AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(9): 31041–31061.
DOI: 10.3934/math.20261228
Received: 09 July 2026
Revised: 10 September 2026
Accepted: 15 September 2026
Published: 22 September 2026

Research article

On reverse super edge trimagic labeling of path-related graphs and cycle-related graphs

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Abstract: A graph is said to admit a reverse super edge trimagic labeling if its vertices and edges can be bijectively labeled with consecutive integers such that the differences between the edge labels and the sums of the labels of their end vertices attain exactly three distinct values, with the vertex labels occupying the smallest integers. In this paper, we establish new results on the existence of reverse super edge trimagic labelings for several families of path- and cycle-related graphs. Specifically, we prove that path graphs, the m-copy path graphs, corona products of a path and a null graph, cycle graphs, m-copy cycle graphs, and corona products of a cycle and a null graph admit reverse super edge trimagic labelings under specified conditions on their parameters. For each graph family, we provide explicit constructions of the corresponding labelings and determine the associated three magic constants. These results extend the known results on reverse super edge trimagic labelings to broader families of path and cycle-related graphs.

Keywords: reverse super edge trimagic labeling; path graph; cycle graph; corona product of a graph
Mathematics Subject Classification: 05C76, 05C78

1. Introduction

Labeling is one of the well-studied topics in graph theory, which has attracted significant attention for both its theoretical importance and practical applications. In general, graph labeling refers to the assignment of integers to the vertices, edges, or both under certain conditions [1]. Over the past decades, numerous types of graph labelings have been introduced and investigated, including various magic-type labelings [2]. These studies have produced many interesting results for different classes of graphs and graph operations. One of the earliest labeling concepts is magic labeling, which originated from the idea of Sedláček [3]. A well-known subclass is edge-magic labeling, where labels are assigned to edges. This concept was later extended by Kotzig and Rosa in 1970 to edge-magic total labeling, where labels are assigned to vertices and edges such that the sum of the label of an edge and the labels of its incident vertices is constant [4].

Another important variation is edge bimagic labeling, introduced by Babujee in 2004 [5]. In this labeling, two distinct constants are obtained. Further developments include super edge bimagic labeling on pyramid graphs and snake graphs [6], graphs with cycles [7], and other graph families [8]. Several recent works have also investigated this labeling on other graph structures, such as $(3, n)-$ kite graphs, bistar graphs $K_{2,N}$, double wheel graphs [9, 10], hibiscus graphs [11], and wheel-related graphs [12]. A further extension is edge trimagic labeling, introduced by Jayasekaran and Regees [13], where three distinct constants are obtained. Many results on super edge trimagic labeling have been reported for various graph families, including disconnected graphs, generalized prism graphs [14, 15], umbrella graphs, dumbbell graphs, circular ladder graphs [16], Möbius ladder graphs, book graphs, dragon graphs [17], and corona products of dovetail graphs and null graphs [18].

These labeling concepts have also been extended into reverse versions. In reverse edge magic labeling, the difference between the label of an edge and the sum of the labels of its incident vertices is considered. Several results on reverse super edge magic (RSEM) labeling have been established for different graph classes. There are $y-$trees, generalized Peterson graphs, $(2m + 1)C_3$ graphs given in [19], and tree graphs, lobster graphs, and banana graphs were investigated in [20]. Furthermore cycle with chords, and cycle and path-related graphs where explore in [21]. RSEM has continued to reverse super edge bimagic (RSEB) in the work of Amuthavalli and Sugapriya. The result of this research are star-related graphs [22]. The other results in reverse edge bimagic labeling are gear graphs, hibiscus graphs, and dovetail graphs [23]. Furthermore, Amuthavalli and Sugapriya [24] introduced reverse super edge trimagic (RSET) labeling. In [24], a graph H is said to admit a reverse edge trimagic labeling if there exists a bijection $\zeta : V(H) \cup E(H) \to \{1, 2, \ldots, |V(H)| + |E(H)|\}$ such that $\{\zeta(wv) - (\zeta(w) + \zeta(v))|wv \in E(H)\} = \{\theta_1, \theta_2, \theta_3\}$ for some distinct constants $\theta_1, \theta_2, \theta_3$. If the vertices are assigned the labels $\{1, 2, \ldots, |V(H)|\}$, and the edges are assigned $\{|V(H)|+1, |V(H)|+2, \ldots, |V(H)|+|E(H)|\}$, then the labeling is called a reverse super edge trimagic labeling. Star-related graphs have been researched in [24]. Recent work has also investigated on dovetail-related graphs [25].

Although several classes of graphs have been studied under reverse super edge trimagic labeling, the existence of such labelings for families of graphs related to paths and cycles has not been thoroughly investigated. In particular, previous studies considered star-related graphs [24] and dovetail-related graphs [25], whereas path-related graphs and cycle-related graph families have not been systematically studied in the context of RSET labeling. Moreover, the result on the corona product of the path P_3 and a null graph $P_3 \odot N_s$ obtained by Amuthavalli provides a specific case for a path corona graph. In this paper, we generalize this result by establishing RSET labelings for the corona product $P_r \odot N_s$ under appropriate conditions on r and s.

Motivated by these observations, this paper investigates the existence of RSET labelings for several families of graphs related to paths and cycles. Specifically, we consider the path P_r, the m-copy path mP_r, the corona product of a path graph and a null graph $P_r \odot N_s$, the cycle C_r, the m-copy cycle mC_r, and the corona product of a cycle graph and a null graph $C_r \odot N_s$. For each graph family, we provide explicit constructions of the corresponding labelings and determine the three associated magic constants. Thus, the main contribution of this work is to extend the existing study of RSET labeling from star-related and dovetail-related graphs to broader families of path-related graphs and cycle-related graphs, while also generalizing the known corona result for $P_3 \odot N_s$ to $P_r \odot N_s$.

2. Results

Here, we provide concepts required for the main results, primarily from [24] and the corona product of two graphs according to [26].

Definition 2.1. *Given a graph $H = (V(H), E(H))$ with order r and size m, a reverse edge trimagic labeling of a graph is a bijective function $\zeta : V(H) \cup E(H) \to \{1, 2, 3, \ldots, r+m\}$ such that $\{\zeta(wv) - (\zeta(w) + \zeta(v)) | wv \in E(H)\} = \{\theta_1,\ \theta_2,\ \theta_3\}$, where $\theta_1,\ \theta_2$, and θ_3 are distinct constants. Furthermore, a reverse edge trimagic labeling is said to be super if $\zeta(V(H)) = \{1, 2, 3, \ldots, r\}$, and $\zeta(E(H)) = \{r+1, r+2, r+3, \ldots, r+m\}$.*

Definition 2.2. *Given a graph $H = (V(H), E(H))$ with order r_1 and size m_1, and a graph $K = (V(K), E(K))$ with order r_2 and size m_2, the corona product of H and K, denoted by $H \odot K$, is the graph obtained from one copy of H together with r_1 copies of K, denoted by K_i for $1 \leq i \leq r_1$. Furthermore, for each vertex $v_i \in V(H)$, the vertex v_i is joined to every vertex of the corresponding copy K_i, namely the ith copy of K.*

The first fundamental structure graph that we investigate is a path graph. The path graph is a graph that isomorphic to H graph with vertex set $V(P_r) = \{w_1, w_2, w_3, \ldots, w_r\}$ and edge set $E(P_r) = \{w_i w_{i+1} | 1 \leq i \leq r-1\}$.

Theorem 2.3. *The path graph P_r is a reverse super edge trimagic graph for any $r \geq 4$.*

Proof. Let P_r be a path graph with vertex set $V(P_r) = \{w_1, w_2, w_3, \ldots, w_r\}$ and edge set $E(P_r) = \{w_i w_{i+1} | 1 \leq i \leq r-1\}$. In order to prove the existence of a reverse super edge trimagic labeling on P_r, we explicitly construct a bijective labeling ζ such that each vertex and edge of the graph is uniquely assigned a label from the prescribed set of labels that fulfills the conditions of reverse super edge trimagic labeling. The construction is divided into two cases according to the parity of r.

