



Research article

The sharp bounds of the third-order Hankel determinant of inverse functions for certain univalent functions

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Abstract: In this paper, we obtain the sharp bounds of third Hankel determinants $|\mathcal{H}_{3,1}(\kappa)|$ of inverse functions for $\kappa \in \mathcal{S}_e^*$, $\kappa \in \mathcal{C}_e$, $\kappa \in \mathcal{S}_{\mathcal{L}}^*$, and $\kappa \in \mathcal{C}_{\mathcal{L}}$, which elevates the function categories of the existing results to the inverse function category. In addition, we extend the second-order Hankel determinant of the logarithmic coefficient of the function class $\kappa \in \mathcal{S}_e^*$ to the class $\mathcal{S}_{\mathcal{L}}^*$ and obtain precise results. Most notably, we improve the third-order Hankel determinant of the function class $\mathcal{C}_{car}$ from $\frac{319}{4374}$ to $\frac{1}{81}$, which resolves the conjecture posed in [10].

Keywords: Hankel determinants; starlike functions; convex functions; inverse functions; cardioid domain; exponential function

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1. Introduction

The set of all complex analytic functions, normalized by the conditions $\kappa(0) = 0$ and $\kappa'(0) = 1$, in the unit disk $\Xi = \{\varepsilon \in \mathbb{C} : |\varepsilon| < 1\}$, whose Taylor series expansions of the form

$$\kappa(\varepsilon) = \varepsilon + \sum_{j=2}^{\infty} \kappa_j \varepsilon^j, \quad (1.1)$$

is denoted by $\mathcal{B}$. Let $\mathcal{S}$ be a subclass of $\mathcal{B}$ consisting of univalent functions.

There are some very common subfamilies of $\mathcal{S}$, some of them are class of convex and starlike

functions, denoted by C and S^* , respectively:

$$C = \left\{ \kappa \in \mathcal{B} : \operatorname{Re} \left(1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} \right) > 0, \quad \varepsilon \in \Xi \right\},$$

$$S^* = \left\{ \kappa \in \mathcal{B} : \operatorname{Re} \left(\frac{\varepsilon \kappa'(\varepsilon)}{\kappa(\varepsilon)} \right) > 0, \quad \varepsilon \in \Xi \right\}.$$

The concept of Hankel determinants, introduced in 1966 (see [1]), continues to be a topic of significant interest for researchers today. The definition of the q^{th} Hankel determinant $\mathcal{H}_{q,j}(\kappa)$ of analytic functions $\kappa \in \mathcal{B}$ is as follows:

$$\mathcal{H}_{q,j}(\kappa) = \begin{vmatrix} \kappa_j & \kappa_{j+1} & \cdots & \kappa_{j+q-1} \\ \kappa_{j+1} & \kappa_{j+2} & \cdots & \kappa_{j+q} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{j+q-1} & \kappa_{j+q} & \cdots & \kappa_{j+2q-2} \end{vmatrix} \quad (j, q \in \mathbb{N}).$$

The expression for the third-order Hankel determinant for the specific values $j = 1$ and $q = 3$ is denoted by $\mathcal{H}_{3,1}(\kappa)$, and is given by

$$\mathcal{H}_{3,1}(\kappa) = 2\kappa_4\kappa_3\kappa_2 - \kappa_3^3 + \kappa_5\kappa_3 - \kappa_5\kappa_2^2 - \kappa_4^2. \quad (1.2)$$

It is extremely difficult to derive a sharp bound for Hankel determinants, which is why numerous researchers have tried to do so for different subclasses of analytic functions. For example, Sun and Kuang [2] obtained the third Hankel determinants for Ozaki close-to-convex functions, Raza et al. [3] derived sharp upper bounds of $|\mathcal{H}_{2,2}(\kappa)|$ and $|\mathcal{H}_{3,1}(\kappa)|$ for starlike functions associated with a cardioid domain, Raza et al. [4] investigated the third Hankel determinant for starlike and convex functions associated with the exponential function. Additionally, upper bounds for the third Hankel determinant for various subclasses of $\mathcal{S}$ were obtained by Kumar and Verma [5], Verma and Kumar [6], Kumar et al. [7], Shi and Arif [8].

For $\kappa \in \mathcal{S}$, κ has an inverse κ^{-1} given by

$$\begin{aligned} \kappa^{-1}(w) &= w + \sum_{m=2}^{\infty} C_m w^m \quad |w| < r_0(\kappa) \quad (r_0(\kappa) \geq \frac{1}{4}); \\ &= w - \kappa_2 w^2 + (2\kappa_2^2 - \kappa_3) w^3 - (5\kappa_2^3 - 5\kappa_2\kappa_3 + \kappa_4) w^4 \\ &\quad + (14\kappa_2^4 + 3\kappa_3^2 - 21\kappa_2^2\kappa_3 + 6\kappa_2\kappa_4 - \kappa_5) w^5 + \cdots. \end{aligned} \quad (1.3)$$

From (1.3) and (1.2), we get

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= 2C_4C_3C_2 - C_3^3 + C_5C_3 - C_5C_2^2 - C_4^2 \\ &= -3\kappa_2^4\kappa_3 + \kappa_2^6 + 2\kappa_2\kappa_3\kappa_4 + 3\kappa_2^2\kappa_3^2 - 2\kappa_3^3 - \kappa_2^2\kappa_5 - \kappa_4^2 + \kappa_3\kappa_5. \end{aligned} \quad (1.4)$$

Let $\kappa \in \mathcal{S}$. The logarithmic coefficients η_j of κ are defined by

$$\log \frac{\kappa(\varepsilon)}{\varepsilon} = 2 \sum_{j=1}^{\infty} \eta_j \varepsilon^j. \quad (1.5)$$

Differentiating (1.5), we yield

$$\begin{cases} \eta_1 = \frac{1}{2}\kappa_2, \\ \eta_2 = \frac{1}{4}(2\kappa_3 - \kappa_2^2), \\ \eta_3 = \frac{1}{6}(3\kappa_4 + \kappa_2^3 - 3\kappa_2\kappa_3), \\ \eta_4 = \frac{1}{8}(4\kappa_5 - 2\kappa_3^2 + 4\kappa_3\kappa_2^2 - 4\kappa_4\kappa_2 - \kappa_2^4). \end{cases} \quad (1.6)$$

Therefore, we have

$$\eta_3\eta_1 - \eta_2^2 = \frac{1}{48}(-12\kappa_3^2 + 12\kappa_2\kappa_4 + \kappa_2^4), \quad (1.7)$$

$$\begin{aligned} \eta_2\eta_4 - \eta_3^2 &= \frac{1}{288}(72\kappa_3\kappa_5 - 6\kappa_2^4\kappa_3 + 72\kappa_2\kappa_3\kappa_4 + 18\kappa_2^2\kappa_3^2 - 36\kappa_3^3 \\ &\quad - 36\kappa_2^2\kappa_5 - 72\kappa_4^2 - 12\kappa_2^3\kappa_4 + \kappa_2^6). \end{aligned} \quad (1.8)$$

Ma and Minda (1992) also utilized this concept and studied the general form of classes $C(\Psi)$ and $\mathcal{S}^*(\Psi)$ as:

$$C(\Psi) = \left\{ \kappa \in \mathcal{B} \quad \text{and} \quad 1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} < \Psi(\varepsilon), \quad \varepsilon \in \Xi \right\},$$

$$\mathcal{S}^*(\Psi) = \left\{ \kappa \in \mathcal{B} \quad \text{and} \quad \frac{\varepsilon \kappa'(\varepsilon)}{\kappa(\varepsilon)} < \Psi(\varepsilon), \quad \varepsilon \in \Xi \right\}.$$

Here, $\Psi(\varepsilon)$ is a univalent analytic function with positive real part in the unit disk Ξ , such that $\Psi(0) = 1$ and $\Psi'(0) > 0$, where Ψ maps the open unit disk onto a region starlike with respect to 1 and symmetric with respect to the real axis.

In recent years, a number of researchers have introduced significant and fascinating function classes by assigning different values to $\Psi(\varepsilon)$. Here, we highlight some of these developments:

(a) For $\Psi(\varepsilon) = \sqrt{1 + \sin(\sin \varepsilon)}$, the class $\mathcal{R}_{apl}$ was defined and investigated by Ahmada and his coauthors [9].

(b) By setting $\Psi(\varepsilon) = 1 + \frac{4}{3}\varepsilon + \frac{2}{3}\varepsilon^2$, the family $C(\Psi)$ reduces to the family C_{car} introduced by Shi and his coauthors [10] given as

$$C_{car} = \left\{ \kappa \in \mathcal{B} : \quad 1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} < 1 + \frac{4}{3}\varepsilon + \frac{2}{3}\varepsilon^2, \quad \varepsilon \in \Xi \right\}. \quad (1.9)$$

(c) For $\Psi(\varepsilon) = e^\varepsilon$, the class $\mathcal{S}^*(\Psi)$ becomes $\mathcal{S}_e^*$ defined and examined by Mendiratta et al. [11] given as:

$$\mathcal{S}_e^* = \left\{ \kappa \in \mathcal{B} : \quad \frac{\varepsilon \kappa'(\varepsilon)}{\kappa(\varepsilon)} < e^\varepsilon, \quad \varepsilon \in \Xi \right\}. \quad (1.10)$$

(d) The family $C_e = C(e^\varepsilon)$ was introduced and studied by Shi et al. [12] given as:

$$C_e = \left\{ \kappa \in \mathcal{B} : 1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} < e^\varepsilon, \quad \varepsilon \in \Xi \right\}. \quad (1.11)$$

(e) Setting $\Psi(\varepsilon) = \varepsilon + \sqrt{1 + \varepsilon^2}$, the class $\mathcal{S}^*(\Psi)$ becomes $\mathcal{S}_\mathcal{L}^*$ introduced by Raina and Sokol [13] given as:

$$\mathcal{S}_\mathcal{L}^* = \left\{ \kappa \in \mathcal{B} : \frac{\varepsilon \kappa'(\varepsilon)}{\kappa(\varepsilon)} < \varepsilon + \sqrt{1 + \varepsilon^2}, \quad \varepsilon \in \Xi \right\}. \quad (1.12)$$

The class $\kappa \in C_\mathcal{L}$ of convex function is defined as

$$C_\mathcal{L} = \left\{ \kappa \in \mathcal{B} : 1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} < \varepsilon + \sqrt{1 + \varepsilon^2}, \quad \varepsilon \in \Xi \right\}. \quad (1.13)$$

The authors of [10] have obtained a non-sharp bound of $\mathcal{H}_{3,1}(\kappa)$ for C_{car} . They have proved that $|\mathcal{H}_{3,1}(\kappa)| \leq \frac{319}{4374}$ for $\kappa \in C_{car}$. It was conjectured that $|\mathcal{H}_{3,1}(\kappa)| \leq \frac{1}{81}$. Shi et al. [12] have obtained the first three sharp coefficients bounds, and gave a non-sharp bounds of $|\mathcal{H}_{3,1}(\kappa)|$ for $\kappa \in \mathcal{S}^*(e)$ and $\kappa \in C_e$. Recently, Raza et al. [4] obtained the sharp upper bounds of $|\mathcal{H}_{3,1}(\kappa)|$ for the classes $\kappa \in \mathcal{S}_e^*$ and $\kappa \in C_e$, and showed that $|\mathcal{H}_{3,1}(\kappa)| \leq \frac{1}{9}$, for $\kappa \in \mathcal{S}_e^*$ and $|\mathcal{H}_{3,1}(\kappa)| \leq \frac{1}{144}$ for $\kappa \in C_e$. For the class $\mathcal{S}_e^*$, Shi et al [14] achieved the sharp bounds of Hankel determinant of logarithmic coefficients, and Naz et al. [15] obtained the third Hankel determinant of inverse coefficient. In [16], Riaz and Raza established results concerning the third Hankel determinant for $\mathcal{S}_\mathcal{L}^*$ and $\kappa \in C_\mathcal{L}$.

Unlike existing estimation methods for Hankel determinants, this paper shifts the estimation objective from the original function itself to its inverse function, and provides sharp bounds for third-order determinants simultaneously for Ma–Minda subclasses including exponential kernel, cardioid domain, and lune domain, which differs from earlier literature that only derives non-sharp bounds or deals exclusively with original functions. In addition, this work also addresses the second-order Hankel determinant of logarithmic coefficients, resulting in a more comprehensive coverage than previous conventional methods. The non-linearity of inverse function coefficients transforms the “single-variable coefficient bounding” problem into “multivariate polynomial sharp bound optimization”, which constitutes the core difficulty that distinguishes this approach from most existing studies.

The current research on the third-order Hankel determinant of inverse functions mainly focuses on estimations for the classical categories such as $\mathcal{S}^*$ and C , etc [17–20]. Further research is needed on subclasses with more complex geometric structures. Early studies mostly employed simple coefficient scaling or univariate optimization, which struggled to handle the coupling effects of high-order cross terms in inverse function coefficients, resulting in only non-sharp bounds. This paper introduces a multi-parameter expansion of the Schwarz function and the construction of extremal functions, and through refined multivariate extremum analysis, successfully overcomes the constraints imposed by cross terms for the first time, accurately determining sharp constants. This highlights the novelty and superiority of the extremal method in decoupling complex coefficient structures. Furthermore, this paper extend the exact bounds of the third-order Hankel determinant of the inverse function to fine subclasses such as exponential, heart-shaped, and crescent-shaped.

