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Research article

Norm-oriented similarity modeling for quasirung orthopair fuzzy sets: a framework for pattern recognition and medical diagnosis

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Abstract: The quasirung orthopair fuzzy sets (QOFSs) provide a flexible framework for representing uncertainty through membership degree (MD) and non-membership degree (NMD). This study developed a unified framework for constructing similarity measures (SMs) under the QOFS environment using cosine-, entropy-, and norm-based structures. The proposed measures incorporate membership, non-membership, and hesitancy information to provide flexible similarity assessment through tunable parameters. The theoretical properties of the proposed measures were established, and a dual-perspective analysis was conducted at microscopic and macroscopic levels to examine their mathematical and numerical behavior. Comparative numerical experiments were performed to investigate the discrimination, sensitivity, stability, and computational characteristics of the proposed measures. Their applicability was further illustrated through pattern recognition and medical decision-making examples. The main contributions are as follows: (1) A systematic framework was developed to characterize similarity relationships between quasirung orthopair fuzzy numbers (QOFNs) through multiple complementary mathematical formulations; (2) treatment of numerical issues, including division-by-zero cases, through appropriate similarity constructions; (3) establishment of essential mathematical properties and a systematic micro–macro evaluation framework; and (4) comprehensive numerical comparison and illustrative applications demonstrating the computational applicability of the proposed measures.

Keywords: similarity measures; quasirung orthopair fuzzy information; uncertainty quantification; decision-making; applications
Mathematics Subject Classification: 03E72, 94A17

1. Introduction

The representation of uncertain information has progressively evolved from classical fuzzy sets (FSs) [1] to more expressive orthopair structures. In an intuitionistic fuzzy set (IFS) [2], each element $a \in A$ is characterized by a membership degree $\mu(a)$ and a non-membership degree $\vartheta(a)$, satisfying $\mu(a) + \vartheta(a) \leq 1$. The corresponding hesitation degree is given by $\pi(a) = 1 - \mu(a) - \vartheta(a)$. Although IFSs provide a mechanism for simultaneously representing positive and negative information, the linear constraint limits the feasible region of (μ, ϑ). For example, the assessment (0.30,0.80) is not admissible because $0.30 + 0.80 = 1.1 > 1$. To enlarge the feasible domain, Yager [3] introduced Pythagorean fuzzy sets (PFSs), characterized by the nonlinear condition $\mu^2(a) + \vartheta^2(a) \leq 1$. The quadratic constraint permits a broader collection of membership–non-membership combinations than the IFS framework. Yager [4] subsequently generalized this structure through q-rung orthopair fuzzy sets (q-ROFSs), whose admissibility condition is expressed as $\mu^q(a) + \vartheta^q(a) \leq 1,\ q \geq 1$. Thus, the parameter q controls the expansion of the feasible orthopair domain, with $q = 1$ and $q = 2$ corresponding to the IFS and PFS constraints, respectively.

Despite this increased flexibility, the q-ROFS framework employs a common exponent for both MD and NMD. Such a restriction may be unsuitable when these two components exhibit different uncertainty characteristics. To address this limitation, Seikh and Mandal [5] introduced quasirung orthopair fuzzy sets (QOFSs), in which membership and non-membership information are regulated independently through two positive parameters p and q with the condition $\mu^q(a) + \vartheta^q(a) \leq 1$, where p and q independently regulate the respective influence of membership and non-membership information, providing greater flexibility in representing different uncertainty characteristics. The distinction between QOFSs and q-ROFSs is mathematically important. When $p = q$, the QOFS constraint reduces to the q-rung orthopair condition $\mu^q(a) + \vartheta^q(a) \leq 1$, whereas $p \neq q$ permits an asymmetric treatment of the two orthopair components. Therefore, p and q provide independent control over the contribution of membership and non-membership information to the admissibility condition. This additional degree of freedom enables QOFSs to represent uncertain assessments that may not be adequately described by models imposing a common exponent. For example, an assessment such as $(0.30,0.80)$, which is inadmissible under the IFS constraint, may satisfy the QOFS condition for appropriate choices of p and q. Consequently, QOFSs provide a parameterized framework capable of accommodating both symmetric configurations ($p = q$) and asymmetric configurations ($p \neq q$). This flexibility makes the QOFS framework particularly suitable for similarity assessment and decision-making problems in which membership and non-membership information may possess different levels of uncertainty. A comparative summary of the principal characteristics of these FS generalizations is presented in Table 1.

Table 1. Comparative characterization of representative fuzzy set and their extended frameworks.

Model	Constraints	Interdependence of MD and NMD	Tunable parameters
FSs	$0 \leq \mu \leq 1$	Symmetric	No
IFSs	$\mu + \vartheta \leq 1$	Symmetric	No
PFSs	$\mu^2 + \vartheta^2 \leq 1$	Symmetric	No
$\boldsymbol{q}$-ROFSs	$\mu^q + \vartheta^q \leq 1\ (q \geq 1)$	Symmetric	Yes (Single)
QOFSs	$\mu^p + \vartheta^q \leq 1\ (p, q \geq 1)$	Both symmetric and asymmetric	Yes (Two)

The comparative analysis indicates that QOFSs offer an adaptable representation of uncertain information through the independent regulation of MD and NMD. In particular, the use of two adjustable parameters provides greater freedom in defining the admissible region than frameworks governed by a single common exponent. This feature permits the model to accommodate both $p = q$ configurations, corresponding to symmetric treatment of the two components, and $p \neq q$ configurations, which allow asymmetric regulation. Under appropriate parameter settings, several established orthopair fuzzy models, including IFSs, PFSs, and q-ROFSs, can be embedded within the QOFS framework. The resulting flexibility has stimulated considerable research on QOFS-based theoretical developments [6,7], aggregation mechanisms [8], similarity measures (SMs) [9], and decision-making methodologies [10,11]. These developments have subsequently been employed in diverse decision-support applications, including supplier selection [12,13] and other intelligent decision-analysis problems [14,15]. Hence, the independent parameterization of QOFSs provides a useful mathematical mechanism for representing heterogeneous and asymmetric uncertain assessments in complex decision-making environments.

Similarity measures constitute an important class of information measures in fuzzy-set theory because they provide a numerical means of assessing the degree of resemblance between fuzzy objects. The initial development of such measures was mainly associated with conventional fuzzy sets, in which similarity was determined from membership information [16,17]. The emergence of intuitionistic fuzzy sets introduced a second dimension, namely non-membership, and consequently enabled similarity assessments to account simultaneously for positive and negative information [18–20]. Subsequent developments based on PFSs and q-ROFSs expanded the admissible space of membership–non-membership pairs and stimulated the formulation of diverse similarity-measure constructions, including cosine-based [21–23], distance-based, and set-theoretic approaches [24,25]. The applicability of these measures has been demonstrated in a variety of contexts, such as pattern recognition [26], clustering analysis [27], medical diagnosis [28], and multi-criteria decision-making [29,30]. Nevertheless, the increasing complexity of contemporary decision environments demands similarity models that can exploit the additional flexibility offered by advanced orthopair structures. In particular, the independent treatment of membership and non-membership information in QOFSs creates an opportunity to formulate SMs with greater adaptability to asymmetric and parameter-dependent uncertainty, thereby motivating the development of the measures presented in the present study.