The first case is for odd r.

We define a labeling $\zeta : V(P_r) \cup E(P_r) \to \{1, 2, 3, \ldots, 2r-1\}$ such that the labels of vertices are provided below:

$$\zeta(w_i) = \begin{cases} \dfrac{i+1}{2}, & 1 \leq i \leq r, \text{odd } i, \\ \dfrac{r+i+1}{2}, & 1 \leq i \leq r, \text{even } i, \end{cases}$$

and the edge labels are given by

$$\zeta(w_i w_{i+1}) = \begin{cases} r+1, & 1 \leq i \leq r-3, \\ r+i+1, & i = r-2, \\ r+i-1, & i = r-1. \end{cases}$$

We now verify that this labeling produces exactly three distinct difference constants.

(1) For edges w_iw_{i+1} with $1 \le i \le r-3$, we obtain

$$\begin{aligned}\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) &= r + i - \Big(\frac{i+1}{2} + \frac{r+(i+1)+1}{2}\Big) \\ &= \frac{r-3}{2} = \theta_2.\end{aligned}$$

(2) Similarly, for $i = r-2$, we obtain

$$\begin{aligned}\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) &= r + i + 1 - \left(\frac{i+1}{2} + \frac{r+(i+1)+1}{2}\right) \\ &= \frac{r-1}{2} = \theta_3.\end{aligned}$$

(3) For $i = r-1$, we obtain

$$\begin{aligned}\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) &= r + i - 1 - \Big(\frac{r+i-1}{2}\frac{(i+1)+1}{2}\Big) \\ &= \frac{r-5}{2} = \theta_1.\end{aligned}$$

Thus the three distinct constants are $\theta_1 = (r-5)/2$, $\theta_2 = (r-3)/2$, and $\theta_3 = (r-1)/2$.

The second case is for even r.

We define the vertex labeling using the same rule as in the first case:

$$\zeta(w_i) = \begin{cases} \dfrac{i+1}{2}, & 1 \le i \le r, \text{odd } i, \\ \dfrac{r+i}{2}, & 1 \le i \le r, \text{even } i, \end{cases}$$

and the edge labeling is constructed differently depending on the value of r:

$$\zeta(w_iw_{i+1}) = \begin{cases} r+i+1, & i = 1, & \multirow{2}{*}{$r=4$} \\ r+2i-3, & i = 2,3, & \\ r+i, & 1 \le i \le \dfrac{r-4}{2}, & \\ & \dfrac{r}{2}+2 \le i \le r-1, & r \ge 6 \\ r+i+2, & i = \dfrac{r}{2}-1, & \\ r+i-1, & \dfrac{r}{2} \le i \le \dfrac{r}{2}+1. & \end{cases}$$

Next, we show that this labeling produces exactly three distinct difference constants, θ_1, θ_2, and θ_3.

(1) For $r = 4$,

a. consider the edge w_1w_2,

$$\zeta(w_1w_2) - (\zeta(w_1) + \zeta(w_2)) = r + 1 + 1 - \Big(\frac{1+1}{2} + \frac{r+(1+1)}{2}\Big) = \frac{r}{2} = \theta_3;$$

b. similarly, for the edge w_2w_3,

$$\zeta(w_2w_3) - (\zeta(w_2) + \zeta(w_3)) = r - 4 = 0 = \theta_1;$$

c. for the edge w_3w_4,

$$\zeta(w_3w_4) - (\zeta(w_3) + \zeta(w_4)) = \frac{r-2}{2} = \theta_2.$$

(2) For $r \geq 6$,

a. consider edges w_iw_{i+1} with $1 \leq i \leq (r-4)/2$,

$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = r + i - \Big(\frac{i+1}{2} + \frac{r+(i+1)}{2}\Big) = \frac{r-2}{2} = \theta_2;$$

b. in the case of edge w_iw_{i+1} with $i = r/2 - 1$, we obtain

$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = \frac{r+2}{2} = \theta_3;$$

c. for edges w_iw_{i+1} with $r/2 \leq i \leq r/2 + 1$,

- $r \equiv 0 \pmod 4$,

$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = \frac{r-4}{2} = \theta_1;$$

- $r \equiv 2 \pmod 4$,

$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = \frac{r-4}{2} = \theta_1;$$

d. for edges w_iw_{i+1} with $r/2 + 2 \leq i \leq r - 1$,

$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = r + i - \Big(\frac{r+i}{2} + \frac{(i+1)+1}{2}\Big) = \frac{r-2}{2} = \theta_2.$$

For the second case, we obtain that for $r = 4$, we have $\theta_1 = 0$, $\theta_2 = (r-2)/2$, $\theta_3 = r/2$, and for $r \geq 6$, we have $\theta_1 = (r-4)/2$, $\theta_2 = (r-2)/2$, $\theta_3 = (r+2)/2$.

Therefore, $\{\zeta(wv) - (\zeta(w) + \zeta(v)) | wv \in E(P_r)\} = \{\theta_1, \theta_2, \theta_3\}$. Thus, the path graph P_r admits a reverse edge trimagic labeling. Because a path graph P_r with r vertices are assigned the labels $1, 2, 3, \ldots, r$, and the edges receive the remaining labels, the labeling is a reverse super edge trimagic labeling. Hence, any path graph is a reverse super edge trimagic graph. □

An illustration of reverse super edge trimagic (RSET) labelings for path graphs P_9 and P_{10} are provided in Figures 1 and 2.

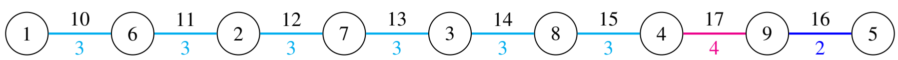

Figure 1. Reverse super edge trimagic labeling of P_9 with $\theta_1 = 2$, $\theta_2 = 3$, $\theta_3 = 4$.

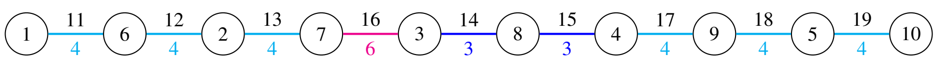

Figure 2. Reverse super edge trimagic labeling of P_{10} with $\theta_1 = 3$, $\theta_2 = 4$, $\theta_3 = 6$.

Theorem 2.3 shows that the path graph P_r admits a reverse super edge trimagic labeling through a balanced arrangement of vertex and edge labels. This labeling pattern naturally leads to a broader question, namely whether the same construction can be adapted to disconnected graphs formed by several copies of P_r. Therefore, we next consider the graph mP_r.

Theorem 2.4. *The m copy of a path graph mP_r is a reverse super edge trimagic graph for $m \geq 2,\ r \geq 5$, and r is odd.*

Proof. Let mP_r be the disjoint union of m copies of the path graph P_r. The vertex set is $V(mP_r) = \{w_i^j | 1 \leq i \leq r, 1 \leq j \leq m\}$ and $E(mP_r) = \{w_i^j w_{i+1}^j | 1 \leq i \leq r-1, 1 \leq j \leq m\}$. We prove the existence of a reverse super edge trimagic labeling by constructing a bijective function $\zeta : V(mP_r) \cup E(mP_r) \rightarrow \{1, 2, \ldots, 2mr-1\}$, that provides the vertex labels as follows:

$$\zeta(w_i^j) = \begin{cases} j, & i = \left\lceil \frac{r}{2} \right\rceil, \\ m(r+1-2i)+j, & 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor, \\ \left(2\left(i - \left\lceil \frac{r}{2} \right\rceil\right) - 1\right)m + j, & \left\lceil \frac{r}{2} \right\rceil + 1 \leq i \leq r, \end{cases} \qquad r \geq 5,\ 1 \leq j \leq m,$$

and the edge labels as follows:

(1) for $1 \leq j \leq m,\ r = 5$,

$$\zeta(w_i^j w_{i+1}^j) = \begin{cases} \frac{14m - i + 1 + 4j}{2}, & i = 1, 3, \\ \frac{10m + i - 4 + 4j}{2}, & i = 2, 4, \end{cases}$$