In Section 2, we obtain the sharp bounds of third Hankel determinants $|\mathcal{H}_{3,1}(\kappa)|$ of inverse functions for $\kappa \in \mathcal{S}_e^*$, $\kappa \in C_e$, $\kappa \in \mathcal{S}_\mathcal{L}^*$, and $\kappa \in C_\mathcal{L}$, which provide a completely new perspective. Although the

precise constant is the same as the existing result, we have elevated the function category to the inverse function category. In Section 3, we extend the second-order Hankel determinant of the logarithmic coefficient of the function class $\kappa \in \mathcal{S}_e^*$ to the class $\mathcal{S}_{\mathcal{L}}^*$ and obtain precise results. In Section 4, we improve the third-order Hankel determinant of the function class C_{car} from $\frac{319}{4374}$ to $\frac{1}{81}$, which resolve the existing conjecture in [10].

Lemma 1.1. ([21]) *Let*

$$\omega(\varepsilon) = \rho_1 \varepsilon + \rho_2 \varepsilon^2 + \rho_3 \varepsilon^3 + \rho_4 \varepsilon^4 + \cdots \quad (1.14)$$

be a Schwarz function. Then,

$$\begin{aligned} \rho_2 &= (1 - |\rho_1|^2)\zeta_1, \\ \rho_3 &= -\bar{\rho}_1 \zeta_1^2 (1 - |\rho_1|^2) + (1 - |\rho_1|^2)(1 - |\zeta_1|^2)\zeta_2, \\ \rho_4 &= \bar{\rho}_1^2 \zeta_1^3 (1 - |\rho_1|^2) - (1 - |\rho_1|^2)(1 - |\zeta_1|^2)(\bar{\zeta}_1 \zeta_2^2 + 2\zeta_2 \bar{\rho}_1 \zeta_1) \\ &\quad + (1 - |\rho_1|^2)(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3 \end{aligned}$$

for some $\zeta_i \in \mathbb{C}$ with $|\zeta_i| \leq 1$ ($i = 1, 2, 3$).

2. The third Hankel determinants of inverse functions for $\mathcal{S}_e^*$, C_e , $\mathcal{S}_{\mathcal{L}}^*$, and $C_{\mathcal{L}}$

Theorem 2.1. *If $\kappa \in \mathcal{S}_e^*$, then*

$$|\mathcal{H}_{3,1}(\kappa^{-1})| \leq \frac{1}{8}.$$

The result is sharp for

$$\kappa_1(\varepsilon) = \varepsilon \exp\left(\int_0^\varepsilon \frac{e^{t^2} - 1}{t} dt\right) = \varepsilon + \frac{1}{2}\varepsilon^3 + \cdots.$$

And the corresponding Schwarz function is $\omega(\varepsilon) = \varepsilon^2$. The image domain of extreme value function is an infinitely spiraling strip-like convex domain generated by the exponential map and symmetric about the real axis.

Proof: Let $\kappa \in \mathcal{S}_e^*$. Then by (1.10), we have

$$\frac{\varepsilon \kappa'(\varepsilon)}{\kappa(\varepsilon)} = e^{\omega(\varepsilon)}, \quad (2.1)$$

where $\omega(\varepsilon)$ is known as Schwarz function.

Utilizing (1.1), we derive

$$\begin{aligned} \frac{\varepsilon \kappa'(\varepsilon)}{\kappa(\varepsilon)} &= 1 + \kappa_2 \varepsilon + (2\kappa_3 - \kappa_2^2)\varepsilon^2 + (\kappa_2^3 - 3\kappa_2 \kappa_3 + 3\kappa_4)\varepsilon^3 \\ &\quad + (-4\kappa_2 \kappa_4 + 4\kappa_2^2 \kappa_3 - \kappa_2^4 - 2\kappa_3^2 + 4\kappa_5)\varepsilon^4 + \cdots \end{aligned} \quad (2.2)$$

Using series form of $\omega(\varepsilon)$ in right hand side of (2.1), we achieve

$$e^{\omega(\varepsilon)} = 1 + \rho_1 \varepsilon + \frac{1}{2}(2\rho_2 + \rho_1^2)\varepsilon^2 + \frac{1}{6}(\rho_1^3 + 6\rho_1 \rho_2 + 6\rho_3)\varepsilon^3$$

$$+ \frac{1}{24}(24\rho_4 + 12\rho_2^2 + 24\rho_1\rho_3 + 12\rho_1^2\rho_2 + \rho_1^4)\varepsilon^4 + \cdots. \quad (2.3)$$

It follows from (2.2) and (2.3), that

$$\kappa_2 = \rho_1, \quad (2.4)$$

$$\kappa_3 = \frac{1}{4}(2\rho_2 + 3\rho_1^2), \quad (2.5)$$

$$\kappa_4 = \frac{1}{36}(17\rho_1^3 + 30\rho_1\rho_2 + 12\rho_3), \quad (2.6)$$

$$\kappa_5 = \frac{1}{72}(18\rho_4 + 42\rho_1\rho_3 + 60\rho_1^2\rho_2 + 18\rho_2^2 + 19\rho_1^4). \quad (2.7)$$

By (1.4) and using the identities (2.4)–(2.7), we achieve

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= \frac{1}{2592}(-288\rho_3^2 + 180\rho_1\rho_2\rho_3 + 102\rho_1^3\rho_3 + 34\rho_1^6 - 324\rho_2^3 + 324\rho_2\rho_4 - 204\rho_1^4\rho_2 \\ &\quad + 306\rho_1^2\rho_2^2 - 162\rho_1^2\rho_4) \\ &= \frac{1}{1296}(-144\rho_3^2 + 90\rho_1\rho_2\rho_3 + 51\rho_1^3\rho_3 + 17\rho_1^6 - 162\rho_2^3 + 162\rho_2\rho_4 - 102\rho_1^4\rho_2 \\ &\quad + 153\rho_1^2\rho_2^2 - 81\rho_1^2\rho_4). \end{aligned} \quad (2.8)$$

From Lemma 1.1 and (2.8), we get

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= \frac{1}{1296}[18\bar{\rho}_1^2(1 - |\rho_1|^2)^2\zeta_1^4 - 162|\zeta_1|^2(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2\zeta_2^2 - 90(1 - |\rho_1|^2)^2|\rho_1|^2\zeta_1^3 \\ &\quad - 36\zeta_2(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2)\bar{\rho}_1\zeta_1^2 + 90\zeta_1\rho_1(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2\zeta_2 + 162\zeta_1(1 - |\zeta_2|^2) \\ &\quad \times (1 - |\rho_1|^2)^2(1 - |\zeta_1|^2)\zeta_3 + 17\rho_1^6 + 153\rho_1^2(1 - |\rho_1|^2)^2\zeta_1^2 - 162(1 - |\rho_1|^2)^3\zeta_1^3 - 102 \\ &\quad \times (1 - |\rho_1|^2)\rho_1^4\zeta_1 - 81(1 - |\rho_1|^2)\rho_1^2\bar{\rho}_1^2\zeta_1^3 - 144\zeta_2^2(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2)^2 + 51(1 - |\rho_1|^2) \\ &\quad \times (1 - |\zeta_1|^2)\rho_1^3\zeta_2 - 81(1 - |\rho_1|^2)(1 - |\zeta_2|^2)(1 - |\zeta_1|^2)\rho_1^2\zeta_3 + 81(1 - |\rho_1|^2)(1 - |\zeta_1|^2)\rho_1^2\bar{\zeta}_1\zeta_2^2 \\ &\quad + 162\zeta_1(1 - |\rho_1|^2)(1 - |\zeta_1|^2)\rho_1^2\bar{\rho}_1\zeta_2 - 51(1 - |\rho_1|^2)\rho_1^3\bar{\rho}_1\zeta_1^2]. \end{aligned}$$

Hence, we have

$$\mathcal{H}_{3,1}(\kappa^{-1}) = \frac{1}{1296}[\sigma_1(\rho_1, \zeta_1) + \sigma_2(\rho_1, \zeta_1)\zeta_2 + \sigma_3(\rho_1, \zeta_1)\zeta_2^2 + \Gamma_1(\rho_1, \zeta_1, \zeta_2)\zeta_3],$$

where

$$\begin{aligned} \sigma_1(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)\{[153\rho_1^2\zeta_1^2 - (162 - 72|\rho_1|^2)\zeta_1^3 + 18\bar{\rho}_1^2\zeta_1^4](1 - |\rho_1|^2) - 81\bar{\rho}_1^2\rho_1^2\zeta_1^3 \\ &\quad - 51\bar{\rho}_1\rho_1^3\zeta_1^2 - 102\zeta_1\rho_1^4\} + 17\rho_1^6, \\ \sigma_2(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)(1 - |\zeta_1|^2)[(90\rho_1\zeta_1 - 36\bar{\rho}_1\zeta_1^2)(1 - |\rho_1|^2) + 162\zeta_1\rho_1^2\bar{\rho}_1 + 51\rho_1^3], \end{aligned}$$

$$\begin{aligned}\sigma_3(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)(1 - |\zeta_1|^2)[(-144 - 18|\zeta_1|^2)(1 - |\rho_1|^2) + 81\bar{\zeta}_1\rho_1^2], \\ \Gamma_1(\rho_1, \zeta_1, \zeta_2) &= (1 - |\zeta_2|^2)(1 - |\zeta_1|^2)(1 - |\rho_1|^2)[-81\rho_1^2 + 162(1 - |\rho_1|^2)\zeta_1].\end{aligned}$$

Replacing $|\rho_1| = \rho$, $|\zeta_1| = \zeta$, $|\zeta_2| = \tau$, and utilizing $|\zeta_3| \leq 1$, we obtain

$$\begin{aligned}|\mathcal{H}_{3,1}(\kappa^{-1})| &\leq \frac{1}{1296} (|\sigma_1(\rho_1, \zeta_1)| + |\sigma_2(\rho_1, \zeta_1)|\tau + |\sigma_3(\rho_1, \zeta_1)|\tau^2 + |\Gamma_1(\rho_1, \zeta_1, \zeta_2)|) \\ &\leq \frac{1}{1296} \Upsilon_1(\rho, \zeta, \tau),\end{aligned}$$

where

$$\Upsilon_1(\rho, \zeta, \tau) = \gamma_1(\rho, \zeta) + \gamma_2(\rho, \zeta)\tau + \gamma_3(\rho, \zeta)\tau^2 + \gamma_4(\rho, \zeta)(1 - \tau^2),$$

with

$$\begin{aligned}\gamma_1(\rho, \zeta) &= 17\rho^6 + (1 - \rho^2) \left[(162\zeta^3 - 72\rho^2\zeta^3 + 18\rho^2\zeta^4 + 153\rho^2\zeta^2)(1 - \rho^2) + 102\rho^4\zeta + 51\rho^4\zeta^2 + 81\rho^4\zeta^3 \right], \\ \gamma_2(\rho, \zeta) &= (1 - \rho^2)(1 - \zeta^2) \left[(90\rho\zeta + 36\rho\zeta^2)(1 - \rho^2) + 51\rho^3 + 162\rho^3\zeta \right], \\ \gamma_3(\rho, \zeta) &= (1 - \zeta^2)(1 - \rho^2) [(144 + 18\zeta^2)(1 - \rho^2) + 81\rho^2\zeta], \\ \gamma_4(\rho, \zeta) &= (1 - \zeta^2)(1 - \rho^2) [162\zeta(1 - \rho^2) + 81\rho^2].\end{aligned}$$

It is obvious that

$$\gamma_3(\rho, \zeta) \leq (1 - \rho^2)(1 - \zeta^2) [(18\zeta^2 + 144)(1 - \rho^2) + 81\rho^2] = \psi_1(\rho, \zeta).$$

Therefore, we obtain

$$\Upsilon_1(\rho, \zeta, \tau) \leq \gamma_1(\rho, \zeta) + \gamma_2(\rho, \zeta)\tau + \psi_1(\rho, \zeta)\tau^2 + \gamma_4(\rho, \zeta)(1 - \tau^2) = \xi_1(\rho, \zeta, \tau).$$

Differentiating $\xi_1(\rho, \zeta, \tau)$ with respect to τ , we have

$$\begin{aligned}\frac{\partial \xi_1}{\partial \tau} &= \gamma_2(\rho, \zeta) + 2\tau[\psi_1(\rho, \zeta) - \gamma_4(\rho, \zeta)] \\ &= \gamma_2(\rho, \zeta) + 36\tau(1 - \rho^2)^2(1 - \zeta^2)(\zeta^2 - 9\zeta + 8) \geq 0.\end{aligned}$$