Although SMs have been extensively investigated within q-ROFSs [31] and related fuzzy frameworks [32–34], the development of dedicated similarity models for QOFSs remains comparatively limited. The contribution in [9] constitutes an important development in this direction; however, its formulation and analysis leave several theoretical and methodological aspects unexplored. In particular, the existing approach is predominantly based on cosine construction, with limited investigation of its fundamental properties. The influence of hesitancy, together with the distinct behavior induced by symmetric and asymmetric parameter configurations [35,36], has not been systematically characterized [37,38]. Moreover, the existing literature provides insufficient distinction between element-wise and set-wise similarity assessment. The occurrence of zero denominators and the resulting undefined values also require explicit mathematical treatment. Further investigation is warranted concerning essential axiomatic properties and the relationships between the resulting measures and established fuzzy-set frameworks. These observations establish a clear motivation for developing a more comprehensive similarity-measure framework within the QOFS environment. The limitations of existing measures are summarized in Table 2.

Table 2. Limitations of existing measures.

Method	Information	Parameters	Divide-by-zero avoided?	Large-sample macro-level validated?
Rahim et al. [9]	(μ,ϑ)	p,q	No	No
Li and Chuntian [18]	(μ,ϑ)	—	Yes	No
Hussain et al. [31]	(μ,ϑ)	q	No	No
Kumar and Parashar [32]	(μ,ϑ)	—	No	No
Wang et al. [33]	(μ,ϑ)	q	No	No
Farhadinia et al. [34]	(μ,ϑ)	q	No	No
Hong and Kim [39]	(μ,ϑ)	—	Yes	No
Ye [40]	(μ,ϑ)	—	Yes	No
Wei and Wei [41]	(μ,ϑ)	—	No	No
Peng and Garg [42]	(μ,ϑ)	—	Partial	No
Proposed study	(μ,ϑ)	p,q,r,λ	Yes	Yes

In response to these limitations, the present study formulates four complementary classes of similarity measures, namely hesitancy-integrated cosine similarity, interaction-aware cosine similarity, entropy-weighted cosine similarity, and norm-induced similarity measures. The corresponding mathematical expressions are constructed intrinsically from the QOFS structure rather than obtained through a direct adaptation of measures originally established for IFSs, PFSs, FFSs, or q-ROFSs. Such a formulation permits membership, non-membership, and hesitancy information to be incorporated coherently while preserving the independent parameterization inherent in QOFSs. Consideration is given to the mathematical domain of each measure to identify and appropriately address cases associated with vanishing denominators. The validity of the resulting constructions is established through rigorous verification of their fundamental axiomatic properties. Their discriminative characteristics are subsequently examined through microscopic and macroscopic analyses, while their practical applicability is demonstrated using pattern-recognition and medical decision-support scenarios. The principal contributions of the study are summarized as follows:

(i) A comprehensive family of QOFS similarity measures is developed, encompassing hesitancy-integrated cosine, weighted hesitancy-integrated cosine, membership–non-membership interaction-based cosine, and norm-based similarity measures, through rigorous mathematical formulations.

(ii) The proposed measures are further examined under fundamental properties such as boundary, identity, monotonicity, and symmetry to ensure their consistency with the underlying QOFS structure.

(iii) Microscopic and macroscopic analyses are conducted to systematically examine discriminative behavior and numerical performance in comparison with existing similarity approaches.

(iv) Pattern-recognition and medical decision-support applications are presented to demonstrate the applicability of the developed framework to practical uncertain-information environments.

The overall organization and methodological progression of the manuscript are depicted in Figure 1.

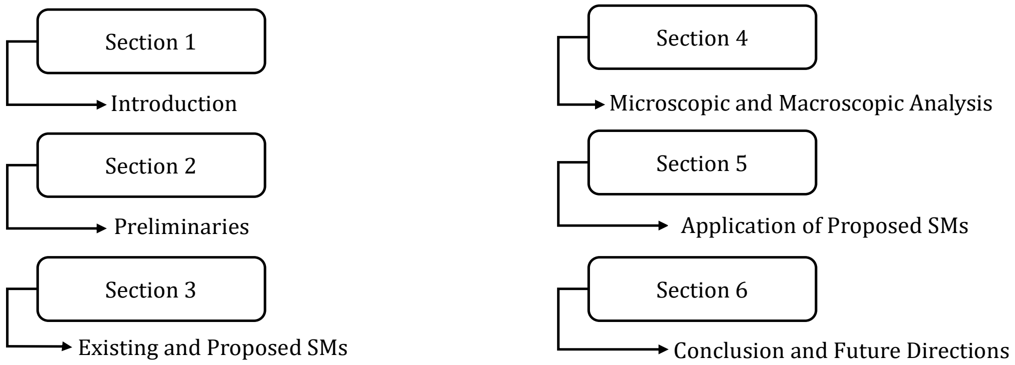

Figure 1. Organizational structure of the study.

2. Preliminaries

In this section, some important definitions and concepts are briefly reviewed to facilitate a clearer understanding of the proposed study.

Definition 1. [5] Let $A = \{a_1, a_2, \dots, a_n\}$ be a finite universe of discourse. A QOFS $\tilde{\alpha}$ defined on A is expressed as $\tilde{\alpha} = \{\langle a, \mu_{\tilde{\alpha}}(a), \vartheta_{\tilde{\alpha}}(a)\rangle \mid a \in A\}$, where the functions $\mu_{\tilde{\alpha}}: A \to [0,1]$ and $\vartheta_{\tilde{\alpha}}: A \to [0,1]$ denote the MD and the NMD of the element $a \in A$, respectively. These values are restricted by parametric condition $\mu_{\tilde{\alpha}}(a)^p + \mu_{\tilde{\alpha}}(a)^q \leq 1,\ \forall a \in A,$ where $p, q \geq 1$ are real-valued parameters controlling the admissible domain of the pair $(\mu_{\tilde{\alpha}}(a), \vartheta_{\tilde{\alpha}}(a))$. For simplicity, the ordered pair $\alpha = (\mu_\alpha, \vartheta_\alpha)$ is referred to as a quasirung orthopair fuzzy number (QOFN). Moreover, the degree of uncertainty associated with the QOFN $\alpha = (\mu_\alpha, \vartheta_\alpha)$ can be defined as $\pi_\alpha = \left(1 - \mu_\alpha^p - \vartheta_\alpha^q\right)^{1/r}$ (where $r = \max(p, q)$. For notational convenience, the QOFN $\alpha = (\mu_\alpha, \vartheta_\alpha)$ may simply be written as α. For the remainder of the manuscript, QROFSs are denoted by $\tilde{\alpha}$ or $\tilde{\beta}$, whereas the corresponding QORNs are represented by α or β.