(2) for $1 \leq j \leq m,\ r \geq 7,\ r \equiv 3 \pmod 4$,

$$\zeta(w_i^j w_{i+1}^j) = \begin{cases} m(2r - 4i + 1) + 2j, & 1 \leq i \leq \left\lceil \frac{r}{4} \right\rceil, \\ m(3r - 4i) - 1 + 2j, & \left\lceil \frac{r}{4} \right\rceil + 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor, \\ m(4i - r - 2) - 1 + 2j, & \left\lfloor \frac{r}{2} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{3r}{4} \right\rfloor, \\ m(4i - 2r - 1) - 1 + 2j, & \left\lfloor \frac{3r}{4} \right\rfloor + 1 \leq i \leq r - 1; \end{cases}$$

(3) for $1 \leq j \leq m,\ r \geq 7,\ r \equiv 1 \pmod 4$,

$$f(w_i^j w_{i+1}^j) = \begin{cases} m(2r - 4i + 1) + 2j, & 1 \leq i \leq \left\lfloor \frac{r}{4} \right\rfloor, \\ m(3r - 4i) - 1 + 2j, & \left\lfloor \frac{r}{4} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{r}{2} \right\rfloor, \\ m(4i - r - 2) - 1 + 2j, & \left\lfloor \frac{r}{2} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{3r}{4} \right\rfloor, \\ m(4i - 2r - 1) - 1 + 2j, & \left\lfloor \frac{3r}{4} \right\rfloor + 1 \leq i \leq r - 1. \end{cases}$$

To complete the verification, we show that the constructed labeling gives rise to exactly three distinct difference constants. We consider three cases based on the value of r.

(1) Case 1 is designated for edges $w_i^j w_{i+1}^j,\ r = 5, 1 \le i \le r-1$.

a. For $i = 1$,

$$\zeta(w_1^j w_2^j) - (\zeta(w_1^j) + \zeta(w_2^j)) = \frac{14m - 1 + 1 + 4j}{2} - (m(r - 2(1) + 1) + j + m(r - 2(1+1) + 1) + j) = m = \theta_1;$$

b. similarly, for $i = 2$,

$$\zeta(w_2^j w_3^j) - (\zeta(w_2^j) + \zeta(w_3^j)) = m(8 - r) - 1 = \theta_2;$$

c. for $i = 3$,

$$\zeta(w_3^j w_4^j) - (\zeta(w_3^j) + \zeta(w_4^j)) = 6m - 1 = \theta_3;$$

d. for $i = 4$,

$$\zeta(w_4^j w_5^j) - (\zeta(w_4^j) + \zeta(w_5^j)) = m = \theta_1.$$

(2) Case 2 corresponds to the edges $w_i^j w_{i+1}^j,\ r \ge 7,\ r \equiv 3 \pmod 4$.

a. Consider edges $w_i^j w_{i+1}^j$ with $1 \le i \le \lceil r/4 \rceil$:

$$\zeta(w_{ij} w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = m(2r - ri + 1) + 2j - (m(r - 2i + 1) + j + m(r - 2(i+1) + 1) + j) = m = \theta_1;$$

b. similarly, for edges $w_i^j w_{i+1}^j$ with $\lceil r/4 \rceil + 1 \le i \le \lfloor r/2 \rfloor$,

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = mr - 1 = \theta_3;$$

c. in the case $w_i^j w_{i+1}^j$ with $i = \lceil r/2 \rceil$, we obtain,

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = m(r-1) - 1 = \theta_2;$$

d. for edges $w_i^j w_{i+1}^j$ with $\lceil r/2 \rceil + 1 \le i \le \lfloor 3r/4 \rfloor$,

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = mr - 1 = \theta_3;$$

e. for edges $w_i^j w_{i+1}^j$ with $\lceil 3r/4 \rceil + 1 \le i \le r - 1$,

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = m = \theta_1.$$

(3) Case 3 is provided for edges $w_i^j w_{i+1}^j,\ r \ge 9,\ r \equiv 1 \pmod 4$.

a. Consider edges $w_i^j w_{i+1}^j$ with $1 \le i \le \left\lfloor \frac{r}{4} \right\rfloor$, we obtain

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = m(2r - 4i + 1) + 2j - (m(r - 2i + 1) + j + m(r - 2(i+1) + 1) + j) = m = \theta_1;$$

b. for edges $w_i^j w_{i+1}^j$ with $\lfloor r/4 \rfloor + 1 \le i \le \lfloor r/2 \rfloor$,

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = m(2r - 4i + 1) + 2j - (m(r - 2(i+1) + j + m(r - 2(i+1) + 1) + j) = mr - 1 = \theta_3;$$

c. similarly, for edges $w_i^j w_{i+1}^j$ with $i = \lceil r/2 \rceil$,

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = m(r-1) - 1 = \theta_2;$$

d. for edges $w_i^j w_{i+1}^j$ with $\lfloor r/2 \rfloor + 2 \le i \le \lfloor 3r/4 \rfloor$, we obtain

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = mr - 1 = \theta_3;$$

e. consider edges $w_i^j w_{i+1}^j$ with $\left\lfloor \frac{3r}{4} \right\rfloor + 1 \le i \le r - 1$; we obtain

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i j) + \zeta(w_{i+1}^j)) = m = \theta_1.$$

Thus, the above construction for every case yields exactly

$$\{\zeta(wv) - (\zeta(w) + \zeta(v)) | wv \in mP_r\} = \{\theta_1,\ \theta_2,\ \theta_3\}$$

for some distinct θ_1, θ_2, θ_3, and mr vertices of mP_r are labeled with the first positive integers $1, 2, 3, \ldots, mr$, which means that mP_r graphs admit a reverse super edge trimagic labeling. Furthermore, mP_r graphs are reverse super edge trimagic graphs for $m \ge 2, r \ge 5$, when r is odd. □

Here, we give an illustration of reverse super edge trimagic labeling of $2P_9$, and $2P_{11}$ in Figures 3 and 4, respectively.

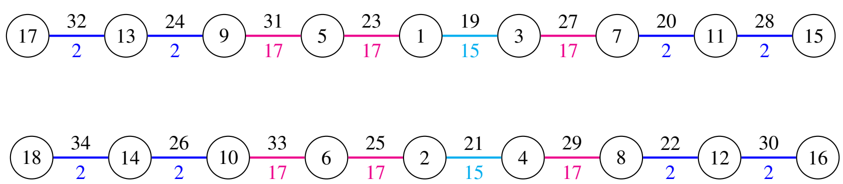

Figure 3. Reverse super edge trimagic labeling of $2P_9$ with $\theta_1 = 2, \theta_2 = 15, \theta_3 = 17$.

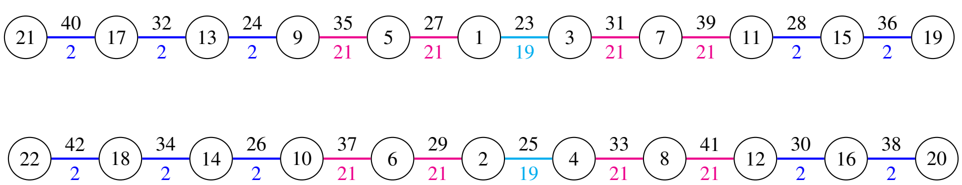

Figure 4. Reverse super edge trimagic labeling of $2P_{11}$ with $\theta_1 = 2, \theta_2 = 19, \theta_3 = 21$.

Theorem 2.4 concerns a family of disconnected graphs formed by several copies of path graphs. We now turn to another family graph derived from path graphs through the corona operation. The following theorem considers the path graph corona null graph, $P_r \odot N_s$.