Thus, $\xi_1(\rho, \zeta, \tau)$ is an increasing function of $\tau \in [0, 1]$, and attains its maximum value at $\tau = 1$. Therefore, we get

$$\begin{aligned}\Upsilon_1(\rho, \zeta, \tau) &\leq \xi_1(\rho, \zeta, \tau) \leq \xi_1(\rho, \zeta, 1) = \gamma_1(\rho, \zeta) + \gamma_2(\rho, \zeta) + \psi_1(\rho, \zeta) \\ &= 17\rho^6 - 51\rho^5 + 63\rho^4 + 51\rho^3 - 207\rho^2 + 144 + (18\rho^6 - 36\rho^5 - 54\rho^4 + 72\rho^3 + 54\rho^2 \\ &\quad - 36\rho - 18)\zeta^4 + (-153\rho^6 + 72\rho^5 + 387\rho^4 + 18\rho^3 - 396\rho^2 - 90\rho + 162)\zeta^3 + (102\rho^6 \\ &\quad + 87\rho^5 - 300\rho^4 - 123\rho^3 + 324\rho^2 + 36\rho - 126)\zeta^2 + (-102\rho^6 - 72\rho^5 + 102\rho^4 - 18\rho^3 \\ &\quad + 90\rho)\zeta = \aleph_1(\rho, \zeta).\end{aligned}$$

The critical points of $\aleph_1(\rho, \zeta)$ satisfy the conditions

$$\begin{cases} \frac{\partial \aleph_1}{\partial \rho} = 102\rho^5 - 255\rho^4 + 252\rho^3 + 153\rho^2 - 414\rho + (108\rho^5 - 180\rho^4 - 216\rho^3 + 216\rho^2 + 108\rho - 36)\zeta^4 \\ + (-918\rho^5 + 360\rho^4 + 1548\rho^3 + 54\rho^2 - 792\rho - 90)\zeta^3 + (612\rho^5 + 435\rho^4 - 1200\rho^3 - 369\rho^2 + 648\rho \\ + 36)\zeta^2 + (-612\rho^5 - 360\rho^4 + 408\rho^3 - 54\rho^2 + 90)\zeta = 0, \\ \frac{\partial \aleph_1}{\partial \zeta} = 90\rho - 18\rho^3 + 102\rho^4 - 72\rho^5 - 102\rho^6 + (72\rho^6 - 144\rho^5 - 216\rho^4 + 288\rho^3 + 216\rho^2 - 144\rho - 72)\zeta^3 \\ + (-459\rho^6 + 216\rho^5 + 1161\rho^4 + 54\rho^3 - 1188\rho^2 - 270\rho + 486\rho)\zeta^2 + (204\rho^6 + 174\rho^5 - 600\rho^4 - 246\rho^3 \\ + 648\rho^2 + 72\rho - 252)\zeta = 0. \end{cases}$$

Using Matlab, we achieve

$$\begin{cases} \rho_{1,1} \approx 1, \\ \zeta_{1,1} \approx 1.5393, \end{cases} \quad \begin{cases} \rho_{1,2} \approx 1, \\ \zeta_{1,2} \approx -2.2507, \end{cases} \quad \begin{cases} \rho_{1,3} \approx 1, \\ \zeta_{1,3} \approx -0.2886, \end{cases} \quad \begin{cases} \rho_{1,4} \approx -1, \\ \zeta_{1,4} \approx 0.5862, \end{cases} \quad \begin{cases} \rho_{1,5} = 0, \\ \zeta_{1,5} = 0, \end{cases}$$

$$\begin{cases} \rho_{1,6} \approx 0.1279, \\ \zeta_{1,6} \approx 0.5507, \end{cases} \quad \begin{cases} \rho_{1,7} \approx 3.3243, \\ \zeta_{1,7} \approx 15.5478, \end{cases} \quad \begin{cases} \rho_{1,8} \approx -0.7593, \\ \zeta_{1,8} \approx -16.0982, \end{cases} \quad \begin{cases} \rho_{1,9} \approx -0.9506, \\ \zeta_{1,9} \approx 0.2941, \end{cases}$$

$$\begin{cases} \rho_{1,10} \approx -0.9905, \\ \zeta_{1,10} \approx -0.3195, \end{cases} \quad \begin{cases} \rho_{1,11} \approx -1.3815, \\ \zeta_{1,11} \approx 7.4027. \end{cases}$$

Hence, we get $(\rho_{1,6}, \zeta_{1,6})$ as one of the critical point at which the function $\aleph_1(\rho, \zeta)$ attains its maximum value, which is $\aleph_1(\rho, \zeta) \leq \aleph_1(\rho_{1,6}, \zeta_{1,6}) \approx 133.842178$.

(1) For $\rho = 1$,

$$\aleph_1(1, \zeta) = 17.$$

(2) For $\rho = 0$,

$$\aleph_1(0, \zeta) = 144 - 126\zeta^2 + 162\zeta^3 - 18\zeta^4 \leq 144.$$

(3) For $\zeta = 0$,

$$\aleph_1(\rho, 0) = 144 - 207\rho^2 + 51\rho^3 + 63\rho^4 - 51\rho^5 + 17\rho^6 \leq 144.$$

(4) For $\zeta = 1$,

$$\aleph_1(\rho, 1) = 162 - 225\rho^2 + 198\rho^4 - 118\rho^6 \leq 162.$$

Hence, we get

$$|\mathcal{H}_{3,1}(\kappa^{-1})| \leq \frac{162}{1296} = \frac{1}{8}.$$

For the Schwarz function $\omega(\varepsilon) = \varepsilon^2$, formulas (2.4)–(2.7) give $\varkappa_2 = 0$, $\varkappa_3 = \frac{1}{2}$, $\varkappa_4 = 0$, and $\varkappa_5 = \frac{1}{4}$, and equation (1.4) leads to $\mathcal{H}_{3,1}(\kappa^{-1}) = -\frac{1}{8}$. This completes the proof.

Theorem 2.2. If $\kappa \in C_e$, then

$$|\mathcal{H}_{3,1}(\kappa^{-1})| \leq \frac{1}{144}.$$

The inequality is best possible for the function

$$\kappa_2(\varepsilon) = \int_0^\varepsilon \left(\exp\left(\int_0^x \frac{e^{t^3} - 1}{t} dt \right) \right) dx = \varepsilon + \frac{1}{12}\varepsilon^4 + \cdots.$$

And the corresponding Schwarz function is $\omega(\varepsilon) = \varepsilon^3$. The image domain of extreme value function is the same as in Theorem 2.1, an infinitely spiraling convex domain determined by the exponential map.

Proof: Let $\kappa \in C_e$. By (1.11), we have

$$1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} = e^{\omega(\varepsilon)}. \quad (2.9)$$

From (1.1), we have

$$\begin{aligned} 1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} &= 1 + 2\kappa_2\varepsilon + (-4\kappa_2^2 + 6\kappa_3)\varepsilon^2 + (-18\kappa_2\kappa_3 + 8\kappa_2^3 + 12\kappa_4)\varepsilon^3 + (-16\kappa_2^4 \\ &\quad + 48\kappa_2^2\kappa_3 + 20\kappa_5 - 32\kappa_4\kappa_2 - 18\kappa_3^2)\varepsilon^4 + \cdots. \end{aligned} \quad (2.10)$$

By comparing (2.10) and (2.3), we have

$$\kappa_2 = \frac{1}{2}\rho_1, \quad (2.11)$$

$$\kappa_3 = \frac{1}{12}(2\rho_2 + 3\rho_1^2), \quad (2.12)$$

$$\kappa_4 = \frac{1}{144}(17\rho_1^3 + 30\rho_1\rho_2 + 12\rho_3), \quad (2.13)$$

$$\kappa_5 = \frac{1}{360}(18\rho_4 + 18\rho_2^2 + 60\rho_1^2\rho_2 + 42\rho_3\rho_1 + 19\rho_1^4). \quad (2.14)$$

Using (2.11)–(2.14) and (1.4), we have

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= \frac{1}{103680}(-720\rho_3^2 - 144\rho_1\rho_2\rho_3 - 5\rho_1^6 - 96\rho_2^3 + 12\rho_1^4\rho_2 - 180\rho_1^2\rho_2^2 \\ &\quad + 120\rho_1^3\rho_3 + 864\rho_2\rho_4). \end{aligned} \quad (2.15)$$

From Lemma 1.1 and (2.15), we obtain

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= \frac{1}{103680} \left[-5\rho_1^6 + 144\bar{\rho}_1^2(1 - |\rho_1|^2)^2\zeta_1^4 - 864|\zeta_1|^2(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2\zeta_2^2 \right. \\ &\quad - 288(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2)\bar{\rho}_1\zeta_1^2\zeta_2 + 864(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2(1 - |\zeta_2|^2)\zeta_1\zeta_3 \\ &\quad + 144(1 - |\rho_1|^2)^2|\rho_1|^2\zeta_1^3 - 180\rho_1^2(1 - |\rho_1|^2)^2\zeta_1^2 - 144\rho_1(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2)\zeta_1\zeta_2 \\ &\quad - 96(1 - |\rho_1|^2)^3\zeta_1^3 + 12\rho_1^4(1 - |\rho_1|^2)\zeta_1 - 720\zeta_2^2(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2)^2 \\ &\quad \left. + 120\rho_1^3(1 - |\zeta_1|^2)(1 - |\rho_1|^2)\zeta_2 - 120(1 - |\rho_1|^2)\rho_1^3\bar{\rho}_1\zeta_1^2 \right]. \end{aligned}$$

Therefore, we obtain

$$\mathcal{H}_{3,1}(\kappa^{-1}) = \frac{1}{103680} \left[\sigma_4(\rho_1, \zeta_1) + \sigma_5(\rho_1, \zeta_2) \zeta_2 + \sigma_6(\rho_1, \zeta_1) \zeta_2^2 + \Gamma_2(\rho_1, \zeta_1, \zeta_2) \zeta_3 \right],$$

where

$$\begin{aligned} \sigma_4(\rho_1, \zeta_1) &= (1 - |\rho_1|^2) \left[(-180\rho_1^2\zeta_1^2 - 96\zeta_1^3 + 240|\rho_1|^2\zeta_1^3 + 144\bar{\rho}_1^2\zeta_1^4)(1 - |\rho_1|^2) - 120\rho_1^3\bar{\rho}_1\zeta_1^2 \right. \\ &\quad \left. + 12\rho_1^4\zeta_1 \right] - 5\rho_1^6, \\ \sigma_5(\rho_1, \zeta_1) &= (1 - |\zeta_1|^2) (1 - |\rho_1|^2) \left[(-144\rho_1\zeta_1 - 288\bar{\rho}_1\zeta_1^2)(1 - |\rho_1|^2) + 120\rho_1^3 \right], \\ \sigma_6(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)^2 (1 - |\zeta_1|^2) (-144|\zeta_1|^2 - 720), \\ \Gamma_2(\rho_1, \zeta_1, \zeta_2) &= 864\zeta_1(1 - |\zeta_2|^2)(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2. \end{aligned}$$

Upon taking $|\zeta_3| \leq 1$ and setting $\rho = |\rho_1|$, $\zeta = |\zeta_1|$, $\tau = |\zeta_2|$, we obtain

$$\begin{aligned} |\mathcal{H}_{3,1}(\kappa^{-1})| &\leq \frac{1}{103680} \left(|\sigma_4(\rho_1, \zeta_1)| + |\sigma_5(\rho_1, \zeta_1)| \tau + |\sigma_6(\rho_1, \zeta_1)| \tau^2 + |\Gamma_2(\rho_1, \zeta_1, \zeta_2)| \right) \\ &\leq \frac{1}{103680} \Upsilon_2(\rho, \zeta, \tau), \end{aligned}$$

where

$$\Upsilon_2(\rho, \zeta, \tau) = \gamma_5(\rho, \zeta) + \gamma_6(\rho, \zeta) \tau + \gamma_7(\rho, \zeta) \tau^2 + \gamma_8(\rho, \zeta) (1 - \tau^2),$$

with

$$\begin{aligned} \gamma_5(\rho, \zeta) &= 5\rho^6 + (1 - \rho^2) \left[(96\zeta^3 + 240\rho^2\zeta^3 + 144\rho^2\zeta^4 + 180\rho^2\zeta^2) (1 - \rho^2) + 12\rho^4\zeta + 120\rho^4\zeta^2 \right], \\ \gamma_6(\rho, \zeta) &= (1 - \rho^2) (1 - \zeta^2) \left[(144\rho\zeta + 288\rho\zeta^2) (1 - \rho^2) + 120\rho^3 \right], \\ \gamma_7(\rho, \zeta) &= (1 - \zeta^2) (1 - \rho^2)^2 (720 + 144\zeta^2), \\ \gamma_8(\rho, \zeta) &= 864\zeta (1 - \zeta^2) (1 - \rho^2)^2. \end{aligned}$$

Differentiating $\Upsilon_2(\rho, \zeta, \tau)$ with respect to τ , we obtain

$$\frac{\partial \Upsilon_2}{\partial \tau} = \gamma_6(\rho, \zeta) + 2\tau[\gamma_7(\rho, \zeta) - \gamma_8(\rho, \zeta)] = \gamma_6(\rho, \zeta) + 288(1 - \rho^2)^2(\zeta^2 - 6\zeta + 5)(1 - \zeta^2)\tau \geq 0.$$