Remark 1. The relationship between QOFNs [5] and classical q-ROFNs [4] can be clarified through their parameter structures. For a QOFN $\alpha = (\mu_\alpha, \vartheta_\alpha)$, where μ_α and ϑ_α are the MD and NMD, the admissible condition is $\mu_\alpha^p + \vartheta_\alpha^q \leq 1,\ p, q \geq 1$, whereas a q-ROFS is characterized by $\mu_\alpha^q + \vartheta_\alpha^q \leq 1$. Thus, QOFSs allow the MD and NMDs to be governed by two potentially different exponents. When $p \neq q$, the transformations $\mu_\alpha \to \mu_\alpha^p$ and $\vartheta_\alpha \to \vartheta_\alpha^q$ affect the two components differently. For a fixed MD μ_α, the maximum admissible NMD is $\underset{\max}{\vartheta_\alpha}(\mu; p, q) = \left(1 - \mu_\alpha^p\right)^{1/q}$ while, for a fixed NMD ϑ,

$\mu_{\alpha_{\max}}(\vartheta; p, q) = \left(1 - \vartheta_\alpha^q\right)^{1/p}$. Hence, p and q independently determine the boundary of the feasible membership–non-membership region. For example, $\vartheta_{\alpha_{\max}}(\mu; 2{,}2) = (1 - \mu_\alpha^2)^{1/2}$, $\vartheta_{\alpha_{\max}}(\mu; 2{,}3) = (1 - \mu_\alpha^2)^{1/3}$, showing that different values of q produce different admissible non-membership ranges for the same μ_α. Moreover, setting $p = q$ gives $\mu_\alpha^q + \vartheta_\alpha^q \leq 1$, and therefore, recovers the conventional q-ROFS. In particular $p = q = 1$ and $p = q = 2$ correspond to the intuitionistic [2] and Pythagorean fuzzy structures [3], respectively. The independent effects of the two parameters are further evident from $\frac{\partial x^p}{\partial p} = x^p \ln x < 0$, $\frac{\partial x^q}{\partial q} = x^q \ln x < 0$, $x \in (0{,}1)$.

Therefore, variations independently modify the transformed membership and non-membership components. In particular, $p \neq q$ enables QOFSs to represent asymmetric membership–non-membership relationships that cannot be captured by a single common parameter in q-ROFSs [4]. This property provides an important basis for constructing similarity measures that reflect the flexible information structure of QOFSs.

Definition 2. [5] Let $\alpha = (\mu_\alpha, \vartheta_\alpha)$ and $\beta = (\mu_\beta, \vartheta_\beta)$ be any QOFNs. The fundamental operations and relationships associated with QOFNs are defined as follows:

(i) If $\alpha = (\mu_\alpha, \vartheta_\alpha)$, then the complement of α is defined as $\alpha^C = (\vartheta_\alpha, \mu_\alpha)$.
(ii) $\alpha \leq \beta$ if and only if $\mu_\alpha \leq \mu_\beta$ and $\mu_\alpha \geq \vartheta_\beta$.
(iii) $\alpha = \beta$ if and only if $\mu_\alpha = \mu_\beta$ and $\vartheta_\alpha = \vartheta_\beta$.
(iv) Boundary cases of QOFNs $1 = (1{,}0)$ and $0 = (0{,}1)$.

The existing q-ROFS similarity measures [35–37] cannot generally reproduce the proposed asymmetric QOFS representation through a single rung parameter because a single parameter q simultaneously controls both μ_α^p and ϑ_α^q. In the QOFS setting, p and q independently determine the transformed membership and non-membership components. Consequently, when $p \neq q$, the resulting similarity calculation is based on an information representation that has no direct counterpart under the q-ROFS structure [38].

Before presenting the proposed similarity measures (SMs) for QOFSs, we first introduce the necessary preliminary functions. Let $\psi: [0{,}1] \to [0{,}1]$ denote a strictly decreasing transformation function. Several admissible forms of ψ are given below:

(a) $\psi_1(\rho) = 1 - \rho$.
(b) $\psi_2(\rho) = 1 - \rho^2$.
(c) $\psi_3(\rho) = \frac{1}{1+\rho}$.
(d) $\psi_4(\rho) = \frac{1-\rho}{1+\rho}$.
(e) $\psi_5(\rho) = e^{-\rho}$.
(f) $\psi_6(\rho) = 1 - \rho e^{\rho - 1}$.

Based on these transformations, comparison mappings are constructed to evaluate the MD and NMD components of two QOFNs $\alpha = (\mu_\alpha, \vartheta_\alpha)$ and $\beta = \left(\mu_\beta, \vartheta_\beta\right)$.

Membership-oriented mapping $\psi_i^\mu(\alpha, \beta) = \psi_i\left(\left|\mu_\alpha^p - \mu_\beta^p\right|\right)$, which gives the following results:

$$\psi_1^\mu(\alpha, \beta) = 1 - \left|\mu_\alpha^p - \mu_\beta^p\right|, \tag{1}$$

$$\psi_2^\mu(\alpha, \beta) = 1 - \left|\mu_\alpha^p - \mu_\beta^p\right|^2, \tag{2}$$

$$\psi_3^{\mu}(\alpha,\beta) = \frac{1}{1-\left|\mu_\alpha^p-\mu_\beta^p\right|}, \tag{3}$$

$$\psi_4^{\mu}(\alpha,\beta) = \frac{1-\left|\mu_\alpha^p-\mu_\beta^p\right|}{1+\left|\mu_\alpha^p-\mu_\beta^p\right|}, \tag{4}$$

$$\psi_5^{\mu}(\alpha,\beta) = e^{-\left|\mu_\alpha^p-\mu_\beta^p\right|}, \tag{5}$$

$$\psi_6^{\mu}(\alpha,\beta) = 1 - \left|\mu_\alpha^p - \mu_\beta^p\right| e^{\left|\mu_\alpha^p-\mu_\beta^p\right|-1}. \tag{6}$$

Following the same approach, we have $\psi_i^{\vartheta}(\alpha,\beta) = \psi_i\left(\left|\vartheta_\alpha - \vartheta_\beta\right|\right)$ and give the following relations:

$$\psi_1^{\vartheta}(\alpha,\beta) = 1 - \left|\vartheta_\alpha^q - \vartheta_\beta^q\right|, \tag{7}$$

$$\psi_2^{\vartheta}(\alpha,\beta) = 1 - \left|\vartheta_\alpha^q - \mu_\beta^q\right|^2, \tag{8}$$

$$\psi_3^{\vartheta}(\alpha,\beta) = \frac{1}{1-\left|\vartheta_\alpha^q-\vartheta_\beta^q\right|}, \tag{9}$$

$$\psi_4^{\vartheta}(\alpha,\beta) = \frac{1-\left|\vartheta_\alpha^q-\vartheta_\beta^q\right|}{1+\left|\vartheta_\alpha^q-\vartheta_\beta^q\right|}, \tag{10}$$

$$\psi_5^{\vartheta}(\alpha,\beta) = e^{-\left|\vartheta_\alpha^q-\vartheta_\beta^q\right|}, \tag{11}$$

$$\psi_6^{\vartheta}(\alpha,\beta) = 1 - \left|\vartheta_\alpha^q - \vartheta_\beta^q\right| e^{\left|\vartheta_\alpha^q-\vartheta_\beta^q\right|-1}. \tag{12}$$

The fundamental mathematical characteristics of these mappings are first investigated to provide a sound basis for constructing the proposed SMs.

