



Research article

Bilateral series weighted by periodic functions

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Abstract: By incorporating the derivative operator θ and hyperbolic cotangent function, a general summation formula is established for the “Dirichlet-like” bilateral Ω -series. Seven classes of infinite series identities are exemplified as consequences. They do not seem to have previously appeared in the literature.

Keywords: “Dirichlet-like” bilateral series; derivative operator; periodic function; binomial sum by telescoping; hyperbolic tangent (cotangent) function

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1. Introduction and outline

Denote by $\mathbb{N}$ and $\mathbb{Z}$ the sets of natural numbers and integers, respectively, with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For a periodic function $\chi(n)$ of period $\ell \in \mathbb{N}$, it is determined by the first ℓ -tuple

$$\{\chi(1), \chi(2), \dots, \chi(\ell)\} \quad \text{since} \quad \chi(k + j\ell) = \chi(k) \quad \text{for} \quad 1 \leq k \leq \ell \quad \text{and} \quad j \in \mathbb{Z}.$$

Then for $m, p \in \mathbb{N}$ and $q \in \mathbb{Z}$, we shall examine the “Dirichlet-like” bilateral series

$$\Omega_m(p, q, x; \chi) := \sum_{n=-\infty}^{\infty} \frac{\chi(n)}{(x^2 + (pn + q)^2)^m}, \quad (1)$$

where m and χ will be suppressed when $m = 1$ and/or $\chi(n) \equiv 1$. To evaluate this series explicitly, the following route path will be followed. First, reduce the series with weight χ to finitely many unweighted residue-class series. Then, evaluate the case $m = 1$ by the partial fraction expansion of $\coth$. Finally, work out higher powers by means of the differential operator $\frac{1}{x} \times \frac{d}{dx}$ together with derivative polynomials for $\tanh$ and $\coth$. This will concretely be realized by employing three crucial lemmas. The first one is a binomial formula as below.

Lemma 1 ($m \in \mathbb{N}$ and $j \in \mathbb{N}_0$).

$$\sum_{i=0}^j \left(\frac{1}{2}\right)^i \binom{m+i}{i} \frac{m-i}{m+i} = \frac{\binom{m+j}{m}}{2^j}.$$

Proof. In fact, this formula follows easily by telescoping. Introducing the sequence T_i and then calculating the difference

$$T_i = \frac{(m+1)_i}{2^i i!} \quad \text{and} \quad T_i - T_{i-1} = \frac{(m+1)_{i-1}}{2^i i!} (m-i) = \left(\frac{1}{2}\right)^i \binom{m+i}{i} \frac{m-i}{m+i};$$

we immediately have

$$\sum_{i=0}^j \left(\frac{1}{2}\right)^i \binom{m+i}{i} \frac{m-i}{m+i} = 1 + \sum_{i=1}^j (T_i - T_{i-1}) = 1 - T_0 + T_j = T_j = \frac{\binom{m+j}{m}}{2^j},$$

which gives rise to the desired formula. $\square$

Let θ stand for the differential operator $\frac{1}{x} \times \frac{d}{dx} = \frac{1}{x} \times \mathcal{D}$. Then it is not hard to verify that

$$\theta \Omega_m(p, q, x) = -2m \Omega_{m+1}(p, q, x) \Leftrightarrow \Omega_{m+1}(p, q, x) = \frac{(-\theta)}{2m} \Omega_m(p, q, x).$$

By iteration, we find that

$$\Omega_{m+1}(p, q, x) = \frac{(-\theta)^m}{2^m m!} \Omega(p, q, x). \quad (2)$$

To show an explicit expression, it is necessary to determine the higher powers of θ as in the second lemma below.

Lemma 2 ($m \in \mathbb{N}$).

$$\theta^m = \sum_{k=0}^m \left(\frac{-1}{2}\right)^k \frac{(m+k)!(m-k)}{m!(m+k)} \binom{m}{k} \frac{\mathcal{D}^{m-k}}{x^{m+k}}.$$

Proof. This can be proven by means of the induction principle on m . For $m = 1$, the equality is obviously true. Suppose the equality is valid for the positive integers $\leq m$. Then we have to verify the equality for $m + 1$. In fact, we have

$$\begin{aligned} \theta^{m+1} &= \theta \times \theta^m = \frac{\mathcal{D}}{x} \sum_{k=0}^m \left(\frac{-1}{2}\right)^k \frac{(m+k)!(m-k)}{m!(m+k)} \binom{m}{k} \frac{\mathcal{D}^{m-k}}{x^{m+k}} \\ &= \sum_{k=0}^m \left(\frac{-1}{2}\right)^k \frac{(m+k)!(m-k)}{m!(m+k)} \binom{m}{k} \frac{\mathcal{D}^{1+m-k}}{x^{1+m+k}} \\ &\quad - \sum_{k=0}^m \left(\frac{-1}{2}\right)^k \frac{(m+k)!(m-k)}{m!} \binom{m}{k} \frac{\mathcal{D}^{m-k}}{x^{2+m+k}}. \end{aligned}$$

Rewriting the latter sum by reindexing $k \rightarrow k - 1$

$$2 \sum_{k=1}^m \left(\frac{-1}{2}\right)^k \frac{(m+k)!k}{m!(m+k)} \binom{m}{k} \frac{\mathcal{D}^{1+m-k}}{x^{1+m+k}},$$

and then unifying it to the former one, we find that

$$\theta^{m+1} = \sum_{k=0}^{m+1} \left(\frac{-1}{2}\right)^k \frac{(m+k+1)!(1+m-k)}{(m+1)!(m+k+1)} \binom{m+1}{k} \frac{\mathcal{D}^{1+m-k}}{x^{1+m+k}},$$

where we have employed the binomial relation

$$\frac{(m+k)!}{m!} \binom{m}{k} \left\{ \frac{m-k}{m+k} + \frac{2k}{m+k} \right\} = \frac{(m+k)!}{m!} \binom{m}{k} = \frac{(m+k+1)!(1+m-k)}{(m+1)!(m+k+1)} \binom{m+1}{k}.$$

This confirms the claimed binomial expression. $\square$

The first five examples are displayed as follows:

$$\begin{aligned}\theta &= \frac{\mathcal{D}}{x}, \\ \theta^2 &= \frac{\mathcal{D}^2}{x^2} - \frac{\mathcal{D}}{x^3}, \\ \theta^3 &= \frac{\mathcal{D}^3}{x^3} - 3\frac{\mathcal{D}^2}{x^4} + 3\frac{\mathcal{D}}{x^5}, \\ \theta^4 &= \frac{\mathcal{D}^4}{x^4} - 6\frac{\mathcal{D}^3}{x^5} + 15\frac{\mathcal{D}^2}{x^6} - 15\frac{\mathcal{D}}{x^7}, \\ \theta^5 &= \frac{\mathcal{D}^5}{x^5} - 10\frac{\mathcal{D}^4}{x^6} + 45\frac{\mathcal{D}^3}{x^7} - 105\frac{\mathcal{D}^2}{x^8} + 105\frac{\mathcal{D}}{x^9}.\end{aligned}$$

Denote by $S(n, k)$ the Stirling number of the second kind (cf. Comtet [4, Chapter 5] and [7, §6.1]). We shall make use of the polynomial $P_n(y)$ of degree $n+1$. They are explicitly defined (cf. Boyadzhiev [1]) as follows:

$$P_0(y) = y \quad \text{and} \quad P_n(y) = 2^n(y-1) \sum_{j=1}^n (-1)^j \frac{j!}{2^j} S(n, j)(1+y)^j. \quad (3)$$

These polynomials can be used to represent higher derivatives of hyperbolic functions as highlighted in the third lemma below.

