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*Research article*

# Function sequences, uniform convergence, and statistical aspects in bipolar metric convergence

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**Abstract:** In this paper, we first develop statistical convergence in bipolar metric spaces. This convergence is characterized both by the corresponding real-valued distance sequence and by the existence of a convergent subsequence indexed by a density-one set. We also show that convergence implies statistical convergence, establish uniqueness when the statistical limit is central, and obtain statistical sequential characterizations of continuity for mappings and comappings. For sequences of mappings, we introduce pointwise, uniform, pointwise statistical, and uniform statistical convergence, and prove that the uniform statistical limit of continuous mappings is a continuous comapping. The mapping–comapping duality thus provides a natural function-space interpretation of these limits and yields a bipolar counterpart of the classical uniform-limit theorem for continuous functions.

**Keywords:** bipolar metric space; uniform convergence; statistical convergence; bisequence; comapping
**Mathematics Subject Classification:** 40A35, 54C30, 54E35

## 1. Introduction

Length and distance, among the most fundamental concepts in nature, are also among oldest fields of interest in mathematics. While there are many alternatives in modern mathematics to meet these notions, metric spaces are perhaps the most elegant representation of distance in mathematics, being both simple and intuitive but neither too restrictive nor too general.

Therefore, after Fréchet's gift of metric spaces to the literature [13], it is not surprising that many generalizations and modifications of this structure have emerged. While it is impossible to give a complete list, some of these structures include Hausdorff, topological, quasi-metric, semi-metric, pseudo-metric, uniform, proximity, 2-metric, probabilistic metric, fuzzy metric, Lawvere metric, approach, $D$-metric, $b$-metric, partial metric, dislocated metric, rectangular metric, intuitionistic fuzzy metric, $G$-metric, cone metric, multiplicative metric, modular metric, $S$-metric, $D^*$-metric, $A$-metric, and bipolar metric spaces [12, 26]. This enrichment process has continued unabated in recent years with structures, such as octonion-valued metric [44], $\star$-metric [47], multimetric [21], and modulative metric spaces [23]. It is worth adding that many of these structures have inherited the popularity of metric spaces in varying but notable proportions, and they are also helpful in many applied fields.

Almost all metric space-like abstract structures have been developed to measure the distance, or other distance-like variables, such as nearness and neighborhood relations, between points within a set. In contrast, there are examples in the social and natural sciences, and particularly in statistics and mathematics, where the distance (or equivalently, closeness) between two things of different kinds is measured.

For example, observable luminosity, roughly speaking the luminosity provided by a star when viewed from a planet, is a measure of proximity between a group of planets and a group of stars visible from all those planets. This way, a different concept of distance from spatial distance can be defined between two separate groups, planets and stars. However, it is challenging to give a similar value between two different planets or between two different stars for technical reasons.

In another area, the pairwise reaction rates between suitable groups of common bases and common acids can also be used to define a measure of distance between them. However, this measure does not allow comparisons between two bases or two acids, at least not directly.

As another example, putting each child to tests, a value of proximity, and thus distance, can be found between a set of measurable abilities and a group of children. However, tests cannot be used directly to measure the distance between two abilities or between two children in an abstract sense, and this is something other than what is aimed at in practice.

Mutlu and Gürdal introduced bipolar metric spaces to formalize such cross-pole measurements and established the basic fixed-point framework [34]. Later work proved Banach-type fixed-point results [14], developed $\alpha\psi$-contractive mappings [22], and established a Meir–Keeler common fixed-point theorem [33]. Coupled fixed points were investigated in [35], while multivalued mappings were treated in [36]. Locally and weakly contractive principles appear in [37], and bipolar metric completeness is characterized in [39]. Caristi-type cyclic contractions and applications were studied in [27]; a fuzzy triple-controlled model was used to solve an integral equation in [30]. Orthogonal fuzzy bipolar structures and their fixed-point applications were developed in [46], whereas bipolar ultrametric spaces were characterized in [6]. Other extensions include fuzzy bipolar metrics [5], bicomplex-valued bipolar metrics [15], bipolar $R$-metrics [29], and neutrosophic triplet bipolar metrics [42]. In recent years, it has been possible to come across various real-life applications of bipolar metric spaces [20,38].

Statistical convergence relaxes ordinary convergence by allowing exceptional indices whose natural density is zero. The asymptotic viewpoint underlying this notion appears in Steinhaus's work [45], and its connection with statistically Cauchy sequences was developed by Rath and Tripathy [41].

Statistical convergence of double sequences of fuzzy numbers is studied in [43]. In intuitionistic fuzzy normed spaces, ordinary sequences are treated in [25] and double sequences in [32]. Generalized statistical convergence in $G$-metric spaces appears in [16]; the corresponding $L$-fuzzy normed theory is developed for single sequences in [49] and for double sequences in [50].

A lacunary generalized variant is investigated in [1], whereas ideal-based extremal statistical limit points are studied in [17]. Hyperspace dynamics are addressed through $\lambda$-statistical convergence in [28], and a statistical approach to fractal analysis is given in [40]. Operator-theoretic questions are examined in [18].

Approximation-theoretic developments include relative uniform convergence for positive linear operators [7], statistical summation of convolution operators [9], and a Korovkin-type theorem on an infinite interval [10]. Power-series methods are investigated in measure [8] and on product time scales [11]; a logarithmic treatment in multiplicative metric spaces is given in [31]. Statistical convergence has also been extensively studied in the context of Riesz spaces [2–4].

Unlike previous work on statistical convergence in metric spaces, $G$-metric spaces, fuzzy normed spaces, and Riesz spaces, this paper establishes the first systematic treatment of statistical convergence in bipolar metric spaces. The key novel contributions are as follows: (i) the definition and characterization of statistical convergence for sequences whose terms belong to one pole and whose limits belong to the opposite pole; (ii) the proof that a statistical limit is unique whenever it is central, together with an example showing that uniqueness may fail for noncentral limits; (iii) the introduction of pointwise, uniform, pointwise statistical, and uniform statistical convergence for sequences of mappings; and (iv) a bipolar version of the classical uniform limit theorem, showing that the uniform statistical limit of continuous mappings is a continuous comapping.

## 2. Material and methods

In the following, some basic terminology and information about bipolar metric spaces and statistical convergence is given.

**Definition 2.1.** [19] A pair of sets is called a bipolar set. Whenever a bipolar set $(A, B)$ is given, the first coordinate is called its left pole, the second coordinate is called its right pole, and the intersection of the left pole and the right pole (i.e., the set $A \cap B$) is called its center. A similar terminology extends to the points of these sets as left points, right points, and central points. Moreover, a purely left point is defined to be a left point which is not a central point, and similarly, a purely right point is a non-central right point. By a nonempty bipolar set, we mean a bipolar set $(A, B)$, such that both poles are nonempty, that is, $A \times B \neq \emptyset$.

Although bipolar sets are ordinary pairs of sets, what makes them interesting is the category $Bip$ they form together with their morphisms (covariant mappings) defined as follows.

**Definition 2.2.** [19] A mapping (map, or more precisely a covariant mapping) from a bipolar set $(A, B)$ to a bipolar set $(C, D)$ is defined to be a function $f : A \cup B \to C \cup D$, carrying left points to left points, and right points to right points (i.e., $f[A] \subseteq C$ and $f[B] \subseteq D$). This case is denoted by $f : (A, B) \rightrightarrows (C, D)$.

In addition to mappings, another interesting tool to correlate bipolar sets is comappings, whose usefulness has been tested in many applications, particularly those related to fixed point theory.