Theorem 1. For any three QOFNs $\alpha = (\mu_\alpha, \vartheta_\alpha)$, $\beta = (\mu_\beta, \vartheta_\beta)$ and $\gamma = (\mu_\gamma, \vartheta_\gamma)$, consider the transformation mappings ψ_i^{μ} and ψ_i^{ϑ} (where $i = 1,2,\dots,6$), as defined in Eqs (1)–(12), satisfy the following properties:
(i) $0 \le \psi_i^{\mu}(\alpha,\beta), \psi_i^{\vartheta}(\alpha,\beta) \le 1$.
(ii) $\psi_i^{\mu}(\alpha,\beta) = \psi_i^{\mu}(\alpha,\beta)$, $\psi_i^{\vartheta}(\alpha,\beta) = \psi_i^{\vartheta}(\beta,\alpha)$ for all $p, q \ge 1$.
(iii) $\psi_i^{\mu}(\alpha,\beta) = \psi_i^{\vartheta}(\alpha,\beta) = 1$ if and only if $\alpha = \beta$.
(iv) If $\alpha \le \beta \le \gamma$, then $\psi_i^{\mu}(\alpha,\gamma) \le \psi_i^{\mu}(\alpha,\beta)$, $\psi_i^{\mu}(\alpha,\gamma) \le \psi_i^{\mu}(\beta,\gamma)$ and $\psi_i^{\vartheta}(\alpha,\gamma) \le \psi_i^{\vartheta}(\alpha,\beta)$, $\psi_i^{\vartheta}(\alpha,\gamma) \le \psi_i^{\vartheta}(\beta,\gamma)$.

Proof. Since $\alpha = (\mu_\alpha, \vartheta_\alpha)$, $\beta = (\mu_\beta, \vartheta_\beta)$ and $\gamma = (\mu_\gamma, \vartheta_\gamma)$, then

(i) The transformation functions ψ_i for all i are strictly decreasing mappings from [0,1] into [0,1], and it follows that the resulting mappings ψ_i^{μ} and ψ_i^{ϑ} also lie within the interval [0,1].

Hence, the boundedness condition holds, i.e., $0 \leq \psi_i^{\mu}(\alpha,\beta), \psi_i^{\vartheta}(\alpha,\beta) \leq 1$.

(ii) Since both mappings $\psi_i^{\mu}(\alpha,\beta), \psi_i^{\vartheta}(\alpha,\beta)$, depend on the absolute differences $\left|\mu_{\alpha}^{p} - \mu_{\beta}^{p}\right|$ and $\left|\vartheta_{\alpha}^{q} - \vartheta_{\beta}^{q}\right|$, the symmetry of the absolute-value terms follows directly from their definition. Thus, $\psi_i^{\mu}(\alpha,\beta) = \psi_i^{\mu}(\beta,\alpha)$ and $\psi_i^{\vartheta}(\alpha,\beta) = \psi_i^{\vartheta}(\beta,\alpha)$ holds.

(iii) For both mappings $\psi_i^{\mu}(\alpha,\beta)$ and $\psi_i^{\vartheta}(\alpha,\beta)$, the upper bound 1 is achieved precisely when $\left|\mu_{\alpha}^{p} - \mu_{\beta}^{p}\right| = 0$ and $\left|\vartheta_{\alpha}^{q} - \vartheta_{\beta}^{q}\right| = 0$. This implies $\mu_{\alpha}^{p} = \mu_{\beta}^{p}$ and $\vartheta_{\alpha}^{q} = \vartheta_{\beta}^{q}$, as a result, $\alpha = \beta$. On the other hand, if $\alpha = \beta$, then both differences $\left|\mu_{\alpha}^{p} - \mu_{\beta}^{p}\right|$ and $\left|\vartheta_{\alpha}^{q} - \vartheta_{\beta}^{q}\right|$ become zero, and hence $\psi_i^{\mu}(\alpha,\beta) = \psi_i^{\vartheta}(\alpha,\beta) = 1$.

(iv) Consider that $\alpha \leq \beta \leq \gamma$, then by Definition 2, we have $0 \leq \mu_{\alpha} \leq \mu_{\beta} \leq \mu_{\gamma} \leq 1$ and $1 \geq \vartheta_{\alpha} \geq \vartheta_{\beta} \geq \vartheta_{\gamma} \geq 0$. This implies $\left|\mu_{\alpha}^{p} - \mu_{\gamma}^{p}\right| \geq \left|\mu_{\alpha}^{p} - \mu_{\beta}^{p}\right|$, $\left|\mu_{\alpha}^{p} - \mu_{\gamma}^{p}\right| \geq \left|\mu_{\beta}^{p} - \mu_{\gamma}^{p}\right|$, $\left|\vartheta_{\alpha}^{q} - \vartheta_{\gamma}^{q}\right| \geq \left|\vartheta_{\alpha}^{q} - \vartheta_{\beta}^{q}\right|$, and $\left|\vartheta_{\alpha}^{q} - \vartheta_{\gamma}^{q}\right| \geq \left|\vartheta_{\beta}^{q} - \vartheta_{\gamma}^{q}\right|$. By virtue of the strict antitonicity of the transformation functions ψ_i, the preceding inequalities imply $\psi_i^{\mu}(\alpha,\gamma) \leq \psi_i^{\mu}(\alpha,\beta)$, $\psi_i^{\mu}(\alpha,\gamma) \leq \psi_i^{\mu}(\beta,\gamma)$, and $\psi_i^{\vartheta}(\alpha,\gamma) \leq \psi_i^{\vartheta}(\alpha,\beta)$, $\psi_i^{\vartheta}(\alpha,\gamma) \leq \psi_i^{\vartheta}(\beta,\gamma)$.

Theorem 2. If $\alpha = (0,1)$ and $\beta = \alpha^{C} = (1,0)$, then the transformation functions $\psi_i^{\mu}(\alpha,\beta) = \psi_i^{\vartheta}(\alpha,\beta) = 0$ for $i = 1,2,4,6$.

Proof. Since $\alpha = (0,1)$ and $\beta = (1,0)$, i.e., $\mu_{\alpha} = 0$, $\vartheta_{\alpha} = 0$, $\mu_{\beta} = 1$, and $\vartheta_{\beta} = 0$. By putting these values in Eqs (1) to (6), we get $\psi_1^{\mu}(\mu_{\alpha},\mu_{\beta}) = 1 - |0-1| = 0$, $\psi_2^{\mu}(\mu_{\alpha},\mu_{\beta}) = 1 - |0-1|^2 = 0$, $\psi_4^{\mu}(\mu_{\alpha},\mu_{\beta}) = \frac{1-|1-0|}{1+|1-0|} = 0$, and $\psi_6^{\mu}(\mu_{\alpha},\mu_{\beta}) = 1 - |1-0|\, e^{|1-0|-1} = 0$. Similarly, the results can be proved for ψ_i^{ϑ} $(i = 1,2,4,6)$.

