



*Research article***Optical solitons in a fractional cubic–quintic Schrödinger equation: traveling wave and modulation instability****Ziyad A. Alhussain***

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Abstract: In the present work, the conformable fractional cubic–quintic nonlinear Schrödinger equation is investigated through traveling-wave analysis, nonlinear dynamics, and stability characterization of optical wave solutions. By applying a suitable traveling-wave transformation, the governing conformable fractional model is reduced to a nonlinear ordinary differential equation. Several analytical approaches, including the variational method, the projective Riccati equation method, and the $(\mathcal{G}'/\mathcal{G}^2)$ -expansion method, are employed to construct different classes of optical wave profiles, distinguished throughout as exact, piecewise classical, or variational. The resulting profiles describe various nonlinear wave structures, including bright, dark, periodic, singular, and localized optical waves. Furthermore, the associated reduced dynamics are analyzed through phase portraits and bifurcation diagrams, while the wave profiles illustrate the influence of the governing parameters on nonlinear propagation. Modulational instability of the continuous-wave background and spectral stability of the traveling-wave families for which a spectral problem on the line is defined are analyzed to identify parameter regimes associated with perturbation growth. Two independent discretization schemes agree to their truncation error, and the numerical benchmark reproduces the predicted growth rate to within about 2%. Grid refinement reveals numerically neutral modes that can appear weakly unstable on a fixed grid, and the apparent ordering of perturbation amplification with the conformable order depends on the observation window rather than on the intrinsic growth rate. The cubic–quintic nonlinearities thus govern the intrinsic instability, while the conformable order affects the temporal representation and finite window observation of the waves.

Keywords: fractional derivatives; Schrödinger equation; cubic–quintic nonlinearity; conformable derivative; optical solitons; stability analysis

Mathematics Subject Classification: 34A08, 35C08, 35Q55, 35R11, 37K40

1. Introduction

The mathematical modeling of numerous complex physical phenomena arising in fluid mechanics, plasma physics, nonlinear optics, quantum mechanics, Bose–Einstein condensation, optical fiber communications, and wave propagation theory is largely based on nonlinear partial differential equations (NLPDEs). Owing to the combined effects of nonlinearity, dispersion, dissipation, and diffraction, these equations exhibit a rich variety of nonlinear wave structures, including solitary waves, optical solitons, shock waves, rogue waves, breather waves, turbulence, and modulational instability phenomena [1]. In nonlinear optical media, NLPDEs play a fundamental role in describing the propagation of ultrashort optical pulses, nonlinear envelope waves, and localized wave structures in dispersive optical fibers and waveguides. Among the classical nonlinear evolution equations, the Korteweg–de Vries (KdV) equation has been widely employed to describe weakly nonlinear dispersive wave propagation in shallow water and plasma physics [2]. The modified Korteweg–de Vries (mKdV) equation extends this framework by incorporating stronger nonlinear interactions and has found important applications in nonlinear optics and plasma wave dynamics [3]. Likewise, the Benjamin–Bona–Mahony (BBM) equation provides an alternative regularized model for long-wave propagation and has been extensively used in the study of nonlinear dispersive systems and optical pulse propagation [4]. These classical models have played a fundamental role in advancing the theoretical understanding of nonlinear wave propagation and have stimulated the development of numerous analytical and numerical techniques for investigating nonlinear evolution equations. Among the most important nonlinear evolution equations arising in nonlinear optics is the nonlinear Schrödinger (NLS) equation, together with its dissipative extensions, which provide fundamental mathematical models for describing the propagation of nonlinear optical pulses and envelope solitons in dispersive media [5]. A representative nonlinear Schrödinger equation with linear loss is given by

$$i\psi_t + a_0\psi_{xx} + b_0|\psi|^2\psi + i\gamma_0\psi = 0, \quad (1.1)$$

where the complex-valued wave function $\psi = \psi(x, t)$ denotes the slowly varying envelope of the optical field, while a_0 , b_0 , and γ_0 represent the group-velocity dispersion, Kerr nonlinearity, and dissipation coefficients, respectively. Owing to the balance between chromatic dispersion and nonlinear self-phase modulation, the NLS equation admits stable localized wave structures, commonly referred to as optical solitons, which play a central role in optical fiber communications, laser systems, photonic crystals, metamaterials, and nonlinear waveguides. Consequently, nonlinear Schrödinger-type equations have become one of the most extensively studied classes of nonlinear evolution equations in mathematical physics and nonlinear optics.

In the absence of dissipation ($\gamma_0 = 0$), Eq (1.1) reduces to the classical nonlinear Schrödinger (NLS) equation

$$i\psi_t + a\psi_{xx} + \beta|\psi|^2\psi = 0, \quad \beta = \pm 1, \quad (1.2)$$

where, under the usual normalization $a > 0$, the values $\beta = 1$ and $\beta = -1$ correspond to the focusing and defocusing nonlinear regimes, respectively. The classical NLS equation serves as one of the fundamental models in nonlinear science and has been extensively employed to describe the evolution of slowly varying wave envelopes in nonlinear optical fibers, water waves, Bose–Einstein condensates, plasma physics, and nonlinear transmission lines. Its ability to support localized optical solitons through the balance between dispersion and Kerr nonlinearity has made it a cornerstone of the

mathematical analysis of nonlinear wave propagation. Nevertheless, in many realistic physical systems, higher-order nonlinear effects become significant, particularly for high-intensity optical pulses, thereby motivating the development of more general models such as the cubic–quintic nonlinear Schrödinger equation.

The NLS equation is also a fundamental integrable nonlinear evolution equation in nonlinear optics and mathematical physics. It provides a unified mathematical framework for describing important nonlinear phenomena, including self-focusing, self-phase modulation, pulse compression, nonlinear wave interactions, and optical soliton propagation in Kerr-type media. Under the same positive-dispersion normalization, the focusing regime ($\beta > 0$) supports bright optical solitons and modulational instability, whereas the defocusing regime ($\beta < 0$) supports dark solitons and modulationally stable continuous-wave backgrounds. Owing to its complete integrability, the NLS equation admits exact analytical soliton solutions through the inverse scattering transform, thereby providing a mathematical foundation for modern soliton theory and nonlinear wave dynamics [5, 6]. The inclusion of higher-order nonlinear effects leads naturally to the cubic–quintic NLS equation, in which cubic and quintic nonlinearities compete to produce a richer variety of localized and periodic wave structures.

Various analytical approaches have been developed for constructing exact traveling-wave solutions of nonlinear Schrödinger-type equations. Representative examples include the $\left(\frac{g'}{g}\right)$ -expansion method [7], He's semi-inverse method [8], analytical ansatz techniques [9], and several modern expansion and trial-function methods that have been successfully applied to optical soliton models [10, 11].

Beyond the classical NLS equation, several generalized Schrödinger-type models have been proposed to incorporate higher-order physical effects arising in ultrashort optical pulse propagation. Among these, the higher-order nonlinear Schrödinger (HNLS) equation proposed by Kodama and Hasegawa [12] is given by

$$i\mathcal{U}_t + \frac{1}{2}k_1\mathcal{U}_{xx} + |\mathcal{U}|^2\mathcal{U} = -i\varepsilon \left[-\frac{1}{6}k_2\mathcal{U}_{xxx} + h_1(|\mathcal{U}|^2\mathcal{U})_x - h_2(|\mathcal{U}|^2)_x\mathcal{U} \right], \quad (1.3)$$

where the higher-order dispersive and nonlinear correction terms account for effects associated with the propagation of ultrashort optical pulses in monomode optical fibers. This model has been widely studied in the context of femtosecond pulse dynamics, nonlinear fiber optics, optical switching, and photonic signal transmission.

Motivated by the rich dynamics of nonlinear Schrödinger-type equations, numerous analytical techniques, including Riccati-equation methods, expansion methods, mapping approaches, and perturbative techniques, have been developed to construct traveling-wave and optical-soliton solutions and to investigate their stability properties [13, 14]. Furthermore, Biswas and Milovic [15] systematically investigated bright and dark optical solitons under Kerr, power-law, parabolic-law, and dual-power-law nonlinear response laws, providing valuable insight into nonlinear optical pulse propagation in dispersive media.

Although these integer-order models have achieved remarkable success in describing nonlinear wave propagation, generalized fractional models have also been investigated to represent anomalous transport, multiscale effects, and complex dynamical behavior that may not be adequately captured by conventional integer-order formulations. Consequently, fractional differential equations have attracted considerable attention in physics, biology, signal processing, control theory, and nonlinear wave

propagation [16–18]. The construction of fractional operators is itself an active subject, and recent proposals differ in kernel structure and in whether memory is retained. The Mittag–Leffler–Caputo–Fabrizio derivative [19] and the Atangana–Baleanu derivative [20] retain a memory kernel. The general fractal derivative [21] and the cotangent derivative with respect to another function [22] instead generalize the differentiation rule. The conformable derivative used here lies at the local end of that range.

Recently, several studies have investigated fractional nonlinear evolution equations from the perspectives of bifurcation theory, chaos analysis, optical soliton dynamics, and modulational instability. Alam et al. [23] studied bifurcation structures and chaotic behavior in a fractional two-mode Nizhnik–Novikov–Veselov equation. Ilali et al. [24] analyzed optical solitons and quasi-periodic dynamics in a fractional cubic–quintic nonlinear pulse propagation model. Malomed [25] examined fractional nonlinear-wave models and stabilization mechanisms for cubic–quintic soliton structures. Khalifa et al. [26] investigated modulational instability and nonlinear wave structures in higher-order nonlinear Schrödinger equations with cubic–quintic nonlinearities. Furthermore, Alazman et al. [27] explored bifurcation phenomena, chaotic dynamics, and multi-soliton solutions of fractional nonlinear Schrödinger equations arising in plasma physics. These studies demonstrate the growing interest in fractional nonlinear models for describing complex wave and dynamical phenomena.

A variety of analytical, variational, and transform-based approaches have also been developed for fractional Schrödinger-type equations. These include path-integral formulations [28], Caputo fractional derivative models [29], Fourier, Laplace, and Mellin transform methods [30, 31], Sumudu transform techniques [32], momentum-space formulations [33], and fractional complex transformations combined with F -expansion methods [34]. These developments have contributed to the analysis of fractional wave propagation, optical solitons, and nonlinear dynamical structures.

More recently, several 2026 studies have further advanced the analysis of fractional and conformable nonlinear Schrödinger models. Amer et al. [35] investigated exact and numerical optical soliton solutions of a fractional quadratic–cubic nonlinear Schrödinger equation formulated with a conformable derivative, showing that the fractional formulation supports distinct localized and periodic wave structures. Ahmad et al. [36] studied a conformable fractional Schrödinger model in connection with ferromagnetic wave dynamics and examined the influence of the fractional order on soliton localization, dispersion, and stability. In addition, Al Amin et al. [37] investigated a fractional Schrödinger–Hirota equation with a β -fractional derivative, focusing on soliton structures, bifurcation, chaos, and stability. These recent developments demonstrate the continuing interest in fractional Schrödinger-type models; however, a unified treatment of the local conformable cubic–quintic NLS equation combining reduced modal dynamics, modulational instability, multiple analytical wave families, and variational characterization remains comparatively limited.

Fractional nonlinear Schrödinger-type models have attracted considerable attention as generalized descriptions of temporal evolution and nonlinear wave interactions in nonlinear optics, plasma physics, fluid dynamics, and quantum mechanics. Motivated by these developments, the present study investigates the fractional temporal dynamics of the cubic–quintic nonlinear Schrödinger equation using the conformable derivative, which provides a local fractional-order temporal scaling. Accordingly, the model considered here is

$$i\mathcal{D}_t^\alpha \psi + a\psi_{xx} + b|\psi|^2\psi + c|\psi|^4\psi + d\psi = 0, \quad 0 < \alpha \leq 1, \quad (1.4)$$

where $\mathcal{D}_t^\alpha$ denotes the conformable derivative defined by

$$\mathcal{D}_t^\alpha f(t) = t^{1-\alpha} \frac{df}{dt}, \quad 0 < \alpha \leq 1. \quad (1.5)$$

The coefficients a , b , c , and d are real parameters associated with dispersion, cubic nonlinearity, quintic nonlinearity, and a constant linear frequency shift, respectively. The coefficient a determines the strength of the dispersive contribution, while b controls the leading-order cubic self-interaction. The coefficient c represents the higher-order quintic nonlinear response and becomes increasingly important at larger wave amplitudes. The competition between the cubic and quintic nonlinearities determines the existence, localization and stability of solitary wave solutions. The model is considered in normalized variables, so the parameter values used below represent relative strengths of the corresponding dispersive and nonlinear effects. In contrast, the constant coefficient d does not constitute an independent localization potential; it represents a uniform linear frequency (phase) shift that can be absorbed into the temporal phase of the wave field.

To make this point explicit, introducing the conformable time variable

$$\tau = \frac{t^\alpha}{\alpha}, \quad (1.6)$$

gives

$$\mathcal{D}_t^\alpha = \frac{d}{d\tau}.$$

Under the phase transformation

$$\psi(x, t) = u(x, \tau) e^{id\tau}, \quad (1.7)$$

we obtain

$$\mathcal{D}_t^\alpha \psi = e^{id\tau} \left(\frac{\partial u}{\partial \tau} + idu \right), \quad (1.8)$$

and hence,

$$i\mathcal{D}_t^\alpha \psi + d\psi = ie^{id\tau} \frac{\partial u}{\partial \tau}.$$

Therefore, Eq (1.4) is transformed into

$$i \frac{\partial u}{\partial \tau} + au_{xx} + b|u|^2 u + c|u|^4 u = 0. \quad (1.9)$$

Thus, d produces only a uniform phase/frequency shift and does not independently modify the spatial localization mechanism or the intrinsic modulational-instability threshold. Its effect can equivalently be absorbed into the carrier frequency through the above gauge transformation.

This study combines a low-order three-mode Galerkin reduction, traveling-wave analysis, the Projective Riccati Equation Method, modulational-instability analysis, and a variational/Ritz formulation for the cubic–quintic nonlinear Schrödinger equation in a conformable temporal setting. The Galerkin reduction is used to describe the leading modal dynamics, whereas the traveling-wave and Projective Riccati approaches provide distinct hyperbolic, singular, and periodic wave families independently of the Galerkin truncation. The variational formulation further provides algebraic conditions for the amplitude and width of localized states.

A central feature of the present formulation is the explicit treatment of the conformable derivative as a local temporal operator, so that α acts as a temporal rescaling parameter rather than as a source of hereditary memory, with consequences for wave phases and perturbation growth that are developed below.

The map $\tau = t^\alpha/\alpha$ carries (1.4) exactly onto a classical cubic–quintic NLS equation in τ . This reduction identifies the role of the conformable order as the relation between the evolution variable and laboratory time. Over a finite window the amplification is $\exp(\Lambda_{\max} D(\alpha))$ with $D(\alpha) = (t^\alpha - t_0^\alpha)/\alpha$, and $\Lambda_{\max}$ does not depend on α . The first integral of the projective Riccati system is derived explicitly, and the resulting algebraic system admits several branches. The identity $A_1 R = -A_0$ holds on Branch A, so that profile is exact and smooth in its signed form, the corner at the origin belonging to its nonnegative modulus. The neutral modes of the discretized operator can acquire a small spurious positive real part, so the growth rate is assessed under grid refinement.

The main features of the present investigation are summarized below:

- (1) A conformable cubic–quintic nonlinear Schrödinger equation is investigated analytically within the fractional-order range

$$0 < \alpha \leq 1,$$

with particular emphasis on the effects of conformable temporal scaling on the resulting wave dynamics.

- (2) A sign-independent modulational-instability criterion is derived for the continuous-wave background,

$$ag > 0, \quad 0 < \mu^2 < \frac{2g}{a}, \quad g = b|\mathcal{A}|^2 + 2c|\mathcal{A}|^4,$$

which reduces to $2b|\mathcal{A}|^2 + 4c|\mathcal{A}|^4 > a\mu^2$ for $a > 0$. It characterizes the parameter regime in which perturbations exhibit exponential growth in the conformable temporal coordinate.

- (3) The influence of the conformable parameter α on the temporal evolution of the wave field and perturbations is examined by comparing the conformable and integer-order cases. The analysis highlights the temporal-rescaling effect induced by the mapping $t \mapsto t^\alpha/\alpha$.
- (4) Distinct analytical wave families, including hyperbolic, singular, and periodic profiles, are generated using the traveling-wave reduction and Projective Riccati Equation Method, subject to explicit regularity and admissibility conditions.
- (5) A low-order three-mode Galerkin approximation is employed to describe the leading modal dynamics of the conformable CQ–NLS system. The associated truncation is explicitly interpreted as a reduced-order spectral approximation, with the possible influence of neglected higher harmonics acknowledged.
- (6) A variational/Ritz formulation is developed for the localized solitary waves, yielding algebraic conditions for the admissible amplitude and inverse width. These conditions provide additional criteria for the existence of localized wave profiles.
- (7) The analytical and reduced-order results are complemented by numerical spectral computations for selected fractional orders, including the integer-order limit $\alpha = 1$. The numerical results are used to illustrate the temporal-rescaling effects and the stability behavior predicted by the analytical framework.
- (8) The present study focuses on analytical wave construction, reduced-order modal dynamics, modulational-instability analysis, variational characterization, and spectral stability within the

conformable CQ–NLS framework. Direct numerical simulations of the original conformable PDE are also performed to verify the analytical and reduced order results, including the predicted perturbation growth and the temporal rescaling associated with the conformable derivative.

Overall, the contribution of this work lies in combining complementary analytical and reduced-order approaches within a single conformable cubic–quintic NLS framework, while explicitly distinguishing the local temporal-rescaling effect of the conformable derivative from the hereditary memory effects associated with nonlocal fractional operators. The resulting formulation provides a combined treatment of localized and periodic wave structures, reduced modal dynamics, modulational instability, and variational characteristics of the conformable CQ–NLS system.

2. Preliminaries

2.1. Physical interpretation of the conformable derivative

The conformable derivative used in this study is the local fractional differential operator (1.5) introduced above. Introducing the effective conformable time

$$\tau = \frac{t^\alpha}{\alpha}, \quad (2.1)$$

gives

$$\mathcal{D}_t^\alpha = \frac{d}{d\tau}. \quad (2.2)$$

Thus, in the present CQ–NLS model, the parameter α should not be interpreted as introducing a hereditary memory kernel. Instead, it modifies the parametrization of temporal evolution through the nonlinear mapping $t \mapsto t^\alpha/\alpha$. The classical integer-order model is recovered when $\alpha = 1$, for which $\tau = t$.

This interpretation is particularly relevant for the wave dynamics considered here. A temporal oscillation of the form $\exp(-i\Omega\tau)$ and a perturbation with spectral exponent λ are expressed in the original time variable as

$$\exp\left(-i\Omega\frac{t^\alpha}{\alpha}\right), \quad \exp\left(\lambda\frac{t^\alpha}{\alpha}\right), \quad (2.3)$$

respectively. Hence, for $0 < \alpha < 1$, the temporal progression of oscillations, modulation, and perturbation growth differs from the classical case when viewed in the original time variable t . This provides the principal dynamical role of α in the present local conformable formulation.

The conformable derivative differs fundamentally from nonlocal fractional operators such as the Caputo and Riemann–Liouville derivatives. For $0 < \alpha < 1$, the Caputo derivative may be written in the form

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, \quad (2.4)$$

whereas the Riemann–Liouville derivative is defined through a fractional integral followed by an integer-order differentiation,

$${}_0^{RL} D_t^\alpha f(t) = \frac{d}{dt} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau \right]. \quad (2.5)$$

Both operators are nonlocal and depend on the history of the function over the interval $[0, t]$. In contrast, the conformable derivative in (1.5) depends only on the local value of $f'(t)$ together with the time-dependent scaling factor $t^{1-\alpha}$. Consequently, the three formulations represent different mathematical descriptions of fractional-order evolution and should not be assigned identical physical interpretations.

For the present optical-wave problem, the conformable formulation is therefore particularly convenient when the objective is to investigate fractional-order temporal rescaling while retaining a local differential structure. Caputo and Riemann–Liouville formulations are more appropriate when nonlocal history dependence and hereditary effects are essential features of the physical process being modeled. The choice of fractional operator should consequently be determined by the underlying physical assumptions rather than by the fractional order alone. The conformable derivative used here is local, which makes the reduction (2.2) available and confines the results below to that setting.

This section presents some fundamental definitions, properties, and analytical tools associated with conformable fractional calculus and Sobolev functional spaces, which are required throughout the subsequent analysis.