**Definition 2.3.** [19] A comapping (comap, or more precisely a contravariant mapping) from a bipolar set $(A, B)$ to a bipolar set $(C, D)$ is a function $f : A \cup B \to C \cup D$, carrying left points to right points, and vice versa (i.e., $f[A] \subseteq D$ and $f[B] \subseteq C$). In this case, we use the notation $f : (A, B) \looparrowright (C, D)$.

In other words, a comapping from $(A, B)$ to $(C, D)$ is equivalent to a mapping from $(A, B)$ to $(D, C)$.

**Definition 2.4.** [19] Let $(A, B)$ be a bipolar set. If $(u_n)$ is a sequence on the set $A$, then we call $(u_n)$ a left sequence on $(A, B)$. If $(u_n)$ is a sequence on $B$, then it is called a right sequence on $(A, B)$. A central sequence on $(A, B)$ is defined to be both a left sequence and a right sequence on the bipolar set $(A, B)$, that is a sequence on the set $A \cap B$.

Occasionally, in the context of bipolar sets, the concept of sequence is also used without specifying direction. When it is said that $(u_n)$ is a sequence on $(A, B)$, it is meant that $(u_n)$ is either a left sequence or a right sequence. We readily understand that it does not possess purely left and purely right terms simultaneously.

When a bipolar metric space (see Definition 2.6) is given over a bipolar set, there is a notion of convergence for sequences. However, to generalize concepts, such as Cauchy sequences, to bipolar metric spaces, the following additional tool is needed.

**Definition 2.5.** [19] Let $(A, B)$ be a bipolar set. Then, a sequence on $A \times B$ is called bisequence on $(A, B)$.

According to this definition, if $(x_n, y_n)$ is a bisequence on $(A, B)$, this means that $(x_n)$ is a left sequence and $(y_n)$ is a right sequence on $(A, B)$.

In the following, we provide the definition of bipolar metric space.

**Definition 2.6.** [34] Given a bipolar set $(X, Y)$ and a function $b : X \times Y \to \mathbb{R}_0^+$ satisfying the conditions

i) $b(x, y) = 0$ iff $x = y$,

ii) $b(x, y) = b(y, x)$, if $x$ and $y$ are central points,

iii) $b(x, y) \leq b(x, y') + b(x', y') + b(x', y)$

for all $(x, y), (x', y') \in X \times Y$, then $b$ is called a bipolar metric, and the triple $(X, Y, b)$ is called a bipolar metric space. If $X \cap Y = \emptyset$, this space is called disjoint, and otherwise it is called joint. If it is allowed for $x \neq y$ that $b(x, y) = 0$, then the corresponding weaker structure is called a bipolar pseudo-metric space.

Here, we note that the second axiom, known as central symmetry, corresponds to vacuous truth in the case of a disjoint (without central point) bipolar metric space, meaning that it imposes no meaningful requirements. The third condition is known as the quadrilateral inequality, and it is reduced to the triangle inequality at the center.

**Example 2.7.** **(i)** Every metric space can be considered as a bipolar metric space. To do this, we identify a metric space $(X, d)$ with the bipolar metric space $(X, X, d)$.
**(ii)** Let $(X, d)$ be a quasi-metric space [48], the variant of the metric spaces where symmetry is not required. Let $\widetilde{X} = \{\widetilde{x} : x \in X\}$ be a disjoint copy of $X$. Then, $\left(X, \widetilde{X}, b\right)$ is a disjoint bipolar pseudo-metric space, where $b$ is defined as $b\left(x_1, \widetilde{x_2}\right) := d\left(x_1, x_2\right)$ for all $x_1, x_2 \in X$.
**(iii)** Let $(X, d)$ be a dislocated metric space [24], one of the metric space variants where the distance of a point from itself can be a positive number. Define the set $U = \{x \in X : d(x, x) = 0\}$ and $\widetilde{U^c}$ be the disjoint copy of $X \backslash U$. Say $Y = U \cup \widetilde{U^c}$. Then, $(X, Y, b)$ is a bipolar metric space, where $b(x, u) = d(x, u)$ for $u \in U$, and $b\left(x, \widetilde{y}\right) = d(x, y)$, whenever $\widetilde{y} \in \widetilde{U^c}$ is the copy of $y \in X \backslash U$.
**(iv)** If we define a function $b : [1, 3]^{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}_0^+$ by $b(f, x) = f(x)$, then $\left([1, 3]^{\mathbb{R}}, \mathbb{R}, b\right)$ becomes a disjoint bipolar metric space, where $[1, 3]^{\mathbb{R}}$ is the set of functions from $\mathbb{R}$ to $[1, 3]$.

Example 2.7 (i) clearly states that the concept of bipolar metric space generalizes metric spaces. As a result, it can be said that every metric space is also a bipolar metric space. In addition, the center of a bipolar metric space is always a metric space. As a principle, each definition given for bipolar metric spaces must be given in a way that it generalizes its namesake in metric spaces in this context. Parts (ii)–(iv) of the example, illustrate the additional flexibility of the bipolar framework: it accommodates asymmetric data, nonzero self-distances, and cross-type measurements that need not arise from a metric on a single set.

We will reiterate this relationship between bipolar metric spaces and metric spaces below. The proposition could be stated so concisely because it is evident from the definitions of metric spaces and bipolar metric spaces that $X$ is a set and $d$ is a nonnegative real valued function defined from $X$.

**Proposition 2.8.** [34] $(X, b)$ is a metric space if and only if $(X, X, b)$ is a bipolar metric space.

**Definition 2.9.** [19] Given a bipolar metric space $(X, Y, b)$. Then, it is possible to define a function $\overline{b} : Y \times X \to \mathbb{R}_0^+$ by $\overline{b}(y, x) := b(x, y)$ for every $(y, x) \in Y \times X$, and this gives another bipolar metric space $(Y, X, \overline{b})$. This is called the opposite of $(X, Y, b)$ and denoted by $\overline{(X, Y, b)} = (Y, X, \overline{b})$.

Obviously, one always has $\overline{\overline{(X, Y, b)}} = (X, Y, b)$. Now, we explain the continuity in terms of bipolar metrics.

**Definition 2.10.** [34] Let $(X, Y, b)$ and $(X', Y', b')$ be bipolar metric spaces.
**(i)** A map $f : (X, Y) \rightrightarrows (X', Y')$ is continuous at the left point $x_0 \in X$, if

$$\forall\varepsilon > 0, \exists\delta > 0, b(x_0, y) < \delta \Rightarrow b'(f(x_0), f(y)) < \varepsilon$$

for all $y \in Y$. Similarly, it is said to be continuous at the right point $y_0 \in Y$, if

$$\forall\varepsilon > 0, \exists\delta > 0, b(x, y_0) < \delta \Rightarrow b'(f(x), f(y_0)) < \varepsilon$$

for all $x \in X$.
**(ii)** Similarly, a comap $f : (X, Y) \looparrowright (X', Y')$ is continuous at the left point $x_0 \in X$, if

$$\forall\varepsilon > 0, \exists\delta > 0, b(x_0, y) < \delta \Rightarrow b'(f(y), f(x_0)) < \varepsilon$$

for all $y \in Y$. Similarly, it is said to be continuous at the right point $y_0 \in Y$, if

$$\forall\varepsilon > 0, \exists\delta > 0, b(x, y_0) < \delta \Rightarrow b'(f(y_0), f(x)) < \varepsilon$$

for all $x \in X$.

Convergence of sequences in bipolar metric spaces is defined as follows.