Lemma 3 (Boyadzhiev [1], see also Chu and Wang [3]).

$$\mathcal{D}^n\{\tanh x\} = P_n(\tanh x) \quad \text{and} \quad \mathcal{D}^n\{\coth x\} = P_n(\coth x).$$

The initial polynomials are given as follows:

$$\begin{aligned}P_0(y) &= y, \\ P_1(y) &= 1 - y^2, \\ P_2(y) &= -2y(1 - y^2), \\ P_3(y) &= (6y^2 - 2)(1 - y^2), \\ P_4(y) &= (16y - 24y^3)(1 - y^2),\end{aligned}$$

$$P_5(y) = (120y^4 - 120y^2 + 16)(1 - y^2).$$

As preliminary results (for $m = 1$), explicit formulae for $\Omega(p, q, x; \chi)$ (Propositions 4 and 5) will be derived in Section 2. Then the preceding three lemmas will be incorporated in Section 3 to prove a very general summation formula for $\Omega_m(p, q, x; \chi)$ as in Theorem 6. Seven classes of infinite series identities will be illustrated as consequences in Section 4. Finally, the paper ends with concluding comments in Section 5.

2. The initial series with $m = 1$

Let us start from the following simple, but important case.

Proposition 4. *Let $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. Then, for an indeterminate $x > 0$, we have the algebraic identities:*

$$\begin{aligned} \Omega(p, q, x) &= \sum_{n=-\infty}^{\infty} \frac{1}{x^2 + (pn + q)^2} = \frac{\pi}{2px} \left\{ \coth\left(\frac{\pi(x + qi)}{p}\right) + \coth\left(\frac{\pi(x - qi)}{p}\right) \right\} \\ &= \frac{\pi \sinh\left(\frac{2\pi x}{p}\right)}{px \left\{ \cosh\left(\frac{2\pi x}{p}\right) - \cos\left(\frac{2\pi q}{p}\right) \right\}} = \frac{\pi \sinh\left(\frac{2\pi x}{p}\right)}{2px \sinh\left(\frac{\pi(x+qi)}{p}\right) \sinh\left(\frac{\pi(x-qi)}{p}\right)}. \end{aligned}$$

Proof. Recalling the partial fraction decomposition [6, §1.42 (4)]

$$\coth(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} = \frac{1}{\pi} \lim_{n \rightarrow \infty} \sum_{k=-n}^n \frac{1}{x + ki},$$

we can deduce the expansion

$$\begin{aligned} \coth\left(\frac{\pi(x + qi)}{p}\right) + \coth\left(\frac{\pi(x - qi)}{p}\right) &= \frac{1}{\pi} \lim_{n \rightarrow \infty} \sum_{k=-n}^n \left\{ \frac{1}{\frac{x+qi}{p} + ki} + \frac{1}{\frac{x-qi}{p} - ki} \right\} \\ &= \frac{2px}{\pi} \lim_{n \rightarrow \infty} \sum_{k=-n}^n \frac{1}{x^2 + (pk + q)^2}, \end{aligned}$$

which is equivalent to the first expression stated in Proposition 4. The second and the third expressions follow from the first one after some routine trigonometric simplification. $\square$

For $n \in \mathbb{Z}$, by writing $n \rightarrow k + j\ell$ with $1 \leq k \leq \ell$ and $j \in \mathbb{Z}$, we have the analytic formula.

Proposition 5. *Let χ be a periodic function of period ℓ . Then, under the same conditions of Proposition 4, the following formula holds:*

$$\begin{aligned} \Omega(p, q, x; \chi) &= \sum_{n=-\infty}^{\infty} \frac{\chi(n)}{x^2 + (pn + q)^2} \\ &= \sum_{k=1}^{\ell} \sum_{j=-\infty}^{\infty} \frac{\chi(k)}{x^2 + (p(k + j\ell) + q)^2} \\ &= \frac{\pi}{p\ell x} \sum_{k=1}^{\ell} \frac{\chi(k) \sinh\left(\frac{2\pi x}{p\ell}\right)}{\cosh\left(\frac{2\pi x}{p\ell}\right) - \cos\left(\frac{2pk\pi + 2q\pi}{p\ell}\right)}. \end{aligned}$$

By specifying function χ , we illustrate seven infinite series identities as below.

- $p = 1$ and $q = 0$ with $\chi(n) \equiv 1$ and $\ell = 1$ (cf. [6, §1.42]):

$$\sum_{n=-\infty}^{\infty} \frac{1}{x^2 + n^2} = \frac{\pi}{x} \frac{\sinh(2\pi x)}{\cosh(2\pi x) - 1} = \frac{\pi}{x} \coth(\pi x).$$

- $p = 1$ and $q = 0$ with $\chi(n) = (-1)^n$ and $\ell = 2$ (cf. [6, §1.42]):

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x^2 + n^2} &= \frac{\pi}{2x} \left\{ \frac{\sinh(\pi x)}{\cosh(\pi x) - 1} - \frac{\sinh(\pi x)}{\cosh(\pi x) + 1} \right\} \\ &= \frac{\pi}{x} \left\{ \frac{\sinh(\pi x)}{\cosh^2(\pi x) - 1} \right\} = \frac{\pi}{x} \operatorname{csch}(\pi x). \end{aligned}$$

- $p = 1$ and $q = 0$ with $\chi(n) = (-1)^{\binom{n}{2}}$. The period of χ equals $\ell = 4$, since it is routine to check that

$$\chi(n+4) = \chi(n) \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline n & 1 & 2 & 3 & 4 \\ \hline \chi(n) & 1 & -1 & -1 & 1 \\ \hline \end{array}. \quad \text{Then}$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{x^2 + n^2} &= \frac{\pi}{4x} \sum_{k=1}^4 \frac{\chi(k) \sinh(\frac{\pi x}{2})}{\cosh(\frac{\pi x}{2}) - \cos(\frac{k\pi}{2})} \\ &= \frac{\pi}{4x} \sinh\left(\frac{\pi x}{2}\right) \left\{ \frac{1}{\cosh(\frac{\pi x}{2})} - \frac{1}{\cosh(\frac{\pi x}{2}) + 1} - \frac{1}{\cosh(\frac{\pi x}{2})} + \frac{1}{\cosh(\frac{\pi x}{2}) - 1} \right\} \\ &= \frac{\pi}{4x} \sinh\left(\frac{\pi x}{2}\right) \frac{2}{\cosh^2(\frac{\pi x}{2}) - 1} = \frac{\pi}{2x} \operatorname{csch}\left(\frac{\pi x}{2}\right). \end{aligned}$$