Definition 2.1 (Conformable fractional derivative). Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $\alpha \in (0, 1]$. The conformable fractional derivative of order α introduced by Khalil et al. [38] and developed further by Abdeljawad [39] is defined by

$$\mathcal{D}_t^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0. \quad (2.6)$$

Lemma 2.2. If f is differentiable on $(0, \infty)$, then the conformable fractional derivative satisfies (1.5), and for the power function $f(t) = t^m$, one has

$$\mathcal{D}_t^\alpha(t^m) = m t^{m-\alpha}, \quad m \in \mathbb{R}. \quad (2.7)$$

Lemma 2.3 (Basic properties). Let f and g be conformably differentiable functions, and let $a, b \in \mathbb{R}$. Then the conformable derivative satisfies the following properties:

$$\mathcal{D}_t^\alpha(af + bg) = a\mathcal{D}_t^\alpha(f) + b\mathcal{D}_t^\alpha(g), \quad (2.8)$$

and

$$\mathcal{D}_t^\alpha(fg) = f\mathcal{D}_t^\alpha(g) + g\mathcal{D}_t^\alpha(f). \quad (2.9)$$

Definition 2.4 (Sobolev Space). Let $\Omega \subseteq \mathbb{R}^N$ be an open domain. The Sobolev space $H^1(\Omega)$ is defined by

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in (L^2(\Omega))^N\}, \quad (2.10)$$

equipped with the norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \right)^{1/2}. \quad (2.11)$$

Lemma 2.5 (Gagliardo–Nirenberg Inequalities). Let $u \in H^1(\mathbb{R})$. Then there exists a positive constant $C > 0$ such that

$$\|u\|_{L^4(\mathbb{R})}^4 \leq C \|u\|_{L^2(\mathbb{R})}^3 \|u_x\|_{L^2(\mathbb{R})}, \quad (2.12)$$

and

$$\|u\|_{L^6(\mathbb{R})}^6 \leq C \|u\|_{L^2(\mathbb{R})}^4 \|u_x\|_{L^2(\mathbb{R})}^2. \quad (2.13)$$

Lemma 2.6 (Sobolev Embedding Inequality). *For every $u \in H^1(\mathbb{R})$, there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty(\mathbb{R})}^2 \leq C\|u\|_{L^2(\mathbb{R})}\|u_x\|_{L^2(\mathbb{R})}. \quad (2.14)$$

Remark 2.7. The conformable derivative is a local operator, and (1.5) transforms the fractional model into an equivalent time-rescaled local formulation. The analysis is therefore carried out in the classical Sobolev framework in x , where (2.12)–(2.14) control the cubic and quintic terms. Global boundedness or stability would additionally require a coercive a priori argument. The parameter α is accordingly a temporal scaling parameter rather than a memory kernel.

Remark 2.8. For $\psi(\cdot, t) \in H^1(\mathbb{R})$ the energy-type functional

$$E(\psi) = a\|\psi_x\|_{L^2}^2 - \frac{b}{2}\|\psi\|_{L^4}^4 - \frac{c}{3}\|\psi\|_{L^6}^6 - d\|\psi\|_{L^2}^2 \quad (2.15)$$

is finite, since (2.12) and (2.13) bound $\|\psi\|_{L^4}^4$ and $\|\psi\|_{L^6}^6$ by $\|\psi\|_{L^2}$ and $\|\psi_x\|_{L^2}$.

3. Traveling wave constructions and variational profiles

Lemma 3.1 (Traveling-wave reduction). *Let $a \neq 0$, $p \neq 0$, $t > 0$, and let $\mathcal{H}$ be real-valued and twice continuously differentiable. Under the traveling-wave transformation*

$$\psi(x, t) = \mathcal{H}(\xi) \exp\left[i\left(qx + \frac{\nu t^\alpha}{\alpha}\right)\right], \quad \xi = px + \frac{\omega t^\alpha}{\alpha}, \quad (3.1)$$

with the conformable fractional derivative (1.5), the conformable fractional cubic–quintic nonlinear Schrödinger equation (1.4) reduces to

$$\left(ap^2 \mathcal{H}'' + b\mathcal{H}^3 + c\mathcal{H}^5 + (d - \nu - aq^2)\mathcal{H}\right) + i(\omega + 2apq)\mathcal{H}' = 0, \quad (3.2)$$

where the prime denotes differentiation with respect to ξ . The imaginary part of Eq (3.2) gives $(\omega + 2apq)\mathcal{H}' = 0$; for a nonconstant profile ($\mathcal{H}' \neq 0$) this forces

$$\omega = -2apq, \quad (3.3)$$

whereas a constant profile satisfies the reduced equation below with ω unconstrained (the exceptional case). The real part gives

$$ap^2 \mathcal{H}'' + b\mathcal{H}^3 + c\mathcal{H}^5 + (d - \nu - aq^2)\mathcal{H} = 0. \quad (3.4)$$

3.1. Solution by the projective Riccati equation method

Using the homogeneous balance principle for Eq (3.4), the terms $\mathcal{H}''$ and $\mathcal{H}^5$ are balanced according to

$$N + 2 = 5N, \quad (3.5)$$

which yields

$$N = \frac{1}{2}. \quad (3.6)$$

Since the resulting value of N is non-integer, we introduce the transformation

$$\mathcal{H}(\xi) = Y^{1/2}(\xi). \quad (3.7)$$

Substituting Eq (3.7) into Eq (3.4) and simplifying gives the transformed equation

$$2ap^2YY'' - ap^2(Y')^2 + 4bY^3 + 4cY^4 + 4(d - \nu - aq^2)Y^2 = 0. \quad (3.8)$$

Applying the homogeneous balance principle to Eq (3.8) then determines the appropriate polynomial degree N for the expansion of $Y(\xi)$. Balancing the highest-order derivative term YY'' , of degree $2N + 2$, against the highest-order nonlinear term Y^4 , of degree $4N$, gives $2N + 2 = 4N$ and hence $N = 1$, so that the expansion of $Y(\xi)$ is linear in the auxiliary functions. Accordingly, the finite-series form is taken as

$$Y(\xi) = A_0 + A_1\sigma(\xi) + B_1\tau(\xi), \quad (3.9)$$

where A_0 , A_1 , and B_1 are constants, while $\sigma(\xi)$ and $\tau(\xi)$ satisfy the projective Riccati system. Consequently, the corresponding traveling-wave amplitude is expressed as

$$\mathcal{H}(\xi) = \sqrt{A_0 + A_1\sigma(\xi) + B_1\tau(\xi)}. \quad (3.10)$$

Substituting Eq (3.10) into Eq (3.4) and collecting the coefficients of the different powers and products of $\sigma(\xi)$ and $\tau(\xi)$ yields a nonlinear algebraic system for the unknown parameters A_0 , A_1 , B_1 , R , μ , p , and ν .

Solutions of the resulting algebraic system provide candidate traveling-wave profiles, subject to the regularity and admissibility conditions of Remark 3.8.

Lemma 3.2 (Solution of the projective Riccati algebraic system). *The auxiliary functions satisfy $\sigma' = -\sigma\tau$, $\tau' = R - \tau^2 - \mu\sigma$ together with the first integral*

$$\tau^2 = R - 2\mu\sigma + \frac{\mu^2 + \varepsilon}{R} \sigma^2, \quad \varepsilon = \pm 1, \quad (3.11)$$

Substituting $Y = A_0 + A_1\sigma + B_1\tau$ into (3.8) and eliminating τ^2 by (3.11) produces a polynomial in σ and τ whose coefficients vanish on the branches recorded below. These are the two branches used in this work, with constants corresponding to $\varepsilon = -1$; the full branch structure is given in Remark 3.3.

Branch A ($B_1 = 0$):

$$\mu = 0, \quad A_0 = -\frac{3b}{8c}, \quad A_1 = \frac{2ap^2}{b}, \quad R = \frac{3b^2}{16acp^2}, \quad \Lambda = \frac{15b^2}{64c}. \quad (3.12)$$

Branch B ($A_1 = 0$):

$$\mu = 0, \quad A_0 = -\frac{3b}{8c}, \quad B_1 = \pm \frac{\sqrt{3}}{2} p \sqrt{-\frac{a}{c}}, \quad R = -\frac{3b^2}{16acp^2}, \quad \Lambda = \frac{3b^2}{16c}. \quad (3.13)$$

Reality of B_1 requires $ac < 0$, which also gives $R > 0$. With $\mu = 0$ and $A_1 = 0$ the variable σ drops out of the reduction, the auxiliary system closes as $\tau' = R - \tau^2$, and ε does not enter.

For the branches used here, Λ is *fixed* by the algebraic constraints and is not a free parameter; since $\Lambda = d - \nu - aq^2$, this determines ν once d and q are prescribed.

Remark 3.3 (Structure of the algebraic system). Substituting $Y = A_0 + A_1\sigma + B_1\tau$ into (3.8) and eliminating τ^2 by the first integral gives nine independent coefficient equations. For $\mu = 0$ their solution set contains, besides the trivial and constant solutions, the branches

$$\begin{aligned} \text{(i)} \quad A_1 &= \pm \frac{2ap^2 \sqrt{-\varepsilon}}{b}, \quad B_1 = 0, \quad \Lambda = \frac{15b^2}{64c}, \quad R = \frac{3b^2}{16acp^2}, \\ \text{(ii)} \quad A_1 &= 0, \quad B_1 = \pm \frac{\sqrt{3}p}{2} \sqrt{-a/c}, \quad \Lambda = \frac{3b^2}{16c}, \quad R = -\frac{3b^2}{16acp^2}, \\ \text{(iii)} \quad A_1 &= \pm \frac{ap^2 \sqrt{\varepsilon}}{2b}, \quad B_1 = \pm \frac{\sqrt{3}p}{4} \sqrt{-a/c}, \quad \Lambda = \frac{3b^2}{16c}, \quad R = -\frac{3b^2}{4acp^2}, \end{aligned}$$

all with $A_0 = -3b/(8c)$. Branch (i) requires $\varepsilon = -1$ and is Branch A of Lemma 3.2; branch (ii) is Branch B; branch (iii) requires $\varepsilon = +1$ and corresponds to the (csch, coth) auxiliary functions, with residual vanishing identically.

Writing the first integral instead as $\tau^2 = R - 2\mu\sigma + (\mu^2 + \varepsilon/R)\sigma^2$ would force ε to depend on R and μ and could incorrectly exclude the (csch, coth) and (csc, cot) families. With the correct grouping $\tau^2 = R - 2\mu\sigma + \frac{\mu^2 + \varepsilon}{R}\sigma^2$, that exclusion does not follow.

The remaining restriction is admissibility. Since $\mathcal{H} = \sqrt{Y}$, a branch gives a real amplitude only where $Y \geq 0$. For branch (iii) at $a = 1$, $b = -2$, $c = -1$, $p = 1$ the outcome depends on the sign choice. With $A_1 = +ap^2/(2b) = -\frac{1}{4}$ reversed in sign to $+\frac{1}{4}$ and $B_1 = +\frac{\sqrt{3}}{4}$, so that $A_0 = -\frac{3}{4}$, $R = 3$ and $\Lambda = -\frac{3}{4}$, the identity $\text{csch } u + \coth u = \coth(u/2)$ gives

$$Y(\xi) = \frac{3}{4} \left[\coth\left(\frac{\sqrt{3}\xi}{2}\right) - 1 \right] > 0, \quad \xi > 0,$$

so the branch furnishes a real singular solution on the half line $\xi > 0$, unbounded as $\xi \rightarrow 0^+$ and decaying to 0 as $\xi \rightarrow \infty$. The three remaining sign combinations give $Y < 0$ for all $\xi > 0$. Branch (iii) is therefore admissible on a half line for one sign choice rather than inadmissible outright, and is excluded here on grounds of boundedness rather than reality.

Remark 3.4 (Consequence for the reported solution families). The restriction on the singular and (csc, cot) families is therefore one of admissibility rather than existence: they satisfy the coefficient system with vanishing residual, and the reality condition $Y(\xi) \geq 0$ selects which of them furnish a wave, checked here at one representative parameter set. These two families are not plotted in Figure 1, which shows the hyperbolic and periodic (sec, tan) profiles, and no entries for them appear in Table 2. Table 1 concerns the variational trial profile and is unaffected.

Remark 3.5 (Regularity of Branch A). On Branch A one has $A_1R = -A_0$ identically, so $Y(0) = 0$ for every admissible parameter set. Expanding about the origin gives $Y(\xi) = -\frac{3bR}{16c}\xi^2 + O(\xi^4)$, hence

$$\mathcal{H}(\xi) = \sqrt{-\frac{3bR}{16c}} |\xi| + O(|\xi|^3),$$

so $\mathcal{H}$ vanishes *linearly*, not like $|\xi|^{1/2}$. The derivative jumps by $2\sqrt{-3bR/(16c)}$ across the origin, so $\mathcal{H} \notin C^1(\mathbb{R})$: the profile has a corner, not a square-root cusp, and $\mathcal{H}'''$ carries a Dirac mass there. The

profile solves (3.4) classically on each of $\xi < 0$ and $\xi > 0$; on a grid of 4×10^5 points, the residual is 3.1×10^{-8} for $|\xi| > 0.5$. The structure is a cornered dark wave on the background $\mathcal{H}_\infty = \sqrt{-3b/(8c)}$, which requires $b/c < 0$, rather than a bright localized pulse. The remaining terms of (3.4) are bounded, so nothing cancels that Dirac mass and the profile is not a weak solution across the origin.

The corner is a property of the nonnegative square root rather than of the wave. Since Y is even and analytic with $Y(0) = 0$ and $Y''(0) > 0$, one may write $Y(\xi) = \xi^2 G(\xi)$ with G analytic and $G(0) > 0$, and because (3.4) contains only odd powers of $\mathcal{H}$ the signed profile

$$\widetilde{\mathcal{H}}(\xi) = \operatorname{sgn}(\xi) \sqrt{A_0[1 - \operatorname{sech}(\sqrt{R}\xi)]} = \xi \sqrt{G(\xi)} \quad (3.14)$$

is analytic at the origin and satisfies (3.4) on all of $\mathbb{R}$, the residual vanishing identically. Its modulus equals $\mathcal{H}$, so every profile plotted and every quantity tabulated below is unchanged, while the phase of ψ advances by π across $\xi = 0$, as for a classical dark wave. The regularity restriction recorded above therefore applies to the nonnegative representative, and the spectral quantities reported for it are retained as the discretized diagnostic described in Section 5.5.

Remark 3.6 (The periodic family admits no spectral problem on the line). For $ac < 0$ one has $R < 0$ and the profile $Y = A_0 + A_1 R \sec(\sqrt{|R|}\xi)$ is singular at $|\xi| = \pi/(2\sqrt{|R|})$, with $Y < 0$ beyond. The wave therefore exists only on a bounded interval and does not decay to a background, so the linearized operator is not defined on $\mathbb{R}$ and no spectral stability statement of the kind made for the hyperbolic family is available. Truncating the profile to a finite box and imposing periodic conditions returns the eigenvalues of the truncated numerical problem rather than of a wave defined on $\mathbb{R}$, and those values depend on where the singularity falls relative to the grid. Accordingly no growth rate is reported for this family.

Remark 3.7 (Discretized spectrum of the Branch A Projective Riccati profile). Forming the discretized linearization about the Branch A profile and refining the grid gives $\Lambda_{\max}$ at the numerical roundoff and conditioning level, of order 10^{-13} to 10^{-12} on the reported grids and therefore zero to working accuracy, for every parameter set in Tables 1 and 2 and for both box sizes $L = 40$ and $L = 60$ with $N = 400, \dots, 1600$. This is consistent with the background limits $\mathcal{L}_- \rightarrow ap^2 \partial_{\xi\xi}$ and $\mathcal{L}_+ \rightarrow ap^2 \partial_{\xi\xi} - 3b^2/(16c)$, for which the essential spectrum is purely imaginary. No eigenvalue with positive real part beyond roundoff was detected for any case tested, so the Branch A profiles are *numerically neutral* for the discretized operator. The signed Branch A wave is smooth; the corner belongs only to its nonnegative modulus. The reported spectrum remains a numerical result subject to domain and grid convergence. The entries in Table 2 should therefore be interpreted only as discretized diagnostics; see Section 5.5.

Remark 3.8 (Admissibility and domain of the traveling-wave solutions). Since the transformation $\mathcal{H}(\xi) = \sqrt{Y(\xi)}$ is employed, the resulting traveling-wave amplitude is real-valued only on the domain where

$$Y(\xi) = A_0 + A_1 \sigma(\xi) + B_1 \tau(\xi) \geq 0. \quad (3.15)$$

Thus, the admissibility of each obtained solution requires the nonnegativity condition (3.15) to hold on the corresponding interval of ξ . In addition, the denominators appearing in the auxiliary functions $\sigma(\xi)$ and $\tau(\xi)$ must remain nonzero. Consequently, the domain of each resulting solution is determined jointly by the condition $Y(\xi) \geq 0$ and the nonsingularity of the corresponding auxiliary functions.

• **Case 1: Hyperbolic solitary-wave solutions**

For $\varepsilon = -1$, $\delta = -1$, $R > 0$, the auxiliary functions are given by

$$\sigma(\xi) = \frac{R \operatorname{sech}(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1}, \quad \tau(\xi) = \frac{\sqrt{R} \tanh(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1}. \quad (3.16)$$

Substitution into the projective ansatz yields the corresponding hyperbolic solitary-wave family

$$\psi_1(x, t) = \sqrt{A_0 + \frac{A_1 R \operatorname{sech}(\sqrt{R}\xi) + B_1 \sqrt{R} \tanh(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1}} e^{i(qx + \frac{y^\alpha}{\alpha})}. \quad (3.17)$$

Domain. The auxiliary functions are regular on all of $\mathbb{R}$ for $\mu > -1$, since $0 < \operatorname{sech}(\sqrt{R}\xi) \leq 1$ and hence $\mu \operatorname{sech}(\sqrt{R}\xi) + 1 > 0$; for $\mu \leq -1$ the denominator vanishes at $\xi = \pm R^{-1/2} \operatorname{arcsech}(-1/\mu)$, which must be excluded. The solution is real-valued on the subset of the resulting domain where $Y(\xi) \geq 0$, in accordance with Remark 3.8. As $|\xi| \rightarrow \infty$ one has $\sigma \rightarrow 0$ but $\tau \rightarrow \pm \sqrt{R}$, so $Y \rightarrow A_0 \pm B_1 \sqrt{R}$ and the two limiting backgrounds differ when $B_1 \neq 0$. On Branch A, where $B_1 = 0$, both limits reduce to A_0 and $\psi_1 \rightarrow \sqrt{A_0}$; the solution is then localized only when $A_0 = 0$ and is otherwise a profile on a single constant background. Since $\mathcal{H} = \sqrt{Y}$ is nonnegative by construction, Branch A approaches the same background $\sqrt{A_0}$ at both ends. Connecting orbits of this general type are studied variationally by minimizing the action over paths joining isolated equilibria [40], and the formulation of Section 3.12 uses the same action functional J ; we note the connection without claiming that the profiles obtained here are heteroclinic.

• **Case 2: Singular solitary-wave solutions**

For $\varepsilon = +1$, $\delta = 1$, $R > 0$, the auxiliary functions take the form

$$\sigma(\xi) = \frac{R \operatorname{csch}(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1}, \quad \tau(\xi) = \frac{\sqrt{R} \coth(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1}. \quad (3.18)$$

Consequently, the corresponding singular solitary-wave family is

$$\psi_2(x, t) = \sqrt{A_0 + \frac{A_1 R \operatorname{csch}(\sqrt{R}\xi) + B_1 \sqrt{R} \coth(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1}} e^{i(qx + \frac{y^\alpha}{\alpha})}. \quad (3.19)$$

Domain. Clearing the common factor gives $\sigma = R/[\mu + \sinh(\sqrt{R}\xi)]$ and $\tau = \sqrt{R} \cosh(\sqrt{R}\xi)/[\mu + \sinh(\sqrt{R}\xi)]$, so the apparent singularity of csch and $\coth$ at $\xi = 0$ is removable whenever $\mu \neq 0$. The only genuine pole is $\xi_* = R^{-1/2} \operatorname{arcsinh}(-\mu)$, reducing to $\xi_* = 0$ when $\mu = 0$. The auxiliary pair is therefore defined on $\mathbb{R} \setminus \{\xi_*\}$, and the wave is real only where $Y(\xi) \geq 0$.

• **Case 3: Periodic trigonometric-wave solutions**

For $\varepsilon = -1$, $\delta = -1$, $R < 0$, with $k = \sqrt{|R|}$, the auxiliary functions are

$$\sigma(\xi) = \frac{R \sec(k\xi)}{\mu \sec(k\xi) + 1}, \quad \tau(\xi) = -\frac{k \tan(k\xi)}{\mu \sec(k\xi) + 1}. \quad (3.20)$$

The corresponding periodic trigonometric-wave family is

$$\psi_3(x, t) = \sqrt{A_0 + \frac{A_1 R \sec(k\xi) - B_1 k \tan(k\xi)}{\mu \sec(k\xi) + 1}} e^{i(qx + \frac{y^\alpha}{\alpha})}. \quad (3.21)$$

Domain. With $k = \sqrt{|R|}$ the quotients simplify to $\sigma = R/[\mu + \cos(k\xi)]$ and $\tau = -k \sin(k\xi)/[\mu + \cos(k\xi)]$, so the apparent sec and tan singularities at $\cos(k\xi) = 0$ are removable whenever $\mu \neq 0$. The genuine poles are the zeros of $\mu + \cos(k\xi)$: for $|\mu| \leq 1$ these form the periodic set $\xi = [\pm \arccos(-\mu) + 2\pi n]/k$, $n \in \mathbb{Z}$, and for $|\mu| > 1$ there are none. The solution is therefore defined on the open intervals between consecutive singular points, and within each such interval it is real-valued only where $Y(\xi) \geq 0$. It represents a periodic array of singular structures rather than a smooth periodic wave.

