sigma^n \mathbf{w}^n}{\tau} + Q^{n+1} \varrho^{n+1} (\mathbf{w}^n \cdot \nabla) \mathbf{w}^n + \frac{Q^{n+1}}{2} \nabla \cdot (\varrho^{n+1} \mathbf{w}^n) \mathbf{w}^n + \mu \nabla s^n - \mu \Delta \widehat{\mathbf{w}}^{n+1} + \lambda Q^{n+1} \Delta \eta^n \nabla \eta^n = 0, \quad (3.7b)$$

$$\widehat{\mathbf{w}}^{n+1}|_{\Gamma} = 0, \quad (3.7c)$$

$$\frac{\eta^{n+1} - \eta^n}{\tau} + Q^{n+1} \mathbf{w}^n \cdot \nabla \eta^n - \gamma \Delta \eta^{n+1} = 0, \quad (3.7d)$$

$$\begin{aligned} \frac{Q^{n+1} - Q^n}{\tau} &= (Q^{n+1} (\mathbf{w}^n \cdot \nabla) \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1}) + \frac{1}{2} (\nabla \cdot (\varrho^{n+1} \mathbf{w}^n) \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1}) \\ &\quad + \lambda (\Delta \eta^n \nabla \eta^n, \widehat{\mathbf{w}}^{n+1}) - \lambda (\mathbf{w}^n \cdot \nabla \eta^n, \Delta \eta^{n+1}), \end{aligned} \quad (3.7e)$$

$$-\nabla \cdot \left(\frac{1}{Q^{n+1}} \nabla \phi^{n+1} \right) = \nabla \cdot \widehat{\mathbf{w}}^{n+1}, \quad (3.7f)$$

$$\partial_n \phi^{n+1}|_{\Gamma} = 0, \quad (3.7g)$$

$$\mathbf{w}^{n+1} = \widehat{\mathbf{w}}^{n+1} + \frac{1}{Q^{n+1}} \nabla \phi^{n+1}, \quad (3.7h)$$

$$s^{n+1} = s^n - \nabla \cdot \widehat{\mathbf{w}}^{n+1}, \quad (3.7i)$$

$$\hat{p}^{n+1} = -\frac{1}{\tau} \phi^{n+1} + \mu s^{n+1}, \quad (3.7j)$$

where $\sigma^{n+1} = \sqrt{Q^{n+1}}$.

The stability proof for Algorithm 1 is presented next.

The final velocity is updated through the correction step:

$$\mathbf{w}^{n+1} = \widehat{\mathbf{w}}^{n+1} + \frac{1}{Q^{n+1}} \nabla \phi^{n+1}.$$

Since $\widehat{\mathbf{w}}^{n+1}|_{\partial\Omega} = 0$ and the pressure correction satisfies the Neumann condition $\partial_n \phi^{n+1}|_{\partial\Omega} = 0$, we have

$$\mathbf{w}^{n+1} \cdot \mathbf{n} = \widehat{\mathbf{w}}^{n+1} \cdot \mathbf{n} + \frac{1}{Q^{n+1}} \partial_n \phi^{n+1} = 0 \quad \text{on } \partial\Omega.$$

Thus, the final velocity $\mathbf{w}^{n+1}$ strictly satisfies the physical boundary condition $\mathbf{w}^{n+1} \cdot \mathbf{n} = 0$, which is exactly the condition under which the boundary terms vanish in the energy stability proof.

Theorem 3.1. *Algorithm 1 is unconditionally stable in the sense that, for all $\tau > 0$, the following a priori bounds hold*

$$\|\sigma^{N+1}\|^2 + \sum_{n=0}^N \|\sigma^{n+1} - \sigma^n\|^2 = \|\sigma^0\|^2 \quad (3.8)$$

and

$$\begin{aligned}
& \left\| \sigma^{N+1} \widehat{\mathbf{w}}^{N+1} \right\|^2 + \lambda \left\| \nabla \eta^{N+1} \right\|^2 + |Q^{N+1}|^2 + \mu\tau \left\| s^{N+1} \right\|^2 + \frac{\mu\tau}{2} \sum_{n=0}^N \left\| \nabla \widehat{\mathbf{w}}^{n+1} \right\|^2 \\
& + 2\tau\gamma\lambda \sum_{n=0}^N \left\| \Delta \eta^{n+1} \right\|^2 + \sum_{n=0}^N \left\| \frac{1}{\sigma^n} \nabla \phi^n \right\|^2 + \sum_{n=0}^N \left\| \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} - \sigma^n \mathbf{w}^n \right\|^2 \\
& + \lambda \sum_{n=0}^N \left\| \nabla \eta^{n+1} - \nabla \eta^n \right\|^2 + \sum_{n=0}^N |Q^{n+1} - Q^n|^2 \\
& \leq \left\| \sigma^0 \widehat{\mathbf{w}}^0 \right\|^2 + \lambda \left\| \nabla \eta^0 \right\|^2 + |Q^0|^2 + \mu\tau \left\| s^0 \right\|^2,
\end{aligned} \tag{3.9}$$

where $\sigma^n = \sqrt{\varrho^n}$ and $0 \leq N \leq T/\tau - 1$.

Proof. The proof of (3.8) is given first.

Taking the inner product of (3.7a) with $2\tau\sigma^{n+1}$, since (2.6), we get

$$\left\| \sigma^{n+1} \right\|^2 + \left\| \sigma^{n+1} - \sigma^n \right\|^2 = \left\| \sigma^n \right\|^2.$$

Summation over $n = 0$ to N leads to (3.8).

Turning now to (3.9), we provide its proof.

Taking the inner product of (3.7b) with $2\tau\widehat{\mathbf{w}}^{n+1}$, we get

$$\begin{aligned}
& 2 \left(\sigma^{n+1} \widehat{\mathbf{w}}^{n+1} - \sigma^n \mathbf{w}^n, \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} \right) + 2\tau Q^{n+1} \left(\varrho^{n+1} (\mathbf{w}^n \cdot \nabla) \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1} \right) + 2\mu\tau \left(\nabla s^n, \widehat{\mathbf{w}}^{n+1} \right) \\
& + 2\mu\tau \left\| \nabla \widehat{\mathbf{w}}^{n+1} \right\|^2 + \tau Q^{n+1} \left(\nabla \cdot \left(\varrho^{n+1} \mathbf{w}^n \right) \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1} \right) + 2\tau\lambda Q^{n+1} \left(\Delta \eta^n \nabla \eta^n, \widehat{\mathbf{w}}^{n+1} \right) \\
& = 0.
\end{aligned} \tag{3.10}$$

Now, let $\sigma^n = \sqrt{\varrho^n}$, then we can denote the first term of the above (3.10) as

$$2 \left(\sigma^{n+1} \widehat{\mathbf{w}}^{n+1} - \sigma^n \mathbf{w}^n, \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} \right) = \left\| \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} \right\|^2 + \left\| \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} - \sigma^n \mathbf{w}^n \right\|^2 - \left\| \sigma^n \mathbf{w}^n \right\|^2. \tag{3.11}$$

We derive from (3.7h) that

$$\begin{aligned}
\left\| \sigma^n \mathbf{w}^n \right\|^2 &= (\varrho^n \mathbf{w}^n, \mathbf{w}^n) = (\varrho^n \widehat{\mathbf{w}}^n + \nabla \phi^n, \mathbf{w}^n) \\
&= (\varrho^n \widehat{\mathbf{w}}^n, \mathbf{w}^n) = \left(\varrho^n \widehat{\mathbf{w}}^n, \widehat{\mathbf{w}}^n + \frac{1}{\varrho^n} \nabla \phi^n \right) \\
&= \left\| \sigma^n \widehat{\mathbf{w}}^n \right\|^2 + \left(\mathbf{w}^n - \frac{1}{\varrho^n} \nabla \phi^n, \nabla \phi^n \right) \\
&= \left\| \sigma^n \widehat{\mathbf{w}}^n \right\|^2 - \left\| \frac{1}{\sigma^n} \nabla \phi^n \right\|^2.
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) into (3.10), we obtain

$$\begin{aligned}
& \left\| \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} \right\|^2 + \left\| \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} - \sigma^n \mathbf{w}^n \right\|^2 - \left\| \sigma^n \widehat{\mathbf{w}}^n \right\|^2 + \left\| \frac{1}{\sigma^n} \nabla \phi^n \right\|^2 + 2\mu\tau \left\| \nabla \widehat{\mathbf{w}}^{n+1} \right\|^2 \\
& = -2\tau Q^{n+1} \left(\varrho^{n+1} (\mathbf{w}^n \cdot \nabla) \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1} \right) - \tau Q^{n+1} \left(\nabla \cdot \left(\varrho^{n+1} \mathbf{w}^n \right) \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1} \right) \\
& \quad - 2\tau\lambda Q^{n+1} \left(\Delta \eta^n \nabla \eta^n, \widehat{\mathbf{w}}^{n+1} \right) + A
\end{aligned} \tag{3.13}$$

with

$$A := 2\mu\tau \left(s^n, \nabla \cdot \widehat{\mathbf{w}}^{n+1} \right). \quad (3.14)$$

We derive from (2.1) and (3.7i) that

$$\begin{aligned} A &= -2\mu\tau \left(s^n, s^{n+1} - s^n \right) = -\mu\tau \left(\|s^{n+1}\|^2 - \|s^{n+1} - s^n\|^2 - \|s^n\|^2 \right) \\ &= -\mu\tau \left(\|s^{n+1}\|^2 - \|s^n\|^2 \right) + \mu\tau \left\| \nabla \cdot \widehat{\mathbf{w}}^{n+1} \right\|^2 \\ &\leq -\mu\tau \left(\|s^{n+1}\|^2 - \|s^n\|^2 \right) + \mu\tau \left\| \nabla \widehat{\mathbf{w}}^{n+1} \right\|^2. \end{aligned} \quad (3.15)$$

Taking the inner product of (3.7d) with $-2\lambda\tau\Delta\eta^{n+1}$, we get

$$\begin{aligned} &\lambda \left(\left\| \nabla \eta^{n+1} \right\|^2 - \left\| \nabla \eta^n \right\|^2 + \left\| \nabla \eta^{n+1} - \nabla \eta^n \right\|^2 \right) + 2\gamma\lambda\tau \left\| \Delta \eta^{n+1} \right\|^2 \\ &= 2\tau\lambda Q^{n+1} (\mathbf{w}^n \cdot \nabla \eta^n, \Delta \eta^{n+1}). \end{aligned} \quad (3.16)$$

Taking the inner product of (3.7e) with $2\tau Q^{n+1}$, we get

$$\begin{aligned} &|Q^{n+1}|^2 - |Q^n|^2 + |Q^{n+1} - Q^n|^2 \\ &= 2\tau Q^{n+1} (Q^{n+1} (\mathbf{w}^n \cdot \nabla) \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1}) + \tau Q^{n+1} (\nabla \cdot (Q^{n+1} \mathbf{w}^n) \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1}) \\ &\quad + 2\tau\lambda Q^{n+1} (\Delta \eta^n \nabla \eta^n, \widehat{\mathbf{w}}^{n+1}) - 2\tau\lambda Q^{n+1} (\mathbf{w}^n \cdot \nabla \eta^n, \Delta \eta^{n+1}). \end{aligned} \quad (3.17)$$

Combining Eqs (3.13)–(3.17), we can obtain

$$\begin{aligned} &\left\| \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} \right\|^2 - \left\| \sigma^n \widehat{\mathbf{w}}^n \right\|^2 + \lambda \left(\left\| \nabla \eta^{n+1} \right\|^2 - \left\| \nabla \eta^n \right\|^2 \right) + |Q^{n+1}|^2 - |Q^n|^2 \\ &\quad + \mu\tau \left(\|s^{n+1}\|^2 - \|s^n\|^2 \right) + \left\| \sigma^{n+1} \widehat{\mathbf{w}}^{n+1} - \sigma^n \mathbf{w}^n \right\|^2 + \lambda \left\| \nabla \eta^{n+1} - \nabla \eta^n \right\|^2 \\ &\quad + 2\tau\gamma\lambda \left\| \Delta \eta^{n+1} \right\|^2 + \left\| \frac{1}{\sigma^n} \nabla \phi^n \right\|^2 + \frac{\mu\tau}{2} \left\| \nabla \widehat{\mathbf{w}}^{n+1} \right\|^2 + |Q^{n+1} - Q^n|^2 \\ &\leq 0. \end{aligned} \quad (3.18)$$

Summing over $n = 0$ to N leads to (3.9). $\square$

3.3. Fully discrete scheme

This subsection develops a finite element discretization for Algorithm 1 and presents the corresponding stability results.

Let $\Gamma_h = \{K\}$ be a regular quasi-uniform mesh of the domain Ω with mesh size h . We define

$$V_h = \left\{ v \in C(\Omega) : v|_K \in \mathcal{P}^k, k \geq 1, \forall K \in \Gamma_h \right\},$$

where $\mathcal{P}^k$ represents the function space composed of polynomials of degree less than or equal to k . Let $\mathbf{V}_h := (V_h)^d$, $\mathbf{M}_h = \mathbf{V}_h \cap \mathbf{H}_0^1$, $P_h = V_h \cap L_0^2(\Omega)$, and $H_0(\text{div}) = \left\{ v \in H(\text{div}) : v \cdot \vec{n}|_{\partial\Omega} = 0 \right\}$.

3.3.1. Finite element scheme

Let σ_{0h} , η_{0h} , and $\mathbf{w}_{0h}$ be the numerical approximations of σ_0 , η_0 , and $\mathbf{w}_0$, respectively.