- $p = 1$ and $q = 0$ with $\chi(n) = (-1)^{\binom{n}{3}}$. The period of χ is also equal to $\ell = 4$ since

$$\chi(n+4) = \chi(n) \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline n & 1 & 2 & 3 & 4 \\ \hline \chi(n) & 1 & 1 & -1 & 1 \\ \hline \end{array}. \quad \text{Then}$$

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{x^2 + n^2} &= \frac{\pi}{4x} \sum_{k=1}^4 \frac{\chi(k) \sinh(\frac{\pi x}{2})}{\cosh(\frac{\pi x}{2}) - \cos(\frac{k\pi}{2})} \\ &= \frac{\pi}{4x} \sinh\left(\frac{\pi x}{2}\right) \left\{ \frac{1}{\cosh(\frac{\pi x}{2})} + \frac{1}{\cosh(\frac{\pi x}{2}) + 1} - \frac{1}{\cosh(\frac{\pi x}{2})} + \frac{1}{\cosh(\frac{\pi x}{2}) - 1} \right\} \\ &= \frac{\pi}{4x} \sinh\left(\frac{\pi x}{2}\right) \frac{2 \cosh(\frac{\pi x}{2})}{\cosh^2(\frac{\pi x}{2}) - 1} = \frac{\pi}{2x} \coth\left(\frac{\pi x}{2}\right). \end{aligned}$$

- $p = 2$ and $q = -1$ with $\chi(n) \equiv 1$ and $\ell = 1$ (cf. [6, §1.42]):

$$\sum_{n=-\infty}^{\infty} \frac{1}{x^2 + (2n-1)^2} = \frac{\pi}{2x} \frac{\sinh(\pi x)}{\cosh(\pi x) + 1} = \frac{\pi}{2x} \tanh\left(\frac{\pi x}{2}\right).$$

- $p = 2$ and $q = -1$ with $\chi(n) = (-1)^{\binom{n}{2}}$ and $\ell = 4$:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{x^2 + (2n-1)^2} &= \frac{\pi}{8x} \sum_{k=1}^4 \frac{\chi(k) \sinh\left(\frac{\pi x}{4}\right)}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{2k\pi - \pi}{4}\right)} \\ &= \frac{\pi}{8x} \sinh\left(\frac{\pi x}{4}\right) \left\{ \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{\pi}{4}\right)} - \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{3\pi}{4}\right)} \right. \\ &\quad \left. - \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{5\pi}{4}\right)} + \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{7\pi}{4}\right)} \right\} \\ &= \frac{\pi}{4x} \sinh\left(\frac{\pi x}{4}\right) \frac{\sqrt{2}}{\cosh^2\left(\frac{\pi x}{4}\right) - \frac{1}{2}} = \frac{\pi \sinh\left(\frac{\pi x}{4}\right)}{\sqrt{2}x \cosh\left(\frac{\pi x}{2}\right)}. \end{aligned}$$

- $p = 2$ and $q = -1$ with $\chi(n) = (-1)^{\binom{n}{3}}$ and $\ell = 4$:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{x^2 + (2n-1)^2} &= \frac{\pi}{8x} \sum_{k=1}^4 \frac{\chi(k) \sinh\left(\frac{\pi x}{4}\right)}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{2k\pi - \pi}{4}\right)} \\ &= \frac{\pi}{8x} \sinh\left(\frac{\pi x}{4}\right) \left\{ \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{\pi}{4}\right)} + \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{3\pi}{4}\right)} \right. \\ &\quad \left. - \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{5\pi}{4}\right)} + \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \cos\left(\frac{7\pi}{4}\right)} \right\} \\ &= \frac{\pi}{4x} \sinh\left(\frac{\pi x}{4}\right) \frac{1}{\cosh\left(\frac{\pi x}{4}\right) - \frac{\sqrt{2}}{2}} = \frac{\pi}{2x} \frac{\sinh\left(\frac{\pi x}{4}\right)}{2 \cosh\left(\frac{\pi x}{4}\right) - \sqrt{2}}. \end{aligned}$$

3. The main theorem and proof

Now we are ready to demonstrate the following general theorem.

Theorem 6. Let $m \in \mathbb{N}_0$ and χ be a periodic function of period ℓ . Then under the same conditions of Proposition 4, the following formula holds:

$$\begin{aligned} \Omega_{m+1}(p, q, x; \chi) &= \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{p\ell}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \sum_{k=1}^{\ell} \chi(k) \\ &\quad \times \left\{ P_j \left(\coth \frac{\pi(x + p\mathbf{i} + q\mathbf{i})}{p\ell} \right) + P_j \left(\coth \frac{\pi(x - p\mathbf{i} - q\mathbf{i})}{p\ell} \right) \right\}. \end{aligned}$$

Proof. When $m = 0$, the assertion is precisely Proposition 5. Hence, we may assume $m \geq 1$. For $n \in \mathbb{Z}$, we can write $n \rightarrow k + j\ell$ with $1 \leq k \leq \ell$ and $j \in \mathbb{Z}$. Then, we can reformulate the bilateral series

$$\begin{aligned} \Omega_m(p, q, x; \chi) &= \sum_{n=-\infty}^{\infty} \frac{\chi(n)}{(x^2 + (pn + q)^2)^m} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\ell} \frac{\chi_k(n)}{(x^2 + (pn + q)^2)^m} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\ell} \sum_{j=-\infty}^{\infty} \frac{\chi(k)}{(x^2 + (p(k+j\ell) + q)^2)^m} \\
&= \sum_{k=1}^{\ell} \chi(k) \Omega_m(p\ell, pk + q, x).
\end{aligned}$$

By combining this with (2), we can proceed with the series

$$\begin{aligned}
\Omega_{m+1}(p, q, x; \chi) &= \frac{(-\theta)^m}{2^m m!} \sum_{k=1}^{\ell} \chi(k) \Omega(p\ell, pk + q, x) \\
&= \frac{(-1)^m}{2^m m!} \sum_{k=1}^{\ell} \frac{\chi(k)}{2p\ell} \theta^m \left\{ \frac{\pi}{x} \coth\left(\frac{\pi(x + p\mathbf{k} + q\mathbf{i})}{p\ell}\right) + \frac{\pi}{x} \coth\left(\frac{\pi(x - p\mathbf{k} - q\mathbf{i})}{p\ell}\right) \right\} \\
&= \frac{(-1)^m}{2^m m!} \sum_{k=1}^{\ell} \frac{\chi(k)}{2p\ell} \sum_{i=0}^m \left(\frac{-1}{2}\right)^i \frac{(m+i)!(m-i)}{m!(m+i)} \binom{m}{i} \\
&\quad \times \frac{\mathcal{D}^{m-i}}{x^{m+i}} \left\{ \frac{\pi}{x} \coth\left(\frac{\pi(x + p\mathbf{k} + q\mathbf{i})}{p\ell}\right) + \frac{\pi}{x} \coth\left(\frac{\pi(x - p\mathbf{k} - q\mathbf{i})}{p\ell}\right) \right\}.
\end{aligned}$$