Theorem 2.5. *For any positive integer $r \geq 4$ and $s \geq 1$, the corona product of a path graph and a null graph, $P_r \odot N_s$ is a reverse super edge trimagic graph.*

Proof. Construct the graph $P_r \odot N_s$ with vertex set $V(P_r \odot N_s) = \{w_i | 1 \leq i \leq r\} \cup \{w_i^j | 1 \leq i \leq r,\ 1 \leq j \leq s\}$ and edge set $E(P_r \odot N_s) = \{w_i w_{i+1} | 1 \leq i \leq r-1\} \cup \{w_i w_i^j | 1 \leq i \leq r,\ 1 \leq j \leq s\}$. We prove the existence of a reverse super edge trimagic labeling of $P_r \odot N_s$ by constructing a bijective mapping $\zeta : V(P_r \odot N_s) \cup E(P_r \odot N_s) \to \{1, 2, 3, ..., 2r(s+1) - 1\}$. The construction is divided into two cases according to the parity of r.

The first case is for even r.

Assume that $r \geq 4$ is even and that $s \geq 1$. Define the bijection $\zeta : V(P_r \odot N_s) \cup E(P_r \odot N_s) \to \{1, 2, 3, \ldots, 2r(s+1) - 1\}$, where the vertex labels are assigned as follows:

$$\zeta(w_i) = \begin{cases} \dfrac{i+1}{2}, & 1 \leq i \leq r, \text{odd } i, \\ \dfrac{r+i}{2}, & 1 \leq i \leq r, \text{even } i. \end{cases}$$

Furthermore,

$$\zeta(w_i^j) = \begin{cases} \dfrac{r(2j+1)+i-1}{2}, & 1 \leq i \leq r-1, \text{odd } i, \\ \dfrac{2rj+i}{2}, & 1 \leq i \leq r-2, \text{even } i, \\ r(j+1), & i = r. \end{cases} \quad 1 \leq j \leq s,\ r \geq 5,$$

The edge labels are defined by

$$\zeta(w_i w_{i+1}) = \begin{cases} r(s+1)+i, & 1 \leq i \leq \dfrac{r}{2}+1, \text{odd } i, & r \geq 4, \\ r(s+2)+i-3, & \dfrac{r}{2}+2 \leq i \leq r-1, \text{even } i, & r \geq 6, \end{cases}$$

and,

$$\zeta(w_i w_i^j) = \begin{cases} r(s+3)+i+j-5, & 1 \leq i \leq 2, & \\ rs + \dfrac{3r}{2} + i + j - 2, & 3 \leq i \leq r-1, & j = 1, \\ r(s+3)+j-2, & i = r, & \\ r(s+j+1)+i-1, & 1 \leq i \leq r, & 2 \leq j \leq s. \end{cases} \quad r \geq 4,$$

Next, we verify that ζ satisfies the reverse super edge trimagic condition by evaluating the difference between the label of each edge and the sum of the labels of its incident vertices.

(1) For edges $w_i w_{i+1}$ with $1 \leq i \leq r/2 + 1$,

$$\begin{aligned} \zeta(w_i w_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) &= r(s+1) + i - \Big(\frac{i+1}{2} + \frac{r+(i+1)}{2}\Big) \\ &= \frac{2rs+r-2}{2} = \theta_2. \end{aligned}$$

(2) Similarly, for edges w_iw_{i+1} with $r/2+2 \le i \le r-1$,

$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1}) = \frac{2rs+3r-8}{2} = \theta_3.$$

(3) In the case edges $w_iw_i^j$ with $1 \le i \le 2,\ j = 1$, we obtain

a. $i = 1$,

$$\zeta(w_1w_1^1) - (\zeta(w_1) + \zeta(w_1^1) = \frac{2rs+3r-8}{2} = \theta_3;$$

b. $i = 2$,

$$\zeta(w_2w_2^1) - (\zeta(w_2) + \zeta(w_2^1) = \frac{2rs+3r-8}{2} = \theta_3.$$

(4) Consider edges $w_iw_i^j$ with $3 \le i \le r-1,\ j = 1$. We obtain

a. odd i,

$$\zeta(w_iw_i^1) - (\zeta(w_i) + \zeta(w_i^1) = rs-1 = \theta_1;$$

b. even i,

$$\zeta(w_iw_i^1) - (\zeta(w_i) + \zeta(w_i^1) = rs-1 = \theta_1.$$

(5) For edges $w_iw_i^j$ with $1 \le i \le r,\ 2 \le j \le s$,

a. odd i,

$$\zeta(w_iw_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+r-2}{2} = \theta_2;$$

b. even i,

$$\zeta(w_iw_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+r-2}{2} = \theta_2.$$

(6) For edge $w_rw_r^1$,

$$\zeta(w_rw_r^1) - (\zeta(w_r) + \zeta(w_r^1) = rs-1 = \theta_1.$$

Hence, in the first case, the three distinct constants are $\theta_1 = rs-1$, $\theta_2 = (2rs+r-2)/2$, and $\theta_3 = (2rs+3r-8)/2$.

The second case is for odd r

Now, suppose that $r \ge 5$ is odd and that $s \ge 1$. Define a bijection $\zeta : V(P_r \odot N_s) \cup E(P_r \odot N_s) \to \{1, 2, 3, ..., 2r(s+1)-1\}$, where the vertex labels are

$$\zeta(w_i) = \begin{cases} \dfrac{i+1}{2}, & 1 \le i \le r, \text{odd } i, \\ \dfrac{r+i+1}{2}, & 1 \le i \le r, \text{even } i, \end{cases}$$

and

$$\zeta(w_i^j) = \begin{cases} \dfrac{r(2j+1)+i}{2}, & 1 \le i \le r, \text{odd } i, \\ \dfrac{2rj+i}{2}, & 1 \le i \le r, \text{even } i, \end{cases} \quad 1 \le j \le s,$$

and the edge labels are given by

$$\zeta(w_i w_{i+1}) = r(s+1) + i,\ 1 \le i \le r-1$$

and

$$\zeta(w_i w_i^j) = \begin{cases} r(s+j+2)+i-3, & r \ge 5,\ 1 \le i \le 2, \\ r(s+j+1)+i-3, & r \ge 5,\ 3 \le i \le r-1. \end{cases} \quad 1 \le j \le s,$$

To prove that ζ is a reverse super edge trimagic labeling, we evaluate the quantity $f(wv)-(f(w)+f(v))$ for every edge $wv \in E(P_r \odot N_s)$.

(1) Consider edges $w_i w_{i+1}$ with $1 \le i \le r-1$:

$$\zeta(w_i w_{i+1} - (\zeta(w_i) + \zeta(w_{i+1})) = \frac{2rs+r-3}{2} = \theta_2.$$

(2) In a similar manner, for edges $w_i w_i^j$ with $r=3,\ 1 \le i \le r,\ 1 \le j \le s$,

a. $i=1$,
$$\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+r-1}{2} = \theta_3;$$

b. $i=2$,
$$\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+r-1}{2} = \theta_3;$$

c. $i=3$,
$$\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+r-1}{2} = \theta_3.$$

(3) Likewise, for edges $w_i w_i^j$ with $1 \le i \le 2,\ j=1$,

a. $i=1$,
$$\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+3r-7}{2}) = \theta_3;$$

b. $i=2$,
$$\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+3r-7}{2} = \theta_3.$$

(4) For edges $w_i w_i^j$ with $3 \le i \le r,\ 1 \le j \le s$,

a. odd i,
$$\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+r-7}{2} = \theta_1;$$

b. even i,
$$\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j) = \frac{2rs+r-7}{2} = \theta_1.$$

Therefore, in the second case, we have three distinct constants $\theta_1 = (2rs + r - 7)/2$, $\theta_2 = (2rs + r - 3)/2$, and $\theta_3 = (2rs + 3r - 7)/2$.

Combining both cases, we conclude that the graph $P_r \odot N_s$ yields three distinct constants, $\{\zeta(wv) - (\zeta(w) + \zeta(v)) | wv \in E(P_r \odot N_s)\} = \{\theta_1, \theta_2, \theta_3\}$. Because the vertex labels are precisely the integers $1, 2, 3, \ldots, r(s + 1)$, the labeling ζ is a reverse super edge trimagic labeling. Consequently, $P_r \odot N_s$ is a reverse super edge trimagic graph. □

Here, we present a visual representation reverse super edge trimagic labelings of $P_5 \odot N_s$ in Figure 5 and $P_6 \odot N_s$ in Figure 6.