**Definition 2.11.** [34] Let $(X, Y, b)$ be a bipolar metric space, $(x_n)$ be a left sequence and $y$ be a right point. Then, the left sequence $(x_n)$ converges to the right point $y \in Y$ if for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$, such that $b(x_n, y) < \varepsilon$ whenever $n \geq n_0$, and this case is denoted by $(x_n) \rightharpoonup y$ or $\lim_{n\to\infty} x_n = y$. Similarly, a right sequence $(y_n)$ converges to a left point $x \in X$, if $\forall\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$, $n \geq n_0 \Rightarrow b(x, y_n) < \varepsilon$, and this is denoted by $(y_n) \rightharpoondown x$ or $\lim_{n\to\infty} y_n = x$. If a central sequence $(u_n)$ converges to a central point $u$, such that both $(u_n) \rightharpoonup u$ and $(u_n) \rightharpoondown u$, then it is said that $(u_n)$ converges to $u$, and this is denoted by $(u_n) \to u$.

In a bipolar metric space, convergence to purely left points is not defined for purely left sequences, and convergence to purely right points is not defined for purely right sequences. So, when it is given, for example, that $(z_n) \rightharpoondown z$, then it is readily understood from the symbol $\rightharpoondown$ that $(z_n)$ is a left sequence and $z$ is a right point.

**Proposition 2.12.** A left sequence $(z_n)$ in a bipolar metric space $(X, Y, b)$ converges to a right point $z$ (i.e., $(z_n) \rightharpoondown z$) if and only if the corresponding right sequence $(z_n)$ on the opposite bipolar metric space $\overline{(X, Y, b)}$ converges to the corresponding left point $z$, that is $(z_n) \rightharpoonup z$.

**Remark 2.13.** In the light of the previous proposition, it is often convenient to consider only left (or only right) sequences when stating and proving general results on convergence theory in bipolar metric spaces, unless otherwise needed. Similar results for right sequences will readily follow by the duality between the bipolar metric space and its opposite.

The following property is an extremely useful characterization of convergence of sequences on bipolar metric spaces.

**Proposition 2.14.** [34] $(x_n) \rightharpoondown y$ on a bipolar metric space $(X, Y, b)$ if and only if $(b(x_n, y)) \to 0$ on $\mathbb{R}$ and $(y_n) \rightharpoonup x$ on $(X, Y, b)$ is equivalent to $(b(x, y_n)) \to 0$, on $\mathbb{R}$.

It is often desirable for convergent sequences to have only one limit. Although bipolar metric spaces generally do not have this property, the uniqueness of the limit can be guaranteed under additional conditions.

**Theorem 2.15.** [34] If a bipolar metric space $(X, Y, b)$ can be embedded as a bipolar subspace into any metric space, that is, if there exists a metric space $(M, d)$ and a map $f : (X, Y, b) \rightrightarrows (M, M, d)$ such that the function $f : X \cup Y \to M$ is injective and $d(f(x), f(y)) = b(x, y)$ for all $(x, y) \in X \times Y$, then every convergent sequence has a unique limit in $(X, Y, b)$.

**Theorem 2.16.** [34] If a sequence converges to a central point, then the limit is unique.

**Definition 2.17.** [34] We call a bisequence $(x_n, y_n)$ on a bipolar metric space convergent, if both the left sequence $(x_n)$ and the right sequence $(y_n)$ are convergent. Moreover, if the limits of $(x_n)$ and $(y_n)$ are equal (a unique central point in this case), then $(x_n, y_n)$ is called biconvergent.

**Definition 2.18.** [34] A bisequence $(x_n, y_n)$ is called a Cauchy bisequence, if for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $b(x_n, y_m) < \varepsilon$ whenever $m, n \geq n_0$.

**Proposition 2.19.** [34] If a bisequence is biconvergent, then it is a Cauchy bisequence. If a Cauchy bisequence is convergent, then it is biconvergent.

Although the distances between purely left (or right) points are undefined, some distances can be defined with the help of distances to the points on the opposite pole. The most prominent of these distances are the inner pseudo-metrics given below.

**Definition 2.20.** [34] When a bipolar metric space $(X, Y, b)$ is given, the function $b_X : X \times X \to \mathbb{R}_0^+$, defined by

$$b_X(x_1, x_2) = \sup_{y \in Y} |b(x_1, y) - b(x_2, y)|,$$

gives a pseudo-metric on $X$, and $b_Y : Y \times Y \to \mathbb{R}_0^+$, defined by

$$b_Y(y_1, y_2) = \sup_{x \in X} |b(x, y_1) - b(x, y_2)|$$

is a pseudo-metric on $Y$.

Being presented some basic concepts on bipolar metric spaces, in this section, we finally present the terminology on statistical convergence [41].

**Definition 2.21.** [49] Let $K$ be a subset of $\mathbb{N}$. If the limit exists, the natural density $\delta(K)$ of $K$ is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{|\{k \leq n : k \in K\}|}{n},$$

where $|\cdot|$ denotes the cardinality of a set.

**Definition 2.22.** For a sequence $(x_n)$ on $\mathbb{R}$, if the natural density of the set

$$K(\varepsilon) := \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$$

is 0 for all $\varepsilon > 0$, that is, if

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - l| \geq \varepsilon\}|}{n} = 0,$$

then $(x_n)$ is said to be statistically convergent to the point $l \in \mathbb{R}$. In this case, we write $(x_n) \to^{st} l$.

## 3. Results

### *3.1. Statistical convergence in bipolar metric spaces*

In this section, we will describe the statistical convergence in bipolar metric spaces and illustrate the relationship of this type of convergence both to convergence in bipolar metric spaces and to statistical convergence in metric spaces.

**Definition 3.1.** Let $(X, Y, b)$ be a bipolar metric space. A left sequence $(x_k)$ is called statistically convergent to a right point $y$ if, for every $\varepsilon > 0$,

$$\delta(\{k : b(x_k, y) \geq \varepsilon\}) = 0,$$

or equivalently,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n,\ b(x_k, y) \geq \varepsilon\}| = 0,$$

and it is denoted by $(x_k) \rightharpoondown^{st} y$. Similarly, a right sequence $(y_k)$ is called statistically convergent to a left point $x$ if, for every $\varepsilon > 0$,

$$\delta(\{k : b(x, y_k) \geq \varepsilon\}) = 0,$$

or

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n,\ b(x, y_k) \geq \varepsilon\}| = 0,$$

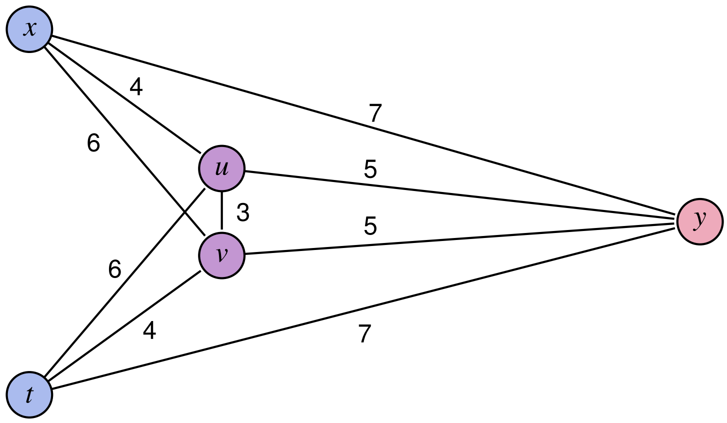
and it is denoted by $(y_k) \rightharpoonup^{st} x$.

A central sequence $(u_n)$ is statistically convergent to a central point $u$ if and only if it is statistically convergent in both pole directions, that is, $(u_n) \rightharpoondown^{st} u$ and $(u_n) \rightharpoonup^{st} u$. In this case, we write $(u_n) \to^{st} u$.