Algorithm 2 Let $\sigma_h^0 = \sigma_{0h}$, $\eta_h^0 = \eta_{0h}$, $\mathbf{w}_h^0 = \mathbf{w}_{0h}$, and $s_h^0 = 0$. Here, $0 \leq n \leq N-1$ with $1 \leq N \leq T/\tau$. We find $\sigma_h^{n+1} \in V_h$, $\widehat{\mathbf{w}}_h^{n+1} \in \mathbf{M}_h$, $\eta_h^{n+1} \in V_h$, $\phi_h^{n+1} \in V_h$, $\mathbf{w}_h^{n+1}$, and $s_h^{n+1} \in V_h$ as the solution of

$$\left(\frac{\sigma_h^{n+1} - \sigma_h^n}{\tau} + \mathbf{w}_h^n \cdot \nabla \sigma_h^{n+1}, \psi_h \right) = 0, \quad (3.19a)$$

$$\left(\sigma_h^{n+1} \frac{\sigma_h^{n+1} \widehat{\mathbf{w}}_h^{n+1} - \sigma_h^n \mathbf{w}_h^n}{\tau}, w_h \right) + Q_h^{n+1} \left(\varrho_h^{n+1} (\mathbf{w}_h^n \cdot \nabla) \mathbf{w}_h^n, w_h \right) + \frac{Q_h^{n+1}}{2} \left(\nabla \cdot (\varrho_h^{n+1} \mathbf{w}_h^n) \mathbf{w}_h^n, w_h \right) - (\mu s_h^n, \nabla \cdot w_h) + (\mu \nabla \widehat{\mathbf{w}}_h^{n+1}, \nabla w_h) + \lambda Q_h^{n+1} (\Delta \eta_h^n \nabla \eta_h^n, w_h) = 0, \quad (3.19b)$$

$$\left(\frac{\eta_h^{n+1} - \eta_h^n}{\tau}, \xi_h \right) + \gamma (\nabla \eta_h^{n+1}, \nabla \xi_h) + Q_h^{n+1} (\mathbf{w}_h^n \cdot \nabla \eta_h^n, \xi_h) = 0, \quad (3.19c)$$

$$\frac{Q_h^{n+1} - Q_h^n}{\tau} = (\varrho_h^{n+1} (\mathbf{w}_h^n \cdot \nabla) \mathbf{w}_h^n, \widehat{\mathbf{w}}_h^{n+1}) + \frac{1}{2} (\nabla \cdot (\varrho_h^{n+1} \mathbf{w}_h^n) \mathbf{w}_h^n, \widehat{\mathbf{w}}_h^{n+1}) + \lambda (\Delta \eta_h^n \nabla \eta_h^n, \widehat{\mathbf{w}}_h^{n+1}) - \lambda (\mathbf{w}_h^n \cdot \nabla \eta_h^n, \Delta \eta_h^{n+1}), \quad (3.19d)$$

$$\left(\frac{1}{Q_h^{n+1}} \nabla \phi_h^{n+1}, \nabla q_h \right) = -(\widehat{\mathbf{w}}_h^{n+1}, \nabla q_h), \quad (3.19e)$$

$$\mathbf{w}_h^{n+1} = \widehat{\mathbf{w}}_h^{n+1} + \frac{1}{Q_h^{n+1}} \nabla \phi_h^{n+1}, \quad (3.19f)$$

$$(s_h^{n+1}, v_h) = (s_h^n - \nabla \cdot \widehat{\mathbf{w}}_h^{n+1}, v_h), \quad (3.19g)$$

$$\hat{p}_h^{n+1} = \mu s_h^{n+1} - \frac{1}{\tau} \phi_h^{n+1}, \quad (3.19h)$$

where $\sigma_h^n = \sqrt{\varrho_h^n}$, for any $(\psi_h, w_h, \xi_h, q_h, v_h) \in (V_h, \mathbf{M}_h, V_h, V_h, V_h)$.

Remark. 3.1. A critical distinction must be made between the semi-discrete and fully discrete settings regarding the incompressibility constraint.

Semi-discrete case (Algorithm 1). The Gauge–Uzawa correction guarantees that the corrected velocity $\mathbf{w}^{n+1}$ satisfies the divergence-free condition pointwise, i.e., $\nabla \cdot \mathbf{w}^{n+1} = 0$ in Ω .

$$-\nabla \cdot \left(\frac{1}{Q_h^{n+1}} \nabla \phi_h^{n+1} \right) = \nabla \cdot \widehat{\mathbf{w}}_h^{n+1}, \quad \nabla \cdot \mathbf{w}^{n+1} = \nabla \cdot \widehat{\mathbf{w}}_h^{n+1} + \nabla \cdot \left(\frac{1}{Q_h^{n+1}} \nabla \phi_h^{n+1} \right) = 0 \quad \text{in } \Omega,$$

with the corresponding boundary conditions. Hence, the skew-symmetry property

$$b(\mathbf{w}^{n+1}, \mathbf{w}^{n+1}, \mathbf{w}^{n+1}) = 0$$

holds exactly for all velocity variables involved. This rigorously justifies all cancellations in the energy stability proof of Theorem 3.1 and in the error analysis for the semi-discrete scheme.

Fully discrete case (Algorithm 2). The velocity $\mathbf{w}_h^{n+1} \in \mathbf{M}_h + \nabla P_h$ satisfies the divergence-free condition only in a weak sense, i.e.,

$$(\nabla \cdot \mathbf{w}_h^{n+1}, q_h) = 0, \quad \forall q_h \in P_h,$$

but in general $\nabla \cdot \mathbf{w}_h^{n+1} \neq 0$ holds pointwise. As a result, the skew-symmetry identity

$$b(\mathbf{w}_h^{n+1}, \mathbf{w}_h^{n+1}, \mathbf{w}_h^{n+1}) = 0$$

does not hold exactly at the discrete level.

In the stability proof, this residual term does not affect the energy estimate. The reason is that the momentum equation (3.19b) uses the intermediate velocity $\mathbf{w}_h^{n+1}$ as the test function, while the right-hand side term of the evolution equation for the auxiliary variable Q_h^{n+1} (3.19d) has the opposite sign, so they cancel exactly when summed together. Therefore, this residual never appears in the final energy inequality.

In the error analysis, when estimating the terms on the right-hand side of the error equations, although $\nabla \cdot \mathbf{w}_h^{n+1} \neq 0$ generates additional errors, these terms can be estimated using finite element estimates to obtain $\|\nabla \cdot \mathbf{w}_h^{n+1}\|_0 \leq Ch^k \|\nabla \mathbf{w}_h^{n+1}\|_0$. Since the stability result guarantees that $\|\nabla \mathbf{w}_h^{n+1}\|_0$ is bounded, it follows that these terms are of order $O(h^k)$, which vanishes as $h \rightarrow 0$. Therefore, this source of error does not affect the final convergence conclusion.

Next, we will give the stability results of the above FEM scheme.

Theorem 3.2. Algorithm 2 is unconditionally stable in the sense that, for all $\tau > 0$, the following priori bounds hold

$$\|\sigma_h^{N+1}\|^2 + \sum_{n=0}^N \|\sigma_h^{n+1} - \sigma_h^n\|^2 = \|\sigma_h^0\|^2 \quad (3.20)$$

and

$$\begin{aligned} & \|\sigma_h^{N+1} \widehat{\mathbf{w}}_h^{N+1}\|^2 + \lambda \|\nabla \eta_h^{N+1}\|^2 + |Q_h^{N+1}|^2 + \mu\tau \|s_h^{N+1}\|^2 + \frac{\mu\tau}{2} \sum_{n=0}^N \|\nabla \widehat{\mathbf{w}}_h^{n+1}\|^2 \\ & + \sum_{n=0}^N \left\| \frac{1}{\sigma_h^n} \nabla \phi_h^n \right\|^2 + 2\tau\gamma\lambda \sum_{n=0}^N \|\Delta \eta_h^{n+1}\|^2 + \sum_{n=0}^N \|\sigma_h^{n+1} \widehat{\mathbf{w}}_h^{n+1} - \sigma_h^n \mathbf{w}_h^n\|^2 \\ & + \lambda \sum_{n=0}^N \|\nabla \eta_h^{n+1} - \nabla \eta_h^n\|^2 + \sum_{n=0}^N |Q_h^{n+1} - Q_h^n|^2 \\ & \leq \|\sigma_h^0 \widehat{\mathbf{w}}_h^0\|^2 + \lambda \|\nabla \eta_h^0\|^2 + |Q_h^0|^2 + \mu\tau \|s_h^0\|^2, \end{aligned} \quad (3.21)$$

where $\sigma_h^n = \sqrt{\varrho_h^n}$ and $1 \leq N \leq T/\tau$.

Proof. Given the resemblance to the stability proof of Theorem 3.1, the analysis is not reproduced here. $\square$

3.3.2. Implementation

We next present an efficient implementation of the fully discrete scheme (3.19). Full decoupling is achieved simply by employing the scalar auxiliary variable Q_h^{n+1} to separate the two variables. We rewrite $\widehat{\mathbf{w}}_h^{n+1}, \eta_h^{n+1}$ as linear combinations:

$$\begin{aligned} \widehat{\mathbf{w}}_h^{n+1} &= \widehat{\mathbf{w}}_{1h}^{n+1} + Q_h^{n+1} \widehat{\mathbf{w}}_{2h}^{n+1}, \\ \eta_h^{n+1} &= \eta_{1h}^{n+1} + Q_h^{n+1} \eta_{2h}^{n+1}. \end{aligned} \quad (3.22)$$

Next, substituting (3.22) into (3.19b)–(3.19c) allows us to perform an inductive classification based on the variable Q_h^{n+1} .

Step 1: Find σ_h^{n+1} , the solution of (3.19a).

Step 2: Find $\widehat{\mathbf{w}}_{ih}^{n+1} \in \mathbf{M}_h$ ($i = 1, 2$) such that

$$\left(\sigma_h^{n+1} \frac{\sigma_h^{n+1} \widehat{\mathbf{w}}_{1h}^{n+1} - \sigma_h^n \mathbf{w}_h^n}{\tau}, w_h \right) - (\mu s_h^n, \nabla \cdot w_h) + (\mu \nabla \widehat{\mathbf{w}}_{1h}^{n+1}, \nabla w_h) = 0, \quad (3.23a)$$

$$\begin{aligned} \left(\sigma_h^{n+1} \frac{\sigma_h^{n+1} \widehat{\mathbf{w}}_{2h}^{n+1}}{\tau}, w_h \right) + (\varrho_h^{n+1} (\mathbf{w}_h^n \cdot \nabla) \mathbf{w}_h^n, w_h) + \frac{1}{2} (\nabla \cdot (\varrho_h^{n+1} \mathbf{w}_h^n) \mathbf{w}_h^n, w_h) \\ + (\mu \nabla \widehat{\mathbf{w}}_{2h}^{n+1}, \nabla w_h) + \lambda (\Delta \eta_h^n \nabla \eta_h^n, w_h) = 0. \end{aligned} \quad (3.23b)$$

Step 3: Find $\eta_i^{n+1} \in V_h$ ($i = 1, 2$) such that

$$\left(\frac{\eta_{1h}^{n+1} - \eta_h^n}{\tau}, \xi_h \right) + \gamma (\nabla \eta_{1h}^{n+1}, \nabla \xi_h) = 0, \quad (3.24a)$$

$$\left(\frac{\eta_{2h}^{n+1}}{\tau}, \xi_h \right) + \gamma (\nabla \eta_{2h}^{n+1}, \nabla \xi_h) + (\mathbf{w}_h^n \cdot \nabla \eta_h^n, \xi_h) = 0. \quad (3.24b)$$

Step 4: Find Q_h^{n+1} . Using the split form for the variables $\widehat{\mathbf{w}}_h^{n+1}, \eta_h^{n+1}$, one can rewrite (3.19d) in the form

$$\begin{aligned} \frac{Q_h^{n+1} - Q_h^n}{\tau} = (\varrho_h^{n+1} (\mathbf{w}_h^n \cdot \nabla) \mathbf{w}_h^n, \widehat{\mathbf{w}}_{1h}^{n+1} + Q_h^{n+1} \widehat{\mathbf{w}}_{2h}^{n+1}) + \frac{1}{2} (\nabla \cdot (\varrho_h^{n+1} \mathbf{w}_h^n) \mathbf{w}_h^n, \widehat{\mathbf{w}}_{1h}^{n+1} + Q_h^{n+1} \widehat{\mathbf{w}}_{2h}^{n+1}) \\ + \lambda (\Delta \eta_h^n \nabla \eta_h^n, \widehat{\mathbf{w}}_{1h}^{n+1} + Q_h^{n+1} \widehat{\mathbf{w}}_{2h}^{n+1}) - \lambda (\mathbf{w}_h^n \cdot \nabla \eta_h^n, \Delta \eta_{1h}^{n+1} + Q_h^{n+1} \Delta \eta_{2h}^{n+1}), \end{aligned} \quad (3.25)$$

which implies

$$Q_h^{n+1} = \frac{\frac{1}{\tau} Q_h^n + J_1}{\frac{1}{\tau} - J_2}, \quad (3.26)$$

where J_i , $i = 1, 2$,

$$\begin{aligned} J_i = (\varrho_h^{n+1} (\mathbf{w}_h^n \cdot \nabla) \mathbf{w}_h^n, \widehat{\mathbf{w}}_{ih}^{n+1}) + \frac{1}{2} (\nabla \cdot (\varrho_h^{n+1} \mathbf{w}_h^n) \mathbf{w}_h^n, \widehat{\mathbf{w}}_{ih}^{n+1}) \\ + \lambda (\Delta \eta_h^n \nabla \eta_h^n, \widehat{\mathbf{w}}_{ih}^{n+1}) - \lambda (\mathbf{w}_h^n \cdot \nabla \eta_h^n, \Delta \eta_{ih}^{n+1}). \end{aligned} \quad (3.27)$$

Since all terms in J_1 and J_2 are already known from Steps 1–3 (the solvability of (3.26) is addressed below), the linear algebraic equation (3.26) can be readily solved.