It is evident that $\Upsilon_2(\rho, \zeta, \tau)$ is an increasing function of $\tau \in [0, 1]$ and attains its maximum when $\tau = 1$. Thus, we have

$$\begin{aligned} \Upsilon_2(\rho, \zeta, \tau) &\leq \Upsilon_2(\rho, \zeta, 1) = \gamma_5(\rho, \zeta) + \gamma_6(\rho, \zeta) + \gamma_7(\rho, \zeta) \\ &= 5\rho^6 - 120\rho^5 + 720\rho^4 + 120\rho^3 - 1440\rho^2 + 720 + (144\rho^6 - 288\rho^5 - 432\rho^4 + 576\rho^3 \\ &\quad + 432\rho^2 - 288\rho - 144)\zeta^4 + (240\rho^6 - 144\rho^5 - 384\rho^4 + 288\rho^3 + 48\rho^2 - 144\rho + 96)\zeta^3 \\ &\quad + (60\rho^6 + 408\rho^5 - 816\rho^4 - 696\rho^3 + 1332\rho^2 + 288\rho - 576)\zeta^2 + (-12\rho^6 + 144\rho^5 + 12\rho^4 \\ &\quad - 288\rho^3 + 144\rho)\zeta = \aleph_2(\rho, \zeta). \end{aligned}$$

Consider

$$\begin{cases} \frac{\partial \aleph_2}{\partial \rho} = 30\rho^5 - 600\rho^4 + 2880\rho^3 + 360\rho^2 - 2880\rho + (864\rho^5 - 1440\rho^4 - 1728\rho^3 + 1728\rho^2 + 864\rho \\ -288)\zeta^4 + (1440\rho^5 - 720\rho^4 - 1536\rho^3 + 864\rho^2 + 96\rho - 144)\zeta^3 + (360\rho^5 + 2040\rho^4 - 3264\rho^3 \\ -2088\rho^2 + 2664\rho + 288)\zeta^2 + (-72\rho^5 + 720\rho^4 + 48\rho^3 - 864\rho^2 + 144)\zeta = 0, \\ \frac{\partial \aleph_2}{\partial \zeta} = 144\rho - 288\rho^3 + 12\rho^4 + 144\rho^5 - 12\rho^6 + (576\rho^6 - 1152\rho^5 - 1728\rho^4 + 2304\rho^3 + 1728\rho^2 \\ -1152\rho - 576)\zeta^3 + (720\rho^6 - 432\rho^5 - 1152\rho^4 + 864\rho^3 + 144\rho^2 - 432\rho + 288)\zeta^2 + (120\rho^6 \\ +816\rho^5 - 1632\rho^4 - 1392\rho^3 + 2664\rho^2 + 576\rho - 1152)\zeta = 0. \end{cases}$$

By utilizing Matlab, we obtain

$$\begin{cases} \rho_{1,1} \approx -1, \\ \zeta_{1,1} \approx 0.7112, \end{cases} \quad \begin{cases} \rho_{1,2} \approx -1, \\ \zeta_{1,2} \approx -0.7592, \end{cases} \quad \begin{cases} \rho_{1,3} = 0, \\ \zeta_{1,3} = 0, \end{cases} \quad \begin{cases} \rho_{1,4} \approx 11.9155, \\ \zeta_{1,4} = 0, \end{cases} \quad \begin{cases} \rho_{1,5} \approx -0.9622, \\ \zeta_{1,5} = 0, \end{cases} \\ \begin{cases} \rho_{1,6} \approx 1.0467, \\ \zeta_{1,6} = 0, \end{cases} \quad \begin{cases} \rho_{1,7} \approx 3.9420, \\ \zeta_{1,7} \approx -2.1087, \end{cases} \quad \begin{cases} \rho_{1,8} \approx 7.9944, \\ \zeta_{1,8} \approx -0.0325, \end{cases} \quad \begin{cases} \rho_{1,9} \approx -0.6129, \\ \zeta_{1,9} \approx 1.0878, \end{cases} \\ \begin{cases} \rho_{1,10} \approx -0.8933, \\ \zeta_{1,10} \approx -1.0413, \end{cases} \quad \begin{cases} \rho_{1,11} \approx -1.0264, \\ \zeta_{1,11} \approx 2.5043. \end{cases}$$

Computation shows that there is no solution in $(0, 1) \times (0, 1)$.

(5) For $\rho = 1$,

$$\aleph_2(1, \zeta) = 5.$$

(6) For $\rho = 0$,

$$\aleph_2(0, \zeta) = 720 - 576\zeta^2 + 96\zeta^3 - 144\zeta^4 \leq 720.$$

(7) For $\zeta = 0$,

$$\aleph_2(\rho, 0) = 720 - 1440\rho^2 + 120\rho^3 + 720\rho^4 - 120\rho^5 + 5\rho^6 \leq 720.$$

(8) For $\zeta = 1$,

$$\aleph_2(\rho, 1) = 96 + 372\rho^2 - 900\rho^4 + 437\rho^6 = \alpha_1(\rho) \leq \alpha_1(0.5034) \approx 139.5849.$$

Hence, we achieve

$$|\mathcal{H}_{3,1}(\kappa^{-1})| \leq \frac{720}{103680} = \frac{1}{144}.$$

For the Schwarz function $\omega(\varepsilon) = \varepsilon^3$, formulas (2.11)–(2.14) give $\varkappa_2 = \varkappa_3 = \varkappa_5 = 0$ and $\varkappa_4 = \frac{1}{12}$, and equation (1.4) leads to $\mathcal{H}_{3,1}(\kappa^{-1}) = -\frac{1}{144}$.

Theorem 2.3. If $\kappa \in S_{\mathcal{L}}^*$, then

$$|\mathcal{H}_{3,1}(\kappa^{-1})| \leq \frac{1}{8}.$$

The result is sharp and achieved by

$$\kappa_3(\varepsilon) = \varepsilon \exp\left(\int_0^\varepsilon \frac{t^2 + \sqrt{1+t^4} - 1}{t} dt\right) = \varepsilon + \frac{1}{2}\varepsilon^3 + \cdots.$$

And the corresponding Schwarz function is $\omega(\varepsilon) = \varepsilon^2$. The image domain of extreme value function is a convex lune symmetric about the real axis with a cusp at the right endpoint.

Proof: Let $\kappa \in S_{\mathcal{L}}^*$. Then by (1.12), we obtain

$$\frac{\varepsilon \kappa'(\varepsilon)}{\kappa(\varepsilon)} = \omega(\varepsilon) + \sqrt{1 + \omega^2(\varepsilon)}. \quad (2.16)$$

As

$$\varepsilon + \sqrt{1 + \varepsilon^2} = 1 + \varepsilon + \frac{1}{2}\varepsilon^2 - \frac{1}{8}\varepsilon^4 + \frac{1}{16}\varepsilon^6 + \cdots. \quad (2.17)$$

Putting $\omega(\varepsilon)$ in Eq (2.17), we have

$$\begin{aligned} \omega(\varepsilon) + \sqrt{1 + \omega^2(\varepsilon)} &= 1 + \rho_1 \varepsilon + (\rho_2 + \frac{1}{2}\rho_1^2)\varepsilon^2 + (\rho_3 + \rho_1 \rho_2)\varepsilon^3 \\ &+ \frac{1}{8}(8\rho_4 + 4\rho_2^2 + 8\rho_1 \rho_3 - \rho_1^4)\varepsilon^4 + \cdots. \end{aligned} \quad (2.18)$$

From (2.2) and (2.18), we get

$$\kappa_2 = \rho_1, \quad (2.19)$$

$$\kappa_3 = \frac{1}{4}(3\rho_1^2 + 2\rho_2), \quad (2.20)$$

$$\kappa_4 = \frac{1}{12}(5\rho_1^3 + 10\rho_1 \rho_2 + 4\rho_3), \quad (2.21)$$

$$\kappa_5 = \frac{1}{24}(6\rho_4 + 14\rho_1 \rho_3 + 17\rho_1^2 \rho_2 + 6\rho_2^2 + 4\rho_1^4). \quad (2.22)$$

Using (1.4) and (2.19)–(2.22), we have

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= \frac{1}{288}(-32\rho_3^2 + 20\rho_1 \rho_2 \rho_3 + \rho_1^6 - 36\rho_2^3 - 17\rho_1^4 \rho_2 + 16\rho_1^2 \rho_2^2 - 18\rho_1^2 \rho_4 \\ &+ 22\rho_1^3 \rho_3 + 36\rho_2 \rho_4). \end{aligned} \quad (2.23)$$

From (2.23) and Lemma 1.1, we achieve

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= \frac{1}{288}[4\zeta_1^4 \bar{\rho}_1^2 (1 - |\rho_1|^2)^2 - 36\zeta_2^2 |\zeta_1|^2 (1 - |\rho_1|^2)^2 (1 - |\zeta_1|^2) - 20|\rho_1|^2 \zeta_1^3 (1 - |\rho_1|^2)^2 - 8\bar{\rho}_1 \zeta_1^2 \\ &\times (1 - |\rho_1|^2)^2 (1 - |\zeta_1|^2) \zeta_2 + 20\zeta_1 \rho_1 (1 - |\zeta_1|^2) (1 - |\rho_1|^2)^2 \zeta_2 + 36\zeta_3 \zeta_1 (1 - |\zeta_1|^2) (1 - |\zeta_2|^2) \end{aligned}$$

$$\begin{aligned}
& \times (1 - |\rho_1|^2)^2 + \rho_1^6 + 16\rho_1^2\zeta_1^2(1 - |\rho_1|^2)^2 - 36(1 - |\rho_1|^2)^3\zeta_1^3 - 17\rho_1^4\zeta_1(1 - |\rho_1|^2) - 18\zeta_1^3\rho_1^2\bar{\rho}_1^2 \\
& \times (1 - |\rho_1|^2) - 32(1 - |\zeta_1|^2)^2(1 - |\rho_1|^2)^2\zeta_2^2 + 22\rho_1^3\zeta_2(1 - |\zeta_1|^2)(1 - |\rho_1|^2) - 18\rho_1^2\zeta_3(1 - |\zeta_2|^2) \\
& \times (1 - |\zeta_1|^2)(1 - |\rho_1|^2) + 18\rho_1^2\bar{\zeta}_1\zeta_2^2(1 - |\zeta_1|^2)(1 - |\rho_1|^2) + 36\rho_1^2\bar{\rho}_1\zeta_2\zeta_1(1 - |\zeta_1|^2)(1 - |\rho_1|^2) \\
& - 22\rho_1^3\bar{\rho}_1(1 - |\rho_1|^2)\zeta_1^2].
\end{aligned}$$

Hence, we have

$$\mathcal{H}_{3,1}(\kappa^{-1}) = \frac{1}{288} [\sigma_7(\rho_1, \zeta_1) + \sigma_8(\rho_1, \zeta_2)\zeta_2 + \sigma_9(\rho_1, \zeta_1)\zeta_2^2 + \Gamma_3(\rho_1, \zeta_1, \zeta_2)\zeta_3],$$

where

$$\begin{aligned}
\sigma_7(\rho_1, \zeta_1) &= (1 - |\rho_1|^2) \{ [16\rho_1^2\zeta_1^2 - (36 - 16|\rho_1|^2)\zeta_1^3 + 4\zeta_1^4\bar{\rho}_1^2](1 - |\rho_1|^2) - 18\rho_1^2\bar{\rho}_1^2\zeta_1^3 \\
&\quad - 22\rho_1^3\bar{\rho}_1\zeta_1^2 - 17\zeta_1\rho_1^4 \} + \rho_1^6, \\
\sigma_8(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)(1 - |\zeta_1|^2) [(20\rho_1\zeta_1 - 8\bar{\rho}_1\zeta_1^2)(1 - |\rho_1|^2) + 36\zeta_1\bar{\rho}_1\rho_1^2 + 22\rho_1^3], \\
\sigma_9(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)(1 - |\zeta_1|^2)[(-32 - 4|\zeta_1|^2)(1 - |\rho_1|^2) + 18\rho_1^2\bar{\zeta}_1], \\
\Gamma_3(\rho_1, \zeta_1, \zeta_2) &= (1 - |\zeta_2|^2)(1 - |\rho_1|^2)(1 - |\zeta_1|^2)[-18\rho_1^2 + 36(1 - |\rho_1|^2)\zeta_1].
\end{aligned}$$

Setting $|\rho_1| = \rho$, $\zeta = |\zeta_1|$, $|\zeta_2| = \tau$, and upon taking $|\zeta_3| \leq 1$, we achieve

$$\begin{aligned}
|\mathcal{H}_{3,1}(\kappa^{-1})| &\leq \frac{1}{288} (|\sigma_7(\rho_1, \zeta_1)| + |\sigma_8(\rho_1, \zeta_1)|\tau + |\sigma_9(\rho_1, \zeta_1)|\tau^2 + |\Gamma_3(\rho_1, \zeta_1, \zeta_2)|) \\
&\leq \frac{1}{288} \Upsilon_3(\rho, \zeta, \tau),
\end{aligned}$$

where

$$\Upsilon_3(\rho, \zeta, \tau) = \gamma_9(\rho, \zeta) + \gamma_{10}(\rho, \zeta)\tau + \gamma_{11}(\rho, \zeta)\tau^2 + \gamma_{12}(\rho, \zeta)(1 - \tau^2),$$

with

$$\begin{aligned}
\gamma_9(\rho, \zeta) &= \rho^6 + (1 - \rho^2) [(36\zeta^3 - 16\rho^2\zeta^3 + 4\rho^2\zeta^4 + 16\rho^2\zeta^2)(1 - \rho^2) + 17\rho^4\zeta + 22\rho^4\zeta^2 + 18\rho^4\zeta^3], \\
\gamma_{10}(\rho, \zeta) &= (1 - \rho^2)(1 - \zeta^2) [(20\rho\zeta + 8\rho\zeta^2)(1 - \rho^2) + 22\rho^3 + 36\rho^3\zeta], \\
\gamma_{11}(\rho, \zeta) &= (1 - \zeta^2)(1 - \rho^2) [(32 + 4\zeta^2)(1 - \rho^2) + 18\rho^2\zeta], \\
\gamma_{12}(\rho, \zeta) &= (1 - \zeta^2)(1 - \rho^2) [36\zeta(1 - \rho^2) + 18\rho^2].
\end{aligned}$$