A second periodic trigonometric family is obtained from

$$\sigma(\xi) = \frac{R \csc(k\xi)}{\mu \csc(k\xi) + 1}, \quad \tau(\xi) = +\frac{k \cot(k\xi)}{\mu \csc(k\xi) + 1}. \quad (3.22)$$

Accordingly, the second periodic trigonometric-wave family is

$$\psi_4(x, t) = \sqrt{A_0 + \frac{A_1 R \csc(k\xi) + B_1 k \cot(k\xi)}{\mu \csc(k\xi) + 1}} e^{i(qx + \frac{v\alpha}{\alpha})}. \quad (3.23)$$

Domain. With $k = \sqrt{|R|}$ the quotients simplify to $\sigma = R/[\mu + \sin(k\xi)]$ and $\tau = k \cos(k\xi)/[\mu + \sin(k\xi)]$, so the apparent csc and cot singularities at $\sin(k\xi) = 0$ are removable whenever $\mu \neq 0$. The genuine poles are the roots of $\sin(k\xi) = -\mu$, which exist for $|\mu| \leq 1$ and must be excluded; for $|\mu| > 1$ there are none. On each remaining interval the solution is real only where $Y(\xi) \geq 0$.

Figure 1 presents profiles generated by the Projective Riccati construction for $\alpha = 0.5$ and $\alpha = 1$. The hyperbolic Branch A profile $|\psi_1(x, t)|$ remains bounded but possesses a corner at the profile minimum, as implied by $A_1 R = -A_0$. The periodic (sec, tan) family is not a bounded wave on $\mathbb{R}$: it blows up at $\xi = |R|^{-1/2}(n + \frac{1}{2})\pi$, $n \in \mathbb{Z}$, and is real only where $Y(\xi) \geq 0$, which selects alternate gaps between consecutive singularities, so the lower row shows the profile on those bands rather than a periodic wave train on the line. The singular families likewise develop unbounded peaks at their singular points. Differences between $\alpha = 0.5$ and $\alpha = 1$ in the physical-time representation arise from the temporal mapping $\tau = t^\alpha/\alpha$ and should therefore be interpreted as temporal reparametrization rather than as a change in the intrinsic traveling-wave orbit.

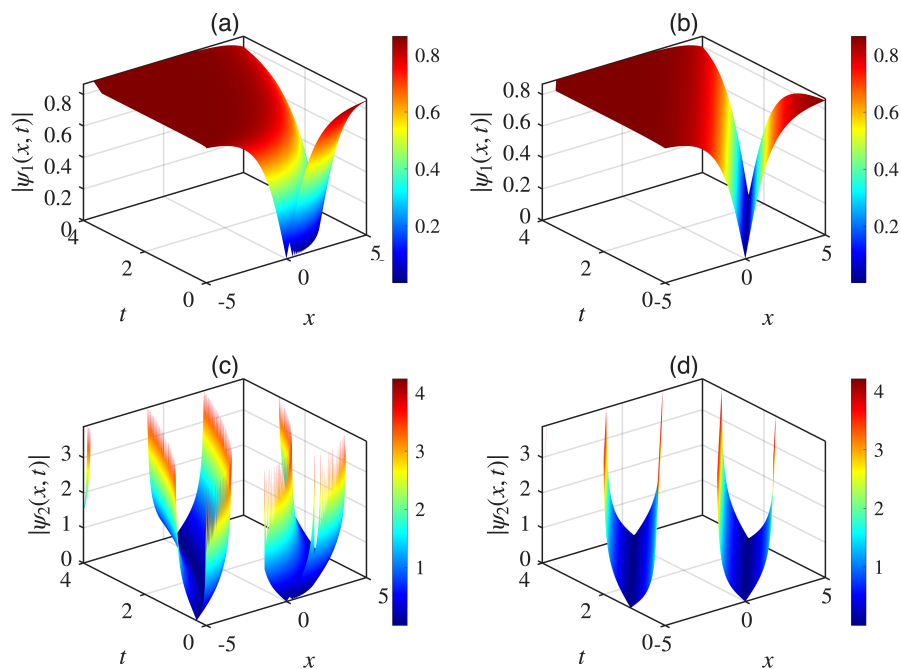


Figure 1. Profiles $|\psi_i(x, t)|$ generated by the Projective Riccati construction (Branch A, $\mu = 0$, $B_1 = 0$) of Lemma 3.2 for $\alpha = 0.5$ (left) and $\alpha = 1$ (right); $a = 1$, $b = -2$, $p = q = 1$. Top: hyperbolic (sech, tanh) wave, $c = +1$ so $R > 0$. Bottom: periodic (sec, tan) wave, $c = -1$ so $R < 0$. The corner at the base of each trough follows from $A_1 R = -A_0$. The hyperbolic profile is bounded on $\mathbb{R}$; the periodic profile is singular at $\xi = |R|^{-1/2}(n + \frac{1}{2})\pi$, $n \in \mathbb{Z}$, and is real only on alternate gaps between consecutive singularities.

Remark 3.9 (Justification of the three-mode Galerkin truncation). The field is projected onto the mean mode and the first pair of conjugate Fourier modes, which retains the background and the fundamental sideband interaction in a tractable reduced system. The cubic and quintic terms generate higher harmonics, so the neglected modes do not vanish identically and the truncation is a low-order spectral projection rather than an exact closure. The traveling-wave reduction and the Projective Riccati Equation Method are applied independently of it, so the two constructions serve complementary purposes.

Lemma 3.10 (Formal linearization of the Projective Riccati profiles). *Let $\mathcal{H}_j(\xi)$ denote a Projective Riccati profile on a domain where it is sufficiently smooth and satisfies the reduced equation classically. To investigate its spectral stability, we introduce the perturbed solution*

$$\psi(x, t) = \left[\mathcal{H}_j(\xi) + \varepsilon(u(\xi) + iv(\xi)) e^{i\lambda t} \right] e^{i\Theta}, \quad 0 < \varepsilon \ll 1, \quad (3.24)$$

where $u(\xi)$ and $v(\xi)$ are real-valued perturbation functions, $\lambda \in \mathbb{C}$ is the spectral parameter, and Θ denotes the phase of the corresponding traveling-wave solution. Here $\Lambda = d - \nu - aq^2$, and the operator is written in the co-moving coordinate $\xi = px + \omega t^\alpha/\alpha$, so spatial differentiation carries the factor p^2 and the carrier phase shifts the linear coefficient from d to Λ . This is the same linearization as (3.42)–(3.43) and is the form used for every spectral computation in this section.

Substituting Eq (3.24) into the governing equation and retaining only the terms of order $O(\varepsilon)$ yields the linearized eigenvalue problem

$$\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -ap^2\partial_{\xi\xi} - (b\mathcal{H}_j^2 + c\mathcal{H}_j^4 + \Lambda) \\ ap^2\partial_{\xi\xi} + (3b\mathcal{H}_j^2 + 5c\mathcal{H}_j^4 + \Lambda) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \quad (3.25)$$

Equivalently, the associated differential operator is

$$\mathcal{L}_j = \begin{pmatrix} 0 & -ap^2\partial_{\xi\xi} - (b\mathcal{H}_j^2 + c\mathcal{H}_j^4 + \Lambda) \\ ap^2\partial_{\xi\xi} + (3b\mathcal{H}_j^2 + 5c\mathcal{H}_j^4 + \Lambda) & 0 \end{pmatrix}. \quad (3.26)$$

For numerical computation, the second-order derivatives are approximated using the standard central finite-difference scheme

$$u_{\xi\xi}(\xi_k) \approx \frac{u_{k+1} - 2u_k + u_{k-1}}{(\Delta\xi)^2}, \quad v_{\xi\xi}(\xi_k) \approx \frac{v_{k+1} - 2v_k + v_{k-1}}{(\Delta\xi)^2}. \quad (3.27)$$

After spatial discretization, Eq (3.25) leads to the finite-dimensional eigenvalue problem

$$\lambda \begin{pmatrix} u_1, \dots, u_N, v_1, \dots, v_N \end{pmatrix}^T = \mathbf{L}_j \begin{pmatrix} u_1, \dots, u_N, v_1, \dots, v_N \end{pmatrix}^T, \quad (3.28)$$

where $\mathbf{L}_j$ is the matrix obtained from the finite-difference discretization of the linearized operator $\mathcal{L}_j$. The eigenvalues of $\mathbf{L}_j$ then provide a numerical spectral diagnostic for these profiles. Defining the spectral growth rate by

$$\Lambda_{\max} = \max_{\lambda \in \sigma(\mathbf{L}_j)} \operatorname{Re}(\lambda), \quad (3.29)$$

for profiles for which a conventional spectral problem is well defined, the eigenvalues of the discretized operator provide the numerical spectral criterion: $\Lambda_{\max} \leq 0$ indicates spectral stability and $\Lambda_{\max} > 0$ instability, subject to the treatment of neutral eigenvalues associated with continuous symmetries. For Branch A the signed profile (3.14) is smooth and the corner belongs only to its nonnegative modulus; the computed eigenvalues serve as a discretized diagnostic, subject to domain and grid convergence.

The amplitude and width of the variational bright solitary wave are reported in Table 1. No spectral quantity is tabulated for this profile, since it is a Ritz trial function and not a solution of the reduced traveling-wave equation. The amplitude is independent of the dispersion coefficient a , which does not enter (3.47), and the width obeys $\mathcal{B} \propto a^{-1/2}$, since a and p enter only through the combination ap^2 , which rescales ξ .

Remark 3.11. The Branch A profile is numerically neutral for the discretized operator, as established in Remark 3.7. No spectral statement is made for the periodic (sec, tan) family, whose profile is singular at finite ξ (Remark 3.6), and the singular and (csc, cot) families are excluded from the present spectral analysis because of their singularity and domain restrictions (Remark 3.3).

3.2. Variational formulation of CQ-NLS

Lemma 3.12 (Variational formulation and Ritz functional). *For*

$$\Lambda = d - v - aq^2,$$

the reduced ordinary differential equation (3.4) admits the variational formulation

$$\delta J = 0, \quad J = \int_{-\infty}^{\infty} \left[-\frac{ap^2}{2} (\mathcal{H}')^2 + \frac{\Lambda}{2} \mathcal{H}^2 + \frac{b}{4} \mathcal{H}^4 + \frac{c}{6} \mathcal{H}^6 \right] d\xi. \quad (3.30)$$

Indeed, the corresponding Euler–Lagrange equation is

$$ap^2 \mathcal{H}'' + \Lambda \mathcal{H} + b\mathcal{H}^3 + c\mathcal{H}^5 = 0, \quad (3.31)$$

which coincides with the reduced equation (3.4).

Following the balancing procedure introduced in the Projective Riccati Equation Method, we employ the transformation

$$\mathcal{H}(\xi) = Y^{1/2}(\xi), \quad (3.32)$$

which is consistent with the polynomial formulation of the reduced equation. Accordingly, the Ritz trial function is chosen as

$$Y(\xi) = A^2 \operatorname{sech}^2(B\xi), \quad \mathcal{H}(\xi) = A \operatorname{sech}(B\xi), \quad (3.33)$$

with

$$\mathcal{H}'(\xi) = -AB \operatorname{sech}(B\xi) \tanh(B\xi). \quad (3.34)$$

Substituting (3.33)–(3.34) into (3.30) gives

$$\begin{aligned} J &= \int_{-\infty}^{\infty} \left[-\frac{ap^2 A^2 B^2}{2} \operatorname{sech}^2(B\xi) \tanh^2(B\xi) + \frac{\Lambda A^2}{2} \operatorname{sech}^2(B\xi) \right. \\ &\quad \left. + \frac{bA^4}{4} \operatorname{sech}^4(B\xi) + \frac{cA^6}{6} \operatorname{sech}^6(B\xi) \right] d\xi \\ &= -\frac{ap^2 A^2 B}{3} + \frac{\Lambda A^2}{B} + \frac{bA^4}{3B} + \frac{8cA^6}{45B}, \end{aligned} \quad (3.35)$$

where we have used the standard integrals

$$\begin{aligned} \int_{-\infty}^{\infty} \operatorname{sech}^2(B\xi) \tanh^2(B\xi) d\xi &= \frac{2}{3B}, \\ \int_{-\infty}^{\infty} \operatorname{sech}^2(B\xi) d\xi &= \frac{2}{B}, \quad \int_{-\infty}^{\infty} \operatorname{sech}^4(B\xi) d\xi = \frac{4}{3B}, \end{aligned}$$

and

$$\int_{-\infty}^{\infty} \operatorname{sech}^6(B\xi) d\xi = \frac{16}{15B}.$$

Lemma 3.13 (Ritz stationary conditions). *The stationary conditions*

$$\frac{\partial J}{\partial A} = 0, \quad \frac{\partial J}{\partial B} = 0, \quad (3.36)$$

associated with the reduced functional (3.35) yield

$$B^2 = \frac{3\Lambda + 2bA^2 + \frac{8}{5}cA^4}{ap^2}, \quad (3.37)$$

and the amplitude condition

$$32cA^4 + 45bA^2 + 90\Lambda = 0. \quad (3.38)$$

Equivalently, for $c \neq 0$,

$$A^2 = \frac{-45b \pm \sqrt{2025b^2 - 11520c\Lambda}}{64c}, \quad (3.39)$$

subject to the admissibility conditions

$$A^2 > 0, \quad B^2 > 0.$$

Figure 2 displays the variational (Ritz) trial profile $\mathcal{H} = \mathcal{A} \operatorname{sech}(\mathcal{B}\xi)$, whose amplitude and width follow from (3.47) and (3.37) and depend on the chosen parameter set. The α dependence in panel (b) is the temporal reparametrization $\tau = t^\alpha/\alpha$ applied to that profile, so it records a change of time variable rather than of localization, and no propagation or stability claim is attached to this profile.

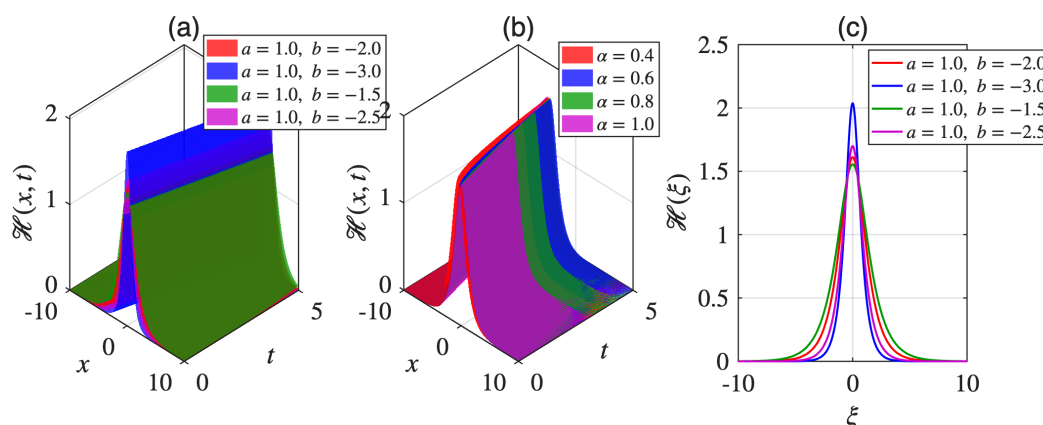


Figure 2. Bright solitary-wave profiles of the variational trial function $\mathcal{H} = \mathcal{A} \operatorname{sech}(\mathcal{B}\xi)$ for the conformable CQ–NLS model (1.4). Panels (a), (c): four parameter sets $(b, c, d) = (-2, 1, 0.20)$, $(-3, 1, 0.10)$, $(-1.5, 0.8, 0.15)$ and $(-2.5, 1.2, 0.05)$, all at $a = 1$. Panel (b): dependence on the conformable order $\alpha \in \{0.4, 0.6, 0.8, 1\}$.

Lemma 3.14 (Spectral stability of the traveling-wave solution). *Consider the traveling-wave solution*

$$\psi(x, t) = \mathcal{H}(\xi) \exp \left[i \left(qx + \frac{\nu t^\alpha}{\alpha} \right) \right], \quad \xi = px + \frac{\omega t^\alpha}{\alpha}, \quad \omega = -2apq, \quad (3.40)$$

where $\mathcal{H}(\xi)$ is a real-valued solitary-wave profile. The linearization about this solution leads to the coupled spectral problem

$$\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (3.41)$$

where

$$L_- = ap^2 \partial_{\xi\xi} + [b\mathcal{H}^2(\xi) + c\mathcal{H}^4(\xi) + d - \nu - aq^2], \quad (3.42)$$

$$L_+ = ap^2 \partial_{\xi\xi} + [3b\mathcal{H}^2(\xi) + 5c\mathcal{H}^4(\xi) + d - \nu - aq^2]. \quad (3.43)$$

For the bright solitary-wave profile

$$\mathcal{H}(\xi) = A \operatorname{sech}(B\xi), \quad (3.44)$$

the operators reduce to

$$L_- = ap^2 \partial_{\xi\xi} + \left[bA^2 \operatorname{sech}^2(B\xi) + cA^4 \operatorname{sech}^4(B\xi) + d - \nu - aq^2 \right], \quad (3.45)$$

$$L_+ = ap^2 \partial_{\xi\xi} + \left[3bA^2 \operatorname{sech}^2(B\xi) + 5cA^4 \operatorname{sech}^4(B\xi) + d - \nu - aq^2 \right]. \quad (3.46)$$

The eigenvalue problem (3.41) is discretized exactly as in (3.27)–(3.28), and the spectral growth rate $\Lambda_{\max}$ is defined as in (3.29). The discretization is not repeated here. The criterion is as before, $\Lambda_{\max} \leq 0$ for spectral stability and $\Lambda_{\max} > 0$ for instability, with the perturbation evolving as $e^{\lambda\tau} = \exp(\lambda\tau/\alpha)$ in physical time. Neutral eigenvalues associated with continuous symmetries are treated separately.

Two stability tables are reported because the two solution families are constrained differently, and a single table cannot describe both. For the Branch A Projective Riccati profile the algebraic system fixes $\Lambda = 15b^2/(64c)$ and $R = 3b^2/(16acp^2)$ in terms of (a, b, c, p) , so Λ is not free, the coefficient d never reaches the linearized operator, and the admissible parameter set is one point for each choice of (a, b, c, p) . For the variational wave $\Lambda = d - \nu - aq^2$ remains free, so d enters directly and the amplitude and width vary continuously with it. Table 2 therefore sweeps a, b, c, p and Table 1 sweeps b, c, d . Both tables are retained because they report different quantities: the Branch A profile admits a discretized spectral calculation and is numerically neutral to working accuracy for the discretized problem, whereas the variational profile yields only the amplitude and width predicted by its construction.

Table 1. Variational (Ritz) trial profile $\mathcal{H} = \mathcal{A} \operatorname{sech}(\mathcal{B}\xi)$, with $\nu = -aq^2$ so that $\Lambda = d$. Amplitude from (3.47), width from $\mathcal{B}^2 = (3\Lambda + 2b\mathcal{A}^2 + \frac{8}{5}c\mathcal{A}^4)/(ap^2)$. The residual $\mathcal{R}$ is of order unity, so no stability classification is attached.

Parameter	Value	$\mathcal{A}$	$\mathcal{B}$	$\max \mathcal{R} $	$\ \mathcal{R}\ _2/\ \mathcal{H}\ _2$
a	0.50	1.6111	1.4127	1.2062	0.5357
	1.00	1.6111	0.9990	1.2062	0.5357
	1.50	1.6111	0.8156	1.2062	0.5357
b	−1.50	1.3402	0.6112	0.4805	0.2565
	−2.00	1.6111	0.9990	1.2062	0.5357
	−3.50	2.1920	1.9759	5.6227	1.8355
c	0.50	2.3275	1.5520	3.7948	1.1666
	1.00	1.6111	0.9990	1.2062	0.5357
	1.50	1.2835	0.7237	0.5805	0.3236
d	0.05	1.6618	1.1427	1.4082	0.6063
	0.20	1.6111	0.9990	1.2062	0.5357
	0.25	1.5922	0.9448	1.1370	0.5110

The numerical diagnostics of the variational bright solitary wave trial profile are reported in Table 1. No spectral stability classification is assigned to it: the sech profile is a Ritz trial function rather than

an exact solution of the reduced traveling-wave equation, so the associated linearized operator is not a spectral linearization about an exact traveling wave. The amplitude satisfies

$$32c\mathcal{A}^4 + 45b\mathcal{A}^2 + 90\Lambda = 0, \quad \Lambda = d - v - aq^2, \quad (3.47)$$

whose admissible root is $\mathcal{A}^2 = (-45b + \sqrt{2025b^2 - 11520c\Lambda})/(64c)$. Tables 1 and 2 characterize distinct objects. The Branch A family is an exact solution: the signed profile (3.14) satisfies the reduced equation on all of $\mathbb{R}$, and the nonnegative representative $\mathcal{H} = \sqrt{Y}$ is its modulus, classical on each of $\xi < 0$ and $\xi > 0$ and carrying a corner at the origin. Since $\mathcal{L}_\pm$ depend on $\mathcal{H}$ only through $\mathcal{H}^2$ and $\mathcal{H}^4$, they are unchanged by the choice of representative and may be read as the linearization about the smooth signed wave. The sech ansatz, by contrast, is a trial function obtained by rendering an averaged action stationary and does not satisfy the reduced equation. Its residual is of order unity, with $\max |\mathcal{R}| = 1.2062$ against $\max |\mathcal{H}| = 1.6111$ at the base parameters and a relative L^2 residual between 0.26 and 1.84 over the twelve cases in Table 1. Operators of the form $\mathcal{L}_\pm$ built around it are therefore not the linearization about any traveling wave of this equation, and their eigenvalues therefore do not constitute a spectral stability result; no stability classification is attached to this table. A stronger statement holds at these parameters. Multiplying (3.4) by $\mathcal{H}'$ and integrating with $\mathcal{H}, \mathcal{H}' \rightarrow 0$ gives

$$(\mathcal{H}')^2 = -\frac{\Lambda}{ap^2}\mathcal{H}^2 - \frac{b}{2ap^2}\mathcal{H}^4 - \frac{c}{3ap^2}\mathcal{H}^6,$$

whose right-hand side is negative as $\mathcal{H} \rightarrow 0$ because every row of Table 1 has $\Lambda = d > 0$ and $a > 0$. No exact bright solitary wave decaying to zero exists at these parameters, so the tabulated amplitude and width characterize the trial function itself and not an approximation to such a wave. What the variational construction does supply is the predicted amplitude and width, and those are what the table reports. The amplitude is independent of a , which does not enter (3.47), and the width obeys $\mathcal{B} \propto a^{-1/2}$, reproduced to four digits by the a sweep.