When $X = Y$, this definition reduces exactly to statistical convergence in an ordinary metric space. The corresponding notion for bisequences is given next.

**Definition 3.2.** Let $(X, Y, b)$ be a bipolar metric space and $(x_n, y_n)$ be a bisequence in $(X, Y, b)$. If the left and the right sequences $(x_n)$ and $(y_n)$ are both statistically convergent, then the bisequence $(x_n, y_n)$ is said to be statistically convergent. If the left sequence $(x_n)$ and the right sequence $(y_n)$ statistically converge to a common point, then it is said that the bisequence $(x_n, y_n)$ is statistically biconvergent to that point and this is denoted by $(x_n, y_n) \to^{st} u$.

**Example 3.3.** Let $X = \{x, u, v, t\}$, $Y = \{y, v, u\}$ and define the bipolar metric $b : X \times Y \to \mathbb{R}_0^+$ with the following diagram,




Consider the bisequence

$$(x_n, y_n) = \begin{cases} (x, y), & \text{if } n \text{ is prime,} \\ (u, v), & \text{otherwise.} \end{cases}$$

By the prime number theorem, the set of prime numbers has natural density zero. Hence, for every $\varepsilon > 0$,

$$\delta(\{k : b(x_k, u) \geq \varepsilon\}) = 0,$$

and

$$\delta(\{k : b(v, y_k) \geq \varepsilon\}) = 0.$$

Therefore, $(x_n) \to^{st} u$ and $(y_n) \rightharpoonup^{st} v$, so the bisequence $(x_n, y_n)$ is statistically convergent. It is not statistically biconvergent because $u \neq v$. This example demonstrates that statistical convergence of both component sequences of a bisequence does not force statistical biconvergence to a common point.

**Proposition 3.4.** Let $(X, Y, b)$ be a bipolar metric space, $(x_n)$ be a left sequence and $y$ be a right point. Then, the following statements are equivalent, for every $\varepsilon > 0$,

1) $(x_n) \to^{st} y$.
2) $\delta(\{k : b(x_k, y) > \varepsilon\}) = 0$.
3) $\delta(\{k : b(x_k, y) \leq \varepsilon\}) = 1$.

The following proposition is the bipolar counterpart of the classical scalar characterization of convergence by distances.

**Proposition 3.5.** Let $(X, Y, b)$ be a bipolar metric space. A left sequence $(x_n)$ is statistically convergent to a right point $y$ if and only if the sequence $(b(x_n, y))$ is statistically convergent to $0 \in \mathbb{R}$.

*Proof.* $(x_n) \rightharpoondown^{st} y$ means that, for each $\varepsilon > 0$, $\delta(\{k : b(x_k, y) \geq \varepsilon\}) = 0$, and this is equivalent to

$$\delta(\{k : |b(x_k, y) - 0| \geq \varepsilon\}) = 0,$$

which corresponds to statistical convergence of $b(x_n, y)$ to 0 in $\mathbb{R}$. □

**Theorem 3.6.** In a bipolar metric space $(X, Y, b)$, every convergent left sequence is also statistically convergent.

*Proof.* If $(x_n) \rightharpoondown y$, then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\{k : b(x_k, y) \geq \varepsilon\} \subseteq \{1, \dots, n_0 - 1\}.$$

The exceptional set is finite and therefore has natural density zero. □

This is the bipolar metric space analogue of the classical result that convergence implies statistical convergence in metric spaces.

**Remark 3.7.** However, as seen in the following example the converse of this theorem is not true.

**Example 3.8.** Consider the function $b : (0, 1) \times \mathbb{Z} \to \mathbb{R}^+$ defined as,

$$b(x, y) = \min\{x, 1 - x\}.$$

Note that this is interestingly independent of $y$ but satisfies all axioms to be a bipolar metric on the bipolar set $((0, 1), \mathbb{Z})$. First, let us make this clearer. If $b(x, y) = 0$, then $\min\{x, 1 - x\} = 0$ and $x = 0$ or $x = 1$. However, none of these elements are in the set $X = (0, 1)$, meaning $b(x, y) = 0$ cannot be true, and the requirement that $b(x, y) = 0 \Rightarrow x = y$ is vacuously true. Conversely, since $X$ and $Y$ are disjoint, $x = y$ is never true, and the requirement that $x = y \Rightarrow b(x, y) = 0$ is vacuously true. Central symmetry is also vacuously satisfied because the intersection is empty. The fact that $b(x, y) \leq b(x, y') + b(x', y') + b(x', y)$ comes from the observation $\min\{x, 1 - x\} \leq \min\{x, 1 - x\} + \min\{x', 1 - x'\} + \min\{x', 1 - x'\}$.

Now, let $a_k = \frac{2k-n^2+n}{4n}$, if $n^2 - n < 2k \leq n^2 + n$, for an $n \in \mathbb{N}^+$, that is

$$(a_n) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{1}{12}, \dots\right).$$

$(a_n)$ is a left sequence in $((0, 1), \mathbb{Z})$; however, we are interested in another left sequence $(x_n) = (\frac{(2a_n)^n}{2})$. We note that, $\frac{1}{2}$ occurs in the left sequence $(x_n)$ infinitely, but with a decreasing frequency, while the remaining terms tend to zero. Since $b(\frac{1}{2}, y) = \min\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$, infinitely many occurrences of $\frac{1}{2}$ prevent the convergence of $(x_n)$ to any right point $y$. On the other hand, the nearest non-$\frac{1}{2}$ terms to $\frac{1}{2}$ in the left sequence $(a_n)$ are $\frac{n^2+3n}{2}$-th terms, which are equal to $\frac{n}{2n+2}$. For the left sequence $(x_n)$, even the corresponding terms

$$x_{\frac{n^2+3n}{2}} = \frac{(2a_{\frac{n^2+3n}{2}})^{\frac{n^2+3n}{2}}}{2} = \frac{1}{2}\left(\frac{n}{n+1}\right)^{\frac{n^2+3n}{2}}$$

are sufficiently small as $n \to \infty$. Then, for any given $\varepsilon > 0$, there exists an $n_0$ such that $b(x_k, y) < \varepsilon$, if $k \geq n_0$ and $2k \neq n^2 + n$, for all $n \in \mathbb{N}$, and for any $y \in \mathbb{Z}$. In this case, $\delta(\{k : b(x_k, y) \geq \varepsilon\}) = 0$, and $(x_n)$ statistically converges to all elements of $\mathbb{Z}$ at the same time, while it is not convergent.

**Remark 3.9.** This example illustrates two facts: (i) statistical convergence does not imply ordinary convergence in bipolar metric spaces and (ii) statistical limits need not be unique when the limit is not a central point.

For uniqueness of a statistical limit of a sequence in a bipolar metric space, we need some extra property. The next theorem characterises one of these implications.

**Theorem 3.10.** Let $(X, Y, b)$ be a bipolar metric space. If a central point is a statistical limit of a sequence, then it is the unique statistical limit of this sequence.