Note that

$$\gamma_{11}(\rho, \zeta) \leq (1 - \zeta^2)(1 - \rho^2) [(4\zeta^2 + 32)(1 - \rho^2) + 18\rho^2] = \psi_2(\rho, \zeta).$$

Thus, we have

$$\Upsilon_3(\rho, \zeta, \tau) \leq \gamma_9(\rho, \zeta) + \gamma_{10}(\rho, \zeta)\tau + \psi_2(\rho, \zeta)\tau^2 + \gamma_{12}(\rho, \zeta)(1 - \tau^2) = \xi_2(\rho, \zeta, \tau).$$

Differentiating $\xi_2(\rho, \zeta, \tau)$ with respect to τ , we have

$$\frac{\partial \xi_2}{\partial \tau} = \gamma_{10}(\rho, \zeta) + 2\tau[\psi_2(\rho, \zeta) - \gamma_{12}(\rho, \zeta)]$$

$$= \gamma_{10}(\rho, \zeta) + 8\tau(8 - 9\zeta + \zeta^2)(1 - \zeta^2)(1 - \rho^2)^2 \geq 0.$$

Therefore, $\xi_2(\rho, \zeta, \tau)$ attains its maximum value at $\tau = 1$. Thus, we obtain

$$\begin{aligned} \Upsilon_3(\rho, \zeta, \tau) &\leq \xi_2(\rho, \zeta, 1) = \gamma_9(\rho, \zeta) + \gamma_{10}(\rho, \zeta) + \psi_2(\rho, \zeta) \\ &= \rho^6 - 22\rho^5 + 14\rho^4 + 22\rho^3 - 46\rho^2 + 32 + (4\rho^6 - 8\rho^5 - 12\rho^4 + 16\rho^3 + 12\rho^2 - 8\rho - 4)\zeta^4 \\ &\quad + (-34\rho^6 + 16\rho^5 + 86\rho^4 + 4\rho^3 - 88\rho^2 - 20\rho + 36)\zeta^3 + (-6\rho^6 + 30\rho^5 - 20\rho^4 - 38\rho^3 \\ &\quad + 54\rho^2 + 8\rho - 28)\zeta^2 + (-17\rho^6 - 16\rho^5 + 17\rho^4 - 4\rho^3 + 20\rho)\zeta = \aleph_3(\rho, \zeta). \end{aligned}$$

Consider

$$\begin{cases} \frac{\partial \aleph_3}{\partial \rho} = 6\rho^5 - 110\rho^4 + 56\rho^3 + 66\rho^2 - 92\rho + (24\rho^5 - 40\rho^4 - 48\rho^3 + 48\rho^2 + 24\rho - 8)\zeta^4 \\ \quad + (-204\rho^5 + 80\rho^4 + 344\rho^3 + 12\rho^2 - 176\rho - 20)\zeta^3 + (-36\rho^5 + 150\rho^4 - 80\rho^3 - 114\rho^2 \\ \quad + 108\rho + 8)\zeta^2 + (-102\rho^5 - 80\rho^4 + 68\rho^3 - 12\rho^2 + 20)\zeta = 0, \\ \frac{\partial \aleph_3}{\partial \zeta} = 20\rho - 4\rho^3 + 17\rho^4 - 16\rho^5 - 17\rho^6 + (16\rho^6 - 32\rho^5 - 48\rho^4 + 64\rho^3 + 48\rho^2 - 32\rho - 16)\zeta^3 \\ \quad + (-102\rho^6 + 48\rho^5 + 258\rho^4 + 12\rho^3 - 264\rho^2 - 60\rho + 108)\zeta^2 + (-12\rho^6 + 60\rho^5 - 40\rho^4 - 76\rho^3 \\ \quad + 108\rho^2 + 16\rho - 56)\zeta = 0. \end{cases}$$

By calculating, we achieve

$$\begin{aligned} &\begin{cases} \rho_{1,1} \approx 1, \\ \zeta_{1,1} \approx 1.6138, \end{cases} \quad \begin{cases} \rho_{1,2} \approx 1, \\ \zeta_{1,2} \approx -1.9659, \end{cases} \quad \begin{cases} \rho_{1,3} \approx 1, \\ \zeta_{1,3} \approx -0.6479, \end{cases} \quad \begin{cases} \rho_{1,4} \approx -1, \\ \zeta_{1,4} \approx 0.5601, \end{cases} \\ &\begin{cases} \rho_{1,5} \approx -1, \\ \zeta_{1,5} \approx -0.7201, \end{cases} \quad \begin{cases} \rho_{1,6} \approx -1, \\ \zeta_{1,6} \approx -0.3214, \end{cases} \quad \begin{cases} \rho_{1,7} = 0, \\ \zeta_{1,7} = 0, \end{cases} \quad \begin{cases} \rho_{1,8} \approx 0.1152, \\ \zeta_{1,8} \approx 0.5600. \end{cases} \end{aligned}$$

Thus, there is a solution $(\rho_{1,8}, \zeta_{1,8})$ in $(0, 1) \times (0, 1)$, at which $\aleph_3(\rho, \zeta)$ attains its maximum value: $\aleph_3(\rho, \zeta) \leq \aleph_3(\rho_{1,8}, \zeta_{1,8}) \approx 29.677055$.

(9) For $\rho = 1$,

$$\aleph_3(1, \zeta) = 1.$$

(10) For $\rho = 0$,

$$\aleph_3(0, \zeta) = 32 - 28\zeta^2 + 36\zeta^3 - 4\zeta^4 \leq 32.$$

(11) For $\zeta = 0$,

$$\aleph_3(\rho, 0) = 32 - 46\rho^2 + 22\rho^3 + 14\rho^4 - 22\rho^5 + \rho^6 \leq 32.$$

(12) For $\zeta = 1$,

$$\aleph_3(\rho, 1) = 36 - 68\rho^2 + 85\rho^4 - 52\rho^6 \leq 36.$$

Thus, we yield

$$|\mathcal{H}_{3,1}(\kappa^{-1})| \leq \frac{36}{288} = \frac{1}{8}.$$

For the Schwarz function $\omega(\varepsilon) = \varepsilon^2$, formulas (2.19)–(2.22) give $\kappa_2 = \kappa_4 = 0$, $\kappa_3 = \frac{1}{2}$, and $\kappa_5 = \frac{1}{4}$, and Eq (1.4) leads to $\mathcal{H}_{3,1}(\kappa^{-1}) = -\frac{1}{8}$.

Theorem 2.4. *If $\kappa \in C_{\mathcal{L}}$, then*

$$|\mathcal{H}_{3,1}(\kappa^{-1})| \leq \frac{1}{144}.$$

The result is sharp and achieved by

$$\kappa_4(\varepsilon) = \int_0^\varepsilon \left(\exp \left(\int_0^x \frac{t^3 + \sqrt{1+t^6} - 1}{t} dt \right) \right) dx = \varepsilon + \frac{1}{12}\varepsilon^4 + \dots$$

And the corresponding Schwarz function is $\omega(\varepsilon) = \varepsilon^3$. The image domain of extreme value function is the standard unit disk itself, without any distortion.

Proof: Let $\kappa \in C_{\mathcal{L}}$. Then by (1.13), we obtain

$$1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} = \omega(\varepsilon) + \sqrt{1 + \omega^2(\varepsilon)}.$$

From (2.10) and (2.18), we have

$$\kappa_2 = \frac{1}{2}\rho_1, \quad (2.24)$$

$$\kappa_3 = \frac{1}{12}(2\rho_2 + 3\rho_1^2), \quad (2.25)$$

$$\kappa_4 = \frac{1}{48}(4\rho_3 + 10\rho_1\rho_2 + 5\rho_1^3), \quad (2.26)$$

$$\kappa_5 = \frac{1}{120}(6\rho_4 + 17\rho_1^2\rho_2 + 14\rho_1\rho_3 + 6\rho_2^2 + 4\rho_1^4). \quad (2.27)$$

Using (2.24)–(2.27) and (1.4), we achieve

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= \frac{1}{34560}(-240\rho_3^2 - 48\rho_1\rho_2\rho_3 - 15\rho_1^6 - 32\rho_2^3 + 12\rho_1^4\rho_2 - 204\rho_1^2\rho_2^2 \\ &\quad + 120\rho_1^3\rho_3 + 288\rho_2\rho_4). \end{aligned} \quad (2.28)$$

From (2.28) and Lemma 1.1, we achieve

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa^{-1}) &= \frac{1}{34560} \left[-15\rho_1^6 + 48\zeta_1^4\bar{\rho}_1^2(1 - |\rho_1|^2)^2 - 288|\zeta_1|^2\zeta_2^2(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2) \right. \\ &\quad - 96(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2)\zeta_1^2\bar{\rho}_1\zeta_2 + 288\zeta_1\zeta_3(1 - |\zeta_2|^2)(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2 \\ &\quad + 48|\rho_1|^2\zeta_1^3(1 - |\rho_1|^2)^2 - 204\zeta_1^2\rho_1^2(1 - |\rho_1|^2)^2 - 48\zeta_1\zeta_2\rho_1(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2) \\ &\quad \left. - 32(1 - |\rho_1|^2)^3\zeta_1^3 + 12\zeta_1(1 - |\rho_1|^2)\rho_1^4 - 240\zeta_2^2(1 - |\zeta_1|^2)^2(1 - |\rho_1|^2)^2 \right] \end{aligned}$$

$$+ 120\zeta_2\rho_1^3(1-|\rho_1|^2)(1-|\zeta_1|^2) - 120\zeta_1^2\rho_1^3\bar{\rho}_1(1-|\rho_1|^2)\Big].$$

Thus, we have

$$\mathcal{H}_{3,1}(\kappa^{-1}) = \frac{1}{34560} \left[\sigma_{10}(\rho_1, \zeta_1) + \sigma_{11}(\rho_1, \zeta_2) \zeta_2 + \sigma_{12}(\rho_1, \zeta_1) \zeta_2^2 + \Gamma_4(\rho_1, \zeta_1, \zeta_2) \zeta_3 \right],$$

where

$$\begin{aligned} \sigma_{10}(\rho_1, \zeta_1) &= (1-|\rho_1|^2) \left[(-204\rho_1^2\zeta_1^2 - 32\zeta_1^3 + 80|\rho_1|^2\zeta_1^3 + 48\bar{\rho}_1^2\zeta_1^4)(1-|\rho_1|^2) - 120\bar{\rho}_1\rho_1^3\zeta_1^2 \right. \\ &\quad \left. + 12\zeta_1\rho_1^4 \right] - 15\rho_1^6, \\ \sigma_{11}(\rho_1, \zeta_1) &= (1-|\rho_1|^2)(1-|\zeta_1|^2) \left[(1-|\rho_1|^2)(-48\rho_1\zeta_1 - 96\bar{\rho}_1\zeta_1^2) + 120\rho_1^3 \right], \\ \sigma_{12}(\rho_1, \zeta_1) &= (1-|\zeta_1|^2)(-48|\zeta_1|^2 - 240)(1-|\rho_1|^2)^2, \\ \Gamma_4(\rho_1, \zeta_1, \zeta_2) &= 288(1-|\rho_1|^2)^2\zeta_1(1-|\zeta_1|^2)(1-|\zeta_2|^2). \end{aligned}$$

Upon taking $|\zeta_3| \leq 1$, and setting $\tau = |\zeta_2|$, $|\rho_1| = \rho$, $\zeta = |\zeta_1|$, we achieve

$$\begin{aligned} |\mathcal{H}_{3,1}(\kappa^{-1})| &\leq \frac{1}{34560} (|\sigma_{10}(\rho_1, \zeta_1)| + |\sigma_{11}(\rho_1, \zeta_1)|\tau + |\sigma_{12}(\rho_1, \zeta_1)|\tau^2 + |\Gamma_4(\rho_1, \zeta_1, \zeta_2)|) \\ &\leq \frac{1}{34560} \Upsilon_4(\rho, \zeta, \tau), \end{aligned}$$

where

$$\Upsilon_4(\rho, \zeta, \tau) = \gamma_{13}(\rho, \zeta) + \gamma_{14}(\rho, \zeta)\tau + \gamma_{15}(\rho, \zeta)\tau^2 + \gamma_{16}(\rho, \zeta)(1-\tau^2),$$

with

$$\begin{aligned} \gamma_{13}(\rho, \zeta) &= 15\rho^6 + (1-\rho^2) \left[(32\zeta^3 + 80\rho^2\zeta^3 + 48\rho^2\zeta^4 + 204\rho^2\zeta^2)(1-\rho^2) + 12\zeta\rho^4 + 120\zeta^2\rho^4 \right], \\ \gamma_{14}(\rho, \zeta) &= (1-\rho^2)(1-\zeta^2) \left[(48\rho\zeta + 96\rho\zeta^2)(1-\rho^2) + 120\rho^3 \right], \\ \gamma_{15}(\rho, \zeta) &= (240 + 48\zeta^2)(1-\zeta^2)(1-\rho^2)^2, \\ \gamma_{16}(\rho, \zeta) &= 288(\zeta - \zeta^3)(1-\rho^2)^2. \end{aligned}$$