The solvability of (3.27) is established by verifying $\frac{1}{\tau} - J_2 \neq 0$. By setting $w_h = \widehat{\mathbf{w}}_{2h}^{n+1}$ in (3.23b) and $\xi_h = -\lambda \Delta \eta_{2h}^{n+1}$ in (3.24b), and combining the two obtained equations, we get

$$\begin{aligned} J_2 = (\varrho_h^{n+1} (\mathbf{w}_h^n \cdot \nabla) \mathbf{w}_h^n, \widehat{\mathbf{w}}_{2h}^{n+1}) + \frac{1}{2} (\nabla \cdot (\varrho_h^{n+1} \mathbf{w}_h^n) \mathbf{w}_h^n, \widehat{\mathbf{w}}_{2h}^{n+1}) \\ + \lambda (\Delta \eta_h^n \nabla \eta_h^n, \widehat{\mathbf{w}}_{2h}^{n+1}) - \lambda (\mathbf{w}_h^n \cdot \nabla \eta_h^n, \Delta \eta_{2h}^{n+1}), \\ = -\frac{1}{\tau} \|\sigma_h^{n+1} \widehat{\mathbf{w}}_{2h}^{n+1}\|^2 - \mu \|\nabla \widehat{\mathbf{w}}_{2h}^{n+1}\|^2 - \frac{\lambda}{\tau} \|\nabla \eta_{2h}^{n+1}\|^2 - \gamma \lambda \|\Delta \eta_{2h}^{n+1}\|^2 < 0. \end{aligned}$$

Therefore, the scheme (3.19) is uniquely solvable.

Step 5: Find $\widehat{\mathbf{w}}_h^{n+1}$ and η_h^{n+1} from (3.22).

Step 6: Find $\phi_h^{n+1} \in V_h$ from (3.19e).

Step 7: Update $\mathbf{w}_h^{n+1}$ by (3.19f), and find $s_h^{n+1} \in V_h^k$ from (3.19g).

Step 8: Find $\hat{p}_h^{n+1}$ from (3.19h).

Finally, solving the above equations yields σ_h^{n+1} , $\mathbf{w}_h^{n+1}$, $\widehat{\mathbf{w}}_h^{n+1}$, $\hat{p}_h^{n+1}$, and η_h^{n+1} . In practice, Algorithm 2 is computationally efficient.

4. Error estimates

The fully decoupled first-order semi-discrete scheme (3.7) is subjected to a rigorous error analysis in this section.

The stability results below follow directly from Theorem 3.1.

$$\begin{aligned} & \|\mathbf{w}^m\| + \|\sigma^m\|^2 + \|\nabla \eta^m\|^2 + |\varrho^m|^2 + \tau \sum_{n=0}^{m-1} \|\nabla \widehat{\mathbf{w}}^{n+1}\|^2 + \tau \sum_{n=0}^{m-1} \|\Delta \eta^{n+1}\|^2 \\ & \leq K, \end{aligned} \quad (4.1)$$

where the positive constant K is dependent on $\mathbf{w}^0$, σ^0 , and $\nabla \eta^0$. The subsequent error analysis relies heavily on (4.1).

We now carry out the error estimation. Let $(\sigma, \varrho, \mathbf{w}, \eta, p)$ be the exact solutions of (3.4), and $(\sigma^{n+1}, \varrho^{n+1}, \widehat{\mathbf{w}}^{n+1}, \mathbf{w}^{n+1}, \eta^{n+1}, p^{n+1})$ are solutions of the scheme (3.7). The definitions of the errors are given

$$\begin{cases} \hat{e}_{\mathbf{w}}^{n+1} = \mathbf{w}(t_{n+1}) - \widehat{\mathbf{w}}^{n+1}, & e_{\mathbf{w}}^{n+1} = \mathbf{w}(t_{n+1}) - \mathbf{w}^{n+1}, \\ e_{\eta}^{n+1} = \eta(t_{n+1}) - \eta^{n+1}, & e_{\varrho}^{n+1} = \varrho(t_{n+1}) - \varrho^{n+1}, \\ \hat{e}_p^{n+1} = \hat{p}(t_{n+1}) - \hat{p}^{n+1}, & e_{\sigma}^{n+1} = \sigma(t_{n+1}) - \sigma^{n+1}. \end{cases} \quad (4.2)$$

The following results can be derived from (3.7c), (3.7f), and (3.7h).

$$\hat{e}_{\mathbf{w}}^n = 0, \quad \text{on } \partial\Omega, \quad (4.3a)$$

$$\nabla \cdot e_{\mathbf{w}}^n = 0, \quad \hat{e}_{\mathbf{w}}^{n+1} = e_{\mathbf{w}}^{n+1} + \frac{1}{\varrho^{n+1}} \nabla \phi^{n+1}, \quad \text{in } \Omega, \quad (4.3b)$$

$$(e_{\mathbf{w}}^n, \nabla \phi^n) = 0, \quad (\varrho^{n+1} \hat{e}_{\mathbf{w}}^n, e_{\mathbf{w}}^n) = (\varrho^{n+1} e_{\mathbf{w}}^n, e_{\mathbf{w}}^n), \quad \text{in } \Omega. \quad (4.3c)$$

Furthermore, we have

$$\|e_{\mathbf{w}}^{n+1}\|_1^2 \leq C \left(\|\hat{e}_{\mathbf{w}}^{n+1}\|_1^2 + \left\| \frac{1}{\varrho^{n+1}} \nabla \phi^{n+1} \right\|_1^2 \right) \quad (4.4)$$

and

$$\begin{aligned} \|\sigma^n e_{\mathbf{w}}^n\|^2 &= (\varrho^n e_{\mathbf{w}}^n, e_{\mathbf{w}}^n) = (\varrho^n \hat{e}_{\mathbf{w}}^n - \nabla \phi^n, e_{\mathbf{w}}^n) = \left(\varrho^n \hat{e}_{\mathbf{w}}^n, e_{\mathbf{w}}^n - \frac{1}{\varrho^n} \nabla \phi^n \right) \\ &= \|\sigma^n \hat{e}_{\mathbf{w}}^n\|^2 - (e_{\mathbf{w}}^n + \frac{1}{\varrho^n} \nabla \phi^n, \nabla \phi^n) = \|\sigma^n \hat{e}_{\mathbf{w}}^n\|^2 - \left\| \frac{1}{\sigma^n} \nabla \phi^n \right\|^2. \end{aligned} \quad (4.5)$$

Moreover, establishing convergence rates for the numerical approximation to the exact solution requires certain solution regularities. Throughout this section, the exact solution is assumed to satisfy the following regularities:

$$\begin{aligned}\sigma &\in L^\infty(0, T; W^{1,\infty}(\Omega)) \cap W^{1,\infty}(0, T; H^2(\Omega)), \\ \varrho &\in L^\infty(0, T; W^{1,\infty}(\Omega)) \cap W^{1,\infty}(0, T; H^2(\Omega)), \\ \mathbf{w} &\in W^{1,\infty}(0, T; \mathbf{H}^2(\Omega)) \cap W^{2,\infty}(0, T; \mathbf{H}^{-1}(\Omega)), \\ \eta &\in W^{1,\infty}(0, T; H^2(\Omega)) \cap W^{2,\infty}(0, T; H^{-1}(\Omega)), \\ \hat{p} &\in W^{1,\infty}(0, T; H^1(\Omega)), \quad Q \in C([0, T]).\end{aligned}\tag{4.6}$$

The following maximum principle for σ^n is derived next. Since its proof mimics Lemma 2 in [18], it is not repeated here.

Lemma 4.1. *Assume that the initial $\sigma^0(\mathbf{x})$ satisfied $0 < \sqrt{\varrho_m} \leq \sigma^0(x) \leq \sqrt{\varrho_M} < \infty$, $\forall x \in \overline{\Omega}$. Then, the numerical density σ^n determined from (3.7a) satisfies:*

$$\sqrt{\varrho_m} \leq \sigma^n(x) \leq \sqrt{\varrho_M}, \quad \forall x \in \overline{\Omega}, \quad \forall n,$$

and we can directly obtain

$$\varrho_m \leq \varrho^n(x) \leq \varrho_M, \quad \forall x \in \overline{\Omega}, \quad \forall n.$$

In the following lemma, we provide estimates of σ^{n+1} . For any sequence of functions $\{\nu^n\}_{n=0}^N = 0$, denote $D_\tau \nu^{n+1} = (\nu^{n+1} - \nu^n)/\tau$ for $0 \leq n \leq N-1$.

Lemma 4.2. *Under the regularity assumptions (4.6), the following holds*

$$\|e_\sigma^{n+1}\|^2 - \|e_\sigma^n\|^2 + \|e_\sigma^{n+1} - e_\sigma^n\|^2 \leq C\tau \left(\|e_\mathbf{w}^n\|^2 + \|e_\sigma^{n+1}\|^2 \right) + C\tau^3\tag{4.7}$$

and

$$\|e_\varrho^{n+1}\| \leq C\|e_\sigma^{n+1}\|.$$

Proof. Subtracting (3.7a) from (3.4a) at t_{n+1} , we have

$$\frac{e_\sigma^{n+1} - e_\sigma^n}{\tau} + e_\mathbf{w}^n \cdot \nabla \sigma(t_{n+1}) + \mathbf{w}^n \cdot \nabla e_\sigma^{n+1} = R_\sigma^{n+1},\tag{4.8}$$

where the truncation error R_σ^{n+1} satisfies

$$\|R_\sigma^{n+1}\| = \|D_\tau \sigma(t_{n+1}) - \sigma_t(t_{n+1}) - \nabla \sigma(t_{n+1}) \cdot (\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n))\| \leq C\tau.\tag{4.9}$$

Taking the inner product of (4.8) with $2\tau e_\sigma^{n+1}$ and using the equation

$$2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2,\tag{4.10}$$

we obtain

$$\begin{aligned}&\|e_\sigma^{n+1}\|^2 - \|e_\sigma^n\|^2 + \|e_\sigma^{n+1} - e_\sigma^n\|^2 \\ &= (R_\sigma^{n+1}, 2\tau e_\sigma^{n+1}) - (e_\mathbf{w}^n \cdot \nabla \sigma(t_{n+1}), 2\tau e_\sigma^{n+1}) - (\mathbf{w}^n \cdot \nabla e_\sigma^{n+1}, 2\tau e_\sigma^{n+1}).\end{aligned}\tag{4.11}$$

Using (4.9), the first term on the right-hand side of (4.11) can be estimated by

$$(R_{\sigma}^{n+1}, 2\tau e_{\sigma}^{n+1}) \leq C\tau \|e_{\sigma}^{n+1}\|^2 + C\tau^3. \quad (4.12)$$

An estimate for the second term on the right-hand side of (4.11) is given by

$$-(e_{\mathbf{w}}^n \cdot \nabla \sigma(t_{n+1}), 2\tau e_{\sigma}^{n+1}) \leq C\tau (\|e_{\mathbf{w}}^n\|^2 + \|e_{\sigma}^{n+1}\|^2). \quad (4.13)$$

Using (2.6), the third term on the right-hand side of (4.11) can be estimated by

$$-(\mathbf{w}^n \cdot \nabla e_{\sigma}^{n+1}, 2\tau e_{\sigma}^{n+1}) = 0. \quad (4.14)$$

Combining the above Eqs (4.11)–(4.14), we have (4.7).

According to $\sigma^{n+1} = \sqrt{\varrho^{n+1}}$, we have

$$e_{\varrho}^{n+1} = e_{\sigma}^{n+1} (\sigma(t_{n+1}) + \sigma^{n+1}). \quad (4.15)$$

Taking L^2 -norm on the both sides of (4.15), we derive by Lemma 4.1 that

$$\|e_{\varrho}^{n+1}\| \leq \|e_{\sigma}^{n+1}\| \|\sigma(t_{n+1}) + \sigma^{n+1}\|_{L^{\infty}} \leq C \|e_{\sigma}^{n+1}\|. \quad (4.16)$$

□

In the following lemma, we provide estimates of $\sigma^{n+1} e_{\mathbf{w}}^{n+1}$.