Table 2. Spectrum of the finite-difference linearization about the Branch A Projective Riccati profile of Lemma 3.2, with $\Lambda_{\max} = \max \operatorname{Re} \lambda$ at two grid sizes. The coefficient functions involve only $\mathcal{H}^2$ and $\mathcal{H}^4$, so they are the same for the smooth signed wave (3.14); the entries remain a discretized diagnostic, subject to the convergence qualifications of Section 5.5. Λ and R are fixed by (a, b, c, p) , so d does not enter. $\Lambda_{\max}$ stays at the roundoff level. Admissibility requires $ac > 0$ and $bc < 0$.

Parameter	Value	A_0	A_1	R	Λ	$\Lambda_{\max}$		Stability
						$N = 800$	$N = 1600$	
a	0.50	0.7500	-0.5000	1.5000	0.9375	1.92×10^{-13}	1.02×10^{-12}	Neutral
	1.00	0.7500	-1.0000	0.7500	0.9375	5.68×10^{-13}	3.74×10^{-12}	Neutral
	1.50	0.7500	-1.5000	0.5000	0.9375	6.68×10^{-13}	3.41×10^{-12}	Neutral
b	-1.50	0.5625	-1.3333	0.4219	0.5273	4.12×10^{-13}	2.16×10^{-12}	Neutral
	-2.00	0.7500	-1.0000	0.7500	0.9375	5.68×10^{-13}	3.74×10^{-12}	Neutral
	-3.50	1.3125	-0.5714	2.2969	2.8711	4.41×10^{-13}	2.27×10^{-12}	Neutral
c	0.50	1.5000	-1.0000	1.5000	1.8750	5.12×10^{-13}	2.96×10^{-12}	Neutral
	1.00	0.7500	-1.0000	0.7500	0.9375	5.68×10^{-13}	3.74×10^{-12}	Neutral
	1.50	0.5000	-1.0000	0.5000	0.6250	5.40×10^{-13}	3.07×10^{-12}	Neutral
p	0.50	0.7500	-0.2500	3.0000	0.9375	1.35×10^{-13}	6.14×10^{-13}	Neutral
	1.00	0.7500	-1.0000	0.7500	0.9375	5.68×10^{-13}	3.74×10^{-12}	Neutral
	1.50	0.7500	-2.2500	0.3333	0.9375	1.14×10^{-12}	6.37×10^{-12}	Neutral

Table 2 replaces the growth rates reported in the original submission. Those values, which ranged from 0.045 to 160, were produced by a linearization about $Y = A_0 + A_1\sigma + B_1\tau$ rather than about $\mathcal{H} = \sqrt{Y}$, with constants $A_0 = 0.2$, $A_1 = 1.0$, $B_1 = 0.5$, $R = 1$, $\mu = 0.5$ that were prescribed independently of (a, b, c, d) and do not satisfy the algebraic system, and with the singular families truncated by $|\mathcal{H}| \leq 20$. The reported magnitudes scaled with that truncation level and are therefore not reproducible. With the Branch A constants and no truncation, the growth rate is zero to working accuracy for every admissible parameter set, in agreement with the analytic essential spectrum computation. The practical consequence is that the Branch A profile is numerically neutral to working accuracy for the discretized problem. No competing instability claim is made for the variational profile of Table 1, which is a trial function rather than a solution; see the discussion following that table.

The computed $\Lambda_{\max}$, of order 10^{-13} to 10^{-12} on the reported grids, *increases* with N in proportion to the conditioning of the larger matrix, which is consistent with a numerically zero eigenvalue rather than a resolved growth rate.

Remark 3.15. Tables 1 and 2 require separate interpretation. Table 1 reports the amplitude, width and residual of a variational trial profile; no stability statement is attached to it, since the profile does not solve the reduced equation and the associated operators are not a linearization about any traveling wave. Table 2 concerns the Branch A profile, whose computed growth rate is at roundoff on every grid, so it is numerically neutral for the discretized problem. Although $A_1R = -A_0$ forces $\mathcal{H} \sim C|\xi|$, the signed profile (3.14) is smooth and the corner belongs only to its nonnegative modulus; the reported spectrum remains a numerical result subject to domain and grid convergence. No parameter set considered here gives a growth rate that survives refinement, so no instability is asserted for either object.

3.3. The extended $\left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)$ -expansion method

Lemma 3.16. Consider, for $\mathcal{G} = \mathcal{G}(\xi)$, the auxiliary Riccati-type equation

$$\left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)' = \mu + \lambda \left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)^2, \quad \lambda \neq 0. \quad (3.48)$$

Then Eq (3.48) admits the following exact auxiliary-function solutions.

1) For $\mu\lambda > 0$, the trigonometric solution is

$$\frac{\mathcal{G}'}{\mathcal{G}^2} = \sqrt{\frac{\mu}{\lambda}} \left[\frac{\delta_1 \cos(\sqrt{\mu\lambda}\xi) + \delta_2 \sin(\sqrt{\mu\lambda}\xi)}{\delta_2 \cos(\sqrt{\mu\lambda}\xi) - \delta_1 \sin(\sqrt{\mu\lambda}\xi)} \right]. \quad (3.49)$$

2) For $\mu\lambda < 0$, the hyperbolic solution becomes

$$\frac{\mathcal{G}'}{\mathcal{G}^2} = -\frac{\sqrt{|\mu\lambda|}}{\lambda} \left[\frac{\delta_1 \cosh(2\sqrt{|\mu\lambda|}\xi) + \delta_1 \sinh(2\sqrt{|\mu\lambda|}\xi) + \delta_2}{\delta_1 \cosh(2\sqrt{|\mu\lambda|}\xi) + \delta_1 \sinh(2\sqrt{|\mu\lambda|}\xi) - \delta_2} \right]. \quad (3.50)$$

3) For $\mu = 0, \lambda \neq 0$, the rational solution is

$$\frac{\mathcal{G}'}{\mathcal{G}^2} = -\frac{\delta_1}{\lambda(\delta_1\xi + \delta_2)}. \quad (3.51)$$

Hence, Eq (3.48) generates trigonometric, hyperbolic and rational candidate structures for the extended $\left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)$ -expansion method.

Lemma 3.17. Consider the reduced nonlinear ordinary differential equation

$$ap^2 \mathcal{H}''' + b\mathcal{H}^3 + c\mathcal{H}^5 + (d - \nu - aq^2)\mathcal{H} = 0. \quad (3.52)$$

According to the extended $\left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)$ -expansion method, the solution is assumed in the form

$$\mathcal{H}(\xi) = s_0 + \sum_{i=1}^N \left[s_i \left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)^i + z_i \left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)^{-i} \right], \quad (3.53)$$

where $\mathcal{G}$ satisfies the auxiliary Riccati equation (3.48).

Following the balancing procedure established previously in the Projective Riccati Equation Method, the balancing number is taken as $N = \frac{1}{2}$. To avoid fractional balancing, we introduce the transformation

$$\mathcal{H}(\xi) = \sqrt{\mathcal{F}(\xi)}, \quad (3.54)$$

which converts Eq (3.52) into a polynomial equation in $\mathcal{F}(\xi)$. Consequently, the balancing order becomes $N = 1$, and hence

$$\mathcal{F}(\xi) = s_0 + s_1 \left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right) + z_1 \left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)^{-1}. \quad (3.55)$$

For $\mu\lambda > 0$, the auxiliary equation admits the trigonometric solution

$$\frac{\mathcal{G}'}{\mathcal{G}^2} = \sqrt{\frac{\mu}{\lambda}} \left[\frac{\delta_1 \cos(\sqrt{\mu\lambda}\xi) + \delta_2 \sin(\sqrt{\mu\lambda}\xi)}{\delta_2 \cos(\sqrt{\mu\lambda}\xi) - \delta_1 \sin(\sqrt{\mu\lambda}\xi)} \right], \quad (3.56)$$

which gives the periodic candidate

$$\begin{aligned} \psi_{\text{per}}(x, t) = & \left[s_0 + s_1 \sqrt{\frac{\mu}{\lambda}} \left(\frac{\delta_1 \cos(\sqrt{\mu\lambda}\xi) + \delta_2 \sin(\sqrt{\mu\lambda}\xi)}{\delta_2 \cos(\sqrt{\mu\lambda}\xi) - \delta_1 \sin(\sqrt{\mu\lambda}\xi)} \right) \right. \\ & \left. + z_1 \sqrt{\frac{\lambda}{\mu}} \left(\frac{\delta_2 \cos(\sqrt{\mu\lambda}\xi) - \delta_1 \sin(\sqrt{\mu\lambda}\xi)}{\delta_1 \cos(\sqrt{\mu\lambda}\xi) + \delta_2 \sin(\sqrt{\mu\lambda}\xi)} \right) \right]^{\frac{1}{2}} e^{i(qx + \frac{v\tau^\alpha}{\alpha})}. \end{aligned} \quad (3.57)$$

For $\mu\lambda < 0$, the hyperbolic solution becomes

$$\frac{\mathcal{G}'}{\mathcal{G}^2} = -\frac{\sqrt{|\mu\lambda|}}{\lambda} \left[\frac{\delta_1 \cosh(2\sqrt{|\mu\lambda|}\xi) + \delta_1 \sinh(2\sqrt{|\mu\lambda|}\xi) + \delta_2}{\delta_1 \cosh(2\sqrt{|\mu\lambda|}\xi) + \delta_1 \sinh(2\sqrt{|\mu\lambda|}\xi) - \delta_2} \right], \quad (3.58)$$

which gives the following hyperbolic candidate

$$\begin{aligned} \psi_{\text{sol}}(x, t) = & \left[s_0 - s_1 \frac{\sqrt{|\mu\lambda|}}{\lambda} \left(\frac{\delta_1 \cosh(2\sqrt{|\mu\lambda|}\xi) + \delta_1 \sinh(2\sqrt{|\mu\lambda|}\xi) + \delta_2}{\delta_1 \cosh(2\sqrt{|\mu\lambda|}\xi) + \delta_1 \sinh(2\sqrt{|\mu\lambda|}\xi) - \delta_2} \right) \right. \\ & \left. - z_1 \frac{\lambda}{\sqrt{|\mu\lambda|}} \left(\frac{\delta_1 \cosh(2\sqrt{|\mu\lambda|}\xi) + \delta_1 \sinh(2\sqrt{|\mu\lambda|}\xi) - \delta_2}{\delta_1 \cosh(2\sqrt{|\mu\lambda|}\xi) + \delta_1 \sinh(2\sqrt{|\mu\lambda|}\xi) + \delta_2} \right) \right]^{\frac{1}{2}} e^{i(qx + \frac{v\tau^\alpha}{\alpha})}. \end{aligned} \quad (3.59)$$

For $\mu = 0, \lambda \neq 0$, the rational solution is

$$\frac{\mathcal{G}'}{\mathcal{G}^2} = -\frac{\delta_1}{\lambda(\delta_1\xi + \delta_2)}, \quad (3.60)$$

which gives the rational candidate

$$\psi_{\text{rat}}(x, t) = \left[s_0 - \frac{s_1\delta_1}{\lambda(\delta_1\xi + \delta_2)} - \frac{z_1\lambda(\delta_1\xi + \delta_2)}{\delta_1} \right]^{\frac{1}{2}} e^{i(qx + \frac{v\tau^\alpha}{\alpha})}. \quad (3.61)$$

Solving the coefficient-matching system shows that the method admits one nontrivial admissible family for the CQ–NLS reduction. Substituting (3.55) into (3.52) and clearing Z^{-1} gives a polynomial of degree eight in Z , whose nine coefficients vanish for

$$z_1 = 0, \quad s_0 = -\frac{3b}{8c}, \quad s_1 = \mp \frac{\sqrt{3}}{2} \lambda p \sqrt{-a/c}, \quad \Lambda = \frac{3b^2}{16c}, \quad \mu = \frac{3b^2}{16ac\lambda p^2}.$$

The coefficient system also admits a reciprocal branch with $s_1 = 0$ and $z_1 \neq 0$. For $a = -1$, $b = -2$, $c = p = \lambda = 1$, $\mu = -3/4$ and $\Lambda = 3/4$, the choice $s_0 = 3/4$, $z_1 = -3\sqrt{3}/8$ with $Z = -(\sqrt{3}/2) \coth(\sqrt{3}\xi/2)$ gives $\mathcal{H}^2 = \frac{3}{4}[1 + \tanh(\sqrt{3}\xi/2)]$, which satisfies the reduced equation identically once the apparent singularity of $\coth$ is removed. It carries the same background $\sqrt{3}/2$ and

coincides with the profile obtained above, so the two branches are reciprocal representations of one family rather than distinct waves. Reality of s_1 requires $ac < 0$; then $\mu\lambda = 3b^2/(16acp^2) < 0$, so only the hyperbolic auxiliary solution is reachable. The trigonometric case needs $\mu\lambda > 0$, hence $ac > 0$ and s_1 imaginary, and the rational case $\mu = 0$ leaves the coefficient system with no nontrivial branch. The periodic and rational forms written above therefore satisfy the auxiliary equation but not (3.52), and are retained only as intermediate steps. Note also that $\Lambda = 3b^2/(16c)$ is fixed by (b, c) , so d is not free: it only re-assigns v . Thus, the only admissible family obtained through the extended $\left(\frac{\mathcal{G}'}{\mathcal{G}^2}\right)$ -expansion method is hyperbolic.

Figure 3 shows the single admissible family. Panel (a) is the profile $\mathcal{H}(\xi) = \sqrt{s_0 + s_1 Z(\xi)}$ for $a = -1, b = -2, c = 1, p = 1, \lambda = 1$, for which $s_0 = 0.75, s_1 = -0.8660, \Lambda = 0.75$, and $\mu = -0.75$. It is a bounded monotone front rising from 0 to the background $\sqrt{3/2} = 1.2247$. Panel (b) is the corresponding $|\psi(x, t)|$ over the (x, t) plane at $\alpha = 0.8$: the front translates according to $\xi = px + \omega t^\alpha/\alpha$ with $\omega = -2apq$. Panel (c) resolves the front at $t = 3$ for $\alpha = 0.65, 0.80$, and 0.95 ; the three curves are translates of one another, located at $x = -2t^\alpha/\alpha$, that is at $-6.28, -6.02$, and -5.98 respectively. The conformable order therefore displaces the front without altering its shape, width or background, which is the same reparametrization statement established in Section 2.1.

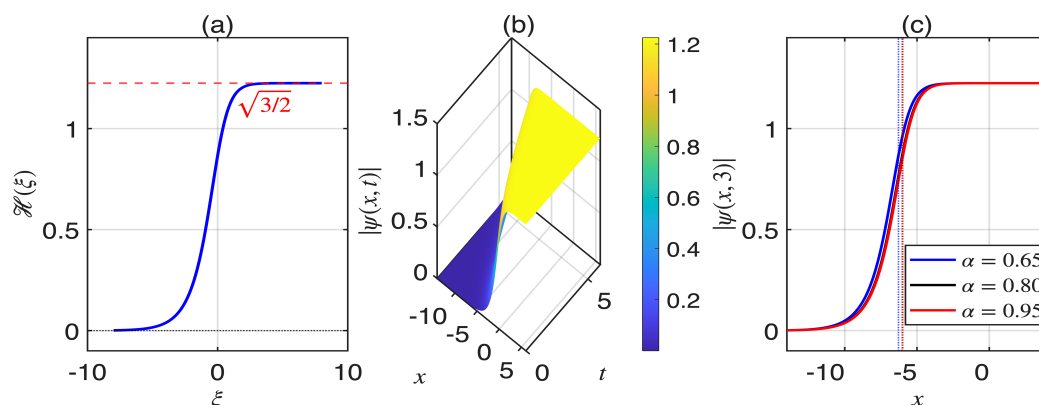


Figure 3. The single admissible exact family of the extended $(\mathcal{G}'/\mathcal{G}^2)$ -expansion, for $a = -1, b = -2, c = 1, p = q = \lambda = 1$. (a) Profile $\mathcal{H}(\xi)$, a bounded front from 0 to $\sqrt{3/2}$. (b) $|\psi(x, t)|$ at $\alpha = 0.8$. (c) The front at $t = 3$ for $\alpha = 0.65, 0.80, 0.95$, located at $x = -2t^\alpha/\alpha$, the curves being translates rather than reshapings.

Lemma 3.18 (Galerkin Reduction and equilibrium analysis). *Consider the conformable fractional cubic–quintic nonlinear Schrödinger equation*

$$i\mathcal{D}_t^\alpha \psi + a\psi_{xx} + b|\psi|^2\psi + c|\psi|^4\psi + d\psi = 0, \quad 0 < \alpha \leq 1, \quad (3.62)$$

with $\mathcal{D}_t^\alpha$ as in (1.5).

Using the truncated Fourier expansion

$$\psi(x, t) = \frac{1}{\sqrt{2}}A(t) + B(t)\cos x + C(t)\sin x, \quad (3.63)$$

we obtain

$$\psi_{xx} = -B \cos x - C \sin x, \quad \mathcal{D}_t^\alpha \psi = \frac{t^{1-\alpha}}{\sqrt{2}} \dot{A} + t^{1-\alpha} \dot{B} \cos x + t^{1-\alpha} \dot{C} \sin x. \quad (3.64)$$

Substituting Eq (3.63) into the nonlinear terms of Eq (3.62) yields

$$|\psi|^2 = \left(\frac{1}{\sqrt{2}} A + B \cos x + C \sin x \right) \left(\frac{1}{\sqrt{2}} A^* + B^* \cos x + C^* \sin x \right).$$

Expanding and collecting harmonics gives

$$\begin{aligned} |\psi|^2 = & \frac{1}{2} |A|^2 + |B|^2 \cos^2 x + |C|^2 \sin^2 x + \frac{1}{\sqrt{2}} (AB^* + A^* B) \cos x + \frac{1}{\sqrt{2}} (AC^* + A^* C) \sin x \\ & + (BC^* + B^* C) \sin x \cos x. \end{aligned} \quad (3.65)$$

Using the trigonometric identities

$$\cos^2 x = \frac{1}{2} (1 + \cos 2x), \quad \sin^2 x = \frac{1}{2} (1 - \cos 2x), \quad \sin x \cos x = \frac{1}{2} \sin 2x,$$

the full nonlinear products are formed first and then projected onto $\{1, \cos x, \sin x\}$; the second and higher harmonics generated in the products are discarded at that projection step. Let Π_1 denote the L^2 projection onto $\text{span}\{1, \cos x, \sin x\}$. The cubic product carries harmonics through order three and the quintic product through order five, so the reduction is

$$\Pi_1(|\psi|^2 \psi) = P_0 + P_1 \cos x + P_2 \sin x, \quad \Pi_1(|\psi|^4 \psi) = Q_0 + Q_1 \cos x + Q_2 \sin x, \quad (3.66)$$

where

$$\begin{aligned} P_0 &= \frac{1}{2\sqrt{2}} (|A|^2 A + 2|B|^2 A + 2|C|^2 A + B^2 A^* + C^2 A^*), \\ P_1 &= \frac{1}{2} A^2 B^* + |A|^2 B + \frac{3}{4} |B|^2 B + \frac{1}{2} |C|^2 B + \frac{1}{4} C^2 B^*, \\ P_2 &= \frac{1}{2} A^2 C^* + |A|^2 C + \frac{1}{4} B^2 C^* + \frac{1}{2} |B|^2 C + \frac{3}{4} |C|^2 C. \end{aligned} \quad (3.67)$$

Applying Galerkin projection onto the orthogonal basis

$$\{1, \cos x, \sin x\},$$

with

$$\int_{-\pi}^{\pi} \cos x \, dx = \int_{-\pi}^{\pi} \sin x \, dx = 0,$$

and

$$\int_{-\pi}^{\pi} \cos^2 x \, dx = \int_{-\pi}^{\pi} \sin^2 x \, dx = \pi.$$

Equation (3.62) reduces to

$$\begin{aligned} it^{1-\alpha} \dot{A} + dA + \sqrt{2} b P_0 + \sqrt{2} c Q_0 &= 0, \\ it^{1-\alpha} \dot{B} - aB + dB + bP_1 + cQ_1 &= 0, \end{aligned}$$

$$it^{1-\alpha}\dot{C} - aC + dC + bP_2 + cQ_2 = 0. \quad (3.68)$$

The reduced dynamical system admits the equilibrium points

$$E_0 = (0, 0, 0), \quad E_1 = (A_0, 0, 0), \quad E_2 = (0, R, R), \quad (3.69)$$

where the factor $\sqrt{2}$ on the nonlinear terms arises from rescaling the constant-mode projection of the ansatz $\psi = A/\sqrt{2} + B\cos x + C\sin x$, in which P_0 and Q_0 are the constant Fourier coefficients of $|\psi|^2\psi$ and $|\psi|^4\psi$. The nontrivial equilibrium amplitude is the admissible root of the resulting algebraic equation $dA + \sqrt{2}bP_0 + \sqrt{2}cQ_0 = 0$ evaluated at $B = C = 0$; written out,

$$|A_0|^2 = \frac{-b \pm \sqrt{b^2 - 4cd}}{c}, \quad R^2 = \frac{-3b \pm \sqrt{9b^2 - 40c(d-a)}}{10c}. \quad (3.70)$$

These follow from the projected equations at $B = C = 0$ and at $A = 0, B = C = R$ respectively. At $B = C = 0$ one has $P_0 = |A|^2 A / (2\sqrt{2})$ and $Q_0 = |A|^4 A / (4\sqrt{2})$, so the zero mode satisfies $d + \frac{b}{2}|A_0|^2 + \frac{c}{4}|A_0|^4 = 0$; at $A = 0, B = C = R$ one has $P_1 = \frac{3}{2}R^3$ and $Q_1 = \frac{5}{2}R^5$, giving $d - a + \frac{3b}{2}R^2 + \frac{5c}{2}R^4 = 0$.