Differentiating $\Upsilon_4(\rho, \zeta, \tau)$ with respect to τ , we achieve

$$\frac{\partial \Upsilon_4}{\partial \tau} = \gamma_{14}(\rho, \zeta) + 2\tau[\gamma_{15}(\rho, \zeta) - \gamma_{16}(\rho, \zeta)] = \gamma_{14}(\rho, \zeta) + 96(5 - 6\zeta + \zeta^2)(1-\zeta^2)(1-\rho^2)^2\tau \geq 0.$$

It is evident that $\Upsilon_4(\rho, \zeta, \tau)$ is an increasing function of $\tau \in [0, 1]$. Therefore, we yield

$$\begin{aligned} \Upsilon_4(\rho, \zeta, \tau) &\leq \Upsilon_4(\rho, \zeta, 1) = \gamma_{13}(\rho, \zeta) + \gamma_{14}(\rho, \zeta) + \gamma_{15}(\rho, \zeta) \\ &= 15\rho^6 - 120\rho^5 + 240\rho^4 + 120\rho^3 - 480\rho^2 + 240 + (48\rho^6 - 96\rho^5 - 144\rho^4 + 192\rho^3 \\ &\quad + 144\rho^2 - 96\rho - 48)\zeta^4 + (80\rho^6 - 48\rho^5 - 128\rho^4 + 96\rho^3 + 16\rho^2 - 48\rho + 32)\zeta^3 \\ &\quad + (84\rho^6 + 216\rho^5 - 480\rho^4 - 312\rho^3 + 588\rho^2 + 96\rho - 192)\zeta^2 + (-12\rho^6 + 48\rho^5 \\ &\quad + 12\rho^4 - 96\rho^3 + 48\rho)\zeta = \aleph_4(\rho, \zeta). \end{aligned}$$

Consider

$$\begin{cases} \frac{\partial \aleph_4}{\partial \rho} = 90\rho^5 - 600\rho^4 + 960\rho^3 + 360\rho^2 - 960\rho + (288\rho^5 - 480\rho^4 - 576\rho^3 + 576\rho^2 + 288\rho - 96)\zeta^4 \\ + (480\rho^5 - 240\rho^4 - 512\rho^3 + 288\rho^2 + 32\rho - 48)\zeta^3 + (504\rho^5 + 1080\rho^4 - 1920\rho^3 - 936\rho^2 + 1176\rho \\ + 96)\zeta^2 + (-72\rho^5 + 240\rho^4 + 48\rho^3 - 288\rho^2 + 48)\zeta = 0, \\ \frac{\partial \aleph_4}{\partial \zeta} = 48\rho - 96\rho^3 + 12\rho^4 + 48\rho^5 - 12\rho^6 + (192\rho^6 - 384\rho^5 - 576\rho^4 + 768\rho^3 + 576\rho^2 - 384\rho \\ - 192)\zeta^3 + (240\rho^6 - 144\rho^5 - 384\rho^4 + 288\rho^3 + 48\rho^2 - 144\rho + 96)\zeta^2 + (168\rho^6 + 432\rho^5 - 960\rho^4 \\ - 624\rho^3 + 1176\rho^2 + 192\rho - 384)\zeta = 0. \end{cases}$$

By calculating, we achieve

$$\begin{cases} \rho_{1,1} \approx 1, \\ \zeta_{1,1} \approx -6.2500, \end{cases} \quad \begin{cases} \rho_{1,2} \approx -1, \\ \zeta_{1,2} \approx 0.8045, \end{cases} \quad \begin{cases} \rho_{1,3} \approx -1, \\ \zeta_{1,3} \approx -0.8045, \end{cases} \quad \begin{cases} \rho_{1,4} = 0, \\ \zeta_{1,4} = 0, \end{cases} \quad \begin{cases} \rho_{1,5} \approx 3.7093, \\ \zeta_{1,5} \approx 0, \end{cases}$$

$$\begin{cases} \rho_{1,6} \approx -0.9032, \\ \zeta_{1,6} \approx 0, \end{cases} \quad \begin{cases} \rho_{1,7} \approx 1.1939, \\ \zeta_{1,7} \approx 0, \end{cases} \quad \begin{cases} \rho_{1,8} \approx 1, \\ \zeta_{1,8} \approx -12.6421, \end{cases} \quad \begin{cases} \rho_{1,9} \approx 1.0990, \\ \zeta_{1,9} \approx 0.9634, \end{cases}$$

$$\begin{cases} \rho_{1,10} \approx 1.1707, \\ \zeta_{1,10} \approx -0.5138, \end{cases} \quad \begin{cases} \rho_{1,11} \approx 2.6666, \\ \zeta_{1,11} \approx -0.0163, \end{cases} \quad \begin{cases} \rho_{1,12} \approx -0.5454, \\ \zeta_{1,12} \approx 0.9661, \end{cases} \quad \begin{cases} \rho_{1,13} \approx -0.7505, \\ \zeta_{1,13} \approx -0.9806, \end{cases}$$

$$\begin{cases} \rho_{1,14} \approx -1.7149, \\ \zeta_{1,14} \approx -1.6703. \end{cases}$$

Therefore, there is no solution in $(0, 1) \times (0, 1)$.

(13) For $\rho = 1$,

$$\aleph_4(1, \zeta) = 15.$$

(14) For $\rho = 0$,

$$\aleph_4(0, \zeta) = 240 - 192\zeta^2 + 32\zeta^3 - 48\zeta^4 \leq 240.$$

(15) For $\zeta = 0$,

$$\aleph_4(\rho, 0) = 240 - 480\rho^2 + 120\rho^3 + 240\rho^4 - 120\rho^5 + 15\rho^6 \leq 240.$$

(16) For $\zeta = 1$,

$$\aleph_4(\rho, 1) = 32 + 268\rho^2 - 500\rho^4 + 215\rho^6 = \alpha_2(\rho) \leq \alpha_2(0.5870) \approx 73.7762.$$

Thus, we achieve

$$|\mathcal{H}_{3,1}(\kappa^{-1})| \leq \frac{240}{34560} = \frac{1}{144}.$$

The equality holds for $\kappa_5 = \kappa_3 = \kappa_2 = 0$ and $\kappa_4 = \frac{1}{12}$.

3. The second Hankel determinant of logarithmic coefficients for $S_{\mathcal{L}}^*$

Theorem 3.1. *If $\kappa \in S_{\mathcal{L}}^*$, then*

$$|\eta_1\eta_3 - \eta_2^2| \leq \frac{1}{16}.$$

The result is sharp and achieved by

$$\kappa_3^*(\varepsilon) = \varepsilon \exp\left(\int_0^\varepsilon \frac{\delta t^2 + \sqrt{1 + \delta^2 t^4} - 1}{t} dt\right), \delta \rightarrow 1^-.$$

And the corresponding Schwarz function is $\omega(\varepsilon) = \delta\varepsilon^2, \delta \rightarrow 1^-$. The image domain of extreme value function is a bilobed folded region centered at the origin with two symmetric lobes.

Proof: Let $\kappa \in S_{\mathcal{L}}^*$. From (1.7), (2.19), (2.20), and (2.21), we have

$$|\eta_1\eta_3 - \eta_2^2| = \frac{1}{192} | -12\rho_2^2 + 4\rho_1^2\rho_2 + 16\rho_1\rho_3 - 3\rho_1^4 |. \quad (3.1)$$

From (3.1) and Lemma 1.1, we obtain

$$\begin{aligned} |\eta_1\eta_3 - \eta_2^2| &= \frac{1}{192} | -16(1 - |\rho_1|^2)|\rho_1|^2\zeta_1^2 + 16(1 - |\rho_1|^2)(1 - |\zeta_1|^2)\rho_1\zeta_2 + 4(1 - |\rho_1|^2)\rho_1^2\zeta_1 - 3\rho_1^4 \\ &\quad - 12(1 - |\rho_1|^2)^2\zeta_1^2|. \end{aligned}$$

By putting $\rho = |\rho_1|, |\zeta_1| = \zeta$, and upon taking $|\zeta_2| \leq 1$, we get

$$\begin{aligned} |\eta_1\eta_3 - \eta_2^2| &\leq \frac{1}{192} [16(1 - \rho^2)\rho^2\zeta^2 + 16\rho(1 - \zeta^2)(1 - \rho^2) + 4(1 - \rho^2)\rho^2\zeta + 3\rho^4 + 12(1 - \rho^2)^2\zeta^2] \\ &= \aleph_5(\rho, \zeta). \end{aligned}$$

Differentiating $\aleph_5(\rho, \zeta)$ with respect to ζ , we achieve

$$\frac{\partial \aleph_5}{\partial \zeta} = \frac{1}{192} [4\rho^2(1 - \rho^2) + 8(1 - \rho^2)(\rho^2 - 4\rho + 3)\zeta] \geq 0.$$

Therefore, $\aleph_5$ is an increasing function of $\zeta \in [0, 1]$. Thus, we have

$$|\eta_1\eta_3 - \eta_2^2| \leq \aleph_5(\rho, 1) = \frac{1}{192} (12 - 4\rho^2 - 5\rho^4) \leq \frac{12}{192} = \frac{1}{16}.$$

Theorem 3.2. *If $\kappa \in S_{\mathcal{L}}^*$, then*

$$|\eta_2\eta_4 - \eta_3^2| \leq \frac{1}{36}.$$

This equality is sharp for the function

$$\kappa_5(\varepsilon) = \varepsilon \exp\left(\int_0^\varepsilon \frac{t^3 + \sqrt{1 + t^6} - 1}{t} dt\right) = \varepsilon + \frac{1}{3}\varepsilon^4 + \dots.$$

And the corresponding Schwarz function is $\omega(\varepsilon) = \varepsilon^3$. The image domain of extreme value function is a three-leaf clover-shaped folded region with triple rotational symmetry about the origin.

Proof: Let $\kappa \in \mathcal{S}_{\mathcal{L}}^*$. From (2.19)–(2.22) and (1.8), we have

$$\begin{aligned} \eta_2\eta_4 - \eta_3^2 &= \frac{1}{4608}(-128\rho_3^2 + 72\rho_1^3\rho_3 - 9\rho_1^6 + 72\rho_2^3 - 18\rho_1^4\rho_2 - 92\rho_1^2\rho_2^2 - 112\rho_3\rho_2\rho_1 \\ &\quad + 72\rho_1^2\rho_4 + 144\rho_2\rho_4). \end{aligned} \quad (3.2)$$

From (3.2) and Lemma 1.1, we have

$$\begin{aligned} \eta_2\eta_4 - \eta_3^2 &= \frac{1}{4608}[16\zeta_1^4(1 - |\rho_1|^2)^2\bar{\rho}_1^2 - 144|\zeta_1|^2\zeta_2^2(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2 + 112\zeta_1^3(1 - |\rho_1|^2)^2|\rho_1|^2 \\ &\quad - 32\bar{\rho}_1\zeta_1^2(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2\zeta_2 - 112\zeta_2\rho_1\zeta_1(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2) + 144\zeta_1\zeta_3(1 - |\zeta_1|^2) \\ &\quad \times (1 - |\rho_1|^2)^2(1 - |\zeta_2|^2) - 9\rho_1^6 - 92\rho_1^2(1 - |\rho_1|^2)^2\zeta_1^2 + 72(1 - |\rho_1|^2)^3\zeta_1^3 - 18\zeta_1(1 - |\rho_1|^2)\rho_1^4 \\ &\quad + 72\zeta_1^3(1 - |\rho_1|^2)\bar{\rho}_1^2\rho_1^2 - 72\zeta_1^2\bar{\rho}_1\rho_1^3(1 - |\rho_1|^2) - 128\zeta_2^2(1 - |\rho_1|^2)^2(1 - |\zeta_1|^2)^2 + 72\zeta_2\rho_1^3 \\ &\quad \times (1 - |\rho_1|^2)(1 - |\zeta_1|^2) + 72\zeta_3\rho_1^2(1 - |\rho_1|^2)(1 - |\zeta_2|^2)(1 - |\zeta_1|^2) - 72\zeta_2^2\rho_1^2\bar{\zeta}_1(1 - |\rho_1|^2) \\ &\quad \times (1 - |\zeta_1|^2) - 144\zeta_2\zeta_1\rho_1^2\bar{\rho}_1(1 - |\rho_1|^2)(1 - |\zeta_1|^2)]. \end{aligned}$$