Definition 3. A binary operation $\tau: [0,1]^2 \to [0,1]$ is said to be a continuous triangular norm (continuous t-norm) if the following fundamental properties satisfy for all υ_1, υ_2, $\upsilon_3 \in [0,1]$:

(a) $\tau(\upsilon_1,\upsilon_2) = \tau(\upsilon_2,\upsilon_1)$.
(b) $\tau(\upsilon_1,\tau(\upsilon_2,\upsilon_3)) = \tau(\tau(\upsilon_1,\upsilon_2),\upsilon_3)$.
(c) If $\upsilon_1^* \leq \upsilon_1$ and $\upsilon_2^* \leq \upsilon_2 \Longrightarrow \tau(\upsilon_1^*,\upsilon_2^*) = \tau(\upsilon_1,\upsilon_2)$.
(d) $\tau(\upsilon_1,1) = \upsilon_1$.

In addition, τ is called continuous if it is continuous on the unit square $[0,1]^2$; that is, whenever $\upsilon_{1n} \longrightarrow \upsilon_1$ and $\upsilon_{2n} \longrightarrow \upsilon_2$, $\tau(\upsilon_{1n},\upsilon_{2n}) \longrightarrow \tau(\upsilon_1,\upsilon_2)$. From these axioms, a t-norm also satisfies $\tau(\upsilon_1,0) = 0$ and $\tau(\upsilon_1,\upsilon_2) \leq \max(\upsilon_1,\upsilon_2)$.

Similarly, a continuous triangular conorm (continuous t-conorm or s-norm) is a binary operation $\sigma: [0,1]^2 \to [0,1]$ satisfying the following axioms υ_1, υ_2, $\upsilon_3 \in [0,1]$:

(a) $\sigma(\upsilon_1,\upsilon_2) = \sigma(\upsilon_2,\upsilon_1)$.
(b) $\sigma(\upsilon_1,\sigma(\upsilon_2,\upsilon_3)) = \sigma(\sigma(\upsilon_1,\upsilon_2),\upsilon_3)$.
(c) If $\upsilon_1^* \leq \upsilon_1$ and $\upsilon_2^* \leq \upsilon_2 \Longrightarrow \sigma(\upsilon_1^*,\upsilon_2^*) = \sigma(\upsilon_1,\upsilon_2)$.
(d) $\sigma(\upsilon_1,0) = \upsilon_1$.

Moreover, σ is called continuous if it is continuous on the unit square $[0,1]^2$, i.e., $\upsilon_{1n} \longrightarrow \upsilon_1$ and $\upsilon_{2n} \longrightarrow \upsilon_2$, $\sigma(\upsilon_{1n},\upsilon_{2n}) \longrightarrow \sigma(\upsilon_1,\upsilon_2)$. Every s-norm satisfies $\sigma(\upsilon_1,1) = 1$ and $\sigma(\upsilon_1,\upsilon_2) \geq \max(\upsilon_1,\upsilon_2)$.

The axiomatic framework above encompasses various well-known t-norm and s-norm formulations commonly used for fuzzy information processing. Their mathematical representations are listed below [38]. For the algebraic operational family, the corresponding t-norm (τ) and s-norm

(σ) are mathematically specified as follows:

$$\begin{cases} \tau(\upsilon_1, \upsilon_2) = \upsilon_1 \upsilon_2 \\ \tau(\upsilon_1, \upsilon_2) = \upsilon_1 + \upsilon_2 - \upsilon_1 \upsilon_2 \end{cases}. \tag{13}$$

Within the Einstein operational framework, the associated t-norm (τ_E) and t-norm (σ_E) are mathematically characterized as follows:

$$\begin{cases} \tau_E(\upsilon_1, \upsilon_2) = \frac{\upsilon_1 \upsilon_2}{1+(1-\upsilon_1)(1-\upsilon_2)} \\ \sigma_E(\upsilon_1, \upsilon_2) = \frac{\upsilon_1 + \upsilon_2}{1+\upsilon_1 \upsilon_2} \end{cases}. \tag{14}$$

Under the Hamacher operational framework, the corresponding t-norm (τ_H) and s-norm (σ_H) are formulated through the following expressions:

$$\begin{cases} \tau_H(\upsilon_1, \upsilon_2) = \frac{\upsilon_1 \upsilon_2}{\lambda+(1-\lambda)(\upsilon_1+\upsilon_2-\upsilon_1 \upsilon_2)} \\ \sigma_H(\upsilon_1, \upsilon_2) = \frac{\upsilon_1+\upsilon_2-(2-\lambda)\upsilon_1 \upsilon_2}{1-(1-\lambda)\upsilon_1 \upsilon_2} \end{cases}. \tag{15}$$

The parameter λ provides additional flexibility by regulating the aggregation behavior of the operator.

Within the Frank operational family, the corresponding t-norm and s-norm are specified by the following mathematical expressions:

$$\begin{cases} \tau_F(\upsilon_1, \upsilon_2) = \log_\theta \left(1 + \left\{\frac{(\theta^{\upsilon_1}-1)(\theta^{\upsilon_2})}{\theta-1}\right\}\right) \\ \sigma_F(\upsilon_1, \upsilon_2) = 1 - \log_\theta \left(1 + \frac{(\theta^{1-\upsilon_1}-1)(\theta^{1-\upsilon_2})}{\theta-1}\right) \end{cases}. \tag{16}$$

In Eq (16), the parameter is a positive real number such that $\theta \neq 1$.

3. Similarity measures for QOFNs

The theoretical framework developed in Section 2, together with the corresponding distance formulations, serves as the basis for constructing a family of similarity measures that quantitatively characterize the proximity between QOFNs.