Lemma 4.3. *Under the assumptions of regularity conditions (4.6), we have*

$$\begin{aligned} & \|\sigma^{n+1} e_{\mathbf{w}}^{n+1}\|^2 - \|\sigma^n e_{\mathbf{w}}^n\|^2 + \frac{\mu\tau}{8} \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\|^2 + \mu\tau \left(\|s^{n+1}\|^2 - \|s^n\|^2 \right) + \left\| \frac{1}{\varrho^{n+1}} \nabla \phi^{n+1} \right\|^2 \\ & \leq C\tau (\|e_{\varrho}^{n+1}\|^2 + \|e_{\mathbf{w}}^n\|^2 + \|e_{\mathbf{w}}^n\|^2 \|e_{\mathbf{w}}^n\|_1^2 + \|\Delta e_{\eta}^n\|^2 + \|\nabla e_{\eta}^n\|^2) \\ & \quad + 2\tau (B_{64}^{n+1} + B_{74}^{n+1} + B_{83}^{n+1}, \hat{e}_{\mathbf{w}}^{n+1}) + C\tau^3. \end{aligned} \quad (4.17)$$

Proof. Subtracting (3.7b) from (3.4b) at t_{n+1} , we have

$$\begin{aligned} & \sigma^{n+1} \frac{\sigma^{n+1} \hat{e}_{\mathbf{w}}^{n+1} - \sigma^n e_{\mathbf{w}}^n}{\tau} - \mu \Delta \hat{e}_{\mathbf{w}}^{n+1} \\ & = -e_{\sigma}^{n+1} D_{\tau}(\sigma(t_{n+1}) \mathbf{w}(t_{n+1})) - \sigma^{n+1} e_{\sigma}^{n+1} D_{\tau}(\mathbf{w}(t_{n+1})) - \sigma^{n+1} D_{\tau}(e_{\sigma}^{n+1}) \mathbf{w}(t_n) \\ & \quad + \mu \nabla s^n - \nabla \hat{p}(t_{n+1}) - Q(t_{n+1})(\varrho(t_{n+1}) \mathbf{w}(t_{n+1}) \cdot \nabla \mathbf{w}(t_{n+1})) + Q^{n+1}(\varrho^{n+1} \mathbf{w}^n \cdot \nabla \mathbf{w}^n) \\ & \quad - \frac{1}{2} Q(t_{n+1})(\nabla \cdot (\varrho(t_{n+1}) \mathbf{w}(t_{n+1})) \mathbf{w}(t_{n+1})) + \frac{1}{2} Q^{n+1}(\nabla \cdot (\varrho^{n+1} \mathbf{w}^n) \mathbf{w}^n) \\ & \quad - \lambda Q(t_{n+1}) \Delta \eta(t_{n+1}) \nabla \eta(t_{n+1}) + \lambda Q^{n+1} \Delta \eta^n \nabla \eta^n + R_{\mathbf{w}}^{n+1} \\ & = \sum_{i=1}^9 B_i^{n+1}, \end{aligned} \quad (4.18)$$

where the truncation error $R_{\mathbf{w}}^{n+1}$ satisfies

$$\|R_{\mathbf{w}}^{n+1}\|_{-1} = \|\sigma(t_{n+1}) (D_{\tau}(\sigma(t_{n+1}) \mathbf{w}(t_{n+1})) - (\sigma \mathbf{w})_t(t_{n+1}))\|_{-1} \leq C\tau. \quad (4.19)$$

Taking the inner product of each term in (4.18) with $2\tau\hat{e}_{\mathbf{w}}^{n+1}$, we have

$$\begin{aligned} & \left\| \sigma^n \hat{e}_{\mathbf{w}}^{n+1} \right\|^2 - \left\| \sigma^n e_{\mathbf{w}}^n \right\|^2 + \left\| \sigma^{n+1} \hat{e}_{\mathbf{w}}^{n+1} - \sigma^n e_{\mathbf{w}}^n \right\|^2 + 2\mu\tau \left\| \nabla \hat{e}_{\mathbf{w}}^{n+1} \right\|^2 \\ &= \left(\sum_{i=1}^9 B_i^{n+1}, 2\tau \hat{e}_{\mathbf{w}}^{n+1} \right). \end{aligned} \quad (4.20)$$

Now let's analyze each term $(B_i^{n+1}, 2\tau \hat{e}_{\mathbf{w}}^{n+1})$ ($1 \leq i \leq 9$) in (4.20) one by one. First, we have

$$\begin{aligned} (B_1^{n+1}, 2\tau \hat{e}_{\mathbf{w}}^{n+1}) &= -(e_{\sigma}^{n+1} D_{\tau}(\sigma(t_{n+1})\mathbf{w}(t_{n+1})), 2\tau \hat{e}_{\mathbf{w}}^{n+1}) \\ &= -(e_{\sigma}^{n+1}(\sigma(t_{n+1})\mathbf{w}(t_{n+1}) - \sigma(t_n)\mathbf{w}(t_n)), 2\tau \hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq C \|e_{\sigma}^{n+1}\| \|(\sigma(t_{n+1})\mathbf{w}(t_{n+1}) - \sigma(t_n)\mathbf{w}(t_n))\|_{L^{\infty}} \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\| \\ &\leq C\tau \|e_{\sigma}^{n+1}\|^2 + \frac{\mu\tau}{8} \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\|^2 \end{aligned} \quad (4.21)$$

and

$$\begin{aligned} (B_2^{n+1}, 2\tau \hat{e}_{\mathbf{w}}^{n+1}) &= -(\sigma^{n+1} e_{\sigma}^{n+1} D_{\tau}(\mathbf{w}(t_{n+1})), 2\tau \hat{e}_{\mathbf{w}}^{n+1}) \\ &= -(\sigma^{n+1} e_{\sigma}^{n+1}(\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)), 2\tau \hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq C \|\sigma^{n+1}\|_{L^{\infty}} \|e_{\sigma}^{n+1}\| \|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\|_{L^{\infty}} \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\| \\ &\leq C\tau \|e_{\sigma}^{n+1}\|^2 + \frac{\mu\tau}{8} \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\|^2. \end{aligned} \quad (4.22)$$

Using (4.8), we can get

$$\begin{aligned} (B_3^{n+1}, 2\tau \hat{e}_{\mathbf{w}}^{n+1}) &= -(\sigma^{n+1} D_{\tau}(e_{\sigma}^{n+1})\mathbf{w}(t_n), 2\tau \hat{e}_{\mathbf{w}}^{n+1}) \\ &= -(\sigma^{n+1} \mathbf{w}(t_n) (-e_{\mathbf{w}}^n \cdot \nabla \sigma(t_{n+1}) - \mathbf{w}^n \cdot \nabla e_{\sigma}^{n+1}), 2\tau \hat{e}_{\mathbf{w}}^{n+1}) \\ &\quad - (\sigma^{n+1} \mathbf{w}(t_n) R_{\sigma}^{n+1}, 2\tau \hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq \|\sigma^{n+1} \mathbf{w}(t_n)\|_{L^{\infty}} (C\tau^2 + C\tau \|e_{\mathbf{w}}^n\| \|\sigma(t_{n+1})\|_2 + C\tau \|e_{\sigma}^{n+1}\|) \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\| \\ &\leq C\tau (\|e_{\mathbf{w}}^n\|^2 + \|e_{\sigma}^{n+1}\|^2) + \frac{\mu\tau}{8} \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau^3. \end{aligned} \quad (4.23)$$

Using (3.7i) and (2.1), we have

$$\begin{aligned} (B_4^{n+1}, 2\tau \hat{e}_{\mathbf{w}}^{n+1}) &= 2\mu\tau (\nabla s^n, \hat{e}_{\mathbf{w}}^{n+1}) = -2\mu\tau (s^n, \nabla \cdot (\mathbf{w}(t_{n+1}) - \hat{\mathbf{w}}^{n+1})) = 2\mu\tau (s^n, \nabla \cdot \hat{\mathbf{w}}^{n+1}) \\ &= 2\mu\tau (s^n, s^n - s^{n+1}) = \mu\tau (\|s^n\|^2 - \|s^{n+1}\|^2 + \|s^n - s^{n+1}\|^2) \\ &\leq \mu\tau (\|s^n\|^2 - \|s^{n+1}\|^2) + \mu\tau \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\|^2. \end{aligned} \quad (4.24)$$

From (4.3b), we can obtain

$$\begin{aligned} (B_5^{n+1}, 2\tau \hat{e}_{\mathbf{w}}^{n+1}) &= -2\tau (\nabla \hat{p}(t_{n+1}), \hat{e}_{\mathbf{w}}^{n+1}) = -2\tau \left(\nabla \hat{p}(t_{n+1}), e_{\mathbf{w}}^{n+1} + \frac{1}{\varrho^{n+1}} \nabla \phi^{n+1} \right) \\ &= -2\tau \left(\nabla \hat{p}(t_{n+1}), \frac{1}{\varrho^{n+1}} \nabla \phi^{n+1} \right) \\ &\leq C\tau^2 + \varepsilon \left\| \frac{1}{\varrho^{n+1}} \nabla \phi^{n+1} \right\|^2. \end{aligned} \quad (4.25)$$

For simplicity, we now handle B_i^{n+1} for $i = 6, 7, 8$ as follows

$$\begin{aligned}
 B_6^{n+1} &= -Q(t_{n+1})(\varrho(t_{n+1})\mathbf{w}(t_{n+1}) \cdot \nabla \mathbf{w}(t_{n+1})) + Q^{n+1}(\varrho^{n+1}\mathbf{w}^n \cdot \nabla \mathbf{w}^n) \\
 &= -Q(t_{n+1})\left(e_\varrho^{n+1}\mathbf{w}(t_{n+1}) \cdot \nabla \mathbf{w}(t_{n+1})\right) - Q(t_{n+1})\left(\varrho^{n+1}(\mathbf{w}(t_{n+1}) - \mathbf{w}^n) \cdot \nabla \mathbf{w}(t_{n+1})\right) \\
 &\quad - Q(t_{n+1})\left(\varrho^{n+1}\mathbf{w}^n \cdot \nabla (\mathbf{w}(t_{n+1}) - \mathbf{w}^n)\right) - e_\varrho^{n+1}\left(\varrho^{n+1}\mathbf{w}^n \cdot \nabla \mathbf{w}^n\right) \\
 &= B_{61}^{n+1} + B_{62}^{n+1} + B_{63}^{n+1} + B_{64}^{n+1}, \\
 B_7^{n+1} &= -\frac{1}{2}Q(t_{n+1})(\nabla \cdot (\varrho(t_{n+1})\mathbf{w}(t_{n+1}))\mathbf{w}(t_{n+1})) + \frac{1}{2}Q^{n+1}(\nabla \cdot (\varrho^{n+1}\mathbf{w}^n)\mathbf{w}^n) \\
 &= -\frac{1}{2}Q(t_{n+1})(\nabla \cdot (e_\varrho^{n+1}\mathbf{w}(t_{n+1}))\mathbf{w}(t_{n+1})) - \frac{1}{2}Q(t_{n+1})(\nabla \cdot (\varrho^{n+1}(\mathbf{w}(t_{n+1}) - \mathbf{w}^n))\mathbf{w}(t_{n+1})) \\
 &\quad - \frac{1}{2}Q(t_{n+1})\left(\nabla \cdot (\varrho^{n+1}\mathbf{w}^n)(\mathbf{w}(t_{n+1}) - \mathbf{w}^n)\right) - \frac{1}{2}e_\varrho^{n+1}\left(\nabla \cdot (\varrho^{n+1}\mathbf{w}^n)\mathbf{w}^n\right) \\
 &= B_{71}^{n+1} + B_{72}^{n+1} + B_{73}^{n+1} + B_{74}^{n+1}, \\
 B_8^{n+1} &= -\lambda Q(t_{n+1})\Delta\eta(t_{n+1})\nabla\eta(t_{n+1}) + \lambda Q^{n+1}\Delta\eta^n\nabla\eta^n \\
 &= -\lambda Q(t_{n+1})(\Delta\eta(t_{n+1}) - \Delta\eta^n)\nabla\eta(t_{n+1}) - \lambda Q(t_{n+1})\Delta\eta^n(\nabla\eta(t_{n+1}) - \nabla\eta^n) \\
 &\quad - \lambda e_\varrho^{n+1}\Delta\eta^n\nabla\eta^n \\
 &= B_{81}^{n+1} + B_{82}^{n+1} + B_{83}^{n+1}.
 \end{aligned}$$

Next, we estimate the term $(B_6^{n+1}, 2\tau\hat{e}_\mathbf{w}^{n+1})$.

Using (2.7) and Lemma 4.1, we have

$$\begin{aligned}
 (B_{61}^{n+1}, 2\tau\hat{e}_\mathbf{w}^{n+1}) &= -Q(t_{n+1})(e_\varrho^{n+1}\mathbf{w}(t_{n+1}) \cdot \nabla \mathbf{w}(t_{n+1}), 2\tau\hat{e}_\mathbf{w}^{n+1}) \\
 &\leq C\tau\|e_\varrho^{n+1}\| \|\mathbf{w}(t_{n+1})\|_{L^\infty} \|\mathbf{w}(t_{n+1})\|_2 \|\nabla \hat{e}_\mathbf{w}^{n+1}\| \\
 &\leq C\tau\|e_\varrho^{n+1}\|^2 + \frac{\mu\tau}{24} \|\nabla \hat{e}_\mathbf{w}^{n+1}\|^2,
 \end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
 (B_{62}^{n+1}, 2\tau\hat{e}_\mathbf{w}^{n+1}) &= -Q(t_{n+1})\left(\varrho^{n+1}(\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n) + e_\mathbf{w}^n) \cdot \nabla \mathbf{w}(t_{n+1}), 2\tau\hat{e}_\mathbf{w}^{n+1}\right) \\
 &\leq C\tau\|\varrho^{n+1}\|_{L^\infty} \|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n) + e_\mathbf{w}^n\| \|\mathbf{w}(t_{n+1})\|_2 \|\nabla \hat{e}_\mathbf{w}^{n+1}\| \\
 &\leq C\tau\|e_\mathbf{w}^n\|^2 + \frac{\mu\tau}{8} \|\nabla \hat{e}_\mathbf{w}^{n+1}\|^2 + C\tau^3.
 \end{aligned} \tag{4.27}$$

According to Lemma 4.1, we have

$$\begin{aligned}
 (B_{63}^{n+1}, 2\tau\hat{e}_\mathbf{w}^{n+1}) &= -Q(t_{n+1})(\varrho^{n+1}\mathbf{w}^n \cdot \nabla (\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n) + e_\mathbf{w}^n), 2\tau\hat{e}_\mathbf{w}^{n+1}) \\
 &\leq C\tau\|\varrho^{n+1}\mathbf{w}^n\| \|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\|_2 \|\nabla \hat{e}_\mathbf{w}^{n+1}\| \\
 &\quad + C\tau\left(\varrho^{n+1}(\mathbf{w}(t_n) - e_\mathbf{w}^n) \cdot \nabla e_\mathbf{w}^n, \hat{e}_\mathbf{w}^{n+1}\right) \\
 &\leq C\tau^3 + \frac{\mu\tau}{72} \|\nabla \hat{e}_\mathbf{w}^{n+1}\|^2 + C\tau\|\varrho^{n+1}\|_{L^\infty} \|\mathbf{w}(t_n)\|_2 \|e_\mathbf{w}^n\| \|\nabla \hat{e}_\mathbf{w}^{n+1}\| \\
 &\quad + C\tau\|\varrho^{n+1}\|_{L^\infty} \|e_\mathbf{w}^n\|^{1/2} \|e_\mathbf{w}^n\|_1^{1/2} \|e_\mathbf{w}^n\|^{1/2} \|e_\mathbf{w}^n\|_1^{1/2} \|\nabla \hat{e}_\mathbf{w}^{n+1}\| \\
 &\leq \frac{\mu\tau}{24} \|\nabla \hat{e}_\mathbf{w}^{n+1}\|^2 + C\tau\|e_\mathbf{w}^n\|^2 + C\tau\|e_\mathbf{w}^n\|^2 \|e_\mathbf{w}^n\|_1^2 + C\tau^3.
 \end{aligned} \tag{4.28}$$

We now combine the estimates corresponding to $(B_{6j}^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1})(j = 1, 2, 3)$ to get

$$(B_6^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) \leq \frac{\mu\tau}{8}\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau(\|e_{\varrho}^{n+1}\|^2 + \|e_{\mathbf{w}}^n\|^2 + \|e_{\mathbf{w}}^n\|^2\|e_{\mathbf{w}}^n\|_1^2) + C\tau^3. \quad (4.29)$$

Next, we estimate the term $(B_7^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1})$.