*Proof.* Suppose that $(x_n) \rightharpoonup^{st} u$ in $(X, Y, b)$, where $u$ is a central point. Assume that another statistical limit $(x_n) \rightharpoonup^{st} y$ exists. Since $x_n \in X$, $y \in Y$, and $u \in X \cap Y$, all of the values $b(x_n, y), b(x_n, u), b(u, y)$ and $b(u, u)$ are defined, and in particular $b(u, u) = 0$. Then, by the quadrilateral inequality,

$$b(u, y) \leq b(u, u) + b(x_n, u) + b(x_n, y) = b(x_n, u) + b(x_n, y).$$

Given an $\varepsilon > 0$. Then, $\delta(\{k : b(x_k, u) \geq \varepsilon\}) = \delta(\{k : b(x_k, y) \geq \varepsilon\}) = 0$, so that

$$\delta(\{k : b(x_k, u) \geq \varepsilon\} \cup \{k : b(x_k, y) \geq \varepsilon\}) = \delta(\{k : \max\{b(x_k, u), b(x_k, y)\} \geq \varepsilon\}) = 0$$

and we have $\delta(\{k : \max\{b(x_k, u), b(x_k, y)\} < \varepsilon\}) = 1$. In particular, there is a $k \in \mathbb{N}$ such that $\max\{b(x_k, u), b(x_k, y)\} < \varepsilon$. Then,

$$b(u, y) \leq b(x_k, u) + b(x_k, y) < 2\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, we conclude that $u = y$. □

Given a bipolar metric space $(X, Y, b)$. Then, the function $b_X : X \times X \to \mathbb{R}_0^+$,

$$b_X(x_1, x_2) = \sup_{y \in Y} |b(x_1, y) - b(x_2, y)|$$

is a pseudo-metric on $X$, and $b_Y : Y \times Y \to \mathbb{R}_0^+$,

$$b_Y(y_1, y_2) = \sup_{x \in X} |b(x, y_1) - b(x, y_2)|$$

is a pseudo-metric on $Y$

A bipolar metric space is called bicharacterized, if both $(X, b_X)$ and $(Y, b_Y)$ are metric spaces. Every bicharacterized bipolar metric space $(X, Y, b)$ is embeddable into a metric space $(X \cup Y, d)$. For a detailed discussion see [34]. The following result is a direct consequence of the known results for metric spaces.

**Proposition 3.11.** In a bicharacterized bipolar metric space, each convergent sequence has a unique limit.

**Theorem 3.12.** Let $(X, Y, b)$ be a bipolar metric space. Then, a left sequence $(x_n)$ is statistically convergent to a right point $y$ (i.e., $(x_n) \rightharpoonup^{st} y$) if and only if there exists a subset

$$K = \{k_1 < k_2 < \ldots < k_n < \ldots\} \subseteq \mathbb{N}$$

such that $\delta(K) = 1$ and $(x_{k_n}) \rightharpoonup y$.

*Proof.* Suppose that $(x_n) \rightharpoondown^{st} y$. For each $j \in \mathbb{N}^+$, let

$$K_j = \{k \in \mathbb{N} : b(x_k, y) < 1/j\}.$$

Then, $\delta(K_j) = 1$ and the sets are nested: $K_{j+1} \subseteq K_j$. The key step is a diagonal selection from a nested family of density-one sets.

Choose integers $1 \leq N_1 < N_2 < \cdots$ such that, for every $j \in \mathbb{N}^+$ and every $n \geq N_j$,

$$\frac{1}{n}\left|K_j \cap \{1, \ldots, n\}\right| > 1 - \frac{1}{j}.$$

Define

$$K = \bigcup_{j=1}^{\infty}\left(K_j \cap \{N_j, N_j + 1, \ldots, N_{j+1} - 1\}\right).$$

If $N_j \leq n < N_{j+1}$, then the nesting gives $K_j \cap \{N_1, \ldots, n\} \subseteq K$. Consequently,

$$\frac{1}{n}\left|K \cap \{1, \ldots, n\}\right| \geq 1 - \frac{1}{j} - \frac{N_1 - 1}{n}.$$

As $n \to \infty$, the corresponding block index $j \to \infty$, and hence $\delta(K) = 1$. Moreover, given $\varepsilon > 0$, choose $j_0$ with $1/j_0 < \varepsilon$. If $k \in K$ and $k \geq N_{j_0}$, then $k \in K_j \subseteq K_{j_0}$ for some $j \geq j_0$, so $b(x_k, y) < \varepsilon$. Thus, after writing $K = \{k_1 < k_2 < \cdots\}$, we obtain $(x_{k_n}) \rightharpoondown y$.

Conversely, suppose that $K = \{k_1 < k_2 < \cdots\}$ satisfies $\delta(K) = 1$ and $(x_{k_n}) \rightharpoondown y$. Given $\varepsilon > 0$, there is an $n_0$ such that $b(x_{k_n}, y) < \varepsilon$ whenever $n \geq n_0$. Hence,

$$K \setminus \{k_1, \ldots, k_{n_0 - 1}\} \subseteq \{k \in \mathbb{N} : b(x_k, y) < \varepsilon\}.$$

The set on the left has density one, so the set on the right has density one. Its complement therefore has density zero, and $(x_n) \rightharpoondown^{st} y$. □

This extends the classical density-one subsequence characterization of statistical convergence to bipolar metric spaces.

**Theorem 3.13.** Let $(X, Y, b)$ and $(X', Y', b')$ be bipolar metric spaces. A mapping $f : (X, Y, b) \rightrightarrows (X', Y', b')$ is left continuous at a point $x_0 \in X$, if and only if, for every right sequence $(y_n)$ such that $(y_n) \rightharpoonup^{st} x_0$, one has $(f(y_n)) \rightharpoonup^{st} f(x_0)$, and it is right continuous at $y_0 \in Y$, if and only if $(x_n) \rightharpoondown^{st} y_0$ implies $(f(x_n)) \rightharpoondown^{st} f(y_0)$.

*Proof.* We show the result for left continuity, the case of right continuity is similar.

Necessity. Suppose that $f$ is left continuous at $x_0 \in X$, and $(y_n) \rightharpoonup^{st} x_0$. Given an $\varepsilon > 0$. There exists a $\delta_1 > 0$ such that $b(x_0, y) < \delta_1$ implies $b'(f(x_0), f(y)) < \varepsilon$, for all $y \in Y$. In this case,

$$\{n \in \mathbb{N} : b(x_0, y_n) < \delta_1\} \subseteq \{n \in \mathbb{N} : b'(f(x_0), f(y_n)) < \varepsilon\}$$

so that

$$\delta(\{n \in \mathbb{N} : b'(f(x_0), f(y_n)) \geq \varepsilon\}) \leq \delta(\{n \in \mathbb{N} : b(x_0, y_n) \geq \delta_1\}) = 0,$$

that is, $(f(y_n)) \rightharpoonup^{st} f(x_0)$.

Sufficiency. Suppose that $(y_n) \rightharpoonup^{st} x_0$ on $(X, Y, b)$ implies $(f(y_n)) \rightharpoonup^{st} f(x_0)$ on $(X', Y', b')$. Assume that $f$ is not left continuous at $x_0 \in X$. Then, there exists an $\varepsilon$, such that there is no $\delta_1$ satisfying $b(x_0, y) < \delta_1$ implies $b'(f(x_0), f(y)) < \varepsilon$ for all $y \in Y$, in other words; for all $\delta_1 > 0$, there exists some $y \in Y$ such that $b(x_0, y) < \delta_1$, but $b'(f(x_0), f(y)) \geq \varepsilon$. Then, for each $n \in \mathbb{N}^+$, we can pick a $y_n \in Y$ such that $b(x_0, y_n) < \frac{1}{n}$, but $b'(f(x_0), f(y_n)) \geq \varepsilon$. In this case, $(y_n) \rightharpoonup x_0$, and therefore in particular $(y_n) \rightharpoonup^{st} x_0$. However, $(f(y_n)) \not\rightharpoonup^{st} f(x_0)$ since

$$\{n \in \mathbb{N} : b'(f(x_0), f(y_n)) \geq \varepsilon\} = \mathbb{N},$$

and thus,

$$\delta(\{n \in \mathbb{N} : b'(f(x_0), f(y_n)) \geq \varepsilon\}) = 1 \neq 0.$$

□

This is the statistical bipolar analogue of the classical sequential criterion for continuity in metric spaces.