3.1. Cosine-based similarity measures in quasirung orthopair fuzzy environments

Definition 4. [9] For any two QOFSs $\tilde{\alpha} = \{a_j, \mu_{\tilde{\alpha}}(a_j), \vartheta_{\tilde{\alpha}}(a_j): a \in A\}$ and $\tilde{\beta} = \{a_j, \mu_{\tilde{\beta}}(a_j), \vartheta_{\tilde{\beta}}(a_j): a \in A\}$, where $\left(\mu_{\tilde{\alpha}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \leq 1$, $\left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q \leq 1$. The similarity measure for QOFSs $\tilde{\alpha}$ and $\tilde{\beta}$ is denoted by $\Xi_R(\tilde{\alpha}, \tilde{\beta})$ and defined as:

$$\Xi_R(\tilde{\alpha}, \tilde{\beta}) = \sum_{j=1}^n \left(\frac{\left(\mu_{\tilde{\alpha}}(a_j)\right)^p \left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q}{\sqrt{\left(\mu_{\tilde{\alpha}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q} \sqrt{\left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q}} \right), \tag{17}$$

where p and q are positive integers such that $p = q$ or $p \neq q$. When a weight vector ω_i is assigned to the evaluation criteria, the corresponding weighted cosine similarity is denoted by $\Xi_{RW}(\alpha, \beta)$ and can be formulated as follows:

$$\Xi_{RW}(\tilde{\alpha},\tilde{\beta}) = \sum_{j=1}^{n} \omega_j \left(\frac{\left(\mu_{\tilde{\alpha}}(a_j)\right)^p \left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q}{\sqrt{\left(\mu_{\tilde{\alpha}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q} \sqrt{\left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q}} \right). \tag{18}$$

3.2. Formulation of similarity measures under QOFSs

Since existing similarity measures for QOFSs as defined in Eqs (17) and (18) generally do not explicitly account for the degree of hesitancy (HD), they may not fully capture the uncertainty embedded in QOFSs. To overcome this limitation, in this subsection, we introduce a hesitancy-integrated cosine similarity measure denoted by $\Xi_H(\tilde{\alpha},\tilde{\beta})$, which jointly considers the MD, NMD, and HD. By incorporating these three components into a unified cosine-based formulation, hesitancy-integrated cosine similarity measure provides a more comprehensive representation of QOFS information and enables a finer assessment of similarity between uncertain objects.

Definition 5. [9] Let $\tilde{\alpha} = \{a_j, \mu_{\tilde{\alpha}}(a_j), \vartheta_{\tilde{\alpha}}(a_j) : a \in A\}$ and $\tilde{\beta} = \{a_j, \mu_{\tilde{\beta}}(a_j), \vartheta_{\tilde{\beta}}(a_j) : a \in A\}$ be any two QOFSs, where $\left(\mu_{\tilde{\alpha}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \leq 1$, $\left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q \leq 1$, $\pi_{\tilde{\alpha}}(a_j) = \sqrt[r]{1 - \left(\mu_{\tilde{\alpha}}(a_j)\right)^p - \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q}$ and $\pi_{\tilde{\beta}}(a_j) = \sqrt[r]{1 - \left(\mu_{\tilde{\beta}}(a_j)\right)^p - \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q}$ $(r = \max(p,q))$. The hesitancy-integrated cosine similarity measure $\Xi_H(\tilde{\alpha},\tilde{\beta})$ for QOFSs $\tilde{\alpha}$ and $\tilde{\beta}$ is defined as

$$\Xi_H(\tilde{\alpha},\tilde{\beta}) = \sum_{j=1}^{n} \left(\frac{\left(\mu_{\tilde{\alpha}}(a_j)\right)^p \left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q + \pi_{\tilde{\alpha}}(a_j)^r \pi_{\tilde{\beta}}(a_j)^r}{\sqrt{\left(\mu_{\tilde{\alpha}}(a_j)\right)^{2p} + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^{2q} + \left(\pi_{\tilde{\alpha}}(a_j)\right)^{2r}} \sqrt{\left(\mu_{\tilde{\beta}}(a_j)\right)^{2p} + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^{2q} + \left(\pi_{\tilde{\beta}}(a_j)\right)^{2r}}} \right), \tag{19}$$

where p and q are positive integers such that $p = q$ or $p \neq q$.

Theorem 3. The proposed $\Xi_H(\tilde{\alpha},\tilde{\beta})$ measure satisfies the following axioms:
(i) $0 \leq \Xi_H(\tilde{\alpha},\tilde{\beta}) \leq 1$.
(ii) $\Xi_H(\tilde{\alpha},\tilde{\beta}) = \Xi_H(\tilde{\beta},\tilde{\alpha})$.
(iii) $\Xi_H(\tilde{\alpha},\tilde{\beta}) = 1$ if and only if $\tilde{\alpha} = \tilde{\beta}$.
(iv) $\Xi_H(\tilde{\alpha},\tilde{\beta}) = 0$ when $\tilde{\alpha} = (0,1)$ and $\tilde{\beta} = (1,0)$ or $\tilde{\alpha} = (1,0)$ and $\tilde{\beta} = (0,1)$.

Proof. Let $\alpha = (\mu_\alpha, \vartheta_\alpha)$, $\beta = (\mu_\beta, \vartheta_\beta)$ be any two QOFNs, with $\pi_\alpha = \left(1 - \mu_\alpha^p - \vartheta_\alpha^q\right)^{1/r}$, $\pi_\beta = \left(1 - \mu_\beta^p - \vartheta_\beta^q\right)^{1/r}$. Assume $u = \left(\mu_\alpha^p, \vartheta_\alpha^q, \pi_\alpha^r\right)$, $v = \left(\mu_\beta^p, \vartheta_\beta^q, \pi_\beta^r\right)$, so that $\Xi_H(\alpha,\beta) = \frac{u \cdot v}{\|u\| \, \|v\|}$.

(i) Every component of u, v is non-negative, since $\mu, \vartheta, \pi \in [0,1]$ and $p, q, r \geq 1$, so $u \cdot v \geq 0$. By the Cauchy–Schwarz inequality, $u \cdot v \leq \| u \| \| v \|$. Hence, $0 \leq \Xi_H(\alpha,\beta) \leq 1$.
(ii) Both $u \cdot v$ and $\| u \| \| v \|$ are invariant under interchanging u and v, so $\Xi_H(\alpha,\beta) = \Xi_H(\beta,\alpha)$.
(iii) If $\alpha = \beta$, then $u = v$, so that $\Xi_H(\alpha,\beta) = \frac{u \cdot u}{u \cdot u} = 1$.

Conversely, suppose $\Xi_H(\alpha,\beta) = 1\Xi$. Equality in Cauchy–Schwarz holds if and only if $u = kv$

for some scalar $k \geq 0$ (assuming $v \neq 0$), then $\mu_\alpha^p = k\mu_\beta^p, \vartheta_\alpha^q = k\vartheta_\beta^q, \pi_\alpha^r = k\pi_\beta^r$. Since $\mu_\alpha^p + \vartheta_\alpha^q + \pi_\alpha^r = 1, \mu_\beta^p + \vartheta_\beta^q + \pi_\beta^r = 1$. This implies that $1 = k \cdot 1 \Longrightarrow k = 1$. Hence, $u = v$, giving $\mu_\alpha^p = \mu_\beta^p$ and $\vartheta_\alpha^q = \vartheta_\beta^q$. Since $p, q \geq 1$ and $\mu, \vartheta \in [0,1]$, taking p^{th} and q^{th} roots gives $\mu_\alpha = \mu_\beta$, $\vartheta_\alpha = \vartheta_\beta$, i.e., $\alpha = \beta$.