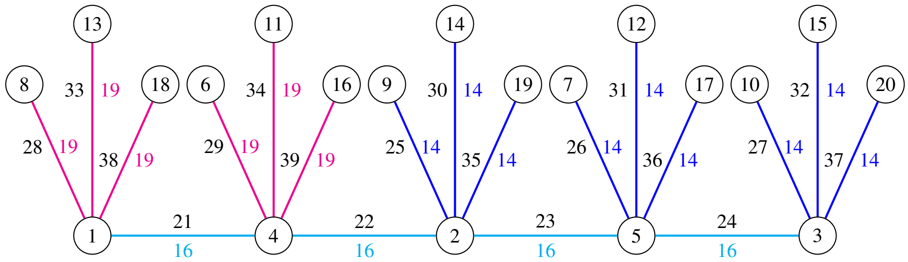

Figure 5. Reverse super edge trimagic labeling of $P_5 \odot N_3$ with $\theta_1 = 14$, $\theta_2 = 16$, $\theta_3 = 19$.

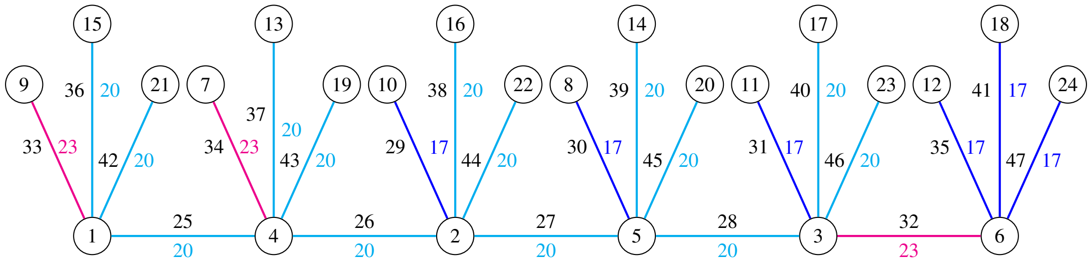

Figure 6. Reverse super edge trimagic labeling of $P_6 \odot N_3$ with $\theta_1 = 17$, $\theta_2 = 20$, $\theta_3 = 23$.

In the three previous theorems, we focus on path related graphs. We now give attention to the other fundamental graphs, namely cycle-related graphs. We start with the cycle graph C_r. Recall that the cycle graph C_r is a graph that isomorphic to G graph with vertex set $V(G) = \{w_i \mid 1 \leq i \leq r\}$ and edge set $E(G) = \{w_i w_{i+1} \mid 1 \leq i \leq r - 1\} \cup \{w_1 w_r\}$.

Theorem 2.6. *For any positive integer $r \geq 5$, the cycle graph C_r is a reverse super edge trimagic graph.*

Proof. Consider the cycle graph C_r with $r \geq 5$, vertex set $V(C_r) = \{w_i | 1 \leq i \leq r\}$ and edge set $E(C_r) = \{w_i w_{i+1} | 1 \leq i \leq r - 1\} \cup \{w_1 w_r\}$. We define a bijection $\zeta : V(C_r) \cup E(C_r) \to \{1, 2, 3..., 2r\}$ such that the condition of reverse super edge trimagic labeling is satisfied. The construction is divided into two cases based on the parity of r.

Case 1 is for odd r.

Assume that $r \geq 5$. Define the bijection $\zeta : V(C_r) \cup E(C_r) \to \{1, 2, 3, ..., 2r\}$ by assigning the vertex labels as $\zeta(w_i) = i, 1 \leq i \leq r$, and label the edges as

$$\zeta(w_i w_{i+1}) = \begin{cases} r + 2i - 1, & 1 \leq i \leq 2, \\ r + 2i - 2, & 3 \leq i \leq \left\lceil \dfrac{r}{2} \right\rceil, \\ 2(i+1), & \left\lceil \dfrac{r}{2} \right\rceil + 1 \leq i \leq r - 1, \end{cases} \quad r \geq 5,$$

$$\zeta(w_1 w_r) = r + 2.$$

The following computations show that the differences edge label with the sum of vertices label are incident to the edge and produce exactly three distinct constants.

(1) For edges $w_i w_{i+1}$ with $1 \leq i \leq 2$,

$$\zeta(w_i w_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1}) = r + 2i - 1 - (i + (i+1)) = r - 2 = \theta_3.$$

(2) For edges $w_i w_{i+1}$ with $3 \leq i \leq \lceil r/2 \rceil$,

$$\zeta(w_i w_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1}) = r + 2i - 2 - (i + (i+1)) = r - 3 = \theta_2.$$

(3) Similarly, for edges $w_i w_{i+1}$ with $\lceil r/2 \rceil + 1 \leq i \leq r - 1$,

$$\zeta(w_i w_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1}) = 2i + 2 - (i + (i+1)) = 1 = \theta_1.$$

(4) For edge $w_1 w_r$,

$$\zeta(w_1 w_r) - (\zeta(w_1) + \zeta(w_r) = r + 2 - (1 + r) = 1 = \theta_1.$$

Furthermore, in Case 1, we have $\theta_1 = 1,\ \theta_2 = r - 3,\ \theta_3 = r - 2$.

Case 2 is for even r.

Now, assume that $r \geq 6$ is even. Construct a bijection $\zeta : V(C_r) \cup E(C_r) \to \{1, 2, ..., 2r\}$ with vertex labels given by

$$\zeta(w_i) = \begin{cases} \dfrac{i+1}{2}, & 1 \leq i \leq r, \text{odd } i, \\ \dfrac{r+i}{2}, & 1 \leq i \leq r, \text{even } i \end{cases} \quad r \geq 6,$$

and the edge labels by

$$\zeta(w_i w_{i+1}) = \begin{cases} r + 2i - 1, & 1 \leq i \leq 2, & \\ r + 2i - 4, & i = 3, & r \geq 6, \\ r + i + 1, & \dfrac{r}{2} + 1 \leq i \leq r - 1, & \\ r + i, & 4 \leq i \leq \dfrac{r}{2}, & r \geq 8, \end{cases}$$

$$\zeta(w_1 w_r) = \frac{3r + 2}{2}.$$

We show below that ζ is a reverse super edge trimagic labeling by computing $\zeta(w_i w_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1}))$ for every edge in $E(C_r)$.

(1) For edges w_iw_{i+1}, $r \geq 6$,

a. for $i = 1$,
$$\zeta(w_1w_2) - (\zeta(w_1) + \zeta(w_2)) = \frac{r-2}{2} = \theta_2;$$

b. moreover, for $i = 2$,
$$\zeta(w_2w_3) - (\zeta(w_2) + \zeta(w_3)) = \frac{r}{2} = \theta_3;$$

c. for $i = 3$,
$$\zeta(w_3w_4) - (\zeta(w_3) + \zeta(w_4)) = \frac{r-4}{2} = \theta_1;$$

d. likewise, for $r/2 + 1 \leq i \leq r - 1$, $r \equiv 0 \pmod 4$,
$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = \frac{r}{2} = \theta_3;$$

e. consider $r/2 + 1 \leq i \leq r - 1$, $r \equiv 2 \pmod 4$,
$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = \frac{r}{2} = \theta_3.$$

(2) In the case edge $w_iw_{i+1}, r \geq 8$, $i = 4$,
$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = \frac{r-2}{2} = \theta_2.$$

(3) Likewise, for edge $w_iw_{i+1}, r \geq 10$, $r/2 + 1 \leq i \leq r - 1$,
$$\zeta(w_iw_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = \frac{r-2}{2} = \theta_2.$$

(4) For edge w_1w_r,
$$\zeta(w_1w_r) - (\zeta(w_1) + \zeta(w_r)) = \frac{r}{2} = \theta_3.$$

Thus, Case 2 yields the three distinct constants $\theta_1 = (r-4)/2$, $\theta_2 = (r-2)/2$, and $\theta_3 = r/2$.