**Theorem 3.14.** Let $(X, Y, b)$ and $(X', Y', b')$ be bipolar metric spaces. Then, a comapping $f : (X, Y, b) \looparrowright (X', Y', b')$ is left continuous at a left point $x_0 \in X$ if and only if for every right sequence $(y_n)$ such that $(y_n) \rightharpoonup^{st} x_0$ one has $(f(y_n)) \rightharpoondown^{st} f(x_0)$, and it is right continuous at $y_0 \in Y$ if and only if $(x_n) \rightharpoondown^{st} y_0$ implies $(f(x_n)) \rightharpoonup^{st} f(y_0)$.

*Proof.* A comapping from $(X, Y, b)$ to $(X', Y', b')$ is a mapping into the opposite bipolar metric space $(Y', X', \overline{b'})$. The assertion therefore follows directly from Theorem 3.13 after reversing the target poles.

□

### 3.2. *Uniform and uniform statistical convergence in bipolar metric spaces*

In this section, we give a notion of uniform convergence for sequences of functions in bipolar metric spaces, and a statistical counterpart of uniform convergence is also presented. Let $(X, Y)$ and $(V, W)$ be bipolar sets. We can denote the set of all mappings from $(X, Y)$ to $(V, W)$ by $(V, W)^{(X,Y)}$. Accordingly, the set of all comappings from $(X, Y)$ to $(V, W)$ is $(V, W)^{(Y,X)}$. However, it is worth noting that $\left((V, W)^{(X,Y)}, (V, W)^{(Y,X)}\right)$ is a bipolar set, which stands out as being suitable for equipping with additional structures. Additionally, we can note that since every set $X$ is also a bipolar set $(X, X)$, the given definition will also be valid for functions defined from or to any set. However, the case of such a mapping would be less interesting. Indeed, one can observe that images of functions defined from a set to a bipolar set always remain within the center of the bipolar set.

Given a bipolar set $(X, Y)$, a bipolar metric space $(V, W, b)$, and a sequence $(f_n)$ on $(V, W)^{(X,Y)}$. In this case, at each point $x$ in $X$, $f_n(x)$ is a left sequence on the bipolar metric space $(V, W, b)$, and $f_n(y)$ is a right sequence at each point y in Y. If each of these left and right sequences converge in bipolar metric space, we can say that this mapping sequence converges pointwise. However, since the limit of every left sequence is a right point and the limit of every right sequence is a left point, the convergence will result in a comapping. Therefore, the functions in the terms of the pointwise convergent sequence and the function that is being converged to are of different kinds, and this fits quite well with the logic of bipolar metric spaces. In this regard, considering the bipolar set $\left((V, W)^{(X,Y)}, (V, W)^{(Y,X)}\right)$ we discussed above, we will denote this convergence with a variant of the symbol $\rightharpoondown$ assuming that this is a left convergence concerning the bipolar set $\left((V, W)^{(X,Y)}, (V, W)^{(Y,X)}\right)$, although no explicit bipolar metric space structure is given on it.

**Definition 3.15.** Given a bipolar set $(X, Y)$, a bipolar metric space $(V, W, b)$ and a sequence $(f_n)$ of mappings $f_n : (X, Y) \rightrightarrows (V, W)$. Then, $(f_n)$ is said to be pointwise convergent to a comapping $f : (X, Y) \looparrowright (V, W)$, if $f_n(x) \rightharpoondown f(x)$ in $(V, W, b)$ for all $x \in X$, and $f_n(y) \rightharpoonup f(y)$ in $(V, W, b)$ for all $y \in Y$. This situation is denoted by $(f_n) \overset{\bullet}{\rightharpoondown} f$. In other words, $(f_n) \overset{\bullet}{\rightharpoondown} f$ if and only if

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists n_x \in \mathbb{N}\ \forall n \geq n_x,\ b(f_n(x), f(x)) < \varepsilon,$$

and

$$\forall \varepsilon > 0,\ \forall y \in Y,\ \exists n_y \in \mathbb{N}\ \forall n \geq n_y,\ b(f(y), f_n(y)) < \varepsilon.$$

On the other hand, a sequence $(g_n)$ of comappings $g_n : (X, Y) \looparrowright (V, W)$ is said to be pointwise convergent to a mapping $g : (X, Y) \rightrightarrows (V, W)$, denoted by $(g_n) \overset{\bullet}{\rightharpoonup} g$, if

$$\forall \varepsilon > 0,\ \forall x \in X,\ \exists n_x \in \mathbb{N}\ \forall n \geq n_x\ \ b(g(x), g_n(x)) < \varepsilon,$$

and

$$\forall \varepsilon > 0,\ \forall y \in Y,\ \exists n_y \in \mathbb{N}\ \forall n \geq n_y,\ b(g_n(y), g(y)) < \varepsilon.$$

If $h_n$ are both mappings and comappings at the same time, or equivalently, if their images are subsets of the center $V \cap W$, and $(h_n) \overset{\bullet}{\rightharpoondown} h$ and $(h_n) \overset{\bullet}{\rightharpoonup} h$, then it is written $(h_n) \overset{\bullet}{\to} h$.

We now present the concept of uniform convergence for bipolar metric spaces in contrast with pointwise convergence.

**Definition 3.16.** Let $(X, Y)$ be a bipolar set, $(V, W, b)$ be a bipolar metric space and $(f_n)$ be a sequence of mappings $f_n : (X, Y) \rightrightarrows (V, W)$. $(f_n)$ is said to be uniformly convergent to a comapping $f : (X, Y) \looparrowright (V, W)$, if

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}\ \forall x \in X,\ \forall y \in Y,\ \forall n \geq n_0,\ \ b(f_n(x), f(x)) < \varepsilon,\ \ b(f(y), f_n(y)) < \varepsilon,$$

and this situation is denoted by $(f_n) \overset{U}{\rightharpoondown} f$. Similarly a sequence $(g_n)$ of comappings is said to be uniformly convergent to a mapping $g$, denoted as $(g_n) \overset{U}{\rightharpoonup} g$, if

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}\ \forall x \in X,\ \forall y \in Y,\ \forall n \geq n_0,\ \ b(g(x), g_n(x)) < \varepsilon,\ \ b(g_n(y), g(y)) < \varepsilon.$$

If for a sequence $(h_n)$ of both mappings and comappings one has $(h_n) \overset{U}{\rightharpoondown} h$ and $(h_n) \overset{U}{\rightharpoonup} h$, then this is denoted by $(h_n) \overset{U}{\to} h$.

In the sequel, we will mostly restrict ourselves to the case of mappings, as the case of comappings is quite similar.