According to the Leibniz rule, the higher derivatives can be determined as below:

$$\begin{aligned}
&\frac{\mathcal{D}^{m-i}}{x^{m+i}} \left\{ \frac{\pi}{x} \coth\left(\frac{\pi(x + p\mathbf{k} + q\mathbf{i})}{p\ell}\right) + \frac{\pi}{x} \coth\left(\frac{\pi(x - p\mathbf{k} - q\mathbf{i})}{p\ell}\right) \right\} \\
&= \frac{\pi}{x^{m+i}} \sum_{j=i}^m \binom{m-i}{j-i} \mathcal{D}^{j-i} \left(\frac{1}{x}\right) \mathcal{D}^{m-j} \left\{ \coth\left(\frac{\pi(x + p\mathbf{k} + q\mathbf{i})}{p\ell}\right) + \coth\left(\frac{\pi(x - p\mathbf{k} - q\mathbf{i})}{p\ell}\right) \right\} \\
&= \sum_{j=i}^m \frac{\pi(i-m)_{j-i}}{x^{1+m+j}} \mathcal{D}^{m-j} \left\{ \coth\left(\frac{\pi(x + p\mathbf{k} + q\mathbf{i})}{p\ell}\right) + \coth\left(\frac{\pi(x - p\mathbf{k} - q\mathbf{i})}{p\ell}\right) \right\}.
\end{aligned}$$

Recalling Lemma 3 and then making substitution, we have further

$$\begin{aligned}
\Omega_{m+1}(p, q, x; \chi) &= \frac{(-1)^m}{2^{m+1} m!} \sum_{i=0}^m \left(\frac{-1}{2}\right)^i \frac{(m+i)!(m-i)}{m!(m+i)} \binom{m}{i} \sum_{j=i}^m \frac{(i-m)_{j-i}}{x^{1+m+j}} \left(\frac{\pi}{p\ell}\right)^{1+m-j} \\
&\quad \times \sum_{k=1}^{\ell} \chi(k) \left\{ P_{m-j} \left(\coth \frac{\pi(x + p\mathbf{k} + q\mathbf{i})}{p\ell} \right) + P_{m-j} \left(\coth \frac{\pi(x - p\mathbf{k} - q\mathbf{i})}{p\ell} \right) \right\} \\
&= \frac{(-1)^m}{2^{m+1} m!} \sum_{j=0}^m \left(\frac{\pi}{p\ell}\right)^{1+m-j} \sum_{i=0}^j \left(\frac{-1}{2}\right)^i \frac{(m+i)!(m-i)}{m!(m+i)} \binom{m}{i} \frac{(i-m)_{j-i}}{x^{1+m+j}} \\
&\quad \times \sum_{k=1}^{\ell} \chi(k) \left\{ P_{m-j} \left(\coth \frac{\pi(x + p\mathbf{k} + q\mathbf{i})}{p\ell} \right) + P_{m-j} \left(\coth \frac{\pi(x - p\mathbf{k} - q\mathbf{i})}{p\ell} \right) \right\},
\end{aligned}$$

where by making use of Lemma 1, the binomial sum with respect to i has been reduced to

$$\sum_{i=0}^j \left(\frac{-1}{2}\right)^i \frac{(m+i)!(m-i)}{m!(m+i)} \binom{m}{i} \frac{(i-m)_{j-i}}{x^{1+m+j}}$$

$$\begin{aligned}
&= \frac{m!(-1)^j}{(m-j)!x^{1+m+j}} \sum_{i=0}^j \left(\frac{1}{2}\right)^i \binom{m+i}{i} \frac{m-i}{m+i} \\
&= \frac{m!(-1)^j}{(m-j)!x^{1+m+j}} \times \frac{\binom{m+j}{m}}{2^j} = \left(\frac{-1}{2}\right)^j \binom{m}{j} \frac{(m+j)!}{m!x^{1+m+j}}.
\end{aligned}$$

Finally, we arrive at the following expression after substitution

$$\begin{aligned}
\Omega_{m+1}(p, q, x; \chi) &= \frac{(-1)^m}{2^{m+1}m!} \sum_{j=0}^m \left(\frac{\pi}{p\ell}\right)^{1+m-j} \left(\frac{-1}{2}\right)^j \binom{m}{j} \frac{(m+j)!}{m!x^{1+m+j}} \\
&\times \sum_{k=1}^{\ell} \chi(k) \left\{ P_{m-j} \left(\coth \frac{\pi(x + pk\mathbf{i} + q\mathbf{i})}{p\ell} \right) + P_{m-j} \left(\coth \frac{\pi(x - pk\mathbf{i} - q\mathbf{i})}{p\ell} \right) \right\}.
\end{aligned}$$

Under the replacement $j \rightarrow m - j$, this becomes the formula displayed in Theorem 6. $\square$

4. Seven classes of infinite series identities

By specifying $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ as well as periodic function $\chi(n)$ in Theorem 6, we are going to explore explicit evaluations for seven classes of infinite series as examples. Following this presentation, the reader can examine more series with different periodic function χ , even though the resulting expressions may be more involved.