Finally, according to Case 1 and Case 2, we find $\{\zeta(wv) - (\zeta(w) + \zeta(v)) | wv \in E(C_r)\} = \{\theta_1, \theta_2, \theta_3\}$ for some distinct $\theta_1, \theta_2, \theta_3$, and every vertex is assigned a unique label from the set $\{1, 2, 3 \ldots, r\}$. Thus, the cycle graph C_r admits a reverse super edge trimagic labeling. Hence, C_r is a reverse super edge trimagic graph for any $r \geq 5$. □

Figures 7 and 8 show examples of reverse super edge trimagic labeling of C_8 and C_9, respectively.

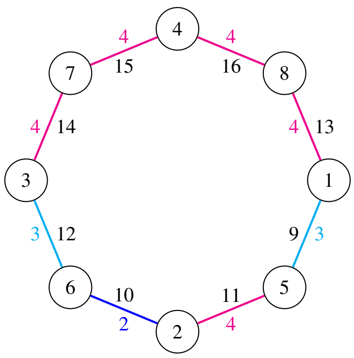

Figure 7. Reverse super edge trimagic labeling of C_8 with $\theta_1 = 2, \theta_2 = 3, \theta_3 = 4$.

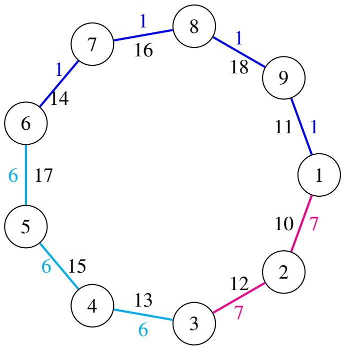

Figure 8. Reverse super edge trimagic labeling of C_9 with $\theta_1 = 1, \theta_2 = 6, \theta_3 = 7$.

Theorem 2.6 has discussed RSET for connected graphs, namely cycle graphs. Next, we investigate disconnected graphs related to the cycle graphs. The graphs are formed by several copies of cycles, specifically disjoint copies of cycles, denoted by mC_r.

Theorem 2.7. *The m copy of a cycle graph mC_r is a reverse super edge trimagic graph for any $m \geq 2$ and for every even integer $r \geq 6$.*

Proof. Let mC_r be the disjoint union of m copies of the cycle graph C_r. The vertex set is $V(mC_r) = \{w_i^j | 1 \leq i \leq r, 1 \leq j \leq m\}$, and the edge set is $E(mC_r) = \{w_i^j w_{i+1}^j | 1 \leq i \leq r-1, 1 \leq j \leq m\} \cup \{w_1^j w_r^j | 1 \leq j \leq m\}$. In order to prove the existence of a reverse super edge trimagic labeling on mC_r, we construct a bijective function $\zeta : V(mC_r) \cup E(mC_r) \to \{1, 2, 3, \ldots, 2mr\}$ that provides the vertex labels by

$$\zeta(w_i^j) = \begin{cases} \left\lfloor \dfrac{i}{2} \right\rfloor m + j, & 1 \leq i \leq r, \text{odd } i, \\ \left(\dfrac{r+i}{2} - 1\right) m + j, & 1 \leq i \leq r, \text{even } i, \end{cases} \quad 1 \leq j \leq m,$$

and the edge labels as follows:

$$\zeta(w_i^j w_{i+1}^j) = \begin{cases} m(r+i-1) - 1 + 2j, & 1 \leq i \leq r-1, \text{odd } i, \\ m(r+i-2) + 2j, & 1 \leq i \leq r-1, \text{even } i. \end{cases} \quad 1 \leq j \leq m,$$

$$\zeta(w_1^j w_r^j) = 2m(r-1) + 2j.$$

Furthermore, we investigate the difference between the edge label and the sum of the labels of its incident vertices, that is, $\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j))$ for every edge $w_i^j w_{i+1}^j \in E(mC_r)$. In this context, we have three cases based on the value of i.

(1) For edges $w_i^j w_{i+1}^j,\ 1 \leq i \leq r-1,$ odd $i,\ 1 \leq j \leq m,$

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = \frac{mr-2}{2} = \theta_2.$$

(2) For edges $w_i^j w_{i+1}^j,\ 1 \leq i \leq r-1,$ even $i,\ 1 \leq j \leq m,$

$$\zeta(w_i^j w_{i+1}^j) - (\zeta(w_i^j) + \zeta(w_{i+1}^j)) = \frac{mr-2m}{2} = \theta_1.$$

(3) For edges $w_1^j w_r^j,\ 1 \le j \le m$,

$$\zeta(w_1^j w_r^j) - (\zeta(w_1^j) + \zeta(w_r^j)) = m(r-1) = \theta_3.$$

Based on the value i, we find that $\{\zeta(wv) - (\zeta(w) + \zeta(v)) | wv \in E(mC_r)\} = \{\theta_1, \theta_2, \theta_3\}$. In this case, every vertex is assigned by integers $\{1, 2, 3, \ldots, mr\}$, and the edges receive labels $\{mr+1, mr+2, \ldots, 2mr\}$. Hence, the mC_r graph admits a reverse super edge trimagic labeling. Therefore, mC_r graph is a reverse super edge trimagic graph for any $m \ge 2,\ r \ge 5$ when r is even. □

Here, we give diagrams of reverse super edge trimagic labelings of $2C_8$ and $3C_8$ in Figures 9 and 10.

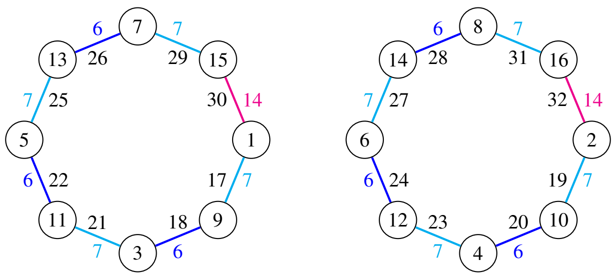

Figure 9. Reverse super edge trimagic labeling of $2C_8$ with $\theta_1 = 6, \theta_2 = 7, \theta_3 = 14$.

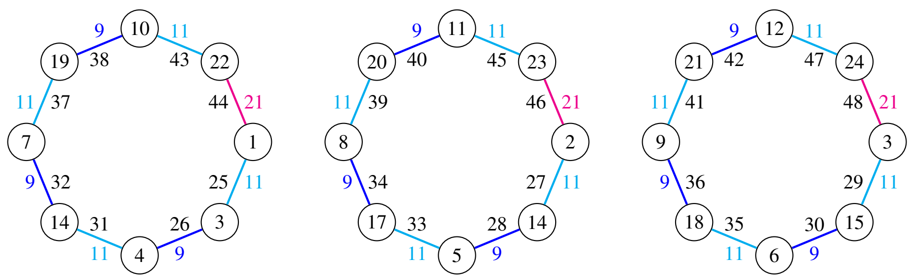

Figure 10. Reverse super edge trimagic labeling of $3C_8$ with $\theta_1 = 9, \theta_2 = 11, \theta_3 = 21$.

In this section, we discuss reverse super edge trimagic labeling of a corona product of two graphs. We recall Definition 2.2 to construct the graph $C_r \odot N_s$. The corona product of a cycle graph and a null graph, $C_r \odot N_s$ is a graph formed from one copy of the cycle graph C_r together with r copies of the null graph N_s in which each vertex $w_i \in V(C_r)$ is adjacent to every vertex w_i^j in the ith copy of N_s, for $1 \le i \le r$ and $1 \le j \le s$.