**Definition 3.17.** Given mappings $f_n : (X, Y) \rightrightarrows (V, W)$, and let $b$ be a bipolar metric on $(V, W)$. Then, the mapping sequence $(f_n)$ is said to be pointwise statistically convergent to a comapping $f : (X, Y) \looparrowright (V, W)$, if $(f_n(x)) \rightharpoondown^{st} f(x)$ for all $x \in X$, and $(f_n(y)) \rightharpoonup^{st} f(y)$ for all $y \in Y$. The pointwise statistical convergence of the mapping sequence $(f_n)$ to the comapping $f$ is denoted by $(f_n) \overset{\bullet}{\rightharpoondown}^{st} f$. $(f_n)$ is said to be uniformly statistically convergent if for every $\varepsilon > 0$,

$$\delta(\{k : b(f_k(x), f(x)) \geq \varepsilon \text{ for some } x \in X\}) = \delta(\{k : b(f(y), f_k(y)) \geq \varepsilon \text{ for some } y \in Y\}) = 0,$$

or equivalently,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : \exists x \in X, \ \ b(f_k(x), f(x)) \geq \varepsilon\}| = \lim_{n\to\infty} \frac{1}{n} |\{k : \exists y \in Y, \ \ b(f(y), f_k(y)) \geq \varepsilon\}| = 0,$$

and this is denoted by $(f_n) \overset{U}{\rightharpoondown}{}^{st} f$.

For convenience, Table 1 summarizes the convergence notation used throughout the paper. The corresponding right-to-left symbols are obtained by reversing the harpoon.

**Table 1.** Summary of convergence notation.

| Symbol | Meaning |
|---|---|
| $(x_n) \rightharpoondown y$, $(y_n) \rightharpoonup x$ | Ordinary convergence of a left or right sequence |
| $(u_n) \to u$ | Ordinary convergence in both directions for a central sequence |
| $(x_n) \rightharpoondown^{st} y$, $(y_n) \rightharpoonup^{st} x$ | Statistical convergence of a left or right sequence |
| $(u_n) \to^{st} u$ | Statistical convergence in both directions for a central sequence |
| $(f_n) \overset{\bullet}{\rightharpoondown} f$ | Pointwise convergence of a sequence of mappings |
| $(f_n) \overset{\bullet}{\rightharpoondown}{}^{st} f$ | Pointwise statistical convergence of a sequence of mappings |
| $(f_n) \overset{U}{\rightharpoondown} f$ | Uniform convergence of a sequence of mappings |
| $(f_n) \overset{U}{\rightharpoondown}{}^{st} f$ | Uniform statistical convergence of a sequence of mappings |

**Remark 3.18.** Every uniformly convergent mapping sequence is both pointwise convergent and uniformly statistically convergent. Each of these two convergence notions implies pointwise statistical convergence. These implications are summarized in Figure 1.

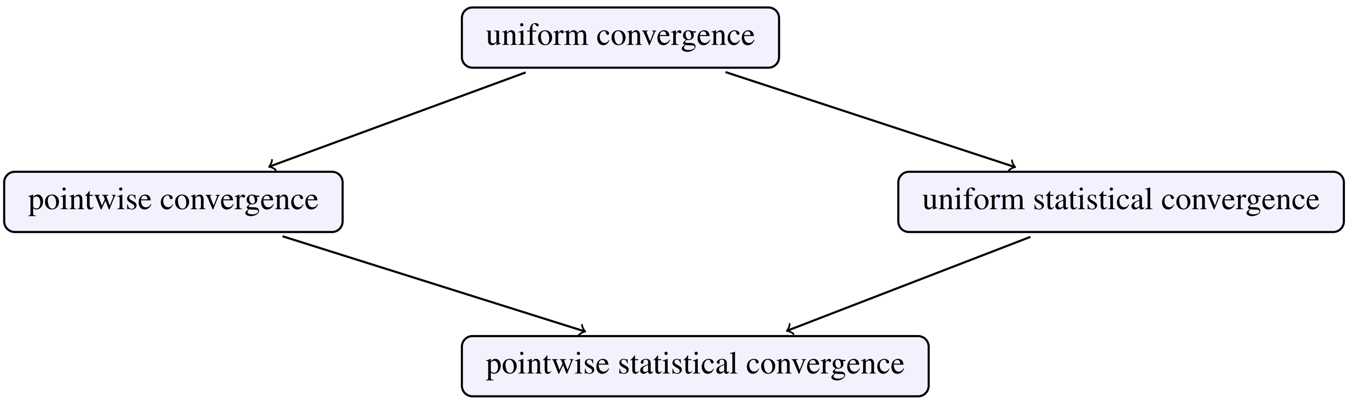



**Figure 1.** Logical implications among the four convergence notions for mapping sequences.

**Example 3.19.** Let

$$X = (-\infty, 0] \qquad \text{and} \qquad Y = [0, \infty),$$

and define

$$b(x, y) = \sqrt{x^2 + y^2} \qquad (x \in X, \ y \in Y).$$

Then, $b : X \times Y \to \mathbb{R}_0^+$ is a bipolar metric on $(X, Y)$. Indeed, $b(x, y) = 0$ if and only if $x = y = 0$, and central symmetry is immediate because $X \cap Y = \{0\}$. Moreover, for $x, x' \in X$ and $y, y' \in Y$,

$$b(x, y) = \sqrt{x^2 + y^2} \leq |x| + |y| \leq \sqrt{x^2 + (y')^2} + \sqrt{(x')^2 + (y')^2} + \sqrt{(x')^2 + y^2} = b(x, y') + b(x', y') + b(x', y),$$

so the quadrilateral inequality is satisfied.

For each $n \in \mathbb{N}^+$, define $f_n : X \cup Y \to X \cup Y$ by

$$f_n(z) = \begin{cases} (\Omega(n+1)^n z)^{1-2n}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where $\Omega(m)$ denotes the total number of prime factors of $m$, counted with multiplicity. For example,

$$\Omega(1) = |\varnothing| = 0,\ \Omega(3) = \left|\{3\}\right| = 1,\ \Omega(4) = 2,\ \Omega(6) = \left|\{2,3\}\right| = 2,\ \Omega(100) = 4.$$

For $z \neq 0$, the first few terms of this sequence are represented by

$$z^{-1}, \quad z^{-3}, \quad (2^3 z)^{-5}, \quad z^{-7}, \quad (2^5 z)^{-9}, \quad z^{-11}, \quad (3^7 z)^{-13},$$
$$(2^8 z)^{-15}, \quad (2^9 z)^{-17}, \quad z^{-19}, \quad (3^{11} z)^{-21}, \quad z^{-23}, \quad (2^{13} z)^{-25}, \ldots.$$

Since $1 - 2n$ is an odd integer, $f_n(z)$ has the same sign as $z$. Consequently,

$$f_n[X] \subseteq X \qquad \text{and} \qquad f_n[Y] \subseteq Y,$$

and hence,

$$f_n : (X, Y) \rightrightarrows (X, Y)$$

is a mapping for every $n \in \mathbb{N}^+$.

Consider the function

$$f(z) = 0 \qquad (z \in X \cup Y).$$

Since $0 \in X \cap Y$, the function $f$ is both a mapping and a comapping; in particular, it may be regarded as a comapping

$$f : (X, Y) \looparrowright (X, Y).$$

We show that $(f_n)$ is pointwise statistically convergent to $f$. Set

$$P^* = \{n \in \mathbb{N}^+ : n + 1 \text{ is prime}\}.$$

The prime number theorem implies

$$\delta(P^*) = 0,$$

since shifting the set of prime numbers by one does not change its natural density.

Fix $z \neq 0$. If $n \notin P^*$, then $n + 1$ is composite and therefore

$$\Omega(n + 1) \geq 2.$$

Consequently,

$$|f_n(z)| = (\Omega(n+1)^n |z|)^{1-2n} = \frac{1}{(\Omega(n+1)^n |z|)^{2n-1}},$$

and, for every sufficiently large $n \notin P^*$,

$$|f_n(z)| \leq \frac{1}{(2^n |z|)^{2n-1}} \longrightarrow 0.$$

Thus, for every fixed $z \neq 0$ and every $\varepsilon > 0$, there exists a finite set $F_{z,\varepsilon} \subseteq \mathbb{N}^+$ such that

$$\{n : |f_n(z)| \geq \varepsilon\} \subseteq P^* \cup F_{z,\varepsilon}.$$

Since $P^*$ has natural density zero, it follows that

$$\delta(\{n : |f_n(z)| \geq \varepsilon\}) = 0.$$

For $z = 0$, the same conclusion is immediate because $f_n(0) = 0$ for every $n$.