Linearizing Eq (3.68) about the trivial equilibrium point E_0 gives the Jacobian matrix

$$J(E_0) = \begin{pmatrix} -d & 0 & 0 \\ 0 & a-d & 0 \\ 0 & 0 & a-d \end{pmatrix}, \quad (3.71)$$

with eigenvalues

$$\lambda_1 = -d, \quad \lambda_2 = \lambda_3 = a - d. \quad (3.72)$$

The reduced system carries the factor i on the time derivative, so the generator is $-iJ(E_0)$, with the purely imaginary eigenvalues id and $i(d-a)$. The linearized dynamics are therefore oscillatory and neutrally stable rather than asymptotically stable, and E_0 is a center. The quantities $-d$ and $a-d$ set the modal frequencies and do not classify the stability type: they identify no saddle, and no homoclinic orbit is associated with E_0 .

Figure 4 shows the modulational instability gain spectrum. The gain $G(Q)$ initially increases with the perturbation wavenumber Q , reaches a maximum value, and then decreases to zero beyond the critical wavenumber.

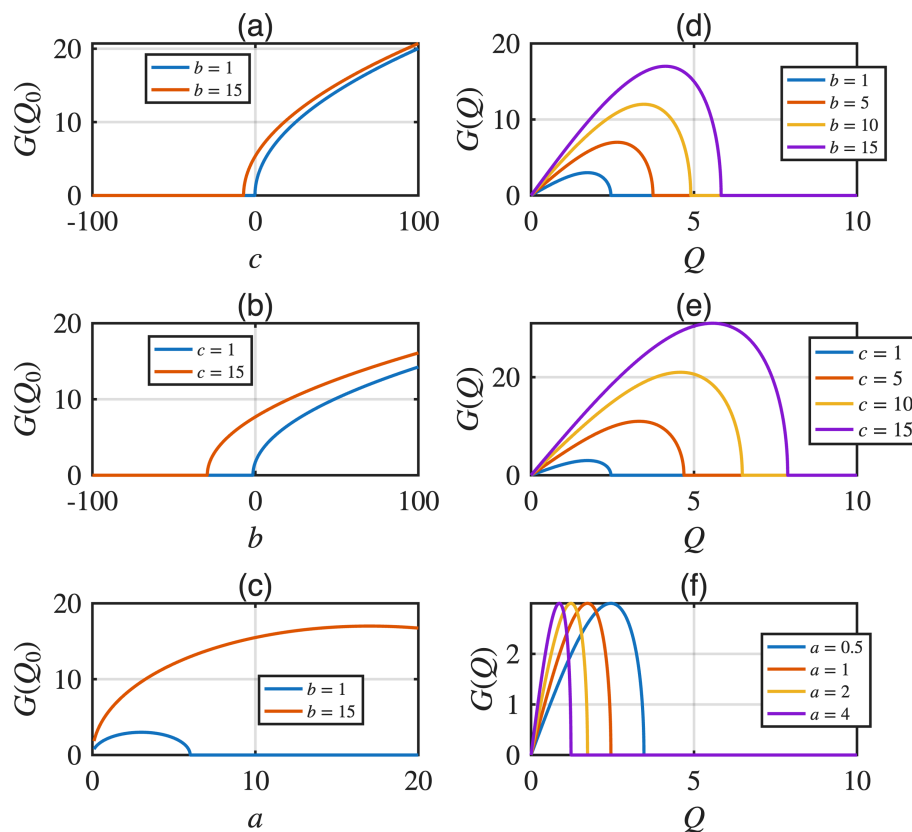


Figure 4. Modulational instability gain dynamics associated with $G(Q) = \sqrt{aQ^2 [2\varrho(b + 2c\varrho) - aQ^2]}$.

4. Bifurcation

Lemma 4.1 (Modulational instability of the conformable CQ–NLS equation). *Consider the conformable cubic–quintic nonlinear Schrödinger equation*

$$i\mathcal{D}_t^\alpha \psi + a\psi_{xx} + b|\psi|^2\psi + c|\psi|^4\psi = 0, \quad 0 < \alpha \leq 1, \quad (4.1)$$

where

$$a, b, c \in \mathbb{R},$$

and $\mathcal{D}_t^\alpha$ is the conformable derivative (1.5). In the effective time (2.1) the model is the classical local CQ–NLS equation, by (2.2).

Consequently, a continuous-wave solution of (4.1) is given by

$$\psi_0(x, t) = \mathcal{A} \exp \left[i \left(kx - \Omega \frac{t^\alpha}{\alpha} \right) \right], \quad (4.2)$$

where $\mathcal{A}$ is a constant background amplitude and the corresponding nonlinear dispersion relation is

$$\Omega = ak^2 - b|\mathcal{A}|^2 - c|\mathcal{A}|^4. \quad (4.3)$$

To examine the stability of the continuous-wave background, consider the small perturbation

$$\psi(x, t) = [\mathcal{A} + \sigma(x, \tau)] \exp[i(kx - \Omega\tau)], \quad |\sigma| \ll 1, \quad (4.4)$$

Introducing the comoving coordinate

$$X = x - 2ak\tau, \quad (4.5)$$

which removes the group velocity term generated by the carrier wavenumber k , the perturbation is represented in the standard sideband form

$$\sigma(X, \tau) = \mathcal{F}(\tau)e^{i\mu X} + \mathcal{G}^*(\tau)e^{-i\mu X}, \quad (4.6)$$

where μ denotes the perturbation wavenumber.

Upon substitution of (4.4)–(4.6) into (4.1), retaining only terms linear in the perturbation, and using (4.3), one obtains the linearized system

$$i \frac{d}{d\tau} \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix} = \begin{pmatrix} \Theta & -\Gamma \\ \Gamma & -\Theta \end{pmatrix} \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix}, \quad (4.7)$$

where

$$\Theta = a\mu^2 - b|\mathcal{A}|^2 - 2c|\mathcal{A}|^4, \quad (4.8)$$

and

$$\Gamma = b|\mathcal{A}|^2 + 2c|\mathcal{A}|^4. \quad (4.9)$$

Seeking normal-mode perturbations of the form

$$\mathcal{F}, \mathcal{G} \sim e^{\lambda\tau},$$

the corresponding characteristic equation becomes

$$\lambda^2 = \Gamma^2 - \Theta^2. \quad (4.10)$$

After simplification, this gives

$$\lambda^2 = a\mu^2 (2b|\mathcal{A}|^2 + 4c|\mathcal{A}|^4 - a\mu^2). \quad (4.11)$$

Writing $g = b|\mathcal{A}|^2 + 2c|\mathcal{A}|^4$, (4.11) reads $\lambda^2 = a\mu^2(2g - a\mu^2)$, and modulational instability ($\lambda^2 > 0$) occurs precisely when

$$ag > 0 \quad \text{and} \quad 0 < \mu^2 < \frac{2g}{a}. \quad (4.12)$$

Condition (4.12) holds for either sign of a . For $a > 0$ it requires $g > 0$, i.e. $2b|\mathcal{A}|^2 + 4c|\mathcal{A}|^4 > a\mu^2$, which is the form quoted in the classical literature and the one used in the figures below, where $a = 1$. For $a < 0$ the same pair of inequalities requires $g < 0$; the quotient $2g/a$ is then again positive, so the unstable band retains the form $0 < \mu^2 < 2g/a$ while the sign requirement on the nonlinear coefficients is reversed. If $ag \leq 0$ the background has no modulationally unstable band. When $ag > 0$, the cutoff wavenumber is $\mu_c = \sqrt{2g/a}$.

In terms of the original time variable t , the perturbation evolves according to

$$e^{\lambda\tau} = \exp\left(\lambda\frac{t^\alpha}{\alpha}\right). \quad (4.13)$$

Hence, $\Re(\lambda) > 0$ corresponds to exponential growth with respect to the conformable time τ , rather than to a hereditary memory-induced growth mechanism. The instability condition (4.12) has the same algebraic structure as the classical cubic–quintic NLS instability criterion because $\mathcal{D}_t^\alpha = d/d\tau$ under the transformation $\tau = t^\alpha/\alpha$. Thus, the conformable parameter α affects the temporal parametrization of the perturbation dynamics rather than introducing a fundamentally new algebraic instability threshold.

Furthermore, the unstable perturbation-wave-number interval is

$$0 < \mu^2 < \frac{2b|\mathcal{A}|^2 + 4c|\mathcal{A}|^4}{a}, \quad (4.14)$$

which is the $a > 0$ form of (4.12); for $a < 0$ the same expression applies with $g < 0$.

Remark 4.2. The modulational instability result above applies to the unforced equation (4.1). The forced planar system of Section 4 is a separate phenomenological model with its own dynamics.

Figure 5 illustrates the modulational-instability behavior of the conformable fractional cubic–quintic nonlinear Schrödinger equation for different fractional orders and perturbation wavenumbers. The modulational instability regions shown in Figure 5(a) are governed by the condition $2b|\mathcal{A}|^2 + 4c|\mathcal{A}|^4 > a\mu^2$, which separates stable and unstable propagation regimes. For the parameter set used in Figure 5, increasing the background amplitude $|\mathcal{A}|$ enlarges the instability region and enhances perturbation growth.

Figure 5(b) presents the instability growth spectrum $\Re(\lambda)$, which is a single curve: the eigenvalue is independent of α . Figure 5(c) plots the perturbation amplification measured in the *physical* time variable t . Because the linearized problem is posed in the conformable time $\tau = t^\alpha/\alpha$, the eigenvalue $\Re(\lambda)$ itself is independent of α ; the entire dependence on the conformable order enters through the effective elapsed time $D(\alpha) = (t^\alpha - t_0^\alpha)/\alpha$ of the observation window, the amplification being $\exp(\Re(\lambda)D(\alpha))$ as established in Eq (5.7). Since the ordering of D in α reverses between windows, the ordering seen here is a property of the window $[t_0, t]$ rather than of the instability mechanism: for the window used in this figure D increases with α , whereas for a window at earlier times it decreases. This is verified numerically in Section 5.7.

Finally, Figure 5(d) plots $D(\alpha)$ for two observation windows. On $[1, 3]$ it increases with α ; on $[0.05, 0.8]$ it decreases. Since the amplification is $\exp[\Re(\lambda)D(\alpha)]$ with $\Re(\lambda)$ fixed, the two windows produce opposite orderings of the growth curves in α from the same instability. The conformable order therefore sets the pace at which the classical CQ–NLS dynamics is traversed, not the strength of the instability itself.

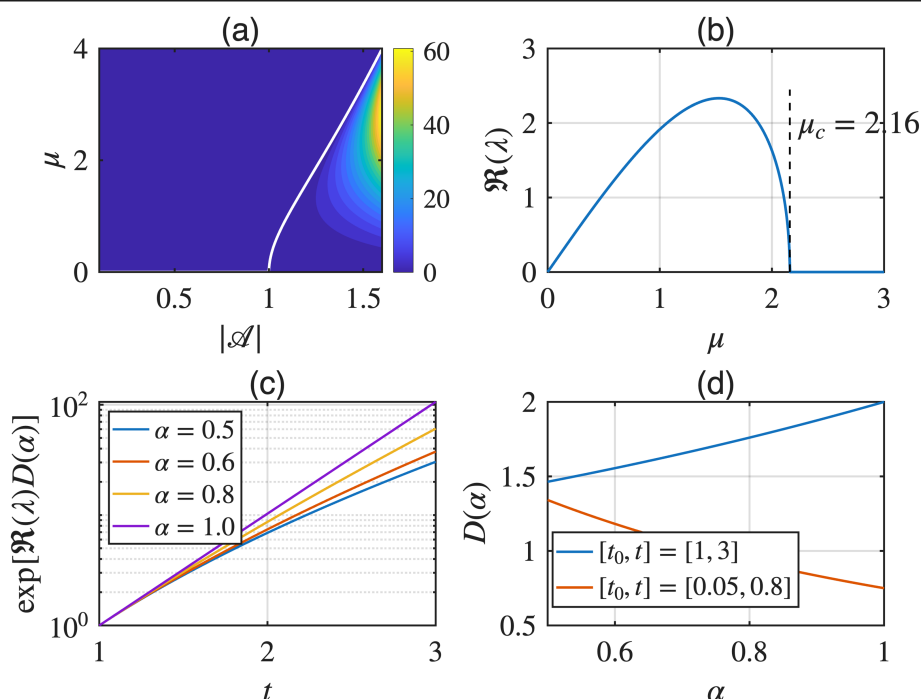


Figure 5. Modulational instability for $a = 1$, $b = -2$, $c = 1$, $|\mathcal{A}| = 1.3$. (a) Instability region in the $(|\mathcal{A}|, \mu)$ plane, boundary $a\mu^2 = 2b|\mathcal{A}|^2 + 4c|\mathcal{A}|^4$. (b) Growth spectrum $\Re(\lambda)$, independent of α , with cutoff $\mu_c = 2.16$. (c) Amplification in physical time, $\exp[\Re(\lambda)D(\alpha)]$. (d) $D(\alpha)$ for two windows, showing that the ordering of D in α reverses between windows.

Lemma 4.3. Introducing $\mathcal{G} = \mathcal{H}'$, Eq. (3.4) reduces to the planar dynamical system

$$\frac{d\mathcal{H}}{d\xi} = \mathcal{G}, \quad \frac{d\mathcal{G}}{d\xi} = \lambda_1 \mathcal{H} + \lambda_2 \mathcal{H}^3 + \lambda_3 \mathcal{H}^5, \quad (4.15)$$

where

$$\lambda_1 = -\frac{d - v - aq^2}{ap^2}, \quad \lambda_2 = -\frac{b}{ap^2}, \quad \lambda_3 = -\frac{c}{ap^2}. \quad (4.16)$$

The corresponding Hamiltonian function is

$$\mathcal{E}(\mathcal{H}, \mathcal{G}) = \frac{\mathcal{G}^2}{2} - \frac{\lambda_1}{2} \mathcal{H}^2 - \frac{\lambda_2}{4} \mathcal{H}^4 - \frac{\lambda_3}{6} \mathcal{H}^6. \quad (4.17)$$

The equilibrium points satisfy

$$\mathcal{G} = 0, \quad \lambda_1 \mathcal{H} + \lambda_2 \mathcal{H}^3 + \lambda_3 \mathcal{H}^5 = 0, \quad (4.18)$$

yielding the trivial equilibrium

$$e_0 = (0, 0), \quad (4.19)$$

together with the nontrivial equilibria

$$e_{\pm}^{(1,2)} = \left(\pm \sqrt{\frac{-\lambda_2 \pm \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{2\lambda_3}}, 0 \right), \quad \lambda_2^2 - 4\lambda_1\lambda_3 \geq 0. \quad (4.20)$$

Linearization about e_0 gives the Jacobian matrix

$$J(e_0) = \begin{pmatrix} 0 & 1 \\ \lambda_1 & 0 \end{pmatrix}, \quad (4.21)$$

with characteristic equation

$$\mu^2 - \lambda_1 = 0, \quad \mu_{1,2} = \pm \sqrt{\lambda_1}. \quad (4.22)$$

Hence, $\lambda_1 > 0$ corresponds to a saddle-type unstable equilibrium, whereas $\lambda_1 < 0$ produces a center surrounded by periodic trajectories. Furthermore, homoclinic phase trajectories generate solitary-wave solutions, while heteroclinic trajectories correspond to kink and anti-kink wave structures.

Figure 6 illustrates the bifurcation behavior of system (4.15) for different parameter signs (λ_1, λ_2) . For $\lambda_1 > 0$, the equilibrium point e_0 generates separatrix trajectories corresponding to solitary-wave structures, whereas $\lambda_1 < 0$ produces closed periodic orbits around the equilibrium point. For the cubic truncation $\lambda_3 = 0$ used in Figure 6, the additional equilibrium points $e_{\pm} = (\pm \sqrt{-\lambda_1/\lambda_2}, 0)$ exist when $\lambda_1/\lambda_2 < 0$, leading to periodic and heteroclinic phase trajectories. Overall, the phase portraits describe the qualitative nonlinear dynamics and geometrical structures of the traveling-wave solutions associated with the conformable fractional cubic–quintic nonlinear Schrödinger equation.

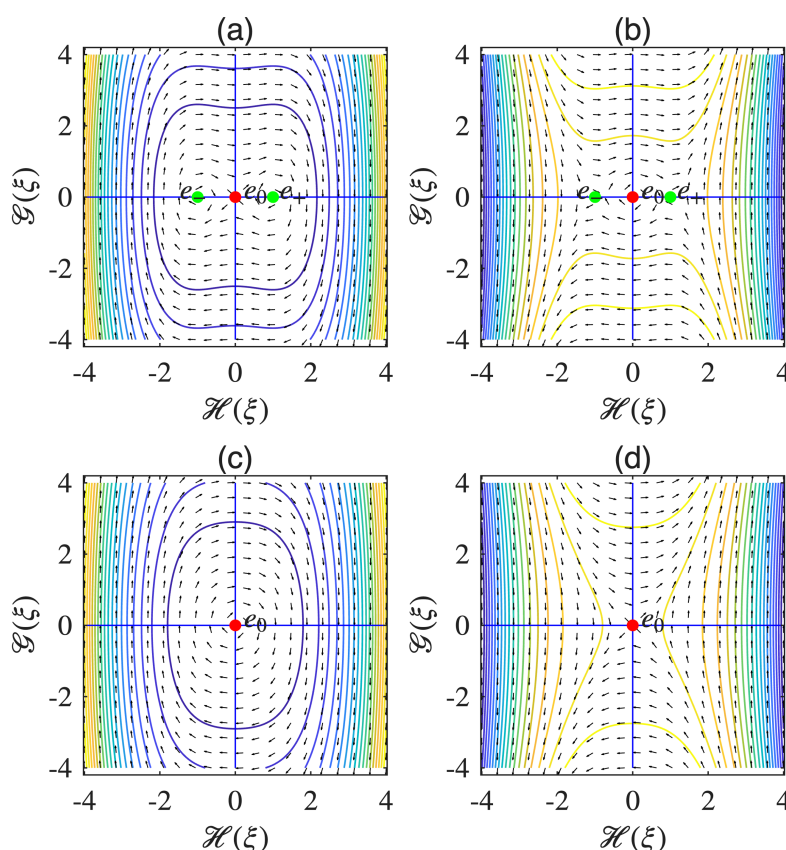


Figure 6. Phase portraits of system (4.15) for different signs of (λ_1, λ_2) . Cases I and IV ($\lambda_1 > 0$) produce saddle-type dynamics and separatrix trajectories, whereas Cases II and III ($\lambda_1 < 0$) generate bounded oscillatory response. The nontrivial equilibria are shown for the cubic truncation $\lambda_3 = 0$, for which $e_{\pm} = (\pm \sqrt{-\lambda_1/\lambda_2}, 0)$ and existence requires $\lambda_1/\lambda_2 < 0$. For $\lambda_3 \neq 0$ the equilibria are given by the quintic condition (4.20).

Figure 7 depicts the nonlinear dynamics of the reduced cubic–quintic planar system for different initial conditions. The time evolution of $\mathcal{H}(\xi)$ and $\mathcal{G}(\xi)$ shows bounded oscillatory behavior, while the corresponding phase portraits produce inward spiraling trajectories in the $(\mathcal{H}, \mathcal{G})$ phase plane. The trajectories shown include a linear damping term $-\gamma\mathcal{G}$ with $\gamma = 0.15$, which is added to (4.15) for this illustration; the undamped system (4.15) itself conserves $\mathcal{E}$. With the damping present, $\mathcal{E}$ decreases monotonically along every trajectory and settles at the value of the attracting equilibrium.

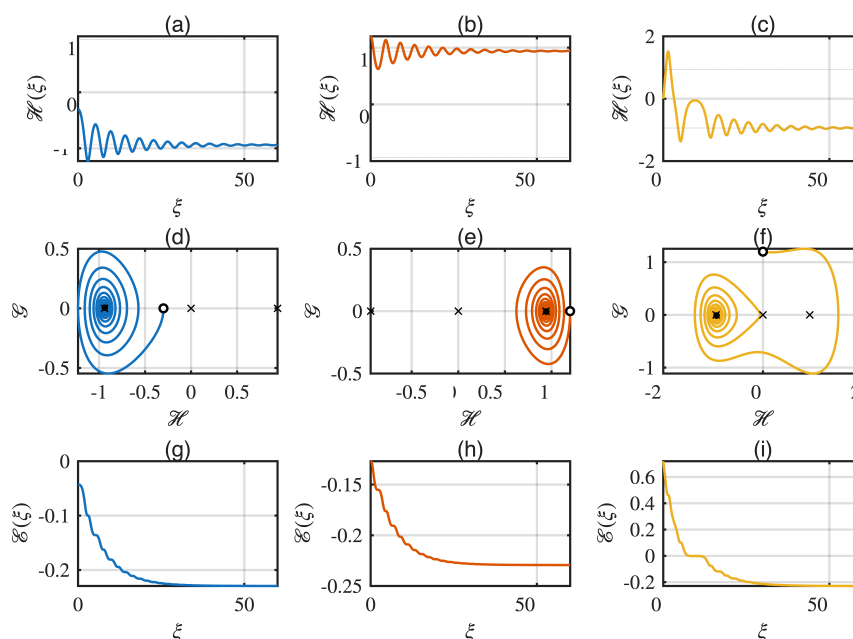


Figure 7. Nonlinear dynamics of the reduced cubic–quintic planar system for different initial conditions. Rows: $\mathcal{H}(\xi)$; phase portraits in the $(\mathcal{H}, \mathcal{G})$ plane; energy $\mathcal{E}(\xi)$. A damping term $-\gamma\mathcal{G}$, $\gamma = 0.15$, is included here, so $\mathcal{E}$ decays monotonically; the undamped reduced system conserves $\mathcal{E}$.