For each formula stated in the corresponding corollary, the specific examples with $m = 1, 2, 3$ are obtained by making use of the following Mathematica commands to simplify the related expressions:

```
FullSimplify[ComplexExpand[Re[uvw], TargetFunctions->Re]]
Factor[Simplify[TrigFactor[%]]]
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4.1. $p = 1$ and $q = 0$ with $\chi(n) \equiv 1$ and $\ell = 1$

In this case, we deduce from Theorem 6

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + n^2)^{m+1}} &= \sum_{j=0}^m (-1)^j \frac{\pi^{1+j}}{(m!)^2} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\
&\times \left\{ P_j(\coth(\pi x + \pi\mathbf{i})) + P_j(\coth(\pi x - \pi\mathbf{i})) \right\}.
\end{aligned}$$

Observing that

$$2 \coth(\pi x) = \coth(\pi x + \pi\mathbf{i}) + \coth(\pi x - \pi\mathbf{i}),$$

we derive, by differentiating j -times with respect to x and then using Lemma 3, the corresponding relation

$$2P_j(\coth(\pi x)) = P_j(\coth(\pi x + \pi\mathbf{i})) + P_j(\coth(\pi x - \pi\mathbf{i})).$$

Thus, we obtain the following simplified formula.

Corollary 7 ($p = 1$ and $q = 0$ with $\chi(n) \equiv 1$ and $\ell = 1$).

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + n^2)^{m+1}} = 2 \sum_{j=0}^m \frac{(-1)^j \pi^{1+j}}{(2x)^{1+2m-j}} \binom{m}{j} \frac{(2m-j)!}{(m!)^2} P_j(\coth(\pi x)).$$

The first three series are highlighted in closed form as follows.

Example 8.

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + n^2)^2} &= \frac{\pi \operatorname{csch}^2(\pi x)}{4x^3} \{2\pi x + \sinh(2\pi x)\}, \\ \sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + n^2)^3} &= \frac{\pi \operatorname{csch}^2(\pi x)}{32x^5} \left\{ 12\pi x + 3 \cosh(3\pi x) \operatorname{csch}(\pi x) + (8\pi^2 x^2 - 3) \coth(\pi x) \right\}, \\ \sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + n^2)^4} &= \frac{\operatorname{csch}^4(\pi x)}{384x^7} \left\{ 32\pi^3 x^3 - 60\pi x + 2(24\pi^2 x^2 - 15) \sinh(2\pi x) \right. \\ &\quad \left. + 4\pi x(4\pi^2 x^2 + 15) \cosh(2\pi x) + 15 \sinh(4\pi x) \right\}. \end{aligned}$$

4.2. $p = 1$ and $q = 0$ with $\chi(n) = (-1)^n$ and $\ell = 2$

According to Theorem 6, the corresponding series becomes

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(x^2 + n^2)^{m+1}} &= \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{2}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\ &\quad \times \left\{ P_j\left(\coth \frac{\pi(x+2\mathbf{i})}{2}\right) - P_j\left(\coth \frac{\pi(x+\mathbf{i})}{2}\right) \right. \\ &\quad \left. + P_j\left(\coth \frac{\pi(x-2\mathbf{i})}{2}\right) - P_j\left(\coth \frac{\pi(x-\mathbf{i})}{2}\right) \right\}. \end{aligned}$$

Taking into account the trigonometric relation

$$\begin{aligned} 2 \coth\left(\frac{\pi x}{2}\right) - 2 \tanh\left(\frac{\pi x}{2}\right) &= \coth\left(\frac{\pi(x+2\mathbf{i})}{2}\right) + \coth\left(\frac{\pi(x-2\mathbf{i})}{2}\right) \\ &\quad - \coth\left(\frac{\pi(x+\mathbf{i})}{2}\right) - \coth\left(\frac{\pi(x-\mathbf{i})}{2}\right), \end{aligned}$$

that can be verified manually (or by Wolfram Mathematica version 11), we derive analogously the following related equation about the P_j -polynomial:

$$\begin{aligned} 2P_j\left(\coth \frac{\pi x}{2}\right) - 2P_j\left(\tanh \frac{\pi x}{2}\right) &= P_j\left(\coth \frac{\pi(x+2\mathbf{i})}{2}\right) + P_j\left(\coth \frac{\pi(x-2\mathbf{i})}{2}\right) \\ &\quad - P_j\left(\coth \frac{\pi(x+\mathbf{i})}{2}\right) - P_j\left(\coth \frac{\pi(x-\mathbf{i})}{2}\right). \end{aligned}$$

Consequently, this leads us to the following analytic formula.

Corollary 9 ($p = 1$ and $q = 0$ with $\chi(n) \equiv (-1)^n$ and $\ell = 2$).

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(x^2 + n^2)^{m+1}} = 2 \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{2}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \left\{ P_j\left(\coth \frac{\pi x}{2}\right) - P_j\left(\tanh \frac{\pi x}{2}\right) \right\}.$$

We display three initial formulae as examples.

Example 10.

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(x^2 + n^2)^2} &= \frac{\pi \operatorname{csch}(\pi x)}{2x^3} \{1 + \pi x \coth(\pi x)\}, \\ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(x^2 + n^2)^3} &= \frac{\pi \operatorname{csch}^3(\pi x)}{16x^5} \left\{ 3\pi^2 x^2 - 3 + (3 + \pi^2 x^2) \cosh(2\pi x) + 3\pi x \sinh(2\pi x) \right\}, \\ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(x^2 + n^2)^4} &= \frac{\pi \operatorname{csch}^3(\pi x)}{192x^7} \left\{ 36\pi^2 x^2 + 60 \sinh^2(\pi x) + \pi x(23\pi^2 x^2 - 15) \coth(\pi x) \right. \\ &\quad \left. + 12\pi^2 x^2 \cosh(2\pi x) + \pi x(15 + \pi^2 x^2) \cosh(3\pi x) \operatorname{csch}(\pi x) \right\}.\end{aligned}$$

4.3. $p = 1$ and $q = 0$ with $\chi(n) \equiv (-1)^{\binom{n}{2}}$ and $\ell = 4$

In this case, we can write down the series corresponding to Theorem 6:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + n^2)^{m+1}} &= \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{4}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\ &\times \left\{ P_j\left(\coth \frac{\pi(x+\mathbf{i})}{4}\right) + P_j\left(\coth \frac{\pi(x-\mathbf{i})}{4}\right) - P_j\left(\coth \frac{\pi(x+2\mathbf{i})}{4}\right) - P_j\left(\coth \frac{\pi(x-2\mathbf{i})}{4}\right) \right. \\ &\quad \left. - P_j\left(\coth \frac{\pi(x+3\mathbf{i})}{4}\right) - P_j\left(\coth \frac{\pi(x-3\mathbf{i})}{4}\right) + P_j\left(\coth \frac{\pi(x+4\mathbf{i})}{4}\right) + P_j\left(\coth \frac{\pi(x-4\mathbf{i})}{4}\right) \right\}.\end{aligned}$$

By invoking an easily proven identity (using Wolfram Mathematica version 11)