By using (2.7), we obtain

$$\begin{aligned} (B_{71}^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) &= -Q(t_{n+1})(\nabla \cdot (e_{\varrho}^{n+1}\mathbf{w}(t_{n+1}))\mathbf{w}(t_{n+1}), \tau\hat{e}_{\mathbf{w}}^{n+1}) \\ &= C\tau((e_{\varrho}^{n+1}\mathbf{w}(t_{n+1})), \nabla\mathbf{w}(t_{n+1}) \cdot \hat{e}_{\mathbf{w}}^{n+1} + \mathbf{w}(t_{n+1}) \cdot \nabla\hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq C\tau\|\mathbf{w}(t_{n+1})\|_{L^\infty}\|\nabla\mathbf{w}(t_{n+1})\|_{L^\infty}\|e_{\varrho}^{n+1}\| \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\| \\ &\quad + C\tau\|\mathbf{w}(t_{n+1})\|_{L^\infty}^2\|e_{\varrho}^{n+1}\| \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\| \\ &\leq C\tau\|e_{\varrho}^{n+1}\|^2 + \frac{\mu\tau}{24}\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2, \end{aligned} \quad (4.30)$$

$$\begin{aligned} (B_{72}^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) &= -Q(t_{n+1})(\nabla \cdot (\varrho^{n+1}(\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n) + e_{\mathbf{w}}^n))\mathbf{w}(t_{n+1}), \tau\hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq C\tau(\|\varrho^{n+1}(\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n))\| \\ &\quad + \|\varrho^{n+1}e_{\mathbf{w}}^n\|)(\|\mathbf{w}(t_{n+1})\|_{L^\infty} + \|\nabla\mathbf{w}(t_{n+1})\|_{L^\infty})\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\| \\ &\leq \frac{\mu\tau}{24}\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau\|e_{\mathbf{w}}^n\|^2 + C\tau^3, \end{aligned} \quad (4.31)$$

$$\begin{aligned} (B_{73}^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) &= -Q(t_{n+1})(\nabla \cdot (\varrho^{n+1}\mathbf{w}^n)(\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n) + e_{\mathbf{w}}^n), \tau\hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq C\tau\|\varrho^{n+1}\mathbf{w}^n\|(\|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\|_{L^\infty} + \|\nabla(\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n))\|_{L^\infty})\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\| \\ &\quad - C\tau(\nabla \cdot (\varrho^{n+1}(\mathbf{w}(t_n) - e_{\mathbf{w}}^n))e_{\mathbf{w}}^n, \hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq \frac{\mu\tau}{24}\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau\|e_{\mathbf{w}}^n\|^2 + C\tau\|e_{\mathbf{w}}^n\|^2\|e_{\mathbf{w}}^n\|_1^2 + C\tau^3, \end{aligned} \quad (4.32)$$

where

$$\begin{aligned} &(\nabla \cdot (\varrho^{n+1}(\mathbf{w}(t_n) - e_{\mathbf{w}}^n))e_{\mathbf{w}}^n, \hat{e}_{\mathbf{w}}^{n+1}) \\ &= -(\varrho^{n+1}(\mathbf{w}(t_n) - e_{\mathbf{w}}^n) \cdot \nabla\hat{e}_{\mathbf{w}}^{n+1}, e_{\mathbf{w}}^n) - (\varrho^{n+1}(\mathbf{w}(t_n) - e_{\mathbf{w}}^n) \cdot \nabla e_{\mathbf{w}}^n, \hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq 2\|\varrho^{n+1}\|_{L^\infty}\|\mathbf{w}(t_n)\|_2\|e_{\mathbf{w}}^n\| \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\| + (\varrho^{n+1}e_{\mathbf{w}}^n \cdot \nabla\hat{e}_{\mathbf{w}}^{n+1}, e_{\mathbf{w}}^n) + (\varrho^{n+1}e_{\mathbf{w}}^n \cdot \nabla e_{\mathbf{w}}^n, \hat{e}_{\mathbf{w}}^{n+1}) \\ &\leq 2\|\varrho^{n+1}\|_{L^\infty}\|\mathbf{w}(t_n)\|_2\|e_{\mathbf{w}}^n\| \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\| + \|\varrho^{n+1}\|_{L^\infty}\|e_{\mathbf{w}}^n\|_{L^4}^2\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\| \\ &\quad + (\|\varrho^{n+1}\|_{L^\infty}\|e_{\mathbf{w}}^n\|^{1/2}\|e_{\mathbf{w}}^n\|_1^{1/2}\|e_{\mathbf{w}}^n\|^{1/2}\|e_{\mathbf{w}}^n\|_1^{1/2}\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|) \\ &\leq \frac{\mu}{48}\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C(\|e_{\mathbf{w}}^n\|^2 + \|e_{\mathbf{w}}^n\|^2\|e_{\mathbf{w}}^n\|_1^2). \end{aligned}$$

We now combine the estimates corresponding to $(B_{7j}^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1})(j = 1, 2, 3)$ to get

$$(B_7^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) \leq \frac{\mu\tau}{8}\|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau(\|e_{\varrho}^{n+1}\|^2 + \|e_{\mathbf{w}}^n\|^2 + \|e_{\mathbf{w}}^n\|^2\|e_{\mathbf{w}}^n\|_1^2) + C\tau^3. \quad (4.33)$$

Next, we estimate the term $(B_8^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1})$.

According to (2.7) and (2.2), we have

$$\begin{aligned}
 (B_{81}^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) &= -\lambda Q(t_{n+1}) \left(Q(t_{n+1})(\Delta\eta(t_{n+1}) - \Delta\eta^n) \nabla\eta(t_{n+1}), 2\tau\hat{e}_{\mathbf{w}}^{n+1} \right) \\
 &\leq C\tau \|\Delta\eta(t_{n+1}) - \Delta\eta(t_n)\| \|\nabla\eta(t_{n+1})\|_{L^\infty} \|\hat{e}_{\mathbf{w}}^{n+1}\| \\
 &\quad + C\tau \|\Delta e_\eta^n\| \|\nabla\eta(t_{n+1})\|_{L^\infty} \|\hat{e}_{\mathbf{w}}^{n+1}\| \\
 &\leq C\tau \|\Delta e_\eta^n\|^2 + \frac{\mu\tau}{16} \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau^3
 \end{aligned} \tag{4.34}$$

and

$$\begin{aligned}
 (B_{82}^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) &= -\lambda Q(t_{n+1}) (\Delta\eta^n (\nabla\eta(t_{n+1}) - \nabla\eta^n), 2\tau\hat{e}_{\mathbf{w}}^{n+1}) \\
 &\leq C\tau \|\Delta\eta^n\| \|\nabla\eta(t_{n+1}) - \nabla\eta(t_n)\|_{L^\infty} \|\hat{e}_{\mathbf{w}}^{n+1}\| \\
 &\quad + C\tau \left((\Delta\eta(t_n) - \Delta e_\eta^n) \nabla e_\eta^n, \hat{e}_{\mathbf{w}}^{n+1} \right) \\
 &\leq \frac{\tau}{48R_e} \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau \|\Delta\eta(t_n)\|_{L^\infty} \|\nabla e_\eta^n\| \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\| \\
 &\quad + C\tau (\Delta e_\eta^n \nabla e_\eta^n, \hat{e}_{\mathbf{w}}^{n+1}) + C\tau^3 \\
 &\leq \frac{\mu\tau}{16} \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau (\|\Delta e_\eta^n\|^2 + \|\nabla e_\eta^n\|^2) + C\tau^3,
 \end{aligned} \tag{4.35}$$

where

$$\begin{aligned}
 (\Delta e_\eta^n \nabla e_\eta^n, \hat{e}_{\mathbf{w}}^{n+1}) &\leq C \|\Delta e_\eta^n\| \|\nabla e_\eta^n\|_{L^4} \|\hat{e}_{\mathbf{w}}^{n+1}\|_{L^4} \\
 &\leq C \|\Delta e_\eta^n\| \|\nabla e_\eta^n\|^{1/2} \|\Delta e_\eta^n\|^{1/2} \|\hat{e}_{\mathbf{w}}^{n+1}\|^{1/2} \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^{1/2} \\
 &\leq \frac{\mu}{48} \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C(\|\nabla e_\eta^n\|^2 + \|\Delta e_\eta^n\|^2).
 \end{aligned}$$

We now combine the estimates corresponding to $(B_{8j}^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1})$ ($j = 1, 2$) to get

$$(B_8^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) \leq \frac{\mu\tau}{8} \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau (\|\Delta e_\eta^n\|^2 + \|\nabla e_\eta^n\|^2) + C\tau^3. \tag{4.36}$$

Next, we deal with the remaining item

$$(B_9^{n+1}, 2\tau\hat{e}_{\mathbf{w}}^{n+1}) = 2\tau (R_{\mathbf{w}}^{n+1}, \hat{e}_{\mathbf{w}}^{n+1}) \leq \frac{\mu\tau}{8} \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + C\tau^3. \tag{4.37}$$

Thus, combining with (4.20)–(4.37), by choosing a sufficiently small value for ε , we can obtain

$$\begin{aligned}
 \|\sigma^{n+1}\hat{e}_{\mathbf{w}}^{n+1}\|^2 - \|\sigma^n e_{\mathbf{w}}^n\|^2 &+ \frac{\mu\tau}{8} \|\nabla\hat{e}_{\mathbf{w}}^{n+1}\|^2 + \mu\tau \left(\|s^{n+1}\|^2 - \|s^n\|^2 \right) \\
 &\leq C\tau (\|e_\rho^{n+1}\|^2 + \|e_{\mathbf{w}}^n\|^2 + \|e_{\mathbf{w}}^n\|^2 \|e_{\mathbf{w}}^n\|_1^2 + \|\Delta e_\eta^n\|^2 + \|\nabla e_\eta^n\|^2) \\
 &\quad + 2\tau (B_{64}^{n+1} + B_{74}^{n+1} + B_{83}^{n+1}, \hat{e}_{\mathbf{w}}^{n+1}) + C\tau^3.
 \end{aligned} \tag{4.38}$$

The desired result (4.17) is then obtained by invoking (4.5). $\square$

We now turn to estimating ∇e_η^{n+1} .

Lemma 4.4. *Under the regularity conditions stated in (4.6), it follows that*

$$\begin{aligned}
 &\lambda \|\nabla e_\eta^{n+1}\|^2 - \lambda \|\nabla e_\eta^n\|^2 + \gamma\lambda\tau \|\Delta e_\eta^{n+1}\|^2 \\
 &\leq C\tau (\|e_{\mathbf{w}}^n\|^2 + \|\nabla e_\eta^n\|^2 + \|\mathbf{w}^n\|_1 \|\Delta e_\eta^n\|^2) + (J_3^{n+1}, -2\lambda\tau \Delta e_\eta^{n+1}) + C\tau^3.
 \end{aligned} \tag{4.39}$$

Proof. Subtracting (3.7d) from (1.3c) leads to

$$\begin{aligned}
 & \frac{e_\eta^{n+1} - e_\eta^n}{\tau} - \gamma \Delta e_\eta^{n+1} \\
 &= -Q(t_{n+1}) \mathbf{w}(t_{n+1}) \cdot \nabla \eta(t_{n+1}) + Q^{n+1} \mathbf{w}^n \cdot \nabla \eta^n + R_\eta^{n+1} \\
 &= -Q(t_{n+1}) (\mathbf{w}(t_{n+1}) - \mathbf{w}^n) \cdot \nabla \eta(t_{n+1}) - Q(t_{n+1}) \mathbf{w}^n \cdot \nabla (\eta(t_{n+1}) - \eta^n) \\
 & \quad - e_Q^{n+1} \mathbf{w}^n \cdot \nabla \eta^n + R_\eta^{n+1} \\
 &= \sum_{i=1}^4 J_i^{n+1},
 \end{aligned} \tag{4.40}$$

where the truncation error R_η^{n+1} satisfies

$$\|R_\eta^{n+1}\| = \|D_\tau \eta(t_{n+1}) - \eta_t(t_{n+1})\| \leq C\tau. \tag{4.41}$$

Taking the inner product of equation (4.40) with $-2\tau\lambda\Delta e_\eta^{n+1}$, we have

$$\begin{aligned}
 & \lambda \left(\|\nabla e_\eta^{n+1}\|^2 - \|\nabla e_\eta^n\|^2 + \|\nabla e_\eta^{n+1} - \nabla e_\eta^n\|^2 \right) + 2\gamma\lambda\tau \|\Delta e_\eta^{n+1}\|^2 \\
 &= \left(\sum_{i=1}^4 J_i^{n+1}, -2\lambda\tau\Delta e_\eta^{n+1} \right).
 \end{aligned} \tag{4.42}$$