Figure 8 depicts the influence of the conformable fractional parameter α on the nonlinear dynamics of the reduced cubic planar system. The temporal profiles differ with α because the conformable order rescales the rate at which the same orbit is traversed, and with it the sampling of the trajectory. The orbit itself is unchanged, as established in Section 2.1. The return maps show bounded, structured recurrence. Moreover, the Fourier spectra are dominated by a single low-frequency component, with no resolvable higher harmonics. Panel (d) shows the phase portrait. The factor $\xi^{\alpha-1}$ multiplies both equations and so cancels from $d\mathcal{G}/d\mathcal{H}$: the two orders trace one common orbit, the $\alpha = 0.5$ curve being the shorter arc of it. Resolving this needs a tight tolerance. At $\text{RelTol} = 10^{-10}$ the first return values differ by 3×10^{-6} and $\max|\mathcal{H}|$ agrees to ten digits, 1.279262542; at the solver default the two orders appear to differ in the third digit.

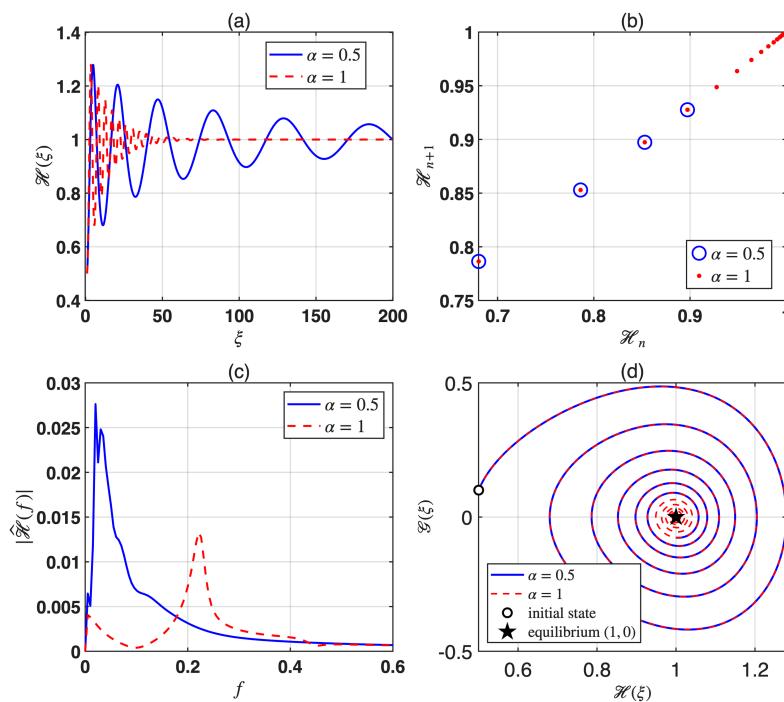


Figure 8. Fractional nonlinear dynamics of the reduced cubic planar system for $\alpha = 0.5$ and $\alpha = 1$. The figure illustrates the temporal evolution, return map and Fourier spectrum, together with panel (d), the phase portrait, in which both orders trace one common orbit.

Figure 9 illustrates the delay-embedded transients of the damped reduced cubic planar system. The equilibrium $\mathcal{H} = \sqrt{-\lambda_1/\lambda_2}$ is attracting, so the reconstructed structures are orbits spiralling into it rather than invariant sets: over the same window the residual deviation is of order 5×10^{-5} at $\alpha = 0.8$ and 5×10^{-11} at $\alpha = 1$, the slower decay at $\alpha < 1$ following from the factor $\xi^{\alpha-1}$, which weakens the damping as ξ grows. The reconstructed phase structures are shown in the delayed coordinates, $(\mathcal{H}(\xi), \mathcal{H}(\xi + \tau), \mathcal{H}(\xi + 2\tau))$ and $(\mathcal{G}(\xi), \mathcal{G}(\xi + \tau), \mathcal{G}(\xi + 2\tau))$. The delay-embedded representations differ with α because the reparametrization $\tau = t^\alpha/\alpha$ changes the rate at which the same orbit is traversed, and therefore the sampling, leaving the phase-space geometry of the autonomous system unchanged.

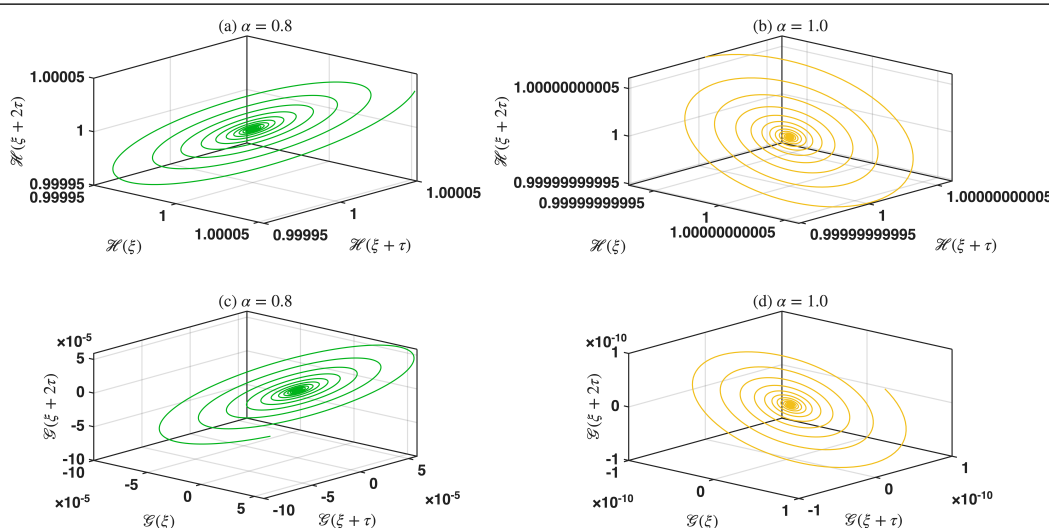


Figure 9. Delay-embedded decaying transients of the damped reduced cubic planar system for $\alpha = 0.8$ and $\alpha = 1$. The delay is one ξ unit and $\xi < 300$ is discarded. Note the axis scales: the deviation from the equilibrium is of order 5×10^{-5} at $\alpha = 0.8$ but only 5×10^{-11} at $\alpha = 1$, so both panels show orbits spiralling into the equilibrium rather than sustained oscillations. The first row represents the delay-embedded representations of $\mathcal{H}(\xi)$, while the second row corresponds to the delayed phase structures of $\mathcal{G}(\xi)$.

4.1. Perturbed fractional planar dynamical system

Lemma 4.4. *The conformable derivative in (1.4) acts on the physical time t . The system below is a phenomenological extension in ξ , obtained by replacing $d/d\xi$ with $\mathcal{D}_\xi^\alpha$ and adding forcing; its α -dependent behavior is therefore a property of that extension:*

$$\mathcal{D}_\xi^\alpha \mathcal{H} = \mathcal{G}, \quad \mathcal{D}_\xi^\alpha \mathcal{G} = \lambda_1 \mathcal{H} + \lambda_2 \mathcal{H}^3 + \lambda_3 \mathcal{H}^5 - \gamma \mathcal{G} + A \sin(\omega_1 \xi) + B \cos(\omega_2 \xi), \quad (4.23)$$

equivalently $d\mathcal{H}/d\xi = \xi^{\alpha-1}\mathcal{G}$ and $d\mathcal{G}/d\xi = \xi^{\alpha-1}[\cdot]$, which is the form integrated numerically from $\xi_0 = 1$, since $\xi^{\alpha-1}$ is singular at the origin for $\alpha < 1$. The factor $\xi^{\alpha-1}$ is a monotone reparametrization of ξ . For the autonomous unforced system it leaves the phase plane orbits unchanged and only relabels the points along them, so $\max |\mathcal{H}|$ from $(\mathcal{H}_0, \mathcal{G}_0) = (0.5, 0.1)$ equals 1.27926 for $\alpha = 0.5, 0.8$ and 1 alike. Once the explicit forcing is present the reparametrization no longer commutes with it and the trajectories depend on α . This is the only mechanism by which α enters the planar dynamics. Here, $\gamma > 0$ is the damping coefficient, while A, B, ω_1, ω_2 denote external forcing amplitudes and frequencies.

The conservative core obtained from system (4.23) by setting $\gamma = A = B = 0$ is Hamiltonian and retains the energy function (4.17) together with the equilibrium condition (4.18) established in the unperturbed analysis above; these are therefore not restated here. Setting $\mathcal{G} = 0$ and $\lambda_1 \mathcal{H} + \lambda_2 \mathcal{H}^3 + \lambda_3 \mathcal{H}^5 = 0$, the equilibria of the unperturbed system are

$$e_0 = (0, 0), \quad e_\pm = \left(\pm \sqrt{\frac{-\lambda_2 \pm \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{2\lambda_3}}, 0 \right), \quad (4.24)$$

with $\lambda_2^2 - 4\lambda_1\lambda_3 \geq 0$. Linearization about e_0 again gives the characteristic Eq (4.22).

Hence, $\lambda_1 > 0$ corresponds to saddle-type instability, whereas $\lambda_1 < 0$ produces a bounded oscillatory response. Furthermore, the forcing terms can generate nonlinear resonance, quasi-periodic oscillations, recurrence transitions, and chaotic attractor dynamics.

Figure 10 shows the effect of the forcing on the quintic phase space. Cases I and IV ($\lambda_1 > 0$) exhibit saddle-type unstable dynamics characterized by separatrix deformation and nonlinear stretching of trajectories, whereas the panels with $\lambda_2 < 0$ retain bounded orbits and those with $\lambda_2 > 0$ show trajectories escaping the plotted region, irrespective of the sign of λ_1 . In these four parameter sets λ_2 and λ_3 carry the same sign, and at large amplitude the restoring or outward force is dominated by $\lambda_3 \mathcal{H}^5$, so the panels record the joint effect of the two nonlinear coefficients and do not isolate λ_2 .

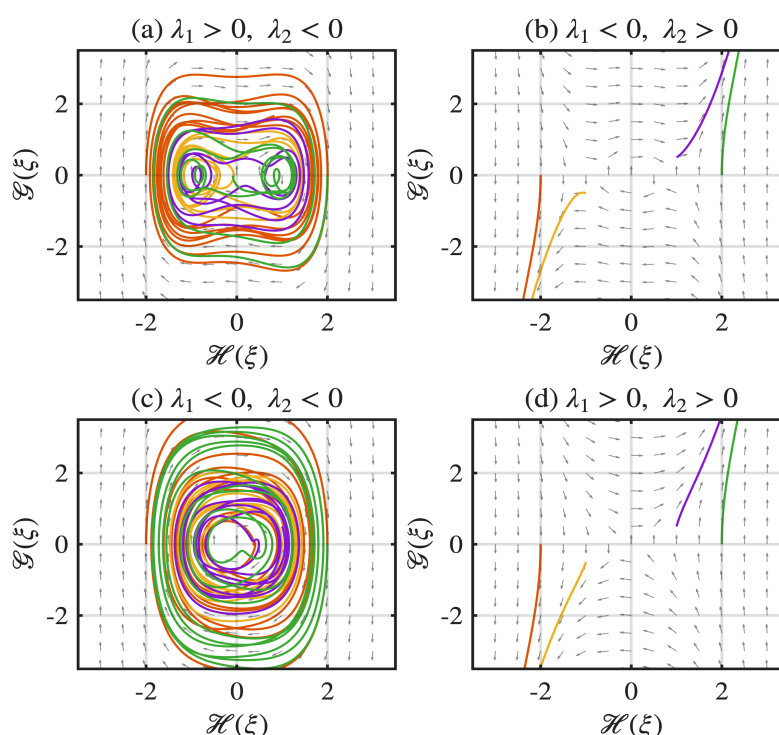


Figure 10. Phase portraits of the perturbed quintic planar system for (a) $(\lambda_1, \lambda_2, \lambda_3) = (1, -1, -0.15)$, (b) $(-1, 1, 0.15)$, (c) $(-1, -1, -0.10)$, (d) $(1, 1, 0.10)$, with $\gamma = 0.08$, $A = 0.35$, $B = 0.25$, $\omega_1 = 1.3$, $\omega_2 = 2.2$. Arrows show the direction field; colored curves are trajectories from different initial conditions.

Figure 11 reports the response of the forced system as λ_1 is varied. Panel (a) collects, for each λ_1 , the local maxima of $\mathcal{H}$ over the sampled response window for an ensemble of initial conditions. At $\alpha = 0.8$ the diagram is flat and close to zero for $\lambda_1 < 0$ and opens into two branches beyond the origin, whereas at $\alpha = 1$ the branches are already separated and unequal in magnitude at $\lambda_1 = -1.2$, so the response is nowhere symmetric about $\mathcal{H} = 0$. This asymmetry is consistent with an imperfect response: a pitchfork requires the symmetry $\mathcal{H} \rightarrow -\mathcal{H}$, whereas the additive forcing $A \sin(\omega_1 \xi) + B \cos(\omega_2 \xi)$ does not change sign under that transformation. Panel (b) gives the half peak-to-peak amplitude $A_{\mathcal{H}} = \frac{1}{2}(\mathcal{H}_{\max} - \mathcal{H}_{\min})$, which at $\alpha = 1$ collapses abruptly near $\lambda_1 \approx -0.52$ and grows again beyond the origin through jagged intervals, while at $\alpha = 0.8$ it rises smoothly from zero. Panel (c) shows two prominent bands of positive finite-time Lyapunov exponents near $\lambda_1 \approx 0.24$ and

1.05 for $\alpha = 1$, together with smaller positive excursions. For $\alpha = 0.8$, the computed exponent remains negative over the plotted range. Panel (d) shows the response at $\lambda_1 = 0.40$: a small slowly contracting orbit at $\alpha = 0.8$ and a thick multiloop band at $\alpha = 1$, both centered near $\mathcal{H} \approx 0.6$ rather than at the origin. At $\alpha = 1$ the orbit is an attractor, its half peak-to-peak width settling at 0.1716 for every integration length tested. For $\alpha < 1$ the forced response instead decreases over the integration window, because the drive keeps its phase in ξ while the system is clocked by $\tau = \xi^\alpha/\alpha$, so relative to that clock it becomes an increasingly rapid chirp against the damping γ . The $\alpha = 0.8$ curves therefore record the response over the window $\xi \leq 600$ rather than an invariant set. The α -dependence is a forcing effect: the forcing is specified in ξ and does not follow the reparametrization of time, so it is modulated relative to the effective conformable time scale of the system. It is not a memory effect, the conformable derivative being local.

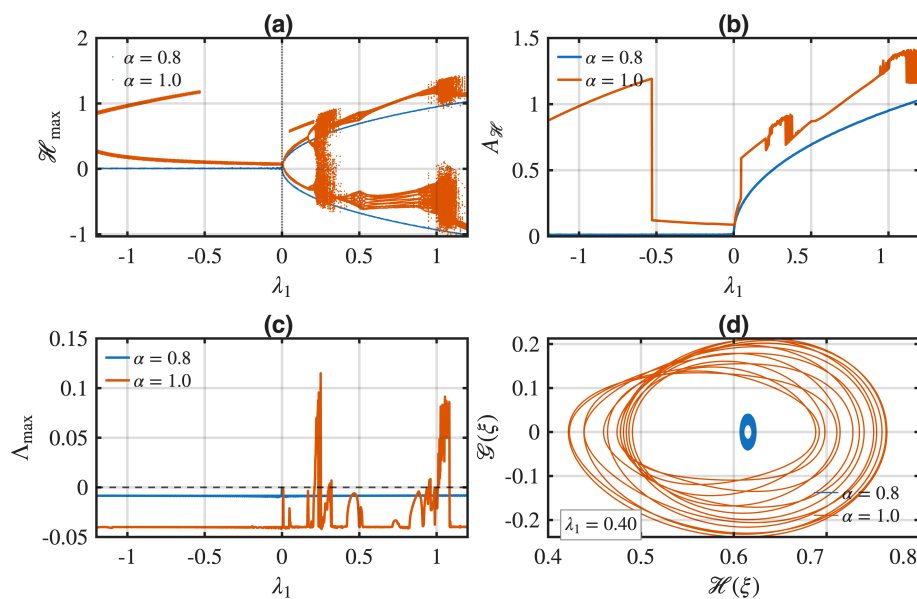


Figure 11. Bifurcation and response structure of the forced cubic–quintic planar system for $\alpha = 0.8$ and $\alpha = 1$. (a) Local maxima of $\mathcal{H}$ over the sampled response window against $\lambda_1 \in [-1.2, 1.2]$; (b) ensemble response half-range $A_{\mathcal{H}} = \frac{1}{2}(\mathcal{H}_{\max} - \mathcal{H}_{\min})$; (c) largest finite-time Lyapunov exponent by the Benettin algorithm from $(\mathcal{H}_0, \mathcal{G}_0) = (0.5, 0.1)$, distinct from the spectral growth rate of Sections 3 and 5; (d) state-space response at $\lambda_1 = 0.40$.

Figure 12 illustrates the delay-reconstructed responses of the perturbed cubic–quintic planar system for different fractional orders α . The embeddings of $\mathcal{H}(\xi)$ and $\mathcal{G}(\xi)$ have a toroidal appearance over the plotted window. The two orders differ by an order of magnitude in extent: the embedding spans about ± 0.01 at $\alpha = 0.8$ and about ± 0.1 at $\alpha = 1$. Both are toroidal.

Figure 13 demonstrates the influence of the external excitation frequencies ω_1 and ω_2 on the nonlinear oscillatory dynamics of the perturbed fractional cubic–quintic planar system. For $\alpha = 0.8$, the system exhibits bounded, generally quasiperiodic, oscillations with reduced nonlinear modulation, whereas the integer-order case $\alpha = 1$ produces sustained oscillatory responses with comparatively larger amplitudes and stronger modulation.

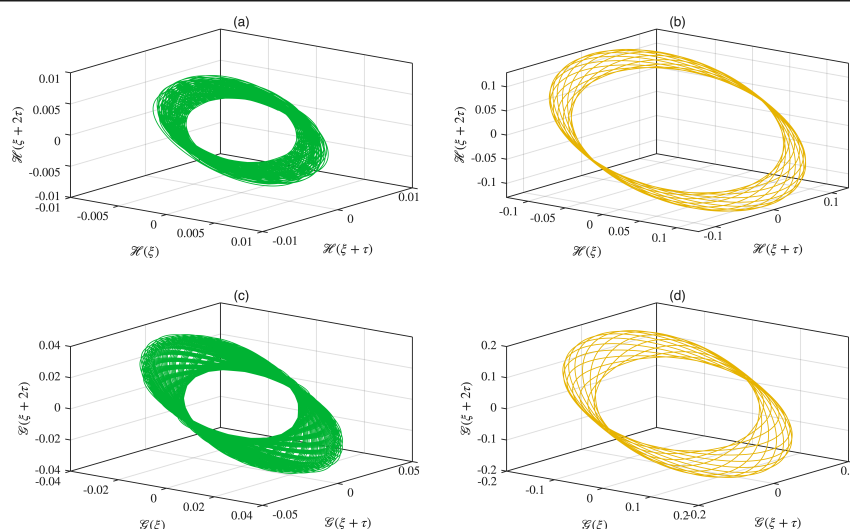


Figure 12. Delay-embedded response of the perturbed cubic–quintic planar system for $\alpha = 0.8$ and $\alpha = 1$. Panels (a)–(b): embedding of $\mathcal{H}$ in $(\mathcal{H}(\xi), \mathcal{H}(\xi + \tau), \mathcal{H}(\xi + 2\tau))$; panels (c)–(d): the same for $\mathcal{G}$.