$$\begin{aligned}2 \coth\left(\frac{\pi x}{4}\right) - 2 \tanh\left(\frac{\pi x}{4}\right) &= \coth\left(\frac{\pi(x+\mathbf{i})}{4}\right) + \coth\left(\frac{\pi(x-\mathbf{i})}{4}\right) \\ &\quad - \coth\left(\frac{\pi(x+2\mathbf{i})}{4}\right) - \coth\left(\frac{\pi(x-2\mathbf{i})}{4}\right) \\ &\quad - \coth\left(\frac{\pi(x+3\mathbf{i})}{4}\right) - \coth\left(\frac{\pi(x-3\mathbf{i})}{4}\right) \\ &\quad + \coth\left(\frac{\pi(x+4\mathbf{i})}{4}\right) + \coth\left(\frac{\pi(x-4\mathbf{i})}{4}\right),\end{aligned}$$

we can show, by differentiating j -times with respect to x and then using Lemma 3, the following formula

$$\begin{aligned}2P_j\left(\coth \frac{\pi x}{4}\right) - 2P_j\left(\tanh \frac{\pi x}{4}\right) &= P_j\left(\coth \frac{\pi(x+\mathbf{i})}{4}\right) + P_j\left(\coth \frac{\pi(x-\mathbf{i})}{4}\right) \\ &\quad - P_j\left(\coth \frac{\pi(x+2\mathbf{i})}{4}\right) - P_j\left(\coth \frac{\pi(x-2\mathbf{i})}{4}\right) \\ &\quad - P_j\left(\coth \frac{\pi(x+3\mathbf{i})}{4}\right) - P_j\left(\coth \frac{\pi(x-3\mathbf{i})}{4}\right) \\ &\quad + P_j\left(\coth \frac{\pi(x+4\mathbf{i})}{4}\right) + P_j\left(\coth \frac{\pi(x-4\mathbf{i})}{4}\right).\end{aligned}$$

Therefore, we have established the general formula as below.

Corollary 11 ($p = 1$ and $q = 0$ with $\chi(n) \equiv (-1)^{\binom{n}{2}}$ and $\ell = 4$).

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + n^2)^{m+1}} = 2 \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{4}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \left\{ P_j \left(\coth \frac{\pi x}{4} \right) - P_j \left(\tanh \frac{\pi x}{4} \right) \right\}.$$

The first three series are given below as applications.

Example 12.

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + n^2)^2} &= \frac{\pi \operatorname{csch}(\frac{\pi x}{2})}{8x^3} \left\{ 2 + \pi x \coth \left(\frac{\pi x}{2} \right) \right\}, \\ \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + n^2)^3} &= \frac{\pi \operatorname{csch}^3(\frac{\pi x}{2})}{128x^5} \left\{ 3\pi^2 x^2 - 12 + (12 + \pi^2 x^2) \cosh(\pi x) + 6\pi x \sinh(\pi x) \right\}, \\ \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + n^2)^4} &= \frac{\pi \operatorname{csch}^3(\frac{\pi x}{2})}{3072x^7} \left\{ 72\pi^2 x^2 + 24\pi^2 x^2 \cosh(\pi x) + \pi x(23\pi^2 x^2 - 60) \coth \left(\frac{\pi x}{2} \right) \right. \\ &\quad \left. + 480 \sinh^2 \left(\frac{\pi x}{2} \right) + (60\pi x + \pi^3 x^3) \cosh \left(\frac{3\pi x}{2} \right) \operatorname{csch} \left(\frac{\pi x}{2} \right) \right\}. \end{aligned}$$

4.4. $p = 1$ and $q = 0$ with $\chi(n) \equiv (-1)^{\binom{n}{3}}$ and $\ell = 4$

By applying Theorem 6, we can explicitly state the corresponding series as

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + n^2)^{m+1}} &= \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{4}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\ &\times \left\{ P_j \left(\coth \frac{\pi(x + \mathbf{i})}{4} \right) + P_j \left(\coth \frac{\pi(x - \mathbf{i})}{4} \right) + P_j \left(\coth \frac{\pi(x + 2\mathbf{i})}{4} \right) + P_j \left(\coth \frac{\pi(x - 2\mathbf{i})}{4} \right) \right. \\ &\left. - P_j \left(\coth \frac{\pi(x + 3\mathbf{i})}{4} \right) - P_j \left(\coth \frac{\pi(x - 3\mathbf{i})}{4} \right) + P_j \left(\coth \frac{\pi(x + 4\mathbf{i})}{4} \right) + P_j \left(\coth \frac{\pi(x - 4\mathbf{i})}{4} \right) \right\}. \end{aligned}$$

Similar to the procedure shown in Subsection 4.3, we have the trigonometric formula (easily verifiable by Wolfram Mathematica version 11)

$$\begin{aligned} 2 \coth \left(\frac{\pi x}{4} \right) + 2 \tanh \left(\frac{\pi x}{4} \right) &= \coth \left(\frac{\pi(x + \mathbf{i})}{4} \right) + \coth \left(\frac{\pi(x - \mathbf{i})}{4} \right) \\ &+ \coth \left(\frac{\pi(x + 2\mathbf{i})}{4} \right) + \coth \left(\frac{\pi(x - 2\mathbf{i})}{4} \right) \\ &- \coth \left(\frac{\pi(x + 3\mathbf{i})}{4} \right) - \coth \left(\frac{\pi(x - 3\mathbf{i})}{4} \right) \\ &+ \coth \left(\frac{\pi(x + 4\mathbf{i})}{4} \right) + \coth \left(\frac{\pi(x - 4\mathbf{i})}{4} \right). \end{aligned}$$

This is equivalent to the identity below (by differentiating j -times with respect to x and then making use of Lemma 3)

$$2P_j \left(\coth \frac{\pi x}{4} \right) + 2P_j \left(\tanh \frac{\pi x}{4} \right) = P_j \left(\coth \frac{\pi(x + \mathbf{i})}{4} \right) + P_j \left(\coth \frac{\pi(x - \mathbf{i})}{4} \right)$$

$$\begin{aligned}
& + P_j\left(\coth \frac{\pi(x+2\mathbf{i})}{4}\right) + P_j\left(\coth \frac{\pi(x-2\mathbf{i})}{4}\right) \\
& - P_j\left(\coth \frac{\pi(x+3\mathbf{i})}{4}\right) - P_j\left(\coth \frac{\pi(x-3\mathbf{i})}{4}\right) \\
& + P_j\left(\coth \frac{\pi(x+4\mathbf{i})}{4}\right) + P_j\left(\coth \frac{\pi(x-4\mathbf{i})}{4}\right).
\end{aligned}$$

Hence, we arrive at the following simplified formula.

Corollary 13 ($p = 1$ and $q = 0$ with $\chi(n) \equiv (-1)^{\binom{n}{3}}$ and $\ell = 4$).

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + n^2)^{m+1}} = 2 \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{4}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \left\{ P_j\left(\coth \frac{\pi x}{4}\right) + P_j\left(\tanh \frac{\pi x}{4}\right) \right\}.$$

The three initial analytic formulae are illustrated as examples.

Example 14.

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + n^2)^2} &= \frac{\pi \operatorname{csch}^2\left(\frac{\pi x}{2}\right)}{8x^3} \{ \pi x + \sinh(\pi x) \}, \\
\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + n^2)^3} &= \frac{\pi \operatorname{csch}^2\left(\frac{\pi x}{2}\right)}{64x^5} \left\{ 6\pi x + (2\pi^2 x^2 - 3) \coth\left(\frac{\pi x}{2}\right) + 3 \cosh\left(\frac{3\pi x}{2}\right) \operatorname{csch}\left(\frac{\pi x}{2}\right) \right\}, \\