We can bound the first term appearing on the right-hand side of (4.42) by

$$\begin{aligned}
 (J_1^{n+1}, -2\lambda\tau\Delta e_\eta^{n+1}) &= 2\lambda\tau Q(t_{n+1}) ((\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n) + e_\mathbf{w}^n) \cdot \nabla \eta(t_{n+1}), \Delta e_\eta^{n+1}) \\
 &\leq C\lambda\tau \|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\| \|\nabla \eta(t_{n+1})\|_{L^\infty} \|\Delta e_\eta^{n+1}\| \\
 & \quad + C\lambda\tau \|e_\mathbf{w}^n\| \|\nabla \eta(t_{n+1})\|_{L^\infty} \|\Delta e_\eta^{n+1}\| \\
 &\leq C\tau^3 + \frac{\gamma\lambda\tau}{4} \|\Delta e_\eta^{n+1}\|^2 + C\tau \|e_\mathbf{w}^n\|^2.
 \end{aligned} \tag{4.43}$$

The second terms on the right-hand side of (4.42) can be estimated by

$$\begin{aligned}
 (J_2^{n+1}, -2\lambda\tau\Delta e_\eta^{n+1}) &= 2\lambda\tau Q(t_{n+1}) (\mathbf{w}^n \cdot \nabla (\eta(t_{n+1}) - \eta(t_n) + e_\eta^n), \Delta e_\eta^{n+1}) \\
 &\leq C\lambda\tau \|\mathbf{w}^n\| \|\nabla \eta(t_{n+1}) - \nabla \eta(t_n)\|_{L^\infty} \|\Delta e_\eta^{n+1}\| \\
 & \quad + C\lambda\tau \|\mathbf{w}^n\|_{L^4} \|\nabla e_\eta^n\|_{L^4} \|\Delta e_\eta^{n+1}\| \\
 &\leq C\tau^3 + \frac{\gamma\lambda\tau}{4} \|\nabla e_\eta^{n+1}\|^2 + C\tau (\|\nabla e_\eta^n\|^2 + \|\mathbf{w}^n\|_1 \|\Delta e_\eta^n\|^2).
 \end{aligned} \tag{4.44}$$

The fourth term on the right-hand side of (4.42) can be bounded by

$$(J_4^{n+1}, -2\lambda\tau\Delta e_\eta^{n+1}) = (R_\eta^{n+1}, -2\lambda\tau\Delta e_\eta^{n+1}) \leq \frac{\gamma\lambda\tau}{4} \|\Delta e_\eta^{n+1}\|^2 + C\tau^3. \tag{4.45}$$

Thus, combining with (4.42)–(4.45), we get (4.39). □

Next, we provide estimates of e_Q^{n+1} .

Lemma 4.5. *Assuming (4.6) holds, we have*

$$\begin{aligned} |e_Q^{n+1}|^2 - |e_Q^n|^2 &\leq C\tau |e_Q^{n+1}|^2 + C\tau (\|e_Q^{n+1}\|^2 + \|e_{\mathbf{w}}^n\|^2 + \|\nabla e_{\eta}^n\|^2 + \|\Delta e_{\eta}^n\|^2) + C\tau^3 \\ &\quad + 2\tau e_Q^{n+1} (M_{14}^{n+1} + M_{24}^{n+1} + M_{33}^{n+1} + M_{43}^{n+1}). \end{aligned} \quad (4.46)$$

Proof. Subtracting (3.7e) from (3.4e) at t_{n+1} , we obtain

$$\begin{aligned} \frac{e_Q^{n+1} - e_Q^n}{\tau} &= (\varrho(t_{n+1})\mathbf{w}(t_{n+1}) \cdot \nabla \mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) - (\varrho^{n+1}\mathbf{w}^n \cdot \nabla \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1}) \\ &\quad + \frac{1}{2} \left((\nabla \cdot (\varrho(t_{n+1})\mathbf{w}(t_{n+1}))\mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) - (\nabla \cdot (\varrho^{n+1}\mathbf{w}^n)\mathbf{w}^n, \widehat{\mathbf{w}}^{n+1}) \right) \\ &\quad + \lambda \left((\Delta \eta(t_{n+1})\nabla \eta(t_{n+1}), \mathbf{w}(t_{n+1})) - (\Delta \eta^n \nabla \eta^n, \widehat{\mathbf{w}}^{n+1}) \right) \\ &\quad + \lambda \left((\mathbf{w}^n \cdot \nabla \eta^n, \Delta \eta^{n+1}) - (\mathbf{w}(t_{n+1}) \cdot \nabla \eta(t_{n+1}), \Delta \eta(t_{n+1})) \right) + R_Q^{n+1} \\ &= \sum_{i=1}^5 M_i^{n+1}, \end{aligned} \quad (4.47)$$

where

$$|R_Q^{n+1}| = |D_{\tau}Q(t_{n+1}) - Q(t_{n+1})| \leq C\tau. \quad (4.48)$$

Multiplying both sides of (4.47) by $2\tau e_Q^{n+1}$ yields

$$|e_Q^{n+1}|^2 - |e_Q^n|^2 + |e_Q^{n+1} - e_Q^n|^2 = \sum_{i=1}^5 2\tau e_Q^{n+1} M_i^{n+1}. \quad (4.49)$$

For simplicity, we now handle M_i^{n+1} for $i = 1, 2, 3, 4$ as follows:

$$\begin{aligned} M_1^{n+1} &= ((\varrho(t_{n+1})\mathbf{w}(t_{n+1}) \cdot \nabla \mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) - (\varrho^{n+1}\mathbf{w}^n \cdot \nabla \mathbf{w}^n, \widehat{\mathbf{w}}^{n+1})) \\ &= (e_Q^{n+1}\mathbf{w}(t_{n+1}) \cdot \nabla \mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) + (\varrho^{n+1}(\mathbf{w}(t_{n+1}) - \mathbf{w}^n) \cdot \nabla \mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) \\ &\quad + (\varrho^{n+1}\mathbf{w}^n \cdot \nabla (\mathbf{w}(t_{n+1}) - \mathbf{w}^n), \mathbf{w}(t_{n+1})) + (\varrho^{n+1}\mathbf{w}^n \cdot \nabla \mathbf{w}^n, \widehat{\mathbf{e}}_{\mathbf{w}}^{n+1}) \\ &= M_{11}^{n+1} + M_{12}^{n+1} + M_{13}^{n+1} + M_{14}^{n+1}, \\ M_2^{n+1} &= \frac{1}{2} \left((\nabla \cdot (\varrho(t_{n+1})\mathbf{w}(t_{n+1}))\mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) - (\nabla \cdot (\varrho^{n+1}\mathbf{w}^n)\mathbf{w}^n, \widehat{\mathbf{w}}^{n+1}) \right) \\ &= \frac{1}{2} \left((\nabla \cdot (e_Q^{n+1}\mathbf{w}(t_{n+1}))\mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) + (\nabla \cdot (\varrho^{n+1}(\mathbf{w}(t_{n+1}) - \mathbf{w}^n))\mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) \right. \\ &\quad \left. + (\nabla \cdot (\varrho^{n+1}\mathbf{w}^n)(\mathbf{w}(t_{n+1}) - \mathbf{w}^n), \mathbf{w}(t_{n+1})) + (\nabla \cdot (\varrho^{n+1}\mathbf{w}^n)\mathbf{w}^n, \widehat{\mathbf{e}}_{\mathbf{w}}^{n+1}) \right) \\ &= M_{21}^{n+1} + M_{22}^{n+1} + M_{23}^{n+1} + M_{24}^{n+1}, \\ M_3^{n+1} &= \lambda \left((\Delta \eta(t_{n+1})\nabla \eta(t_{n+1}), \mathbf{w}(t_{n+1})) - (\Delta \eta^n \nabla \eta^n, \widehat{\mathbf{w}}^{n+1}) \right) \\ &= \lambda \left((\Delta (\eta(t_{n+1}) - \eta^n)\nabla \eta(t_{n+1}), \mathbf{w}(t_{n+1})) + (\Delta \eta^n \nabla (\eta(t_{n+1}) - \eta^n), \mathbf{w}(t_{n+1})) \right. \\ &\quad \left. + (\Delta \eta^n \nabla \eta^n, \widehat{\mathbf{e}}_{\mathbf{w}}^{n+1}) \right) \\ &= M_{31}^{n+1} + M_{32}^{n+1} + M_{33}^{n+1}, \end{aligned}$$

$$\begin{aligned}
M_4^{n+1} &= -\lambda((\mathbf{w}(t_{n+1}) \cdot \nabla \eta(t_{n+1}), \Delta \eta(t_{n+1})) - (\mathbf{w}^n \cdot \nabla \eta^n, \Delta \eta^{n+1})) \\
&= -\lambda(((\mathbf{w}(t_{n+1}) - \mathbf{w}^n) \cdot \nabla \eta(t_{n+1}), \Delta \eta(t_{n+1})) + (\mathbf{w}^n \cdot \nabla (\eta(t_{n+1}) - \eta^n), \Delta \eta(t_{n+1})) \\
&\quad + (\mathbf{w}^n \cdot \nabla \eta^n, \Delta \eta^{n+1})) \\
&= M_{41}^{n+1} + M_{42}^{n+1} + M_{43}^{n+1}.
\end{aligned}$$

An estimate for the last term on the right of (4.49) is given first as follows

$$2\tau e_Q^{n+1} M_5^{n+1} = 2\tau R_Q^{n+1} e_Q^{n+1} \leq C\tau |e_Q^{n+1}|^2 + C\tau^3. \quad (4.50)$$

Then, we can estimate $2\tau e_Q^{n+1} M_1^{n+1}$.

$$\begin{aligned}
2\tau e_Q^{n+1} M_{11}^{n+1} &= 2\tau e_Q^{n+1} (e_Q^{n+1} \mathbf{w}(t_{n+1}) \cdot \nabla \mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) \\
&\leq C\tau |e_Q^{n+1}| |e_Q^{n+1}| |\mathbf{w}(t_{n+1})|_1 |\mathbf{w}(t_{n+1})|_{L^\infty}^2 \\
&\leq C\tau (|e_Q^{n+1}|^2 + \|e_Q^{n+1}\|^2), \\
2\tau e_Q^{n+1} M_{12}^{n+1} &= 2\tau e_Q^{n+1} (\varrho^{n+1} (\mathbf{w}(t_{n+1}) - \mathbf{w}^n) \cdot \nabla \mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) \\
&\leq C\tau |e_Q^{n+1}| (\|\varrho^{n+1}\|_{L^\infty} \|\mathbf{w}(t_{n+1})\| |\mathbf{w}(t_{n+1})|_2 \|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\|_1 \\
&\quad + \|\varrho^{n+1}\|_{L^\infty} \|e_{\mathbf{w}}^n\| |\mathbf{w}(t_{n+1})|_1 \|\mathbf{w}(t_{n+1})\|_2) \\
&\leq C\tau (|e_Q^{n+1}|^2 + \|e_{\mathbf{w}}^n\|^2) + C\tau^3, \\
2\tau e_Q^{n+1} M_{13}^{n+1} &= 2\tau e_Q^{n+1} (\varrho^{n+1} \mathbf{w}^n \cdot \nabla (\mathbf{w}(t_{n+1}) - \mathbf{w}^n), \mathbf{w}(t_{n+1})) \\
&\leq C\tau |e_Q^{n+1}| \|\varrho^{n+1}\|_{L^\infty} \|\mathbf{w}^n\| |\mathbf{w}(t_{n+1})|_2 \|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\|_1 \\
&\quad + C\tau |e_Q^{n+1}| \|\varrho^{n+1}\|_{L^\infty} \|\mathbf{w}^n\|_1 \|e_{\mathbf{w}}^n\| |\mathbf{w}(t_{n+1})|_2 \\
&\leq C\tau (|e_Q^{n+1}|^2 + \|e_{\mathbf{w}}^n\|^2) + C\tau^3.
\end{aligned}$$

Next, we can estimate $2\tau e_Q^{n+1} M_2^{n+1}$.

$$\begin{aligned}
2\tau e_Q^{n+1} M_{21}^{n+1} &= \tau e_Q^{n+1} (\nabla \cdot (e_Q^{n+1} \mathbf{w}(t_{n+1})) \mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) \\
&\leq C\tau |e_Q^{n+1}| |e_Q^{n+1}| |\mathbf{w}(t_{n+1})|_{L^\infty} |\mathbf{w}(t_{n+1})|_1 |\mathbf{w}(t_{n+1})|_2 \\
&\leq C\tau (|e_Q^{n+1}|^2 + \|e_Q^{n+1}\|^2), \\
2\tau e_Q^{n+1} M_{22}^{n+1} &= \tau e_Q^{n+1} (\nabla \cdot (\varrho^{n+1} (\mathbf{w}(t_{n+1}) - \mathbf{w}^n)) \mathbf{w}(t_{n+1}), \mathbf{w}(t_{n+1})) \\
&\leq C\tau |e_Q^{n+1}| \|\varrho^{n+1}\|_{L^\infty} (\|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\| + \|e_{\mathbf{w}}^n\|) \|\mathbf{w}(t_{n+1})\|_1 \|\mathbf{w}(t_{n+1})\|_2 \\
&\leq C\tau (|e_Q^{n+1}|^2 + \|e_{\mathbf{w}}^n\|^2) + C\tau^3, \\
2\tau e_Q^{n+1} M_{23}^{n+1} &= \tau e_Q^{n+1} (\nabla \cdot (\varrho^{n+1} \mathbf{w}^n) (\mathbf{w}(t_{n+1}) - \mathbf{w}^n), \mathbf{w}(t_{n+1})) \\
&\leq C\tau |e_Q^{n+1}| \|\varrho^{n+1}\|_{L^\infty} \|\mathbf{w}^n\|_1 (\|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\| + \|e_{\mathbf{w}}^n\|) \|\mathbf{w}(t_{n+1})\|_2 \\
&\leq C\tau (|e_Q^{n+1}|^2 + \|e_{\mathbf{w}}^n\|^2) + C\tau^3.
\end{aligned}$$