Equivalently, for each $x \in X$,

$$\delta(\{n : b(f_n(x), f(x)) \geq \varepsilon\}) = 0,$$

and for each $y \in Y$,

$$\delta(\{n : b(f(y), f_n(y)) \geq \varepsilon\}) = 0.$$

Therefore,

$$(f_n) \xrightarrow{\bullet}{}^{st} f.$$

The behavior on $P^*$ also explains why ordinary pointwise convergence may fail. If $n \in P^*$, then $\Omega(n + 1) = 1$, and hence

$$|f_n(z)| = |z|^{1-2n}.$$

Accordingly, along the indices in $P^*$,

$$|f_n(z)| \longrightarrow \begin{cases} 0, & |z| > 1, \\ 1, & |z| = 1, \\ +\infty, & 0 < |z| < 1. \end{cases}$$

For example, $f_n(-1) = -1$ whenever $n \in P^*$, whereas $f_n(-1) \to 0$ along the complementary composite indices. Thus, $(f_n)$ is not pointwise convergent to $f$.

We finally show that the convergence is not uniformly statistical. Fix $\varepsilon > 0$. For each $k \in \mathbb{N}^+$, define

$$x_k = -\varepsilon^{\frac{1}{1-2k}} \Omega(k + 1)^{-k}.$$

Since $k + 1 \geq 2$, one has $\Omega(k + 1) \geq 1$; therefore, $x_k$ is well defined and belongs to $X \setminus \{0\}$. Moreover,

$$\begin{aligned} f_k(x_k) &= \left(\Omega(k + 1)^k \left(-\varepsilon^{\frac{1}{1-2k}} \Omega(k + 1)^{-k}\right)\right)^{1-2k} \\ &= \left(-\varepsilon^{\frac{1}{1-2k}}\right)^{1-2k} \\ &= -\varepsilon. \end{aligned}$$

Consequently,

$$b(f_k(x_k), f(x_k)) = \varepsilon$$

for every $k \in \mathbb{N}^+$. Hence,

$$\{k : b(f_k(x), f(x)) \geq \varepsilon \text{ for some } x \in X\} = \mathbb{N}^+,$$

and therefore, this exceptional set has natural density one. Thus, we do not have $(f_n) \xrightarrow{U}{}^{st} f$.

This example shows that pointwise statistical convergence of a mapping sequence need not imply either ordinary pointwise convergence or uniform statistical convergence.

There is a well-known relationship between uniform convergence and continuity in analysis. Next, it is shown that a similar relationship exists between uniform statistical convergence and continuity in bipolar metric spaces, that is, uniform limit of continuous mappings is continuous.

**Theorem 3.20.** Uniform statistical limit of a sequence of continuous mappings between two bipolar metric spaces, is a continuous comapping.

*Proof.* The key idea is that failure of continuity of the limit would force every index into one of two exceptional sets, both of which have natural density zero.

Let $(X, Y, b)$ and $(V, W, \beta)$ be bipolar metric spaces, let $f_n : (X, Y, b) \rightrightarrows (V, W, \beta)$ be continuous mappings, and suppose that $(f_n) \xrightarrow{U}{}^{st} f$. Then, $f$ is a comapping. We prove left continuity; right continuity is analogous.

Fix $x_0 \in X$ and suppose, to the contrary, that $f$ is not continuous at $x_0$. Then, there exists $\varepsilon > 0$ such that, for every $\rho > 0$, one can find $y \in Y$ satisfying

$$b(x_0, y) < \rho \qquad \text{and} \qquad \beta(f(y), f(x_0)) \geq 3\varepsilon.$$

For each $n \in \mathbb{N}$, continuity of $f_n$ at $x_0$ yields a number $\rho_n > 0$ such that

$$b(x_0, y) < \rho_n \implies \beta(f_n(x_0), f_n(y)) < \varepsilon \qquad (y \in Y).$$

Using the failure of continuity of $f$, choose $y_n \in Y$ such that $b(x_0, y_n) < \rho_n$ and $\beta(f(y_n), f(x_0)) \geq 3\varepsilon$. The quadrilateral inequality gives

$$\beta(f(y_n), f(x_0)) \leq \beta(f(y_n), f_n(y_n)) + \beta(f_n(x_0), f_n(y_n)) + \beta(f_n(x_0), f(x_0)).$$

The middle term is smaller than $\varepsilon$. Therefore, for every $n$, either $\beta(f(y_n), f_n(y_n)) \geq \varepsilon$ or $\beta(f_n(x_0), f(x_0)) \geq \varepsilon$. Set

$$A_\varepsilon = \{k : \beta(f_k(x), f(x)) \geq \varepsilon \text{ for some } x \in X\}, \qquad B_\varepsilon = \{k : \beta(f(y), f_k(y)) \geq \varepsilon \text{ for some } y \in Y\}.$$

It follows that $A_\varepsilon \cup B_\varepsilon = \mathbb{N}$. On the other hand, uniform statistical convergence gives $\delta(A_\varepsilon) = \delta(B_\varepsilon) = 0$, and hence $\delta(A_\varepsilon \cup B_\varepsilon) = 0$, a contradiction. Thus, $f$ is left continuous. The proof of right continuity follows by the same argument. □

Since every uniformly convergent sequence of mappings is also uniformly statistically convergent, we can express the following corollary.

**Corollary 3.21.** Uniform convergence preserves continuity on bipolar metric spaces.

This recovers the classical uniform limit theorem when the two poles coincide, that is, when the bipolar metric space reduces to an ordinary metric space.

## 4. Discussions and conclusions

In this study, we have considered statistical convergence as the first alternative concept of convergence that has ever been addressed in bipolar metric spaces. Besides, we have introduced the interesting-at-first-sight theory of sequences of mappings on bipolar metric spaces. We have explained pointwise and uniform convergences for these sequences, as well as using the newly defined concept of uniform statistical convergence to show the relationship between uniform convergence and continuity on bipolar metric spaces. As a final impression, one can express that bipolar metric spaces are structures where the concept of convergence emerges in an interesting way. Some of the remarkable aspects to cite here are that a convergent sequence whose terms are of one type has a limit of the other type, a non-central (ultimately) constant sequence is not convergent, and the limit of a convergent sequence need not be unique.

Several directions remain open for systematic investigation. First, the notions of statistical Cauchy bisequences and statistical completeness would provide a foundation for completeness theory in bipolar metric spaces. Second, ideal, lacunary, weighted, and $\lambda$-statistical convergence variants could be developed, together with suitable Tauberian recovery conditions that allow passage from statistical to ordinary convergence. Third, the function-space structure of bipolar metric spaces merits further study, including Arzelà–Ascoli-type compactness results for families of mappings and comappings. Finally, applications to genuinely bipartite data—such as ligand-target interaction networks, bipartite similarity systems, and cross-type measurement frameworks—would demonstrate the practical utility of the theory developed here. These questions may clarify both the topological structure of bipolar metric spaces and the practical role of sparse exceptional observations.

## Author contributions

Reha Yapalı: Conceptualization, Formal analysis, Investigation, Validation, Writing–original draft; Utku Gürdal: Formal analysis, Investigation, Validation, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare that they have no conflict of interest.

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