\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + n^2)^4} &= \frac{\pi \operatorname{csch}^4\left(\frac{\pi x}{2}\right)}{768x^7} \left\{ 4\pi^3 x^3 + 2\pi^3 x^3 \cosh(\pi x) + 60\pi x \sinh^2\left(\frac{\pi x}{2}\right) \right. \\
&\quad \left. + 2(6\pi^2 x^2 - 15) \sinh(\pi x) + 15 \sinh(2\pi x) \right\}.
\end{aligned}$$

4.5. $p = 2$ and $q = -1$ with $\chi(n) \equiv 1$ and $\ell = 1$

In view of Theorem 6, we have the corresponding series

Corollary 15 ($p = 2$ and $q = -1$ with $\chi(n) \equiv 1$ and $\ell = 1$).

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + (2n-1)^2)^{m+1}} &= \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{2}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\
&\quad \times \left\{ P_j\left(\coth \frac{\pi(x+\mathbf{i})}{2}\right) + P_j\left(\coth \frac{\pi(x-\mathbf{i})}{2}\right) \right\} \\
&= \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \frac{\pi^{1+j}}{2^{1+2m}} \binom{m}{j} \frac{(2m-j)!}{x^{1+2m-j}} P_j\left(\tanh \frac{\pi x}{2}\right).
\end{aligned}$$

The first three series are produced as consequences.

Example 16.

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + (2n-1)^2)^2} = \frac{\pi \operatorname{sech}^2\left(\frac{\pi x}{2}\right)}{8x^3} \{ \sinh(\pi x) - \pi x \},$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + (2n-1)^2)^3} = \frac{2\pi \operatorname{sech}^2(\frac{\pi x}{2})}{64x^5} \left\{ 3 \sinh(\pi x) - 3\pi x - \pi^2 x^2 \tanh\left(\frac{\pi x}{2}\right) \right\},$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + (2n-1)^2)^4} = \frac{\pi \operatorname{sech}^4(\frac{\pi x}{2})}{768x^7} \left\{ 4\pi^3 x^3 - 2\pi x(15 + \pi^2 x^2) \cosh(\pi x) \right. \\ \left. - 30\pi x + 2(15 - 6\pi^2 x^2) \sinh(\pi x) + 15 \sinh(2\pi x) \right\}.$$

4.6. $p = 2$ and $q = -1$ with $\chi(n) \equiv (-1)^{\binom{n}{2}}$ and $\ell = 4$

By means of Theorem 6, the corresponding series reads as

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + (2n-1)^2)^{m+1}} = \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{8}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\ \times \left\{ P_j\left(\coth \frac{\pi(x+i)}{8}\right) + P_j\left(\coth \frac{\pi(x-i)}{8}\right) - P_j\left(\coth \frac{\pi(x+3i)}{8}\right) - P_j\left(\coth \frac{\pi(x-3i)}{8}\right) \right. \\ \left. - P_j\left(\coth \frac{\pi(x+5i)}{8}\right) - P_j\left(\coth \frac{\pi(x-5i)}{8}\right) + P_j\left(\coth \frac{\pi(x+7i)}{8}\right) + P_j\left(\coth \frac{\pi(x-7i)}{8}\right) \right\}.$$

By employing four almost trivial identities

$$P_j\left(\coth \frac{\pi(x \pm i)}{8}\right) = P_j\left(\coth \frac{\pi(x \mp 7i)}{8}\right), \quad (4)$$

$$P_j\left(\coth \frac{\pi(x \pm 3i)}{8}\right) = P_j\left(\coth \frac{\pi(x \mp 5i)}{8}\right);$$

we derive the general formula as stated in the following corollary.

Corollary 17 ($p = 2$ and $q = -1$ with $\chi(n) \equiv (-1)^{\binom{n}{2}}$ and $\ell = 4$).

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + (2n-1)^2)^{m+1}} = 2 \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{8}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\ \times \left\{ P_j\left(\coth \frac{\pi(x+i)}{8}\right) + P_j\left(\coth \frac{\pi(x-i)}{8}\right) - P_j\left(\coth \frac{\pi(x+3i)}{8}\right) - P_j\left(\coth \frac{\pi(x-3i)}{8}\right) \right\}.$$

The three initial series are recorded as follows.

Example 18.

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + (2n-1)^2)^2} = \frac{\pi}{16\sqrt{2}x^3 \cosh^2(\frac{\pi x}{2})} \left\{ 4 \sinh\left(\frac{3\pi x}{4}\right) - 4 \sinh\left(\frac{\pi x}{4}\right) + \pi x \cosh\left(\frac{3\pi x}{4}\right) - 3\pi x \cosh\left(\frac{\pi x}{4}\right) \right\},$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + (2n-1)^2)^3} = \frac{\pi}{512\sqrt{2}x^5 \cosh^3(\frac{\pi x}{2})} \left\{ (\pi^2 x^2 + 48) \sinh\left(\frac{5\pi x}{4}\right) - 3(3\pi^2 x^2 + 16) \sinh\left(\frac{3\pi x}{4}\right) \right. \\ \left. - 2(11\pi^2 x^2 - 48) \sinh\left(\frac{\pi x}{4}\right) + 12\pi x \left(\cosh\left(\frac{5\pi x}{4}\right) - 3 \cosh\left(\frac{3\pi x}{4}\right) - 2 \cosh\left(\frac{\pi x}{4}\right) \right) \right\},$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{2}}}{(x^2 + (2n-1)^2)^4} = \frac{\pi}{24576\sqrt{2}x^7 \cosh^4(\frac{\pi x}{2})} \left\{ 24(\pi^2 x^2 + 40) \sinh\left(\frac{7\pi x}{4}\right) - 24(9\pi^2 x^2 + 40) \sinh\left(\frac{5\pi x}{4}\right) \right. \\ \left. - 72(7\pi^2 x^2 - 40) \sinh\left(\frac{3\pi x}{4}\right) + 24(13\pi^2 x^2 - 120) \sinh\left(\frac{\pi x}{4}\right) + \pi x(\pi^2 x^2 + 240) \cosh\left(\frac{7\pi x}{4}\right) \right. \\ \left. - 9\pi x(3\pi^2 x^2 + 80) \cosh\left(\frac{5\pi x}{4}\right) - \pi x(121\pi^2 x^2 + 240) \cosh\left(\frac{3\pi x}{4}\right) + 5\pi x(47\pi^2 x^2 - 240) \cosh\left(\frac{\pi x}{4}\right) \right\}.$$

4.7. $p = 2$ and $q = -1$ with $\chi(n) \equiv (-1)^{\binom{n}{3}}$ and $\ell = 4$

Considering Theorem 6, we can express the corresponding series as

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + (2n-1)^2)^{m+1}} = \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{8}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\ \times \left\{ P_j\left(\coth \frac{\pi(x+i)}{8}\right) + P_j\left(\coth \frac{\pi(x-i)}{8}\right) + P_j\left(\coth \frac{\pi(x+3i)}{8}\right) + P_j\left(\coth \frac{\pi(x-3i)}{8}\right) \right. \\ \left. - P_j\left(\coth \frac{\pi(x+5i)}{8}\right) - P_j\left(\coth \frac{\pi(x-5i)}{8}\right) + P_j\left(\coth \frac{\pi(x+7i)}{8}\right) + P_j\left(\coth \frac{\pi(x-7i)}{8}\right) \right\}.$$

This can further be simplified, by using (4), into the formula as below.