Then, we can estimate $2\tau e_Q^{n+1} M_3^{n+1}$.

$$\begin{aligned} 2\tau e_Q^{n+1} M_{31}^{n+1} &= 2\tau \lambda e_Q^{n+1} (\Delta(\eta(t_{n+1}) - \eta^n) \nabla \eta(t_{n+1}), \mathbf{w}(t_{n+1})) \\ &\leq C\tau |e_Q^{n+1}| (|\Delta\eta(t_{n+1}) - \Delta\eta(t_n)| + \|\Delta e_\eta^n\|) \|\nabla \eta(t_{n+1})\| \|\mathbf{w}(t_{n+1})\|_{L^\infty} \\ &\leq C\tau (|e_Q^{n+1}|^2 + \|\Delta e_\eta^n\|^2) + C\tau^3, \\ 2\tau e_Q^{n+1} M_{32}^{n+1} &= 2\tau \lambda e_Q^{n+1} (\Delta\eta^n \nabla (\eta(t_{n+1}) - \eta^n), \mathbf{w}(t_{n+1})) \\ &\leq C\tau |e_Q^{n+1}| (\|\Delta\eta^n\| \|\nabla (\eta(t_{n+1}) - \eta(t_n))\| \|\mathbf{w}(t_{n+1})\|_{L^\infty} + \|\Delta\eta^n\| \|\nabla e_\eta^n\| \|\mathbf{w}(t_{n+1})\|_{L^\infty}) \\ &\leq C\tau (|e_Q^{n+1}|^2 + \|\nabla e_\eta^n\|^2) + C\tau^3. \end{aligned}$$

Next, we can estimate $2\tau e_Q^{n+1} M_4^{n+1}$.

$$\begin{aligned} 2\tau e_Q^{n+1} M_{41}^{n+1} &= -2\tau \lambda e_Q^{n+1} ((\mathbf{w}(t_{n+1}) - \mathbf{w}^n) \cdot \nabla \eta(t_{n+1}), \Delta\eta(t_{n+1})) \\ &\leq C\tau |e_Q^{n+1}| (\|\mathbf{w}(t_{n+1}) - \mathbf{w}(t_n)\| + \|e_\mathbf{w}^n\|) \|\nabla \eta(t_{n+1})\|_{L^\infty} \|\Delta\eta(t_{n+1})\| \\ &\leq C\tau (|e_Q^{n+1}|^2 + \|e_\mathbf{w}^n\|^2) + C\tau^3, \\ 2\tau e_Q^{n+1} M_{42}^{n+1} &= -2\tau \lambda e_Q^{n+1} (\mathbf{w}^n \cdot \nabla (\eta(t_{n+1}) - \eta^n), \Delta\eta(t_{n+1})) \\ &\leq C\tau |e_Q^{n+1}| (\|\mathbf{w}^n\| \|\nabla (\eta(t_{n+1}) - \eta(t_n))\| \|\Delta\eta(t_{n+1})\|_{L^\infty} + \|\mathbf{w}^n\| \|\nabla e_\eta^n\| \|\Delta\eta(t_{n+1})\|_{L^\infty}) \\ &\leq C\tau (|e_Q^{n+1}|^2 + \|\nabla e_\eta^n\|^2) + C\tau^3. \end{aligned}$$

Finally, combining the above equations, we get (4.46). □

Theorem 4.6. By virtue of (4.6), we obtain

$$\begin{aligned} &\|e_\sigma^N\|^2 + \|\sigma^N \hat{e}_\mathbf{w}^N\|^2 + \|\sigma^N e_\mathbf{w}^N\|^2 + \lambda \|\nabla e_\eta^N\|^2 + \mu\tau \|s^N\|^2 + |e_Q^N|^2 + \mu\tau \sum_{n=0}^{N-1} \|\nabla \hat{e}_\mathbf{w}^{n+1}\|^2 \\ &\quad + \mu\tau \sum_{n=0}^{N-1} \|\nabla e_\mathbf{w}^{n+1}\|^2 + \sum_{n=0}^{N-1} \left(\left\| \frac{1}{\sigma^{n+1}} \nabla \phi^{n+1} \right\|^2 + \gamma\lambda\tau \|\Delta e_\eta^{n+1}\|^2 \right) \\ &\leq C\tau^2, \quad \forall 1 \leq N \leq T/\tau. \end{aligned} \tag{4.51}$$

Proof. According to

$$\begin{aligned} &2\tau (B_{64}^{n+1} + B_{74}^{n+1} + B_{83}^{n+1}, \hat{e}_\mathbf{w}^{n+1}) - 2\lambda\tau (J_3^{n+1}, \Delta e_\eta^{n+1}) \\ &\quad + 2\tau e_Q^{n+1} (M_{14}^{n+1} + M_{24}^{n+1} + M_{33}^{n+1} + M_{43}^{n+1}) = 0, \end{aligned}$$

and using the above Lemmas 4.2–4.5 lead to

$$\begin{aligned} &\|e_\sigma^{n+1}\|^2 - \|e_\sigma^n\|^2 + \|\sigma^{n+1} e_\mathbf{w}^{n+1}\|^2 - \|\sigma^n e_\mathbf{w}^n\|^2 + \lambda \|\nabla e_\eta^{n+1}\|^2 - \lambda \|\nabla e_\eta^n\|^2 + |e_Q^{n+1}|^2 - |e_Q^n|^2 \\ &\quad + \mu\tau (|s^{n+1}|^2 - |s^n|^2) + \mu\tau \|\nabla \hat{e}_\mathbf{w}^{n+1}\|^2 + \gamma\lambda\tau \|\Delta e_\eta^{n+1}\|^2 + \left\| \frac{1}{\sigma^{n+1}} \nabla \phi^{n+1} \right\|^2 \\ &\leq C\tau (|e_\sigma^{n+1}|^2 + \|e_\mathbf{w}^n\|^2 + \|\nabla e_\eta^n\|^2 + \|\Delta e_\eta^n\|^2 + |e_Q^{n+1}|^2 + \|e_Q^{n+1}\|^2) \\ &\quad + C\tau (\|e_\mathbf{w}^n\|^2 \|e_\mathbf{w}^n\|_1^2 + \|\mathbf{w}^n\|_1 \|\Delta e_\eta^n\|^2) + C\tau^3. \end{aligned} \tag{4.52}$$

We observe from (2.8) that $\mathbf{w}^{n+1} = P_{\mathbf{H}} \widehat{\mathbf{w}}^{n+1}$. Hence, (2.8) implies that $\|\mathbf{w}^{n+1}\|_1 \leq C \|\widehat{\mathbf{w}}^{n+1}\|_1$. By summing (4.52) from $n = 0$ to $n = N - 1$, then applying the Lemma 2.1 to (4.52), yields

$$\begin{aligned} & \|e_\sigma^N\|^2 + \|\sigma^N e_{\mathbf{w}}^N\|^2 + \lambda \|\nabla e_\eta^N\|^2 + \mu\tau \|s^N\|^2 + |e_Q^N|^2 + \sum_{n=0}^{N-1} \left(\left\| \frac{1}{\sigma^{n+1}} \nabla \phi^{n+1} \right\|^2 \right. \\ & \quad \left. + \mu\tau \|\nabla \hat{e}_{\mathbf{w}}^{n+1}\|^2 + \gamma\lambda\tau \|\Delta e_\eta^{n+1}\|^2 \right) \\ & \leq C\tau^2, \end{aligned} \quad (4.53)$$

for all $0 \leq N \leq T/\tau$. Finally, utilizing (4.4) and (4.5), we obtain

$$\|\sigma^N \hat{e}_{\mathbf{w}}^N\|^2 + \mu\tau \sum_{n=0}^{N-1} \|\nabla e_{\mathbf{w}}^{n+1}\|^2 \leq C\tau^2, \quad \forall 1 \leq N \leq T/\tau, \quad (4.54)$$

which completes the proof. $\square$

5. Numerical experiments

This section validates the fully decoupled first-order scheme (3.7) for the variable-density Ericksen–Leslie system through numerical experiments demonstrating its accuracy and efficiency. The finite element spaces $(P_1, P_1, P_2, P_1, P_1)$ are adopted for $(\sigma, \varrho, \mathbf{w}, \hat{p}, \eta)$ throughout.

5.1. Temporal convergence

The computational domain Ω is chosen to be the unit circle in the following simulation:

$$\Omega = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}.$$

A uniform triangular mesh with $M = 50$ nodes is employed on the unit circle, generated using the mesh generator in FreeFem++ [43]. The final time is set to $T = 1$. The required initial conditions for the system are specified as follows

$$\begin{cases} \sigma(x, y, 0) = \sqrt{2+x}, \\ \mathbf{w}(x, y, 0) = (-y, x)^T, \\ \eta(x, y, 0) = 2\pi(\cos(x) - \sin(y)). \end{cases} \quad (5.1)$$

A decreasing time step size $\tau = 1/2^i$ for $i = 5, \dots, 9$ is adopted. The spatial grid is refined enough that the spatial error is negligible relative to the temporal error.

In the first numerical example, we report the errors and temporal convergence rates $O(\tau)$ of Algorithm 1 under various parameter choices. Accordingly, three distinct sets of physical parameter values are considered as follows

$$\begin{aligned} (F1): & \mu = 0.1, \gamma = 0.1, \lambda = 0.1, \\ (F2): & \mu = 0.05, \gamma = 0.05, \lambda = 0.05, \\ (F3): & \mu = 0.01, \gamma = 0.01, \lambda = 0.01. \end{aligned}$$

We report the L^2 errors and temporal convergence orders of $(\sigma, \varrho, \mathbf{w}, \hat{p}, p)$ under parameter conditions $F1$, $F2$, and $F3$ in Tables 2, 4, and 6, respectively. The H^1 errors and temporal convergence orders of $(\sigma, \mathbf{w}, \eta)$ under $F1$, $F2$, and $F3$ are shown in Tables 3, 5, and 7, respectively. It can be seen from Tables 2–7 that all unknowns converge at $O(\tau)$, in agreement with the theoretical order of the proposed first-order scheme. Higher accuracy is also observed under $F3$ compared to the other two conditions.

Table 2. Errors and temporal convergence rates of the scheme in L^2 with $F1$.

τ	$\ e_\sigma\ _{L^2}$	Rate	$\ e_\varrho\ _{L^2}$	Rate	$\ e_{\mathbf{w}}\ _{L^2}$	Rate	$\ e_{\hat{p}}\ _{L^2}$	Rate	$\ e_p\ _{L^2}$	Rate
2^{-5}	4.99e-03	—	1.38e-02	—	1.06e-02	—	3.99e-03	—	1.78e-01	—
2^{-6}	2.50e-03	1.00	6.90e-03	1.00	5.47e-03	0.96	2.38e-03	0.75	1.37e-01	0.38
2^{-7}	1.25e-03	1.00	3.45e-03	1.00	2.73e-03	1.00	1.66e-03	0.52	1.00e-01	0.45
2^{-8}	6.24e-04	1.00	1.73e-03	1.00	1.30e-03	1.06	1.27e-03	0.39	6.91e-02	0.54
2^{-9}	3.19e-04	0.97	8.83e-04	0.97	6.16e-04	1.08	9.38e-04	0.43	4.49e-02	0.62

Table 3. Errors and temporal convergence rates in H^1 with $F1$.

τ	$\ e_\sigma\ _{H^1}$	Rate	$\ e_{\mathbf{w}}\ _{H^1}$	Rate	$\ e_\eta\ _{H^1}$	Rate
2^{-5}	1.14e-02	—	2.22e-01	—	5.01e-01	—
2^{-6}	5.69e-03	1.00	1.33e-01	0.74	3.82e-01	0.39
2^{-7}	2.84e-03	1.00	7.51e-02	0.82	2.80e-01	0.45
2^{-8}	1.42e-03	1.00	3.85e-02	0.96	1.91e-01	0.55
2^{-9}	7.27e-04	0.97	1.78e-02	1.11	1.22e-01	0.65

Table 4. Errors and temporal convergence rates in L^2 with $F2$.

τ	$\ e_\sigma\ _{L^2}$	Rate	$\ e_\varrho\ _{L^2}$	Rate	$\ e_{\mathbf{w}}\ _{L^2}$	Rate	$\ e_{\hat{p}}\ _{L^2}$	Rate	$\ e_p\ _{L^2}$	Rate
2^{-5}	4.99e-03	—	1.38e-02	—	8.55e-03	—	2.80e-03	—	6.86e-02	—
2^{-6}	2.50e-03	1.00	6.90e-03	1.00	4.27e-03	1.00	1.53e-03	0.87	5.04e-02	0.45
2^{-7}	1.25e-03	1.00	3.45e-03	1.00	2.10e-03	1.02	9.04e-04	0.76	3.47e-02	0.54
2^{-8}	6.24e-04	1.00	1.73e-03	1.00	1.02e-03	1.04	5.54e-04	0.71	2.20e-02	0.66
2^{-9}	3.19e-04	0.97	8.83e-04	0.97	5.07e-04	1.01	3.42e-04	0.70	1.32e-02	0.74

Table 5. Errors and temporal convergence rates in H^1 with $F2$.

τ	$\ e_\sigma\ _{H^1}$	Rate	$\ e_{\mathbf{w}}\ _{H^1}$	Rate	$\ e_\eta\ _{H^1}$	Rate
2^{-5}	1.14e-02	—	1.56e-01	—	4.21e-01	—
2^{-6}	5.69e-03	1.00	8.58e-02	0.87	2.94e-01	0.52
2^{-7}	2.84e-03	1.00	4.37e-02	0.97	1.96e-01	0.59
2^{-8}	1.42e-03	1.00	2.02e-02	1.11	1.21e-01	0.70
2^{-9}	7.27e-04	0.97	8.81e-03	1.20	7.11e-02	0.77

Table 6. Errors and temporal convergence rates in L^2 with $F3$.