(iv) (Case 1) If $\alpha = (\mu_\alpha, \vartheta_\alpha) = (0,1)$ and $\beta = (\mu_\beta, \vartheta_\beta) = (1,0)$, then

$$\pi_\alpha = \sqrt[r]{1 - \mu_\alpha^p - \vartheta_\alpha^q} = \sqrt[r]{1 - 0^p - 1^q} = 0,\ \pi_\beta = \sqrt[r]{1 - \mu_\beta^p - \vartheta_\beta^q} = \sqrt[r]{1 - 1^p - 0^q} = 0,$$

$$\frac{\mu_\alpha^p \mu_\beta^p + \vartheta_\alpha^q \vartheta_\beta^q + \pi_\alpha^r \pi_\beta^r}{\sqrt{\mu_\alpha^{2p} + \vartheta_\alpha^{2q} + \pi_\alpha^{2r}}\sqrt{\mu_\beta^{2p} + \vartheta_\beta^{2q} + \pi_\beta^{2r}}} = \frac{0^p \times 1^p + 1^q \times 0^q + 0 \times 0}{\sqrt{0^{2p} + 1^{2q} + 0}\sqrt{1^{2p} + 0^{2q} + 0}} = \frac{0}{1} = 0.$$

Similarly, Case 2 can be verified.

Definition 6. Let $\tilde{\alpha} = \{a_j, \mu_{\tilde{\alpha}}(a_j), \vartheta_{\tilde{\alpha}}(a_j) : a \in A\}$ and $\tilde{\beta} = \{a_j, \mu_{\tilde{\beta}}(a_j), \vartheta_{\tilde{\beta}}(a_j) : a \in A\}$ be any two QOFSs, where

$$\left(\mu_{\tilde{\alpha}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \leq 1,\ \left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q \leq 1,$$
$$\pi_{\tilde{\alpha}}(a_j) = \sqrt[r]{1 - \left(\mu_{\tilde{\alpha}}(a_j)\right)^p - \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q},$$

and

$$\pi_{\tilde{\beta}}(a_j) = \sqrt[r]{1 - \left(\mu_{\tilde{\beta}}(a_j)\right)^p - \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q} (r = \max(p, q)).$$

Suppose ω_j ($j = 1,2, \dots, n$) is the corresponding weight of $a_j \in A$ with the conditions $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^n \omega_j = 1$. The weighted hesitancy-integrated cosine similarity measure $\Xi_{HW}(\tilde{\alpha}, \tilde{\beta})$ for QOFSs $\tilde{\alpha}$ and $\tilde{\beta}$ is defined as:

$$\Xi_{HW}(\tilde{\alpha}, \tilde{\beta}) = \sum_{j=1}^n \omega_j \left(\frac{\left(\mu_{\tilde{\alpha}}(a_j)\right)^p \left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q + \pi_{\tilde{\alpha}}(a_j)^r \pi_{\tilde{\beta}}(a_j)^r}{\sqrt{\left(\mu_{\tilde{\alpha}}(a_j)\right)^{2p} + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^{2q} + \left(\pi_{\tilde{\alpha}}(a_j)\right)^{2r}}\sqrt{\left(\mu_{\tilde{\beta}}(a_j)\right)^{3p} + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^{2q} + \left(\pi_{\tilde{\beta}}(a_j)\right)^{2r}}} \right), \tag{20}$$

where p and q are positive integers such that $p = q$ or $p \neq q$.

Theorem 4. The hesitancy-integrated cosine similarity measure $\Xi_{HW}(\tilde{\alpha}, \tilde{\beta})$ exhibits the following fundamental axioms:

(i) $0 \leq \Xi_{HW}(\tilde{\alpha}, \tilde{\beta}) \leq 1$.

(ii) $\Xi_{HW}(\tilde{\alpha}, \tilde{\beta}) = \Xi_{HW}(\tilde{\beta}, \tilde{\alpha})$.

(iii) $\Xi_{HW}(\tilde{\alpha}, \tilde{\beta}) = 1$ if and only if $\tilde{\alpha} = \tilde{\beta}$.

(iv) $\Xi_{HW}(\tilde{\alpha}, \tilde{\beta}) = 0$ when $\tilde{\alpha} = (0,1)$ and $\tilde{\beta} = (1,0)$ or $\tilde{\alpha} = (1,0)$ and $\tilde{\beta} = (0,1)$.

Proof. For each attribute a_j, define

$$s_j(\alpha, \beta) = \frac{\mu_\alpha^p \mu_\beta^p + \vartheta_\alpha^q \vartheta_\beta^q + \pi_\alpha^r \pi_\beta^r}{\sqrt{\mu_\alpha^{2p} + \vartheta_\alpha^{2q} + \pi_\alpha^{2r}}\sqrt{\mu_\beta^{2p} + \vartheta_\beta^{2q} + \pi_\beta^{2r}}},\ \Xi_{HW}(\alpha, \beta) = \sum_{j=1}^n \omega_j\, s_j(\alpha, \beta).$$

(i) By the identical Cauchy–Schwarz argument used for Ξ_H for every j, since $\omega_j \geq 1$ and $\sum_j \omega_j = 1$, Ξ_{HW} is a convex combination of terms in [0,1], so that $0 \leq \Xi_{HW}(\alpha,\beta) \leq 1$.

(ii) Each $s_j(\alpha,\beta) = S_j(\beta,\alpha)$ (proved as for Ξ_H so $\Xi_{HW}(\alpha,\beta) = \Xi_{HW}(\beta,\alpha)$.

(iii) If $\alpha = \beta$, then $S_j(\alpha,\beta) = 1$, for every j, so $\Xi_{HW}(\alpha,\beta) = \sum_j \omega_j.1 = 1$.

Conversely, suppose that $\Xi_{HW}(\alpha,\beta) = 1$. Since each $S_j \leq 1$ and $\sum_j \omega_j = 1$ with every $\omega_j \geq 0$, $\Xi_{HW}(\alpha,\beta) = \sum_{j=1}^{n} \omega_j s_j \leq \sum_j \omega_j.1 = 1$ with equality if and only if $s_j(\alpha,\beta) = 1$ for every j with $\omega_j > 0$, if any single $s_k < 1$ for an attribute with $\omega_k > 0$, the sum $\sum_{j=1}^{n} \omega_j s_j$ would be strictly less than 1. For each such s_j , the identity $\mu_\alpha^p(a_j) + \vartheta_\alpha^q(a_j) + \pi_\alpha^r(a_j) = 1$ (by the Cauchy–Schwarz equality-condition argument) used for Ξ_H, $k_j = 1$ exactly, giving $\mu_\alpha(a_j) = \mu_\beta(a_j)$ and $\vartheta_\alpha(a_j) = \vartheta_\beta(a_j)$. As this holds for every attribute $a_j \in A$, $\tilde{\alpha} = \tilde{\beta}$.

(iv) For each a_j with $\alpha(a_j) = (0,1)$, $\beta(a_j) = (1,0)$ (or vice versa), $\pi_\alpha(a_j) = \pi_\beta(a_j) = 0$ and $s_j(\alpha,\beta) = 0$ (as shown for Ξ_H), so $\Xi_{HW}(\alpha,\beta) = \sum_j \omega_j\,.0 = 0 \sum_j \omega_j \cdot 0$.