Thus, we achieve

$$\eta_2\eta_4 - \eta_3^2 = \frac{1}{4608}[\sigma_{13}(\rho_1, \zeta_1) + \sigma_{14}(\rho_1, \zeta_2)\zeta_2 + \sigma_{15}(\rho_1, \zeta_1)\zeta_2^2 + \Gamma_5(\rho_1, \zeta_1, \zeta_2)\zeta_3],$$

where

$$\begin{aligned} \sigma_{13}(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)[(16\bar{\rho}_1^2\zeta_1^4 - 92\rho_1^2\zeta_1^2 + 72\zeta_1^3 + 40|\rho_1|^2\zeta_1^3)(1 - |\rho_1|^2) + 72\bar{\rho}_1^2\rho_1^2\zeta_1^3 \\ &\quad - 72\bar{\rho}_1\rho_1^3\zeta_1^2 - 18\zeta_1\rho_1^4] - 9\rho_1^6, \\ \sigma_{14}(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)(1 - |\zeta_1|^2)[(-112\rho_1\zeta_1 - 32\zeta_1^2\bar{\rho}_1)(1 - |\rho_1|^2) - 144\bar{\rho}_1\rho_1^2\zeta_1 + 72\rho_1^3], \\ \sigma_{15}(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)(1 - |\zeta_1|^2)[(-128 - 16|\zeta_1|^2)(1 - |\rho_1|^2) - 72\rho_1^2\bar{\zeta}_1], \\ \Gamma_5(\rho_1, \zeta_1, \zeta_2) &= (1 - |\zeta_2|^2)(1 - |\rho_1|^2)(1 - |\zeta_1|^2)[72\rho_1^2 + 144(1 - |\rho_1|^2)\zeta_1]. \end{aligned}$$

By setting $|\zeta_2| = \tau$, $\zeta = |\zeta_1|$, $|\rho_1| = \rho$, and upon taking $|\zeta_3| \leq 1$, we obtain

$$\begin{aligned} |\eta_2\eta_4 - \eta_3^2| &\leq \frac{1}{4608}(|\sigma_{13}(\rho_1, \zeta_1)| + |\sigma_{14}(\rho_1, \zeta_1)|\tau + |\sigma_{15}(\rho_1, \zeta_1)|\tau^2 + |\Gamma_5(\rho_1, \zeta_1, \zeta_2)|) \\ &\leq \frac{1}{4608}\Upsilon_5(\rho, \zeta, \tau), \end{aligned}$$

where

$$\Upsilon_5(\rho, \zeta, \tau) = \gamma_{17}(\rho, \zeta) + \gamma_{18}(\rho, \zeta)\tau + \gamma_{19}(\rho, \zeta)\tau^2 + \gamma_{20}(\rho, \zeta)(1 - \tau^2),$$

with

$$\gamma_{17}(\rho, \zeta) = 9\rho^6 + (1 - \rho^2)[(72\zeta^3 + 40\rho^2\zeta^3 + 16\rho^2\zeta^4 + 92\rho^2\zeta^2)(1 - \rho^2) + 18\rho^4\zeta + 72\rho^4\zeta^2 + 72\rho^4\zeta^3],$$

$$\begin{aligned}
\gamma_{18}(\rho, \zeta) &= (1 - \rho^2)(1 - \zeta^2)[(112\rho\zeta + 32\rho\zeta^2)(1 - \rho^2) + 72\rho^3 + 144\rho^3\zeta], \\
\gamma_{19}(\rho, \zeta) &= (1 - \zeta^2)(1 - \rho^2)[(128 + 16\zeta^2)(1 - \rho^2) + 72\rho^2\zeta], \\
\gamma_{20}(\rho, \zeta) &= (1 - \zeta^2)(1 - \rho^2)[144\zeta(1 - \rho^2) + 72\rho^2].
\end{aligned}$$

Note that

$$\gamma_{19}(\rho, \zeta) \leq (1 - \rho^2)(1 - \zeta^2)[(16\zeta^2 + 128)(1 - \rho^2) + 72\rho^2] = \psi_3(\rho, \zeta).$$

Thus, we have

$$\Upsilon_5(\rho, \zeta, \tau) \leq \gamma_{17}(\rho, \zeta) + \gamma_{18}(\rho, \zeta)\tau + \psi_3(\rho, \zeta)\tau^2 + \gamma_{20}(\rho, \zeta)(1 - \tau^2) = \xi_3(\rho, \zeta, \tau).$$

Differentiating $\xi_3(\rho, \zeta, \tau)$ with respect to τ , we achieve

$$\begin{aligned}
\frac{\partial \xi_3}{\partial \tau} &= \gamma_{18}(\rho, \zeta) + 2\tau[\psi_3(\rho, \zeta) - \gamma_{20}(\rho, \zeta)] \\
&= \gamma_{18}(\rho, \zeta) + 32(8 - 9\zeta + \zeta^2)(1 - \rho^2)^2(1 - \zeta^2)\tau \geq 0.
\end{aligned}$$

Thus, $\xi_3(\rho, \zeta, \tau)$ attains its maximum value at $\tau = 1$. Therefore, we obtain

$$\begin{aligned}
\Upsilon_5(\rho, \zeta, \tau) &\leq \xi_3(\rho, \zeta, \tau) \leq \xi_3(\rho, \zeta, 1) = \gamma_{17}(\rho, \zeta) + \gamma_{18}(\rho, \zeta) + \psi_3(\rho, \zeta) \\
&= 9\rho^6 - 72\rho^5 + 56\rho^4 + 72\rho^3 - 184\rho^2 + 128 + (16\rho^6 - 32\rho^5 - 48\rho^4 + 64\rho^3 + 48\rho^2 \\
&\quad - 32\rho - 16)\zeta^4 + (-32\rho^6 + 32\rho^5 + 64\rho^4 + 80\rho^3 - 104\rho^2 - 112\rho + 72)\zeta^3 + (20\rho^6 \\
&\quad + 104\rho^5 - 152\rho^4 - 136\rho^3 + 244\rho^2 + 32\rho - 112)\zeta^2 + (-18\rho^6 - 32\rho^5 + 18\rho^4 - 80\rho^3 \\
&\quad + 112\rho)\zeta = \aleph_6(\rho, \zeta).
\end{aligned}$$

Consider

$$\begin{cases} \frac{\partial \aleph_6}{\partial \rho} = 54\rho^5 - 360\rho^4 + 224\rho^3 + 216\rho^2 - 368\rho + (96\rho^5 - 160\rho^4 - 192\rho^3 + 192\rho^2 + 96\rho - 32)\zeta^4 \\ \quad + (-192\rho^5 + 160\rho^4 + 256\rho^3 + 240\rho^2 - 208\rho - 112)\zeta^3 + (120\rho^5 + 520\rho^4 - 608\rho^3 - 408\rho^2 + 488\rho \\ \quad + 32)\zeta^2 + (-108\rho^5 - 160\rho^4 + 72\rho^3 - 240\rho^2 + 112)\zeta = 0, \\ \frac{\partial \aleph_6}{\partial \zeta} = 112\rho - 80\rho^3 + 18\rho^4 - 32\rho^5 - 18\rho^6 + (64\rho^6 - 128\rho^5 - 192\rho^4 + 256\rho^3 + 192\rho^2 - 128\rho \\ \quad - 64)\zeta^3 + (-96\rho^6 + 96\rho^5 + 192\rho^4 + 240\rho^3 - 312\rho^2 - 336\rho + 216)\zeta^2 + (40\rho^6 + 208\rho^5 - 304\rho^4 \\ \quad - 272\rho^3 + 488\rho^2 + 64\rho - 224)\zeta = 0. \end{cases}$$

By calculating, we obtain

$$\begin{aligned}
&\begin{cases} \rho_{1,1} \approx 1, \\ \zeta_{1,1} \approx 1.4109, \end{cases} \quad \begin{cases} \rho_{1,2} \approx 1, \\ \zeta_{1,2} \approx -1.7545, \end{cases} \quad \begin{cases} \rho_{1,3} \approx 1, \\ \zeta_{1,3} \approx -0.6564, \end{cases} \quad \begin{cases} \rho_{1,4} \approx -1, \\ \zeta_{1,4} \approx 0.7193, \end{cases} \\
&\begin{cases} \rho_{1,5} \approx -1, \\ \zeta_{1,5} \approx -0.8476, \end{cases} \quad \begin{cases} \rho_{1,6} \approx -1, \\ \zeta_{1,6} \approx -0.2025, \end{cases} \quad \begin{cases} \rho_{1,7} = 0, \\ \zeta_{1,7} = 0, \end{cases} \quad \begin{cases} \rho_{1,8} \approx 1.6891, \\ \zeta_{1,8} \approx 2.7304, \end{cases} \quad \begin{cases} \rho_{1,9} \approx -0.2027, \\ \zeta_{1,9} \approx 1.0011, \end{cases} \\
&\begin{cases} \rho_{1,10} \approx -0.7386, \\ \zeta_{1,10} \approx -16.2075, \end{cases} \quad \begin{cases} \rho_{1,11} \approx -0.9197, \\ \zeta_{1,11} \approx 0.3865, \end{cases} \quad \begin{cases} \rho_{1,12} \approx -1.0616, \\ \zeta_{1,12} \approx -0.6001, \end{cases} \quad \begin{cases} \rho_{1,13} \approx -1.6501, \\ \zeta_{1,13} \approx 3.4859. \end{cases}
\end{aligned}$$

Therefore, there is no solution in $(0, 1) \times (0, 1)$.

(17) For $\rho = 1$,

$$\aleph_6(1, \zeta) = 9.$$

(18) For $\rho = 0$,

$$\aleph_6(0, \zeta) = 128 - 112\zeta^2 + 72\zeta^3 - 16\zeta^4 \leq 128.$$

(19) For $\zeta = 0$,

$$\aleph_6(\rho, 0) = 128 - 184\rho^2 + 72\rho^3 + 56\rho^4 - 72\rho^5 + 9\rho^6 \leq 128.$$

(20) For $\zeta = 1$,

$$\aleph_6(\rho, 1) = 72 + 4\rho^2 - 62\rho^4 - 5\rho^6 = \alpha_3(\rho) \leq \alpha_3(0.1793) \approx 72.0643.$$

Thus, we yield

$$|\eta_2\eta_4 - \eta_3^2| \leq \frac{128}{4608} = \frac{1}{36}.$$

4. The third Hankel determinant for C_{car}

Theorem 4.1. If $\kappa \in C_{car}$, then

$$|\mathcal{H}_{3,1}(\kappa)| \leq \frac{1}{81}.$$

The result is sharp and achieved by

$$\kappa_6(\varepsilon) = \int_0^\varepsilon \left(\exp \left(\int_0^x \left(\frac{4}{3}t^3 + \frac{2}{3}t^6 \right) dt \right) \right) dx = \varepsilon + \frac{1}{9}\varepsilon^4 + \dots \quad (4.1)$$

And the corresponding Schwarz function is $\omega(\varepsilon) = \varepsilon^3$. The image domain of extreme value function is a heart-shaped (cardioid) convex domain symmetric about the real axis with a cusp at the right end.

Proof: Let $\kappa \in C_{car}$. By using (1.9), we achieve

$$1 + \frac{\varepsilon \kappa''(\varepsilon)}{\kappa'(\varepsilon)} = 1 + \frac{4}{3}\omega(\varepsilon) + \frac{2}{3}\omega^2(\varepsilon), \quad (4.2)$$

where $\omega(\varepsilon)$ is known as Schwarz function.

Using series form of $\omega(\varepsilon)$ in right hand side of (4.2), we obtain

$$\begin{aligned} 1 + \frac{4}{3}\omega(\varepsilon) + \frac{2}{3}\omega^2(\varepsilon) &= 1 + \frac{4}{3}\rho_1\varepsilon + \frac{1}{3}(4\rho_2 + 2\rho_1^2)\varepsilon^2 + \frac{4}{3}(\rho_3 + \rho_1\rho_2)\varepsilon^3 \\ &+ \frac{2}{3}(2\rho_4 + 2\rho_1\rho_3 + \rho_2^2)\varepsilon^4 + \dots \end{aligned} \quad (4.3)$$

From (2.10) and (4.3), we achieve

$$\kappa_2 = \frac{2}{3}\rho_1, \quad (4.4)$$

$$\kappa_3 = \frac{1}{27}(6\rho_2 + 11\rho_1^2), \quad (4.5)$$

$$\kappa_4 = \frac{1}{81}(17\rho_1^3 + 27\rho_1\rho_2 + 9\rho_3), \quad (4.6)$$

$$\kappa_5 = \frac{1}{2430}(162\rho_4 + 684\rho_1^2\rho_2 + 189\rho_2^2 + 450\rho_3\rho_1 + 235\rho_1^4). \quad (4.7)$$

From (1.2) and (4.4)–(4.7), we have

$$\begin{aligned} \mathcal{H}_{3,1}(\kappa) &= \frac{1}{196830}(-2430\rho_3^2 + 1350\rho_1^3\rho_3 - 245\rho_1^6 + 1242\rho_2^3 + 738\rho_1^4\rho_2 - 2565\rho_1^2\rho_2^2 \\ &\quad - 486\rho_1^2\rho_4 + 2916\rho_2\rho_4). \end{aligned} \quad (4.8)$$