τ	$\ e_\sigma\ _{L^2}$	Rate	$\ e_\rho\ _{L^2}$	Rate	$\ e_w\ _{L^2}$	Rate	$\ e_{\hat{p}}\ _{L^2}$	Rate	$\ e_p\ _{L^2}$	Rate
2^{-5}	4.99e-03	—	1.38e-02	—	7.63e-03	—	2.32e-03	—	7.19e-03	—
2^{-6}	2.50e-03	1.00	6.90e-03	1.00	3.81e-03	1.00	1.16e-03	1.00	4.28e-03	0.75
2^{-7}	1.25e-03	1.00	3.45e-03	1.00	1.90e-03	1.00	5.79e-04	1.00	2.40e-03	0.84
2^{-8}	6.24e-04	1.00	1.73e-03	1.00	9.51e-04	1.06	2.91e-04	1.00	1.28e-03	0.91
2^{-9}	3.19e-04	0.97	8.83e-04	0.97	4.86e-04	0.97	1.52e-04	0.93	6.77e-04	0.92

Table 7. Errors and temporal convergence rates in H^1 with $F3$.

τ	$\ e_\sigma\ _{H^1}$	Rate	$\ e_w\ _{H^1}$	Rate	$\ e_\eta\ _{H^1}$	Rate
2^{-5}	1.14e-02	—	9.70e-02	—	2.67e-01	—
2^{-6}	5.69e-03	1.00	4.78e-02	1.02	1.46e-01	0.87
2^{-7}	2.84e-03	1.00	2.35e-02	1.03	7.75e-02	0.91
2^{-8}	1.42e-03	1.00	1.16e-02	1.02	4.02e-02	0.95
2^{-9}	7.27e-04	0.97	5.89e-03	0.98	2.10e-02	0.94

Note that for the recovered pressure and the H^1 -error of η , the observed convergence rates are slightly below 1.0 under some parameter sets. This is attributed to:

- (i) the fixed spatial grid ($M = 50$) may not fully eliminate spatial errors as τ becomes extremely small, leading to a pre-asymptotic regime;
- (ii) the H^1 -norm strongly amplifies numerical oscillations in η .

Nevertheless, the L^2 -errors of the primary variables (σ, w, η) converge at $O(\tau)$, which is consistent with the theoretical first-order prediction.

Using parameter condition $F3$ in the second example, we plot the error against the time step size. Figure 1 displays the L^2 errors for (σ, w, η) and the H^1 errors for $(\sigma, w, \hat{p}, p)$ as the time step varies. Evidently, the errors drop to zero as the time step shrinks.

In the third example, we choose the parameters $\mu = 1, \gamma = 1, \lambda = 1$, and then plot snapshots of the director field increasing with time. Figure 2 shows snapshots of the director field at moments $t = 0, 0.1, 0.5$, and 0.8 , respectively. The singularities are whirled around with the flow at $t = 0$. The singularities clearly continue to move with the flow at $t = 0.1$. After $t = 0.5$, the change trend begins to move in one direction.

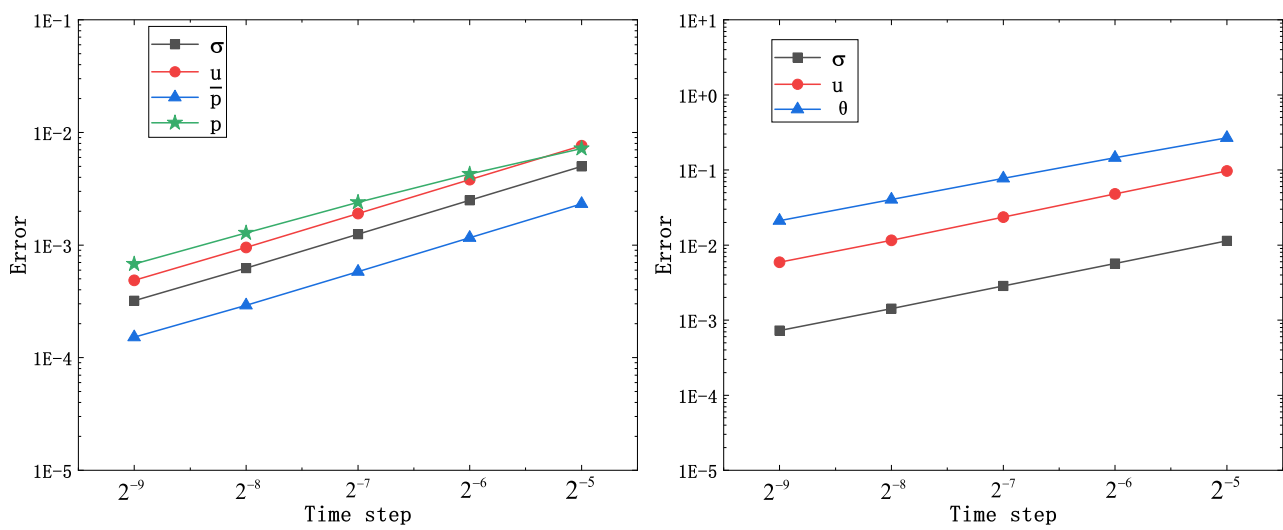


Figure 1. Errors at different time steps with $F3$: L^2 errors (left), H^1 errors (right).



Figure 2. Snapshots of the orientation field: $T = 0$ (top left), $T = 0.1$ (top right), $T = 0.5$ (bottom left), $T = 0.8$ (bottom right).

5.2. Spatial convergence

In this subsection, we give the errors and spatial convergence orders of the modified pressure $\hat{p}$ and the original pressure p under different parameter conditions $F1$, $F2$, as shown in Tables 8 and 9. The computational domain, initial conditions, and final time are the same as in the previous subsection. We choose the spatial grid nodes $M = 8, 16, 32, 64$ and the time step $\tau = 0.0001$.

Table 8. Errors and spatial convergence rates with $F1$.

M	$\ e_{\hat{p}}\ _{L^2}$	Rate	$\ e_p\ _{L^2}$	Rate
8	3.15e-01	—	1.18e-00	—
16	8.74e-02	1.85	5.59e-01	1.08
32	4.56e-02	0.94	3.11e-01	0.85
64	3.18e-02	0.52	1.22e-01	1.34

Table 9. Errors and spatial convergence rates with $F2$.

M	$\ e_{\hat{p}}\ _{L^2}$	Rate	$\ e_p\ _{L^2}$	Rate
8	3.14e-01	—	7.12e-01	—
16	8.48e-02	1.89	2.75e-01	1.37
32	2.95e-02	1.53	1.59e-01	0.79
64	1.66e-02	0.83	6.07e-02	1.39

5.3. Evolutions of energy and behavior of Q^n

To verify the numerical performance of the fully decoupled scheme (Algorithm 2), we plot the energy curves and the evolution of Q^n for different time steps. The numerical experiments are conducted with parameters $T = 1$, $\mu = 0.1$, $\lambda = 0.1$, $\gamma = 0.1$, and a spatial mesh with 50×50 uniform divisions. As shown in the Figures 3–5, for all considered time steps, the energy curves exhibit a smooth monotonic decay over time, which confirms the energy stability of the proposed scheme. Moreover, throughout the entire time evolution, Q^n fluctuates negligibly around 1, strictly maintaining the discrete constraint condition as theoretically expected. These observations collectively confirm the high accuracy and practical feasibility of the proposed method for simulating the variable-density Ericksen–Leslie system.

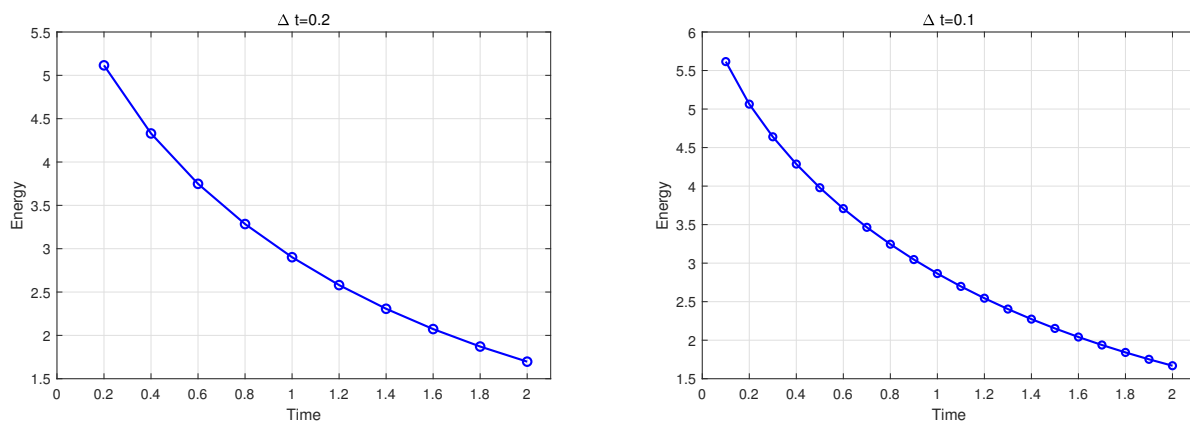


Figure 3. Energy decay for time steps $\Delta t = 0.2$ and $\Delta t = 0.1$.

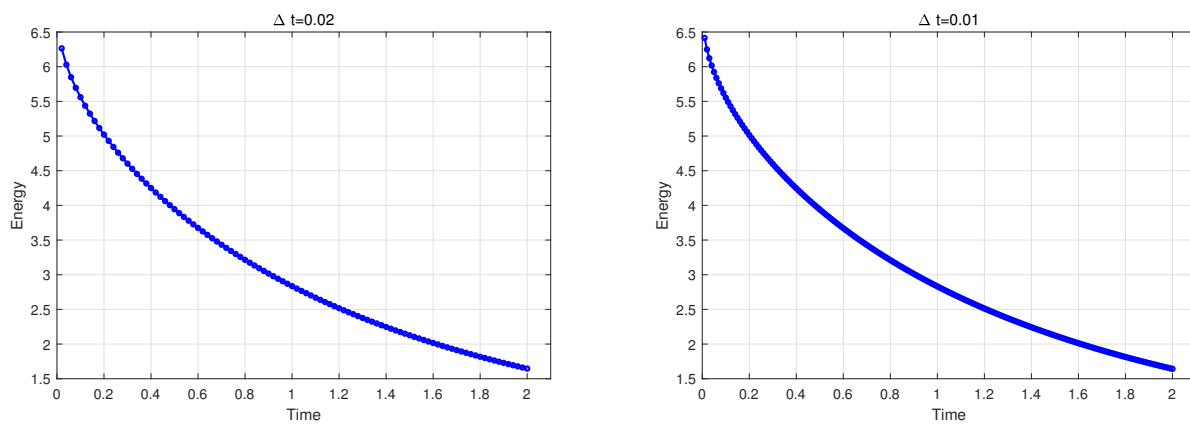


Figure 4. Energy decay for time steps $\Delta t = 0.02$ and $\Delta t = 0.01$.

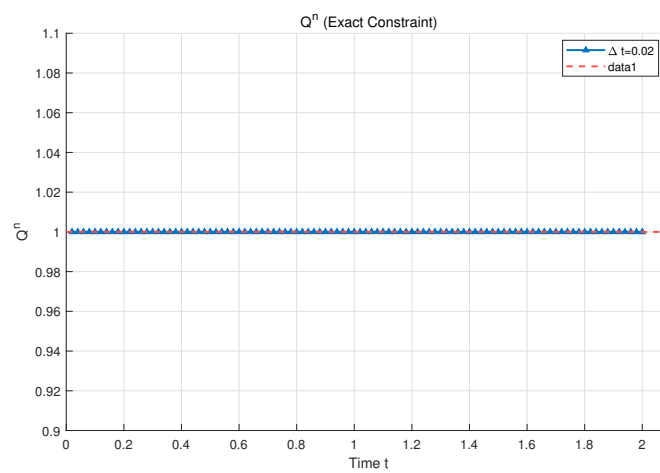


Figure 5. Behavior of the constraint multiplier Q^n .

5.4. Efficiency comparison with a standard coupled solver

To highlight the main advantage of the proposed algorithm—reducing the complex nonlinear system into a sequence of decoupled, linear symmetric positive definite (SPD) subproblems at each time step—we compare its CPU time against a standard coupled Picard iteration. Both methods use the same mesh, finite element spaces (P2/P1), linear solver (GMRES), and tolerance (10^{-6}). Table 10 reports the CPU times and speedups (coupled/decoupled) for different mesh sizes.

Table 10. Efficiency comparison with a standard coupled solver.

M (boundary points)	Elements	Decoupled (s)	Coupled (s)	Speedup
8	10	0.028	0.033	1.18
16	42	0.033	0.098	2.97
32	186	0.077	0.503	6.53
64	718	0.292	2.528	8.66

The speedup increases with mesh refinement, confirming that the decoupled approach becomes increasingly advantageous for larger problems, as it avoids solving a large saddle-point system and instead solves several small SPD systems per time step.

6. Conclusions

A fully decoupled first-order numerical scheme is developed herein for the variable-density Ericksen–Leslie system using the Gauge–Uzawa and SAV methods. Unconditional energy stability is proved, along with rigorous error estimates for the time-semidiscrete scheme. Numerical experiments further validate the method’s accuracy and effectiveness. To the best of our knowledge, no prior work has offered both a fully decoupled scheme and its error analysis for the variable-density Ericksen–Leslie system—two contributions that this paper makes.

Author contributions

Jiaoyan Hou: Writing – original draft. Zhaowei Wang: Validation, Investigation, Formal analysis. Danxia Wang: Supervision, Writing, editing, Funding acquisition.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

All authors have no conflicts of interest.

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