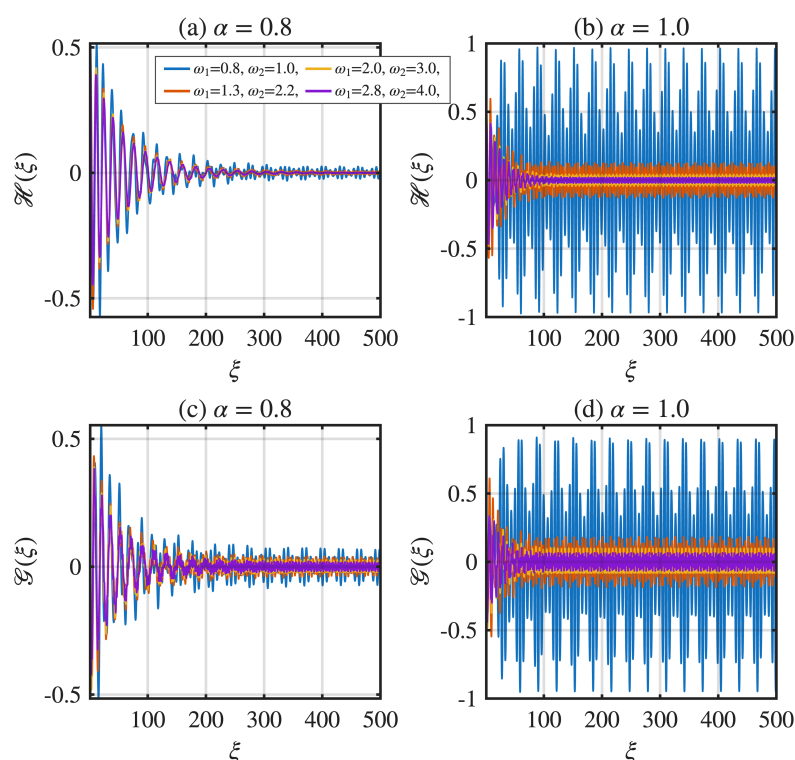


Figure 13. Frequency sensitivity of the forced cubic–quintic planar system for $(\omega_1, \omega_2) = (0.8, 1.0), (1.3, 2.2), (2.0, 3.0), (2.8, 4.0)$. Panels (a) and (b) show $\mathcal{H}(\xi)$ at $\alpha = 0.8$ and $\alpha = 1$; panels (c) and (d) show $\mathcal{G}(\xi)$ at the same two orders. Parameters $\lambda_1 = -0.6$, $\lambda_2 = -1$, $\lambda_3 = -0.15$, $\gamma = 0.08$, $A = 0.12$, $B = 0.08$, $(\mathcal{H}_0, \mathcal{G}_0) = (0.5, 0.1)$, shown over $\xi \in [1, 500]$.

Figure 14 illustrates the first return maps on the section $\mathcal{G} = 0$ of the perturbed fractional cubic–

quintic planar system for periodic and chaotic regimes at $\alpha = 0.8$ and $\alpha = 1$. The periodic cases give compact clustered point distributions, whereas the cases identified as chaotic give more dispersed irregular distributions. In these panels, increasing the order from $\alpha = 0.8$ to $\alpha = 1$ enlarges the distribution and the oscillatory amplitude.

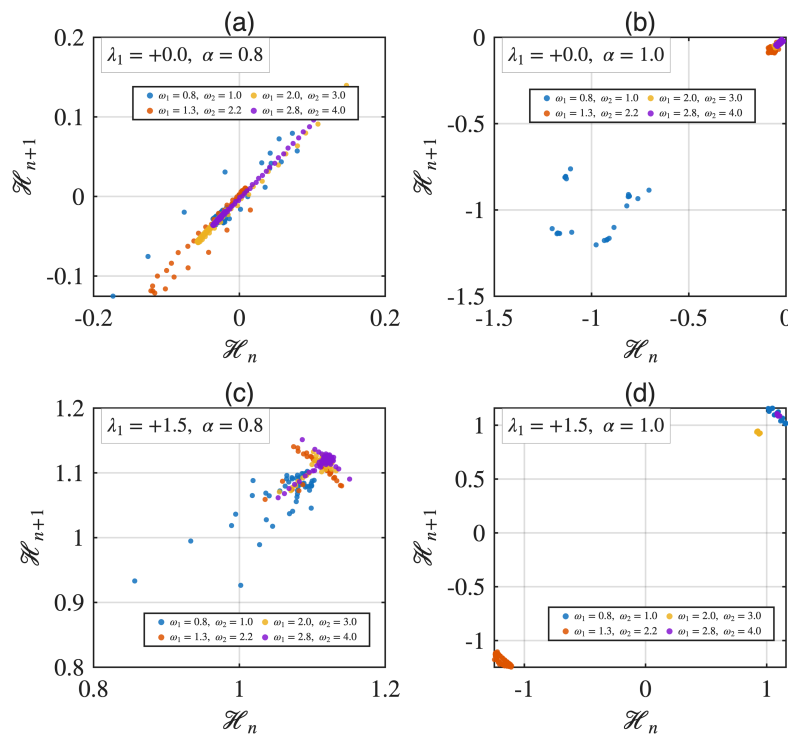


Figure 14. Poincaré return maps of the forced cubic–quintic planar system with $\lambda_2 = -1$, $\lambda_3 = -0.15$, $\gamma = 0.08$, $A = 0.12$, $B = 0.08$ and $(\omega_1, \omega_2) = (0.8, 1.0)$, $(1.3, 2.2)$, $(2.0, 3.0)$, $(2.8, 4.0)$. Panels (a)–(b) are at $\lambda_1 = 0$ and (c)–(d) at $\lambda_1 = +1.5$; left column $\alpha = 0.8$, right column $\alpha = 1$. Each map plots $\mathcal{H}_{n+1}$ against $\mathcal{H}_n$ at successive upward crossings of $\mathcal{G} = 0$.

Figure 15 summarizes several numerical diagnostics for the forced system. The basin panel is computed at $\lambda_1 = +0.8$, where two attractors coexist; at lower values for which only the origin attracts the sampled initial conditions, the diagram becomes uniform. Initial conditions are classified by the mean of $\mathcal{H}$ over the last part of the run rather than by its endpoint, since with periodic forcing the endpoint depends on the forcing phase rather than on which attractor the orbit has reached. The classification threshold is part of the diagnostic, so the panel represents a basin slice for the specified threshold and forcing phase. The energy panel plots the conservative energy along trajectories of the forced system, where it is not conserved. Its drift is therefore the quantity of interest, and it varies with α because the forcing does not follow the reparametrization of time. The governing identity and the separate contributions of forcing and damping are given with Figure 17; the parameters here are weaker forcing in the single-well regime, $\lambda_1 = -0.6$, $A = 0.12$, $B = 0.08$.

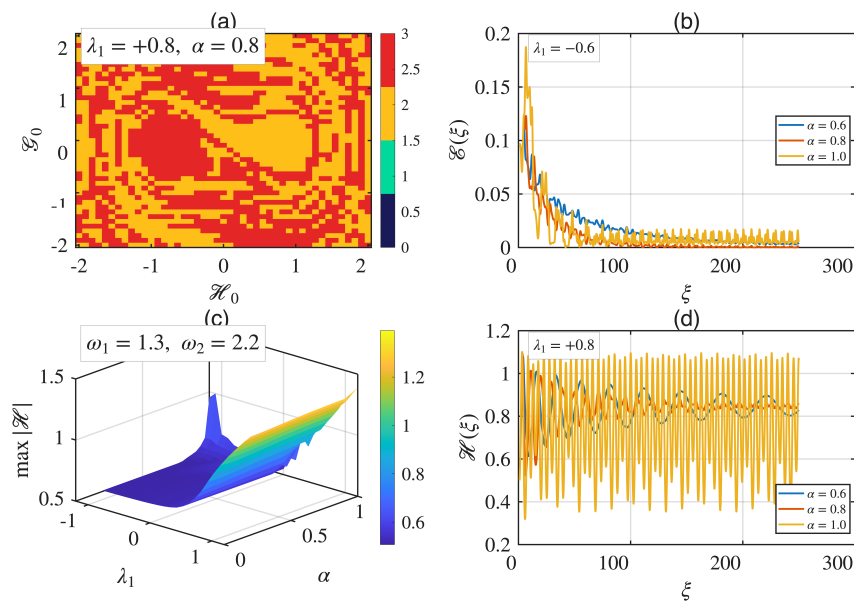


Figure 15. Nonlinear structures of the forced cubic–quintic planar system. Panel (a) is a basin slice at $\lambda_1 = +0.8$, $\alpha = 0.8$, initial states classified by the time-averaged $\mathcal{H}$ over the tail of the run; only the positive (2) and negative (3) branches occur here; panel (b) the energy along trajectories at $\lambda_1 = -0.6$ for $\alpha = 0.6, 0.8, 1$; panel (c) $\max |\mathcal{H}|$ over (λ_1, α) at $\omega_1 = 1.3$, $\omega_2 = 2.2$; panel (d) $\mathcal{H}(\xi)$ at $\lambda_1 = +0.8$ for the same three orders, all with $\lambda_2 = -1$, $\lambda_3 = -0.15$, $\gamma = 0.08$, $A = 0.12$ and $B = 0.08$.

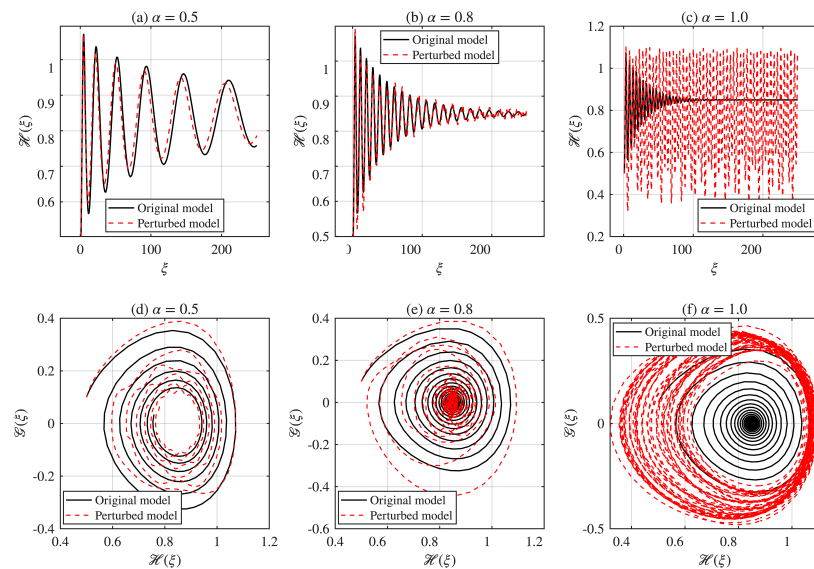


Figure 16. Unforced damped conformable cubic–quintic planar system (solid black) and its forced counterpart (red dashed) for $\alpha = 0.5, 0.8$ and 1 . Top row: $\mathcal{H}$ against ξ ; bottom row: phase portraits in the $(\mathcal{H}, \mathcal{L})$ plane. Both systems are damped, and only the forcing distinguishes them. $\omega_1 = 1.3$, $\omega_2 = 2.2$, $\lambda_1 = 0.8$, $\lambda_2 = -1$, $\lambda_3 = -0.15$, $\gamma = 0.08$, $A = 0.12$, $B = 0.08$, $(\mathcal{H}(1), \mathcal{L}(1)) = (0.5, 0.1)$, integrated from $\xi_0 = 1$.

Figure 16 compares the unforced damped and forced conformable cubic–quintic planar systems for different fractional orders α . Both systems carry the damping $-\gamma\mathcal{G}$, $\gamma = 0.08$, and differ only in the forcing. The unforced one relaxes, spiraling to the equilibrium $\mathcal{H} \approx 0.85$, while the forced one sustains oscillations that grow as $\alpha \rightarrow 1$. The abscissa is ξ rather than $\tau = \xi^\alpha/\alpha$ because the forcing $F(\xi) = A \sin \omega_1 \xi + B \cos \omega_2 \xi$ is periodic in ξ ; expressed in τ it acquires an α dependent frequency, and the unforced orbits of every order would collapse onto a single curve.

Figure 17 separates what the fractional order can and cannot change in the energy description. The energy of the conservative core is a function of $(\mathcal{H}, \mathcal{G})$ alone, so its landscape is the same at every α : the factor $\xi^{\alpha-1}$ multiplies both equations and therefore reparametrizes the independent variable without moving the level sets. Panel (a) shows that landscape. What the order changes is the rate at which an orbit is traversed and hence the energy exchanged along it. Differentiating $\mathcal{E}$ along the governing flow gives $d\mathcal{E}/d\xi = \xi^{\alpha-1}\mathcal{G}[F(\xi) - \gamma\mathcal{G}]$, the first term representing energy exchange with the forcing, which may be of either sign, and the second removing energy. Panel (b) accumulates the two separately, signed so that they sum to $\mathcal{E}(\xi) - \mathcal{E}(\xi_0)$ and their relative size can be read off directly. Integrated on $\xi \in [1, 120]$ at RelTol = 10^{-10} the balance closes to better than 1.1×10^{-7} at every order, which is the quadrature level. The three orders leave the common value $\mathcal{E} = -0.0790$ and arrive at -0.1344 , -0.1451 and -0.1370 . Without forcing the dynamics are order independent, the turning points agreeing to 8×10^{-14} across $\alpha = 0.5, 0.8$ and 1 ; with forcing they differ by about 0.31 .

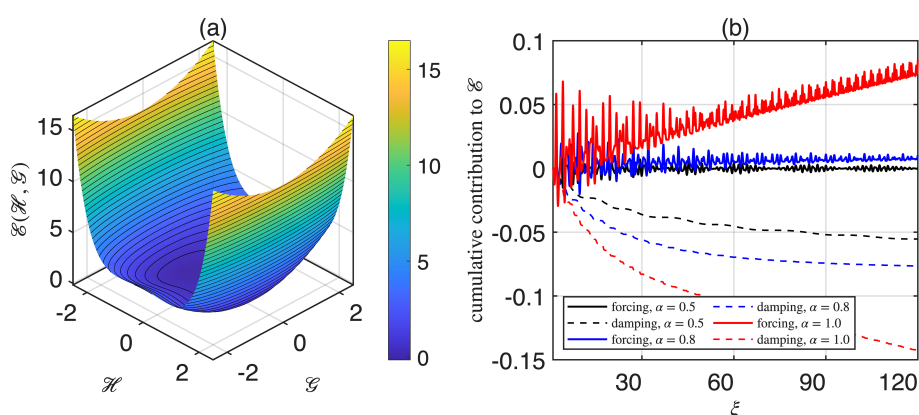


Figure 17. Effective energy landscape of the cubic–quintic planar system. (a) Conservative core $\mathcal{H}'' = \lambda_1 \mathcal{H} + \lambda_2 \mathcal{H}^3 + \lambda_3 \mathcal{H}^5$, for which $\mathcal{E}$ is conserved; the surface depends on $(\mathcal{H}, \mathcal{G})$ alone and is independent of α . (b) Cumulative contributions to $\mathcal{E}$ from the forcing (solid) and from the damping (dashed) along the trajectories of the governing forced and damped system for $\alpha = 0.5, 0.8$ and 1 , from $(\mathcal{H}_0, \mathcal{G}_0) = (0.5, 0.1)$ at $\xi_0 = 1$ with $\lambda_1 = 0.8$, $\lambda_2 = -1$, $\lambda_3 = -0.15$, $\gamma = 0.08$, $A = 0.45$, $B = 0.35$, $\omega_1 = 4.5$ and $\omega_2 = 7$. Each curve is the running integral of the corresponding term of $d\mathcal{E}/d\xi$, signed so that the two sum to $\mathcal{E}(\xi) - \mathcal{E}(\xi_0)$.

5. Numerical verification of the conformable CQ–NLS model

The analysis presented so far is based on traveling-wave reduction, a low-order Galerkin projection and linear spectral theory. This section reports direct numerical solutions of the full conformable partial differential equation (3.62), obtained with two independent discretizations, and uses them to (i) verify

the exact traveling-wave solutions, (ii) quantify the accuracy and convergence of the schemes, (iii) test the linear spectral stability predictions against the nonlinear evolution, and (iv) delimit the validity of the three-mode Galerkin truncation.

5.1. Benchmark traveling-wave solution

Integrating the reduced Eq (3.4) once under the localization conditions $\mathcal{H}, \mathcal{H}' \rightarrow 0$ as $|\xi| \rightarrow \infty$ gives the first integral

$$\frac{ap^2}{2}(\mathcal{H}')^2 + \frac{\Lambda}{2}\mathcal{H}^2 + \frac{b}{4}\mathcal{H}^4 + \frac{c}{6}\mathcal{H}^6 = 0, \quad \Lambda = d - v - aq^2. \quad (5.1)$$

Setting $\mathcal{W} = \mathcal{H}^{-2}$ reduces (5.1) to $(\mathcal{W}')^2 = P\mathcal{W}^2 + Q\mathcal{W} + R$ with

$$P = -\frac{4\Lambda}{ap^2}, \quad Q = -\frac{2b}{ap^2}, \quad R = -\frac{4c}{3ap^2}, \quad (5.2)$$

whose bounded solution yields the closed-form bright solitary wave

$$\mathcal{H}(\xi) = \left[-\frac{Q}{2P} + \frac{\sqrt{Q^2 - 4PR}}{2P} \cosh(\sqrt{P}\xi) \right]^{-1/2}, \quad a\Lambda < 0, \quad Q^2 - 4PR \geq 0, \quad \mathcal{W}(0) > 0, \quad (5.3)$$

Remark 5.1 (Verification of the analytical solutions). Substitution of (5.3) into (3.4) satisfies the equation identically, and the corresponding field $\psi = \mathcal{H}(\xi)e^{i(qx+vt^\alpha/\alpha)}$ with $\xi = px + \omega t^\alpha/\alpha$ and $\omega = -2apq$ satisfies (3.62) exactly. The Branch A Projective Riccati family has likewise been verified by direct substitution on each smooth interval $\xi < 0$ and $\xi > 0$, subject to its algebraic parameter constraints. The remaining auxiliary branches are discussed separately in the admissibility analysis and are not used as numerical benchmarks. Numerically, the residual of (5.3) evaluated on a spectral grid with $N = 2048$ points is 5.1×10^{-8} , which is the numerical differentiation error at this resolution. Solution (5.3) is used below as the benchmark against which both numerical schemes are assessed.

Throughout this section the reference parameter set is $a = 1, b = -2, c = 1, d = 0.2, p = 1, \Lambda = -0.5$, with q chosen commensurate with the periodic computational box so that e^{iqx} is box periodic. The coefficients a, b, c, d and p coincide with the base parameter values used in Table 1, while here $\Lambda = -0.5$ is selected for the exact numerical benchmark.

5.2. Two independent discretizations

(S1) Split-step Fourier scheme in the conformable time. Because the conformable derivative is local, the substitution $\tau = t^\alpha/\alpha$ converts (3.62) into

$$i\psi_\tau + a\psi_{xx} + b|\psi|^2\psi + c|\psi|^4\psi + d\psi = 0, \quad (5.4)$$

which is integrated by Strang splitting [41, 42]. The linear half steps are advanced exactly in Fourier space, $\hat{\psi} \mapsto e^{i(-ak^2+d)\Delta\tau/2}\hat{\psi}$, and the nonlinear step is exact in physical space because $|\psi|$ is invariant under it, $\psi \mapsto \psi e^{i(b|\psi|^2+c|\psi|^4)\Delta\tau}$. The scheme is spectrally accurate in x and second-order accurate in $\Delta\tau$, and conserves $\int |\psi|^2 dx$ to machine precision (the measured drift over the runs reported below is 3×10^{-12}).

(S2) Conformable Crank–Nicolson finite-difference scheme. To avoid relying on the transformation $\tau = t^\alpha/\alpha$, a second scheme is applied directly in the physical time variable. On a uniform grid $t_n = t_0 + n\Delta t$, $x_j = x_0 + j\Delta x$, the conformable derivative is discretized at the midpoint $t_{n+1/2}$ as

$$\mathcal{D}_t^\alpha \psi \Big|_{t_{n+1/2}} \approx t_{n+1/2}^{1-\alpha} \frac{\psi_j^{n+1} - \psi_j^n}{\Delta t}, \quad (5.5)$$

and the remaining terms are evaluated at the same midpoint using the central stencil (3.27), giving

$$i t_{n+1/2}^{1-\alpha} \frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} + a \delta_x^2 \psi_j^{n+1/2} + \left(b |\psi_j^{n+1/2}|^2 + c |\psi_j^{n+1/2}|^4 + d \right) \psi_j^{n+1/2} = 0, \quad (5.6)$$

with $\psi^{n+1/2} = \frac{1}{2}(\psi^n + \psi^{n+1})$. The resulting nonlinear system is solved at each step by Picard iteration to a tolerance of 10^{-13} ; the linear system is periodic tridiagonal and is inverted by the Sherman–Morrison formula. Scheme (S2) is second-order accurate in both Δt and Δx and is formulated without any reference to τ , so agreement between (S1) and (S2) provides independent numerical support of the temporal rescaling interpretation of the conformable operator.

5.3. Convergence and step-size tests

Tables 3 and 4 report the refinement study over the window $\tau \in [1, 1.5]$, measured in the L^∞ norm against a resolved reference solution. Scheme (S1) shows second-order convergence in $\Delta \tau$ to three significant digits and spectral convergence in Δx ; scheme (S2) shows second-order convergence in both variables.

Table 3. Temporal refinement, split-step Fourier scheme (S1), $N = 2048$, $\tau \in [1, 1.5]$.

M	$\Delta \tau$	L^∞ error	observed rate
500	1.00×10^{-3}	5.2458×10^{-4}	—
1000	5.00×10^{-4}	1.3119×10^{-4}	2.000
2000	2.50×10^{-4}	3.2776×10^{-5}	2.001
4000	1.25×10^{-4}	8.1704×10^{-6}	2.004
8000	6.25×10^{-5}	2.0187×10^{-6}	2.017

Table 4. Spatial refinement at $\alpha = 0.8$, the data of Figure 19(b). Each error is taken against an $N = 4096$ reference computed with the same scheme. The $N = 128$ row is excluded from the (S2) fit, its error exceeding the solution amplitude; the finest (S1) entry is at the floating-point floor.