Theorem 2.8. *For any positive integer $r \ge 3,\ \ s \ge 1$, the corona product of a cycle graph and a null graph, $C_r \odot N_s$, is a reverse super edge trimagic graph.*

Proof. Construct the graph $C_r \odot N_s$ with vertex set

$$V(C_r \odot N_s) = \{w_i | 1 \le i \le r\} \cup \{w_i^j | 1 \le i \le r,\ 1 \le j \le s\}$$

and edge set $E(C_r \odot N_s) = \{w_i w_{i+1} | 1 \le i \le r-1\} \cup \{w_1 w_r\} \cup \{w_i w_i^j | 1 \le i \le r,\ 1 \le j \le s\}$. We construct a bijective mapping $\zeta : V(C_r \odot N_s) \cup E(C_r \odot N_s) \to \{1, 2, \ldots, 2r(s+1)\}$ to prove the existence of reverse super edge trimagic of $C_r \odot N_s$ with vertex labels and edge labels that are divided into two cases, odd

r and even r.

The first case is for odd r

Construct a bijective mapping $\zeta : V(C_r \odot N_s) \cup E(C_r \odot N_s) \to \{1, 2, ..., 2r(s+1)\}$ in the following way. Define the vertex labels $\zeta(w_i) = i$, $1 \le i \le r$, and $\zeta(w_i^j) = rj+i$, $1 \le i \le r$, $1 \le j \le s$. Furthermore, assign the edge labels as follows: $\zeta(w_i w_{i+1}) = (s+1)r + 2i$, $1 \le i \le r-1$, $\zeta(w_1 w_r) = (s+1)r + 1$, and

$$\zeta(w_i w_i^j) = \begin{cases} (s+j+1)r + i, & r = 3,\ 1 \le i \le 2, \quad 2 \le j \le s, \\ (s+1)r + 2i + 1, & r \ge 3,\ 1 \le i \le r-1, \quad j = 1, \\ (s+j+2)r, & r \ge 3,\ i = r, \quad 1 \le j \le s \\ (s+j+1)r + 2i - 1, & r \ge 5,\ 1 \le i \le \left\lfloor \dfrac{r}{2} \right\rfloor, \quad 2 \le j \le s, \\ (s+j)r + 2i + 1, & r \ge 5\ ,\ \left\lceil \dfrac{r}{2} \right\rceil \le i \le r-1. \end{cases}$$

We now verify that ζ defines a reverse super edge trimagic labeling on $C_r \odot N_s$ by considering the differences between the label of each edge and the sum of the labels of its incident vertices. The resulting values are as follows:

(1) For edges $w_i w_{i+1}$, $1 \le i \le r-1$

$$\zeta(w_i w_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = (s+1)r + 2i - (i + (i+1)) = (s+1)r - 1 = \theta_3.$$

(2) For edge $w_1 w_r$,

$$\zeta(w_1 w_r) - (\zeta(w_1) + \zeta(w_r)) = (s+1)r + 1 - (1+r) = rs = \theta_1.$$

(3) Considering the edges $w_i w_i^j$, two subcases arise:

a. for $r = 3$,
   i. if $1 \le i \le 2$, $j = 1$, $\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j)) = rs + 1 = \theta_2$;
   ii. if $i = 1$, $2 \le j \le s$, $\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j)) = (s+1)r - 1 = \theta_3$;
   iii. if $i = 2$, $2 \le j \le s$, $\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j)) = rs + 1 = \theta_2$;
   iv. if $i = 3$, $1 \le j \le s$, $\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j)) = rs = \theta_1$;

b. for $r \ge 5$,
   i. if $1 \le i \le \lfloor r/2 \rfloor$, $2 \le j \le s$, $\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j)) = (s+1)r - 1 = \theta_3$,
   ii. if $\lceil r/2 \rceil \le i \le r-1$, $2 \le j \le s$, $\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j)) = rs + 1 = \theta_2$,
   iii. if $i = r$, $1 \le j \le s$, $\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j)) = rs = \theta_1$.

In the first case, the labeling yields three distinct constants given by $\theta_1 = rs$, $\theta_2 = rs + 1$, and $\theta_3 = (s+1)r - 1$.

The second case is for even r

Assume that r is even with $r \ge 3$ and $s \ge 1$. Define a bijective mapping $\zeta : V(C_r \odot N_s) \cup E(C_r \odot N_s) \to \{1, 2, ..., 2r(s+1)\}$ by assigning the vertex labels as follows:

$$\zeta(w_i) = \begin{cases} \dfrac{i+1}{2}, & 1 \le i \le r, \text{odd } i, \\ \dfrac{r+i}{2}, & 1 \le i \le r, \text{even } i, \end{cases}$$

and

$$\zeta(w_i^j) = \begin{cases} rj + \dfrac{i+1}{2}, & 1 \le j \le s \quad 1 \le i \le r, \text{odd } i, \\ rj + \dfrac{r+i}{2}, & 1 \le j \le s \quad 1 \le i \le r, \text{even } i. \end{cases}$$

The edge labels are defined by

$$\begin{aligned} &\zeta(w_i w_{i+1}) = (s+1)r + i, 1 \le i \le r-1, \\ &\zeta(w_1 w_r) = (s+2)r, \\ &\zeta(w_i w_i^j) = (s+1+j)r + i1 \le i \le r, \ 1 \le j \le s. \end{aligned}$$

We now show that the labeling ζ fulfills the reverse super edge trimagic property by considering $\zeta(wv) - (\zeta(w) + \zeta(v)), wv \in E(C_r \odot N_s)$.

(1) For edges $w_i w_{i+1}, 1 \le i \le r-1$,

a. if i is odd,

$$\zeta(w_i w_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = (s+1)r + i - \Big(\frac{i+1}{2} + \frac{r+(i+1)}{2}\Big) = \frac{2rs+r-2}{2} = \theta_2;$$

b. if i is even,

$$\zeta(w_i w_{i+1}) - (\zeta(w_i) + \zeta(w_{i+1})) = (s+1)r + i - \Big(\frac{r+i}{2} + \frac{(i+1)+1}{2}\Big) = \frac{2rs+r-2}{2} = \theta_2.$$

(2) Likewise, for edge $w_1 w_r$,

$$\zeta(w_1 w_r) - (\zeta(w_1) + \zeta(w_r)) = (s+1)r - 1 = \theta_3.$$

(3) Similarly, for edges $w_i w_i^j, 1 \le j \le s, \ 1 \le i \le r$,

a. if i is odd, $\zeta(w_i w_i^j) - (\zeta(w_i) + \zeta(w_i^j)) = (s+1)r - 1 = \theta_3$;

b. if i is even, $\zeta(w_i w_i^j) - \zeta(w_i) + \zeta(w_i^j)) = rs = \theta_1$.

In the second case, the labeling produces the three distinct constants $\theta_1 = rs, \ \theta_2 = (2rs+r-2)/2, \ \theta_3 = (s+1)r - 1$.

The two cases above show that $\{\zeta(wv) - (\zeta(w) + \zeta(v)) | wv \in E(C_r \odot N_s)\} = \{\theta_1, \theta_2, \theta_3\}$, for some distinct $\theta_1, \theta_2, \theta_3$, and every vertex is assigned a unique label from the set $\{1, 2, \ldots, r(s+1)\}$. Consequently, the graph $C_r \odot N_s$ admits a reverse super edge trimagic labeling. Hence, the corona product of $C_r \odot N_s$ graph is a reverse super edge trimagic graph for any $r \ge 3, s \ge 1$. □

Illustrations of reverse super edge trimagic labeling on $C_6 \odot N_3$ and $C_7 \odot N_3$ can be found respectively in Figures 11 and 12.

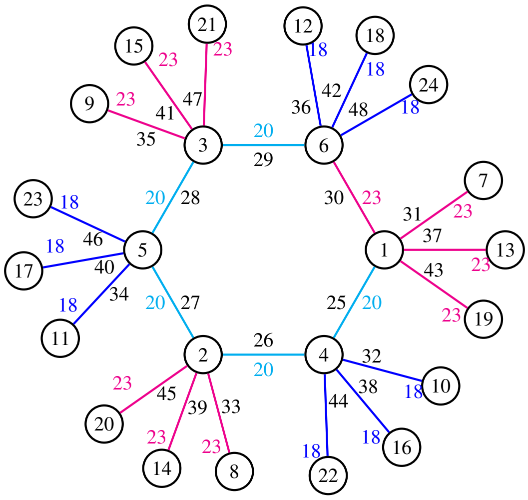

Figure 11. Reverse super edge trimagic labeling of $C_6 \odot N_3$ with $\theta_1 = 18$, $\theta_2 = 20$, $\theta_3 = 23$.