Corollary 19 ($p = 2$ and $q = -1$ with $\chi(n) \equiv (-1)^{\binom{n}{3}}$ and $\ell = 4$).

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + (2n-1)^2)^{m+1}} = 2 \sum_{j=0}^m \frac{(-1)^j}{(m!)^2} \left(\frac{\pi}{8}\right)^{1+j} \binom{m}{j} \frac{(2m-j)!}{(2x)^{1+2m-j}} \\ \times \left\{ P_j\left(\coth \frac{\pi(x+i)}{8}\right) + P_j\left(\coth \frac{\pi(x-i)}{8}\right) \right\}.$$

The three initial series are explicitly evaluated in closed form.

Example 20.

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + (2n-1)^2)^2} = \frac{\pi}{16x^3 \{2 \cosh(\frac{\pi x}{4}) - \sqrt{2}\}^2} \left\{ \sqrt{2}\pi x \cosh\left(\frac{\pi x}{4}\right) - 2\pi x - 4\sqrt{2} \sinh\left(\frac{\pi x}{4}\right) + 4 \sinh\left(\frac{\pi x}{2}\right) \right\}, \\ \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + (2n-1)^2)^3} = \frac{\pi}{256x^5 \{2 \cosh(\frac{\pi x}{4}) - \sqrt{2}\}^3} \left\{ 12\pi x \left(3\sqrt{2} + \sqrt{2} \cosh\left(\frac{\pi x}{2}\right) - 6 \cosh\left(\frac{\pi x}{4}\right)\right) \right. \\ \left. + 6(24 - \pi^2 x^2) \sinh\left(\frac{\pi x}{4}\right) - \sqrt{2}(96 - \pi^2 x^2) \sinh\left(\frac{\pi x}{2}\right) + 48 \sinh\left(\frac{3\pi x}{4}\right) \right\}, \\ \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n}{3}}}{(x^2 + (2n-1)^2)^4} = \frac{\pi}{6144x^7 \{2 \cosh(\frac{\pi x}{4}) - \sqrt{2}\}^4} \left\{ 24\pi x (\pi^2 x^2 - 120) - 8\pi x (\pi^2 x^2 + 240) \cosh\left(\frac{\pi x}{2}\right) \right. \\ \left. - \sqrt{2}\pi x (11\pi^2 x^2 - 3120) \cosh\left(\frac{\pi x}{4}\right) + \sqrt{2}\pi x (\pi^2 x^2 + 240) \cosh\left(\frac{3\pi x}{4}\right) + 960 \sinh(\pi x) \right. \\ \left. + 24\sqrt{2}(7\pi^2 x^2 - 200) \sinh\left(\frac{\pi x}{4}\right) - 192(\pi^2 x^2 - 40) \sinh\left(\frac{\pi x}{2}\right) + 24\sqrt{2}(\pi^2 x^2 - 120) \sinh\left(\frac{3\pi x}{4}\right) \right\}.$$

5. Conclusions

For the bilateral series $\Omega_m(p, q, x; \chi)$ introduced in (1), we succeeded in finding their analytic formulae when

$$\left\{ \begin{array}{l} p = 1 \text{ \& } q = 0 \\ p = 2 \text{ \& } q = -1 \end{array} \right\} \text{ with periodic function factors } \chi(n) = (\pm 1)^n, (-1)^{\binom{n}{2}}, (-1)^{\binom{n}{3}}.$$

Some of these formulae were shown by means of the residue calculus (cf. [5]) in the initially submitted version. For instance, the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x^2 + n^2)^{m+1}} \quad \text{in Corollary 7,}$$

was evaluated by considering the contour integration of the meromorphic function

$$f(z) := \frac{\pi \cot(\pi z)}{(x^2 + z^2)^{m+1}} = \frac{\pi \cos(\pi z)}{\sin(\pi z)(x^2 + z^2)^{m+1}}$$

along the rectangle characterized by four vertices $(\pm(M + \frac{1}{2}), \pm M\mathbf{i})$ oriented anti-clockwise (where $M \in \mathbb{N}$ subject to $M > x$). Then the formula displayed in Corollary 7 can be confirmed by determining the residues of $f(z)$ at simple pole $z = n \in \mathbb{Z}$

$$\operatorname{Res}_{z=n} f(z) = \lim_{z \rightarrow n} (z - n)f(z) = \frac{1}{(x^2 + n^2)^{m+1}},$$

and at two poles $z = \pm x\mathbf{i}$ of order $m + 1$

$$\operatorname{Res}_{z=\pm x\mathbf{i}} f(z) = \sum_{k=0}^m \frac{(-2\pi x)^{k+1}}{(2x)^{2m+2}} \binom{m}{k} \frac{(1+m)_{m-k}}{m!} P_k(\coth(\pi x)). \quad \square$$

To our knowledge, almost all the identities presented in this paper seem new, since the series with powers of quadratic functions in denominators are rare in the literature. The only related reference that we located is due to Wright and Osler [9]. For two complex numbers a and b subject to conditions $a \neq b$ and $a, b \notin \mathbb{Z}$, they found that

$$\frac{\pi \sin(a\pi - b\pi)}{(a - b) \sin(a\pi) \sin(b\pi)} = \pi \left\{ \frac{\cot(a\pi)}{b - a} + \frac{\cot(b\pi)}{a - b} \right\} = \sum_{n=-\infty}^{\infty} \frac{1}{(a + n)(b + n)}. \quad (5)$$

This may serve as an extension of Euler's famous solution of the Basel problem (cf. [2, 8])

$$\zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

since the latter results equivalently in the limiting case $a, b \rightarrow \frac{1}{2}$. In fact, letting $a \rightarrow \frac{1}{2}$ and $b \rightarrow \frac{1}{2}$ across (5), we deduce that

$$\begin{aligned} \pi^2 &= \sum_{n=-\infty}^{\infty} \frac{4}{(1 + 2n)^2} = \sum_{n=0}^{\infty} \frac{4}{(1 + 2n)^2} + \sum_{n=-\infty}^{-1} \frac{4}{(1 + 2n)^2} \\ &= \sum_{n=0}^{\infty} \frac{8}{(1 + 2n)^2} \quad \boxed{n \rightarrow -1 - n \text{ for the second series}} \\ &= \sum_{n=0}^{\infty} \left\{ \frac{8}{(1 + n)^2} - \frac{8}{(2 + 2n)^2} \right\} = 6\zeta(2). \end{aligned}$$

When $a, b \rightarrow \pm i x$, the series in (5) becomes $\Omega(1, 0, x)$, a particular case of Proposition 4.

In addition, there remain three simpler series below

$$\Omega_m(1, 0, x; (-1)^{\binom{n+1}{2}}) = \Omega_m(1, 0, x; (-1)^{\binom{n}{2}}) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n+1}{2}}}{(x^2 + n^2)^m},$$

$$\Omega_m(2, -1, x; (-1)^n) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(x^2 + (2n-1)^2)^m} = 0,$$

$$\Omega_m(2, -1, x; (-1)^{\binom{n+1}{2}}) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{\binom{n+1}{2}}}{(x^2 + (2n-1)^2)^m} = 0.$$

The former is justified by the replacement $n \rightarrow -n$ and evaluated as in Corollary 11. The two latter identities follow easily by inverting the summation index $n \rightarrow 1 - n$.

In general for $\ell, m \in \mathbb{N}$, one may examine similar series with denominators

$$(x^{2\ell} + n^{2\ell})^m \quad \text{and} \quad (x^{2\ell} + (2n-1)^{2\ell})^m.$$

Their evaluation would be more involved and worthy of further investigation.

Author contributions

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

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