N	Δx	(S1) split-step Fourier		(S2) Crank–Nicolson	
		L^∞ error	order	L^∞ error	order
128	4.69×10^{-1}	3.8417×10^0	—	2.2160×10^0	—
256	2.34×10^{-1}	4.3967×10^{-3}	9.77	8.7228×10^{-1}	1.35
512	1.17×10^{-1}	3.1450×10^{-7}	13.77	3.0379×10^{-1}	1.52
1024	5.86×10^{-2}	$\approx 5 \times 10^{-11}$	12.64	5.8810×10^{-2}	2.37

Comparing the two schemes over the same physical window $t \in [1, 1.4]$ with $N = 2048$ gives $\|\psi^{(S1)} - \psi^{(S2)}\|_\infty = 8.16 \times 10^{-3}$, 9.11×10^{-3} and 9.84×10^{-3} for $\alpha = 0.5$, 0.8 and 1 respectively. Measured

against the exact solution on the same grid, scheme (S2) has an error of 9.11×10^{-3} at $N = 2048$, so in each case the discrepancy between the two schemes is at the level of the truncation error of (S2) itself. This confirms that the two formulations converge to the same solution of the conformable model. This is a scheme-to-scheme comparison at fixed resolution and is separate from the refinement study of Table 4, which uses a refined-grid reference for each scheme.

5.4. Long-time evolution and validation of the spectral analysis

Figure 18(a) shows the propagation of the benchmark solitary wave over the resolved window. Panel (b) quantifies the growth of the seeded perturbation, which is the quantity that tests the spectral prediction $\Lambda_{\max} > 0$ of Lemma 3.14. Figure 18(b) compares the paired deviation between the seeded and unseeded runs with the eigenvalue prediction: an exponential fit over the linear stage gives a growth rate of 4.200 per unit τ , against $\Lambda_{\max} = 4.128$ obtained from the linearized operator, a discrepancy of 1.7%. The measurement is paired. Let ψ_δ denote the run started from $(1 + \delta)\psi_{\text{exact}}$ and ψ_0 the run started from ψ_{exact} , both integrated on the same grid with the same step; the reported quantity is $\|\psi_\delta - \psi_0\|_\infty$. Differencing the two reduces the common discretization error instead of counting it as growth. Without that control the fitted rate is 4.38 and the discrepancy 6%, most of which is discretization error rather than physics. The residual margin reflects the finite fitting window and the projection of the seed onto more than one mode. The departure is independent of $\Delta\tau$ under refinement, the fitted rate changing by less than 0.1% when the step is halved, confirming that it is the physical instability rather than a time-discretization effect. The snapshots in Figure 18(a) stop at $\tau = 3.4$, where halving $\Delta\tau$ still moves the peak amplitude by only 0.05%. Beyond saturation it does not converge at all: at $\tau = 5$ the peak moves from 1.24 to 5.38 under the same refinement, with the spatial grid resolved throughout. Those profiles record the discretization rather than the dynamics and are not displayed.

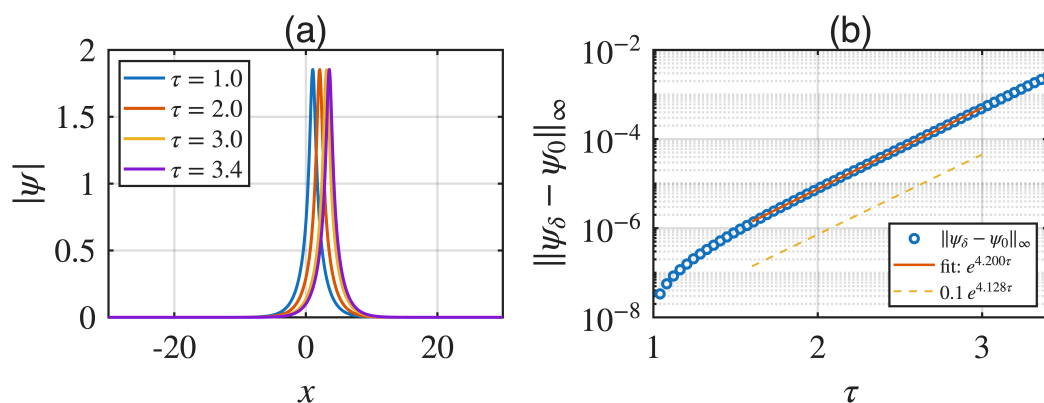


Figure 18. Direct numerical solution of the conformable CQ-NLS equation for $a = 1$, $b = -2$, $c = 1$, $d = 0.2$, $\Lambda = -0.5$. (a) Modulus of the field at $\tau = 1, 2, 3$ and 3.4 , the window over which the peak amplitude is converged under time-step refinement. (b) $\|\psi_\delta - \psi_0\|_\infty$, the difference between the run seeded with $\delta = 10^{-8}$ and an unperturbed control on the same grid and step, on a logarithmic scale, compared with the growth rate $\Lambda_{\max} = 4.128$ predicted by the linearized spectral problem.

5.5. Convergence of the numerical schemes

Before any eigenvalue of the discretized operator is read as an instability, it must be examined under grid refinement [43]: the neutral modes associated with the phase and translation symmetries are perturbed by the finite-difference discretization and can acquire a small spurious positive real part, so a fixed-grid positive real part should not by itself be read as instability. All spectral values reported in Table 2 were obtained after such refinement and lie at the numerical roundoff and conditioning level, of order 10^{-13} to 10^{-12} on the reported grids, which is numerically negligible at the reported working accuracy. The convergence of the two schemes themselves is shown in Figure 19: panel (a) confirms the design order $O(\Delta\tau^2)$ of the split-step scheme in time, and panel (b) the spectral convergence in space of scheme (S1) against the second-order convergence of scheme (S2). Both are measured the same way and on the same problem: $\alpha = 0.8$, a common window and a common number of time steps, with each error taken against a refined-grid reference computed by the same scheme, so that the common temporal error is reduced and each curve isolates the spatial convergence order. Measured this way the fitted order for (S2) is 1.95 over the resolved grids $N = 256, 512, 1024$, against the design order 2, while for (S1) the observed orders increase over the tested grids, which is the signature of spectral accuracy. The coarsest grid, $N = 128$, is excluded from the fit and shown in grey: there $\Delta x = 0.469$ against a profile width of about 0.71 and the error exceeds the peak amplitude itself, so that computation lies outside the convergence regime. Comparing either scheme with the exact solution instead would add its temporal error and mask the spatial order. Panel (c) separates the two cases directly: at $b = -2$ the largest real part is independent of $\Delta\xi$, while at $b = +2$ it falls linearly with $\Delta\xi$ and so is an artifact of the discretization rather than a growth rate.

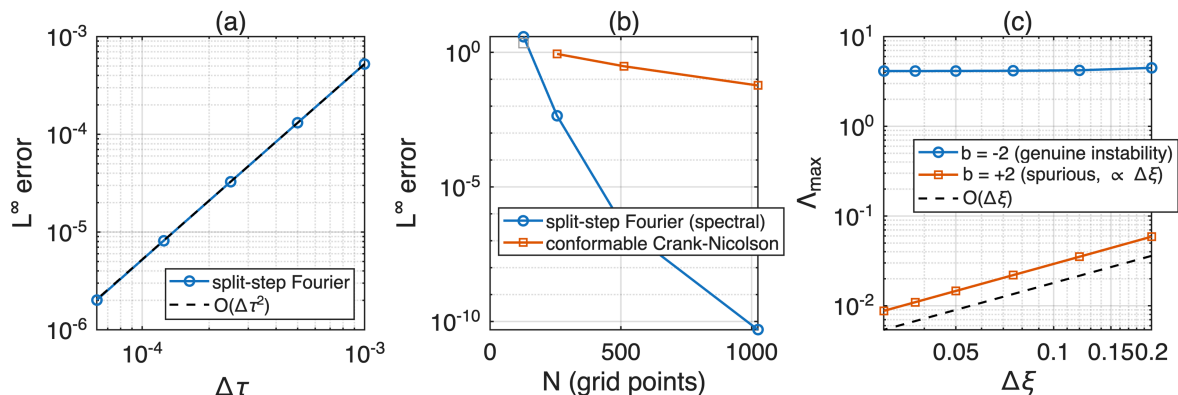


Figure 19. (a) Temporal convergence of the split-step Fourier scheme, showing the design order $O(\Delta\tau^2)$. (b) Spatial convergence of the two schemes. (c) Grid dependence of $\Lambda_{\max}$ for the finite-difference linearization: at $b = -2$ it is flat under refinement, so the growth rate is a property of the operator, whereas at $b = +2$ it decreases as $O(\Delta\xi)$ and therefore vanishes in the limit, which is the signature of a spurious mode.

5.6. Harmonic order convergence of the Galerkin truncation

The validity of the three-mode reduction discussed in Remark 3.9 is quantified by retaining $2K + 1$ Fourier harmonics of the modulated background $\psi_0 = \mathcal{A}(1 + \epsilon \cos \mu x)$, $\mathcal{A} = 1.3$, $\epsilon = 10^{-3}$, $\mu = 0.6$. The three-mode reduction above is written in a scaled coordinate in which the sideband wavenumber is unity, so that $\cos x$ and $\sin x$ have Laplacian eigenvalue -1 ; the test here restores the physical

wavenumber μ , which rescales $a\mu^2$ but leaves the modal structure unchanged. For these values, $2b|\mathcal{A}|^2 + 4c|\mathcal{A}|^4 = 4.664 > a\mu^2 = 0.36$, so the background is modulationally unstable. The measured sideband growth rate is 1.297, against the analytical prediction $\sqrt{a\mu^2(2b|\mathcal{A}|^2 + 4c|\mathcal{A}|^4 - a\mu^2)} = 1.245$, a difference of 4.2%.

Table 5 reports the truncation error against the untruncated computation. During the linear stage the truncation converges superalgebraically and the three-mode model is already accurate to 5×10^{-4} . Once the sidebands saturate the convergence is slower but remains geometric in K . The three windows span the linear, late-linear and saturating stages: the peak modulus of the untruncated field is 1.308, 1.332 and 1.529 at $\tau = 2, 3$ and 4, and the truncation error falls by a factor of about 0.12 per added harmonic order at $\tau = 3$ against about 0.47 at $\tau = 4$. Windows beyond saturation are excluded because their reference is not converged under time-step refinement: a sweep over $\tau = 1, \dots, 6$ comparing 8000 and 16000 steps gives a relative drift of 2.4×10^{-11} , 4.6×10^{-10} , 6.4×10^{-9} and 2.9×10^{-7} at $\tau = 1, \dots, 4$, but 1.7 and 1.2 at $\tau = 5$ and 6. No post-saturation conclusion is therefore drawn. The three-mode reduction is quantitatively reliable for the linear and early nonlinear stages of the modulational instability and is therefore intended for the pre-saturation regime described in Remark 3.9.

Table 5. Relative L^∞ error of the $(2K + 1)$ -mode Fourier–Galerkin truncation against the untruncated solution. All three references are step converged: halving the time step changes the untruncated field by 4.6×10^{-10} , 6.4×10^{-9} and 2.9×10^{-7} at $\tau = 2, 3$ and 4.

K (modes $2K + 1$)	$\tau = 2$ (linear)	$\tau = 3$ (late linear)	$\tau = 4$ (saturating)
1 (3)	4.947×10^{-4}	7.885×10^{-3}	1.708×10^{-1}
2 (5)	1.464×10^{-5}	9.210×10^{-4}	7.937×10^{-2}
3 (7)	4.571×10^{-7}	1.100×10^{-4}	3.690×10^{-2}
4 (9)	1.448×10^{-8}	1.327×10^{-5}	1.713×10^{-2}
6 (13)	1.506×10^{-11}	2.011×10^{-7}	3.725×10^{-3}
8 (17)	6.988×10^{-14}	3.173×10^{-9}	8.279×10^{-4}

5.7. Role of the conformable order in physical time

Because the eigenvalue problem is posed in the conformable time τ , the growth rate $\Lambda_{\max}$ is independent of α ; the whole influence of the conformable order enters through the map $t \mapsto \tau = t^\alpha/\alpha$. A perturbation seeded at t_0 is therefore amplified by

$$\frac{|\delta\psi(t)|}{|\delta\psi(t_0)|} \sim \exp(\Lambda_{\max} D(\alpha)), \quad D(\alpha) = \tau(t) - \tau(t_0) = \frac{t^\alpha - t_0^\alpha}{\alpha}, \quad (5.7)$$

where $D(\alpha)$ is the effective elapsed time of the observation window $[t_0, t]$ measured in the conformable variable.

The ordering of D with α is set by the observation window: whether the amplification observed at a fixed physical instant increases or decreases with the conformable order is a property of the window, not of the instability. For the window $[1, 3]$ one has $D = 1.4641, 1.7603, 2.0000$ for $\alpha = 0.5, 0.8, 1$, so the amplification increases with α ; for the window $[0.05, 0.8]$ one has $D = 1.3416, 0.9319, 0.7500$, so it decreases with α . This explains the α -ordering seen in Figure 5(c).

Direct simulation confirms (5.7) quantitatively. The measurement is paired: with ψ_δ started from $(1 + \delta)\psi_{\text{exact}}$ at t_0 , $\delta = 10^{-8}$, and ψ_0 started from ψ_{exact} on the same grid with the same step, the quantity

measured at t is $\|\psi_\delta - \psi_0\|_\infty$. Differencing the two reduces the common discretization error, which a comparison with the exact profile would otherwise count as growth; the reduction is not exact, since the two trajectories are not identical. This gives, for the window $[1, 3]$, 5.380×10^{-5} , 1.833×10^{-4} , 4.925×10^{-4} for $\alpha = 0.5, 0.8, 1$, and for the window $[0.05, 0.8]$, 3.235×10^{-5} , 5.766×10^{-6} , 2.619×10^{-6} — opposite orderings, exactly as $D(\alpha)$ requires. Dividing by $\delta \exp(\Lambda_{\max} D(\alpha))$ leaves a projection constant $C = 12.77, 12.81, 12.80$ in the first window, i.e., independent of α to within 0.31%, which confirms that the entire α -dependence resides in $D(\alpha)$ and none of it in the growth rate. On the window $[0.05, 0.8]$ the corresponding constants are 12.73, 12.31, 11.85, a spread of 7.2%, so the 0.31% figure is quoted for $[1, 3]$ only. Figure 20 summarizes this behavior.

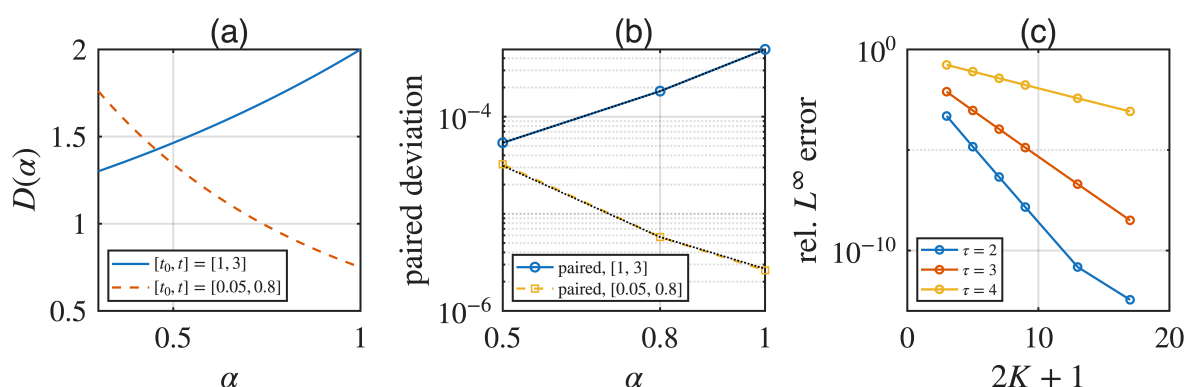


Figure 20. (a) Effective elapsed conformable time $D(\alpha) = (t^\alpha - t_0^\alpha)/\alpha$ for two observation windows; D increases with α on $[1, 3]$ and decreases on $[0.05, 0.8]$. (b) $\|\psi_\delta - \psi_0\|_\infty$ for a seed $\delta = 10^{-8}$, following $C\delta \exp(\Lambda_{\max} D(\alpha))$ (dotted) with a constant C that is α -independent to 0.31% on $[1, 3]$. (c) Harmonic order convergence of the Fourier–Galerkin truncation at $\tau = 2, 3$ and 4 , spanning the linear, late-linear and saturating stages. Windows beyond saturation are excluded because their reference is not converged under time-step refinement; see Section 5.6.

6. Conclusions

This work has analyzed the conformable cubic–quintic nonlinear Schrödinger equation using a framework that combines traveling-wave construction, a low-order Galerkin reduction, variational characterization, modulational instability analysis, spectral stability and direct numerical simulation.

The central structural result is the exact reduction under $\tau = t^\alpha/\alpha$, which turns the model into a classical CQ–NLS equation in a rescaled time. The modulational instability threshold (4.12) and the spectral growth rate $\Lambda_{\max}$ are therefore computed once in τ . For the unforced CQ–NLS problem the conformable order acts through the reparametrization $\exp(\Lambda_{\max}(t^\alpha - t_0^\alpha)/\alpha)$ of the evolution in physical time. Because this effective elapsed time has no universal monotonic ordering in α , the ordering of the growth curves in α is a property of the observation window and not of the instability; direct simulation reproduces both orderings, with an α -independent prefactor.

The status of the constructed profiles differs by family, and each is stated individually. The extended $(\mathcal{G}'/\mathcal{G}^2)$ hyperbolic family and the closed-form bright solitary wave (5.3) are exact solutions verified by substitution, the latter serving as the numerical benchmark. The Branch A Projective Riccati family is also exact; its signed form (3.14) is smooth, and only the nonnegative modulus is cornered. The

sech Ritz profile is a trial function with an $O(1)$ residual. The finite-interval singular periodic family admits no conventional spectral problem on $\mathbb{R}$. Two independent discretizations, split-step Fourier in τ and conformable Crank–Nicolson in t , agree to their truncation error, and the numerical benchmark reproduces the spectral growth rate to within about 2%, confirming the spectral prediction at the level of the full partial differential equation.

Two numerical points are important for the stability interpretation. First, the computed $\Lambda_{\max}$ must be checked under grid refinement, since the neutral symmetry modes can acquire a small spurious positive real part; under grid refinement, the Branch A Projective Riccati profile is numerically neutral to working accuracy for the discretized problem, on every admissible parameter set examined. The signed Branch A wave is smooth; the corner belongs only to its nonnegative modulus, and the reported spectrum remains a numerical result subject to domain and grid convergence. Second, the three-mode Galerkin truncation is quantitatively accurate during the linear and early nonlinear stages of the modulational instability but loses validity beyond sideband saturation.

Several extensions of the present work remain open. These include higher-dimensional formulations, nonlocal Caputo and Riemann–Liouville models, and the recently proposed operators [19, 21, 22], for which the present temporal reparametrization is not available. The chaotic regimes of the forced system may also motivate future studies of continuous-time signal generation for secure communication, subject to dedicated statistical and key-sensitivity analyses.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

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A. Quintic Galerkin coefficients

Writing $\psi = A/\sqrt{2} + B \cos x + C \sin x$, forming the full product and applying the projection Π_1 of (3.66) gives $\Pi_1(|\psi|^4\psi) = Q_0 + Q_1 \cos x + Q_2 \sin x$ with

$$Q_0 = \frac{\sqrt{2}}{16} \left(2|A|^4 A + 2A^3(B^*)^2 + 2A^3(C^*)^2 + 12|A|^2 A B B^* + 12|A|^2 A C C^* \right. \\ \left. + 6A(A^*)^2 B^2 + 9A|B|^4 + 3AB^2(C^*)^2 + 12ABC B^* C^* + 6A(A^*)^2 C^2 \right. \\ \left. + 3AC^2(B^*)^2 + 9A|C|^4 + 6A^* B^3 B^* + 6A^* B^2 C C^* + 6A^* B C^2 B^* + 6A^* C^3 C^* \right), \quad (\text{A.1})$$

$$Q_1 = \frac{1}{8} \left(4A^3 A^* B^* + 6A^2 B(A^*)^2 + 9A^2 B(B^*)^2 + 3A^2 B(C^*)^2 + 6A^2 C B^* C^* \right. \\ \left. + 18AB^2 A^* B^* + 12ABCA^* C^* + 6AC^2 A^* B^* + 3B^3(A^*)^2 + 5B^3(B^*)^2 \right. \\ \left. + B^3(C^*)^2 + 6B^2 C B^* C^* + 3BC^2[(A^*)^2 + (B^*)^2 + (C^*)^2] + 2C^3 B^* C^* \right), \quad (\text{A.2})$$

$$Q_2 = \frac{1}{8} \left(4A^3 A^* C^* + 6A^2 B B^* C^* + 6A^2 C(A^*)^2 + 3A^2 C(B^*)^2 + 9A^2 C(C^*)^2 \right. \\ \left. + 6AB^2 A^* C^* + 12ABCA^* B^* + 18AC^2 A^* C^* + 2B^3 B^* C^* \right. \\ \left. + 3B^2 C[(A^*)^2 + (B^*)^2 + (C^*)^2] + 6BC^2 B^* C^* + 3C^3(A^*)^2 + C^3(B^*)^2 + 5C^3(C^*)^2 \right). \quad (\text{A.3})$$

At $B = C = 0$, these reduce to $Q_0 = |A|^4 A / (4\sqrt{2})$, and at $A = 0$, $B = C = R$ real to $Q_1 = \frac{5}{2}R^5$, the values used for the equilibria in Lemma 3.18.



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