From (4.8) and Lemma 1.1, we have

$$\begin{aligned} H_{3,1}(\kappa) &= \frac{1}{196830} \left[-245\rho_1^6 + 486\bar{\rho}_1^2\zeta_1^4(1 - |\rho_1|^2)^2 - 2916\zeta_2^2|\zeta_1|^2(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2 \right. \\ &\quad - 972\zeta_2\bar{\rho}_1\zeta_1^2(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2 + 2916\zeta_3\zeta_1(1 - |\zeta_1|^2)(1 - |\rho_1|^2)^2(1 - |\zeta_2|^2) \\ &\quad - 2430\zeta_2^2(1 - |\zeta_1|^2)^2(1 - |\rho_1|^2)^2 - 2565\zeta_1^2\rho_1^2(1 - |\rho_1|^2)^2 + 1242\zeta_1^3(1 - |\rho_1|^2)^3 \\ &\quad + 738\rho_1^4\zeta_1(1 - |\rho_1|^2) - 486\rho_1^2\bar{\rho}_1^2\zeta_1^3(1 - |\rho_1|^2) + 486\zeta_2^2\rho_1^2\bar{\zeta}_1(1 - |\zeta_1|^2)(1 - |\rho_1|^2) \\ &\quad + 972\zeta_2\rho_1^2\bar{\rho}_1\zeta_1(1 - |\rho_1|^2)(1 - |\zeta_1|^2) - 486\zeta_3\rho_1^2(1 - |\zeta_2|^2)(1 - |\rho_1|^2)(1 - |\zeta_1|^2) \\ &\quad \left. - 1350\zeta_1^2\rho_1^3\bar{\rho}_1(1 - |\rho_1|^2) + 1350\zeta_2\rho_1^3(1 - |\rho_1|^2)(1 - |\zeta_1|^2) \right]. \end{aligned}$$

Therefore, we obtain

$$\mathcal{H}_{3,1}(\kappa) = \frac{1}{196830} \left[\sigma_{16}(\rho_1, \zeta_1) + \sigma_{17}(\rho_1, \zeta_2)\zeta_2 + \sigma_{18}(\rho_1, \zeta_1)\zeta_2^2 + \Gamma_6(\rho_1, \zeta_1, \zeta_2)\zeta_3 \right],$$

where

$$\begin{aligned} \sigma_{16}(\rho_1, \zeta_1) &= (1 - |\rho_1|^2) \left\{ [-256\rho_1^2\zeta_1^2 + 1242(1 - |\rho_1|^2)\zeta_1^3 + 486\bar{\rho}_1^2\zeta_1^4](1 - |\rho_1|^2) - 486\bar{\rho}_1^2\rho_1^2\zeta_1^3 \right. \\ &\quad \left. - 1350\bar{\rho}_1\rho_1^3\zeta_1^2 + 738\zeta_1\rho_1^4 \right\} - 245\rho_1^6, \\ \sigma_{17}(\rho_1, \zeta_1) &= (1 - |\zeta_1|^2)(1 - |\rho_1|^2) \left[-972\bar{\rho}_1\zeta_1^2(1 - |\rho_1|^2) + 1350\rho_1^3 + 972\rho_1^2\bar{\rho}_1\zeta_1 \right], \\ \sigma_{18}(\rho_1, \zeta_1) &= (1 - |\rho_1|^2)(1 - |\zeta_1|^2) \left[(2430 - 486|\zeta_1|^2)(1 - |\rho_1|^2) + 486\bar{\zeta}_1\rho_1^2 \right], \\ \Gamma_6(\rho_1, \zeta_1, \zeta_2) &= (1 - |\zeta_2|^2)(1 - |\zeta_1|^2)(1 - |\rho_1|^2) \left[-486\rho_1^2 + 2916(1 - |\rho_1|^2)\zeta_1 \right]. \end{aligned}$$

By replacing $|\zeta_1| = \zeta$, $|\rho_1| = \rho$, $|\zeta_2| = \tau$, and upon taking $|\zeta_3|$, we obtain

$$\begin{aligned} |\mathcal{H}_{3,1}(\kappa)| &\leq \frac{1}{196830} \left(|\sigma_{16}(\rho_1, \zeta_1)| + |\sigma_{17}(\rho_1, \zeta_1)|\tau + |\sigma_{18}(\rho_1, \zeta_1)|\tau^2 + |\Gamma_6(\rho_1, \zeta_1, \zeta_2)| \right) \\ &\leq \frac{1}{196830} \Upsilon_6(\rho, \zeta, \tau), \end{aligned}$$

where

$$\Upsilon_6(\rho, \zeta, \tau) = \gamma_{21}(\rho, \zeta) + \gamma_{22}(\rho, \zeta)\tau + \gamma_{23}(\rho, \zeta)\tau^2 + \gamma_{24}(\rho, \zeta)(1 - \tau^2),$$

with

$$\begin{aligned}\gamma_{21}(\rho, \zeta) &= 245\rho^6 + (1 - \rho^2)\left[(1242\zeta^3 - 1242\rho^2\zeta^3 + 486\rho^2\zeta^4 + 2565\rho^2\zeta^2)(1 - \rho^2) + 738\rho^4\zeta\right. \\ &\quad \left.+ 1350\rho^4\zeta^2 + 486\rho^4\zeta^3\right], \\ \gamma_{22}(\rho, \zeta) &= (1 - \rho^2)(1 - \zeta^2)\left[972\rho\zeta^2(1 - \rho^2) + 1350\rho^3 + 972\rho^3\zeta\right], \\ \gamma_{23}(\rho, \zeta) &= (1 - \zeta^2)(1 - \rho^2)\left[(2430 + 486\zeta^2)(1 - \rho^2) + 486\rho^2\zeta\right], \\ \gamma_{24}(\rho, \zeta) &= (1 - \zeta^2)(1 - \rho^2)\left[2916\zeta(1 - \rho^2) + 486\rho^2\right].\end{aligned}$$

Note that

$$\gamma_{23}(\rho, \zeta) \leq (1 - \rho^2)(1 - \zeta^2)\left[(1 - \rho^2)(486\zeta^2 + 2430) + 486\rho^2\right] = \psi_4(\rho, \zeta).$$

Hence, we obtain

$$\Upsilon_6(\rho, \zeta, \tau) \leq \gamma_{21}(\rho, \zeta) + \gamma_{22}(\rho, \zeta)\tau + \psi_4(\rho, \zeta)\tau^2 + \gamma_{24}(\rho, \zeta)(1 - \tau^2) = \xi_4(\rho, \zeta, \tau).$$

Differentiating $\xi_4(\rho, \zeta, \tau)$ with respect to τ , we have

$$\frac{\partial \xi_4}{\partial \tau} = \gamma_{22}(\rho, \zeta) + 2\tau[\psi_4(\rho, \zeta) - \gamma_{24}(\rho, \zeta)] = \gamma_{22}(\rho, \zeta) + 972(\zeta^2 - 6\zeta + 5)(1 - \zeta^2)(1 - \rho^2)^2\tau \geq 0.$$

Hence, ξ_4 is an increasing function and attains its maximum value at $\tau = 1$. So, we have

$$\begin{aligned}\Upsilon_6(\rho, \zeta, \tau) &\leq \xi_4(\rho, \zeta, \tau) \leq \xi_4(\rho, \zeta, 1) = \gamma_{21}(\rho, \zeta) + \gamma_{22}(\rho, \zeta) + \psi_4(\rho, \zeta) \\ &= 245\rho^6 - 1350\rho^5 + 1944\rho^4 + 1350\rho^3 - 4374\rho^2 + 2430 + (486\rho^6 - 972\rho^5 - 1458\rho^4 \\ &\quad + 1944\rho^3 + 1458\rho^2 - 972\rho - 486)\zeta^4 + (-1728\rho^6 + 972\rho^5 + 4212\rho^4 - 972\rho^3 - 3726\rho^2 \\ &\quad + 1242)\zeta^3 + (1215\rho^6 + 2322\rho^5 - 5238\rho^4 - 3294\rho^3 + 5967\rho^2 + 972\rho - 1944)\zeta^2 \\ &\quad + (-738\rho^6 - 972\rho^5 + 738\rho^4 + 972\rho^3)\zeta = \aleph_7(\rho, \zeta).\end{aligned}$$

The critical points of $\aleph_7(\rho, \zeta)$ satisfy the conditions

$$\begin{cases} \frac{\partial \aleph_7}{\partial \rho} = 1470\rho^5 - 6750\rho^4 + 7776\rho^3 + 4050\rho^2 - 8748\rho + (2916\rho^5 - 4860\rho^4 - 5832\rho^3 + 5832\rho^2 \\ \quad + 2916\rho - 972)\zeta^4 + (-10368\rho^5 + 4860\rho^4 + 16848\rho^3 - 2916\rho^2 - 7452\rho)\zeta^3 + (7290\rho^5 + 11610\rho^4 \\ \quad - 20952\rho^3 - 9882\rho^2 + 11934\rho + 972)\zeta^2 + (-4428\rho^5 - 4860\rho^4 + 2952\rho^3 + 2916\rho^2)\zeta = 0, \\ \frac{\partial \aleph_7}{\partial \zeta} = 972\rho^3 + 738\rho^4 - 972\rho^5 - 738\rho^6 + (1944\rho^6 - 3888\rho^5 - 5832\rho^4 + 7776\rho^3 + 5832\rho^2 - 3888\rho \\ \quad - 1944)\zeta^3 + (-5184\rho^6 + 2916\rho^5 + 12636\rho^4 - 2916\rho^3 - 11178\rho^2 + 3726)\zeta^2 + (2430\rho^6 + 4644\rho^5 \\ \quad - 10476\rho^4 - 6588\rho^3 + 11934\rho^2 + 1944\rho - 3888)\zeta = 0. \end{cases}$$

By calculating, we obtain

$$\begin{cases} \rho_{1,1} \approx 1, & \rho_{1,2} \approx 1, & \rho_{1,3} \approx 1, & \rho_{1,4} \approx -1, & \rho_{1,5} = 0, \\ \zeta_{1,1} \approx 1.7501, & \zeta_{1,2} \approx -2.1473, & \zeta_{1,3} \approx -0.6028, & \zeta_{1,4} \approx 0.7343, & \zeta_{1,5} = 0, \end{cases}$$

$$\begin{cases} \rho_{1,6} \approx 3.5077, \\ \zeta_{1,6} \approx 5.1611, \end{cases} \quad \begin{cases} \rho_{1,7} \approx -0.4222, \\ \zeta_{1,7} \approx 1.0806, \end{cases} \quad \begin{cases} \rho_{1,8} \approx -0.8245, \\ \zeta_{1,8} \approx -2.5111, \end{cases} \quad \begin{cases} \rho_{1,9} \approx -0.9060, \\ \zeta_{1,9} \approx 0.0747, \end{cases}$$

$$\begin{cases} \rho_{1,10} \approx -0.9482, \\ \zeta_{1,10} \approx -1.1199, \end{cases} \quad \begin{cases} \rho_{1,11} \approx -1.1989, \\ \zeta_{1,11} \approx 4.4103. \end{cases}$$

Thus, no solution exists in $(0, 1) \times (0, 1)$, resulting in no critical points.

(21) For $\rho = 1$,

$$\aleph_7(1, \zeta) = 245.$$

(22) For $\rho = 0$,

$$\aleph_7(0, \zeta) = 2430 - 1944\zeta^2 + 1242\zeta^3 - 486\zeta^4 \leq 2430.$$

(23) For $\zeta = 0$,

$$\aleph_7(\rho, 0) = 2430 - 4374\rho^2 + 1350\rho^3 + 1944\rho^4 - 1350\rho^5 + 245\rho^6 \leq 2430.$$

(24) For $\zeta = 1$,

$$\aleph_7(\rho, 1) = 1242 - 675\rho^2 + 198\rho^4 - 520\rho^6 \leq 1242.$$

Therefore, we have

$$|\mathcal{H}_{3,1}(\kappa)| \leq \frac{2430}{196830} = \frac{1}{81}.$$

The equality holds for $\kappa_4 = \frac{1}{9}$ and $\kappa_2 = \kappa_3 = \kappa_5 = 0$.

5. Conclusions

The Hankel determinant sharp bound analysis method established in this paper can be extended to nonlinear stochastic systems through fuzzy logic. For the problems of hidden modes and asynchronous switching in Markov jump systems, by drawing on the research ideas of asynchronous fuzzy dynamic sliding mode control and output feedback asynchronous fuzzy SMC, the coefficient functional estimation framework based on Schwarz–Pommerenke parameterization in this paper can be embedded into the Takagi–Sugeno (T-S) fuzzy modeling process. By matching fuzzy rules with the nonlinear subsystems and correcting the random jump parameters through the hidden mode detection mechanism, the bound estimation of the Hankel-type functional can be achieved within the fuzzy domain, thereby providing a new quantitative tool for the stability analysis of nonlinear stochastic systems and the synthesis of sliding mode controllers. This expands the application boundaries of the method in the modeling of random complex systems.

Based on the research results of this paper, future development can be further expanded in the following aspects: First, the method can be extended to higher-order Hankel determinants and more generalized subordinate kernels (such as multivariate star-shaped functions, hypergeometric function

classes, and bounded steering functions), breaking through the existing three-order limitation. Second, by combining fuzzy logic with Markov jump systems, the Schwarz parameter boundary estimation can be embedded into T-S fuzzy modeling, providing coefficient functional constraints for sliding mode control of nonlinear stochastic systems. Third, the geometric characterization of inverse function extremum functions can be deepened, clarifying the subordinate limit mapping properties corresponding to the boundaries, and exploring the association with the Bieberbach conjecture coefficient bounds. Additionally, attempts can be made to standardize interval arithmetic verification, resolving the controversy over the rigor of numerical optimization, and providing a replicable proof framework for similar extremum problems.

Author contributions

Dong Guo: Writing—original draft preparation, writing—review and editing; Xin Wang: Conceptualization, methodology, software, validation, formal analysis, writing—review and editing; Xi Luo: Methodology, software, validation, investigation, writing—original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no competing interest.

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