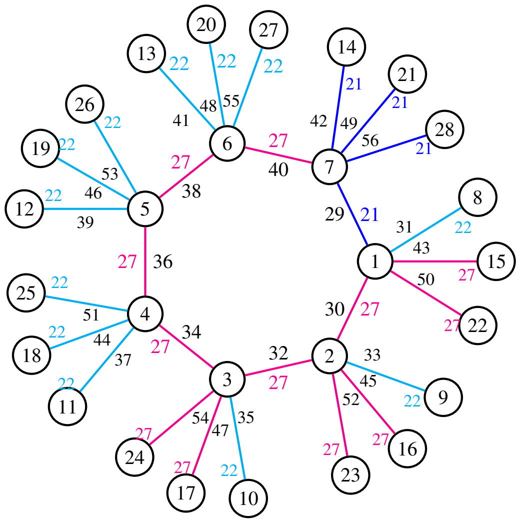

Figure 12. Reverse super edge trimagic labeling of $C_7 \odot N_3$ with $\theta_1 = 21$, $\theta_2 = 22$, $\theta_3 = 27$.

3. Conclusions

In this paper, we determined reverse super edge trimagic labelings for several families of path-related and cycle-related graphs, including path graphs, multiple copies of path graphs, and the corona product of path graphs with null graphs, cycle graphs, multiple copies of cycle graphs, the corona product of cycle with null graphs. Hence, we demonstrate that several families of path-related and cycle-related graphs are reverse super edge trimagic graphs.

Author contributions

Titin Srimartini: Conceptualization, Validation, Writing–original draft; Yeni Susanti: Supervision, Validation, Writing–review and editing; Budi Surodjo: Supervision, Writing–review and editing. All authors have read and approved the final version.

Use of generate AI tools declaration

The authors used artificial intelligence (AI) ChatGPT (OpenAI), GPT-5.6 Luna to assist with language editing including paraphrasing and grammar correction. These tools were not used in the development of the mathematical results, proofs, or conclusions. All AI-assisted text was carefully reviewed and validated by the authors, who assume full responsibility for the accuracy and integrity of the manuscript.

Acknowledgments

The authors gratefully acknowledge the financial support provided by the research grant scheme Penelitian Desertasi Doktor Universitas Sebelas Maret, with contract number 460/UN27.22/PT.01.03/2026.

Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

References

1. W. D. Wallis, *Magic Graphs*, 2 Eds, Basel: Birkhauser, 2013.
2. J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Comb.*, **6** (2023), DS6.
3. J. Sedláček, Theory of graphs and its applications, *Proc. Symp.*, 1963, 163–164.
4. A. Kotzig, A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451–461. https://doi.org/10.4153/CMB-1970-084-1
5. J. B. Babujee, On edge bimagic labeling, *Int. J. Comb. Inform. Syst. Sci.*, **28** (2004), 239–244. http://doi.org/10.22457/apam.v13n1a8
6. J. B. Babujee, R. Jagadesh, Superior edge bimagic labeling, *Int. J. Math Comb.*, **1** (2008), 107–116.
7. J. B. Babujee, R. Jagadesh, Super edge bimagic labeling for graph with cycles, *Pacific-Asian J. Math.*, **2** (2008), 113–122.
8. A. A. Jothi, N. G. David, J. B. Babujee, On edge magic and bimagic labeling of graphs, *Int. J. Pure Appl. Math.*, **101** (2015), 711–718.
9. J. B. Babujee, R. Jagadesh, Super edge bimagic labeling for some classes of connected graphs derived from fundamental graphs, *Int. J. Comb. Graph Theory Appl.*, **4** (2019), 119–126.
10. S. Subbulakshmi, R. Kokila, Edge bimagic labeling of graphs K2,N, KN,2,N, Bistar, nC4, and double wheel, *J. Phys.: Conf. Ser.*, **1377** (2019), 012013. http://doi.org/10.1088/1742-6596/1377/1/012013
11. A. N. Fadhilah, T. S. Martini, Super edge bimagic and super edge trimagic total labeling of hibiscus graph, *AIP Conf. Proc.*, **3285** (2022), 020003. https://doi.org/10.1063/5.0262408

12. M. R. Z. El Deen, I. M. Hanafi, S. G. Shehda, N. A. Omar, Super edge bimagic and trimagic total labeling for some graphs, *Alfarama J. Basic Appl. Sci.*, **6** (2025), 431–442. http://doi.org/10.21608/ajbas.2025.403080.1267
13. C. Jayasekaran, M. Reeges, C. Davidraj, Edge trimagic labeling of some graphs, *Int. J. Comb. Graph Theory Appl.*, **6** (2013), 175–186.
14. M. Reeges, C. Jayasekaran, Edge trimagic total labeling of disconnedted graphs, *Int. J. Math. Trenda Technol.*, **6** (2014), 44–53. https://doi.org/10.14445/22315373/IJMTT-V6P504
15. M. Reeges, C. Jayasekaran, Super edge trimagic total labeling of generalized prism and web graphs, *J. Discrete Math. Sci. Cryptogr.*, **19** (2016), 81–92. https://doi.org/10.1080/09720529.2015.1101880
16. C. Jayasekaran, J. L. Flower, On edge trimagic labeling of umbrella, dumb bell and circular ladder graphs, *J. Ann. Pure Appl. Math.*, **13** (2017), 73–87. http://doi.org/10.22457/apam.v13n1a8
17. C. Jayasekaran, J. L. Flower, Edge trimagic total labeling of mobius ladder, book and dragon graphs, *J. Ann. Pure Appl. Math.*, **13** (2017), 151–163. http://doi.org/10.22457/apam.v13n2a1
18. A. J. Nawawi, T. S. Martini, T. A. Kusmayadi, Super edge trimagic total labeling of Dovetail Corona Null Graph $D_n \odot N_m$, *AIP Conf. Proc.*, **3285** (2025), 020006. https://doi.org/10.1063/5.0262388
19. N. S. Hungund, D. G. Akka, Reverse super edge-magic strength of some new classes of graphs, *J. Discrete Math. Sci. Cryptogr.*, **16** (2013), 19–29. https://doi.org/10.1080/09720529.2013.778462
20. K. A. Reddy, S. S. Basha, New classes of reverse super edge magic graphs, *AIMS Math.*, **7** (2021), 3590–3602. https://doi.org/10.3934/math.2022198
21. S. S. Basha, K. A. Reddy, Reverse edge magic labeling of a cycle with chords, unions of cycles and unions of paths, *Adv. Math.: Sci. J.*, **10** (2021), 571–581. https://doi.org/10.37418/amsj.10.1.56
22. K. Amuthavalli, P. Sugapriya, Reverse super edge–BI magic labeling of star related graphs, *IJREAM*, **5** (2019), 480–484. https://doi.org/10.35291/2454-9150.2019.0078
23. T. S. Martini, D. Indriati, T. A. Kusmayadi, On reverse super edge bimagic labeling of gear graph, hibiscus graph and dovetail graph, *Int. J. Comput. Sci. Appl. Math.*, **10** (2024), 51–55.
24. K. Amuthavalli, P. Sugapriya, Reverse super edge-trimagic labeling for star related graphs, *MJM*, **7** (2019), 519–525. https://doi.org/10.26637/MJM0703/0025
25. T. S. Martini, Y. Susanti, On reverse super edge trimagic labeling of dove tail related graphs, *AIP Conf. Proc.*, **3201** (2023), 030009. https://doi.org/10.1063/5.0230966
26. R. Frucht, F. Harary, On the corona of two graphs, *Aeq. Math.*, **4** (1970), 322–325. http://doi.org/10.1007/BF01844162

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