Definition 7. Let $\tilde{\alpha} = \{a_j, \mu_{\tilde{\alpha}}(a_j), \vartheta_{\tilde{\alpha}}(a_j) : a \in A\}$ and $\tilde{\beta} = \{a_j, \mu_{\tilde{\beta}}(a_j), \vartheta_{\tilde{\beta}}(a_j) : a \in A\}$ be any two QOFSs, where $\left(\mu_{\tilde{\alpha}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \leq 1$ and $\left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q \leq 1$. The cosine similarity with membership–non-membership interaction between $\tilde{\alpha}$ and $\tilde{\beta}$ is denoted by $\Xi_{SI}(\tilde{\alpha},\tilde{\beta})$ and defined as:

$$\Xi_{SI}(\tilde{\alpha},\tilde{\beta}) = \sum_{j=1}^{n} \left(\frac{\left(\mu_{\tilde{\alpha}}(a_j)\right)^p \left(\mu_{\tilde{\beta}}(a_j)\right)^p + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q + \eta X_1 X_2}{\sqrt{\left(\mu_{\tilde{\alpha}}(a_j)\right)^{2p} + \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^{2q} + \eta (X_1)^2}\sqrt{\left(\mu_{\tilde{\beta}}(a_j)\right)^{2p} + \left(\vartheta_{\tilde{\beta}}(a_j)\right)^{2q} + \eta (X_2)^2}} \right). \tag{21}$$

In Eq (21), $X_1 = \left(\mu_{\tilde{\alpha}}(a_j)\right)^p \left(\vartheta_{\tilde{\alpha}}(a_j)\right)^q$ and $X_2 = \left(\mu_{\tilde{\beta}}(a_j)\right)^p \left(\vartheta_{\tilde{\beta}}(a_j)\right)^q$.

Parameter η is introduced as a nonnegative interaction coefficient that controls the relative contribution of the joint membership–non–membership information to the similarity evaluation. Specifically, η determines the degree to which the interaction term influences the proposed cosine formulation. When $\eta = 0$, the interaction effect vanishes and the measure reduces to a conventional cosine-based construction. For $\eta \in (0,1)$, the interaction component contributes moderately, whereas $\eta = 1$ assigns a unit weight to the interaction effect. Values of $\eta > 1$ place greater emphasis on the joint interaction between membership and non-membership information. Hence, η provides an adjustable mechanism for controlling the sensitivity of the similarity measure to the coupled behavior of the two QOFS components.

Theorem 5. The cosine similarity with membership–non-membership interaction $\Xi_{SI}(\tilde{\alpha},\tilde{\beta})$ exhibits the following fundamental axioms.

(i) $0 \leq \Xi_{SI}(\tilde{\alpha},\tilde{\beta}) \leq 1$.

(ii) $\Xi_{SI}(\tilde{\alpha},\tilde{\beta}) = \Xi_{SI}(\tilde{\beta},\tilde{\alpha})$.

(iii) If $\Xi_{SI}(\tilde{\alpha},\tilde{\beta}) = 1$ if and only if $\tilde{\alpha} = \tilde{\beta}$ (where $\mu_{\tilde{\alpha}}$, $\mu_{\tilde{\beta}}$, $\beta_{\tilde{\alpha}}$, $\vartheta_{\tilde{\beta}}$ are all strictly positive).

(iv) $\Xi_{SI}(\tilde{\alpha},\tilde{\beta}) = 0$ when $\tilde{\alpha} = (0,1)$ and $\tilde{\beta} = (1,0)$ or $\tilde{\alpha} = (1,0)$ and $\tilde{\beta} = (0,1)$.

Proof. (i) Let $\alpha = (\mu_\alpha, \vartheta_\alpha)$, $\beta = (\mu_\beta, \vartheta_\beta)$ be any two QOFNs. Then from Eq (21), we get

$$\Xi_{SI}(\alpha,\beta)=\frac{(\mu_\alpha)^p(\mu_\beta)^p+(\vartheta_\alpha)^q(\vartheta_\beta)^q+\eta X_1X_2}{\sqrt{(\mu_\alpha)^{2p}+(\vartheta_\alpha)^{2q}+\eta(X_1)^2}\sqrt{(\mu_\beta)^{2p}+(\vartheta_\beta)^{2q}+\eta(X_2)^2}}.$$

Assuming $A_i=(\mu_\beta)^p$, $B_i=(\vartheta_\alpha)^q$, $C_i=(\mu_\beta)^p$ and $D_i=\left(\vartheta_\beta(a_j)\right)^q$. Then, we get

$$(\mu_\alpha)^p(\mu_\beta)^p+(\vartheta_\alpha)^q\left(\vartheta_\beta(a_j)\right)^q+\eta X_1X_2=A_iC_i+B_iD_i+\eta A_iB_iC_iD_i,$$

$$\sqrt{(\mu_\alpha)^{2p}+(\vartheta_\alpha)^{2q}+\eta(X_1)^2}=\sqrt{A_i^2+B_i^2-\eta A_i^2B_i^2},$$

and

$$\sqrt{(\mu_\beta)^{2p}+(\vartheta_\beta)^{2q}+\eta(X_2)^2}=\sqrt{C_i^2+D_i^2-\eta C_i^2D_i^2}.$$

Since $A_iC_i+B_iD_i+\eta A_iB_iC_iD_i\geq 0$ because $\eta>0$. Also, $A_i^2+B_i^2-\eta A_i^2B_i^2$ and $C_i^2+D_i^2-\eta C_i^2D_i^2\geq 0$ are non-negative. Therefore, $0\leq\Xi_{SI}(\alpha,\beta)\leq 1$.

(ii) Since $(\mu_\alpha)^p(\mu_\beta)^p=(\mu_\beta)^p(\mu_\alpha)^p$, $(\vartheta_\alpha)^q(\vartheta_\beta)^q=(\vartheta_\beta)^q(\vartheta_\alpha)^q$, $\eta X_1X_2=\eta X_2X_1$, and

$$\sqrt{(\mu_\alpha)^{2p}+(\vartheta_\alpha)^{2q}+\eta(X_1)^2}\sqrt{(\mu_\beta)^{2p}+(\vartheta_\beta)^{2q}+\eta(X_2)^2}$$

$$=\sqrt{(\mu_\beta)^{2p}+(\vartheta_\beta)^{2q}+\eta(X_1)^2}\sqrt{(\mu_\alpha)^{2p}+(\vartheta_\alpha)^{2q}+\eta(X_2)^2}$$

$$\Longrightarrow\frac{(\mu_\alpha)^p(\mu_\beta)^p+(\vartheta_\alpha)^q(\vartheta_\beta)^q+\eta X_1X_2}{\sqrt{(\mu_\alpha)^{2p}+(\vartheta_\alpha)^{2q}+\eta(X_1)^2}\sqrt{(\mu_\beta)^{2p}+(\vartheta_\beta)^{2q}+\eta(X_2)^2}}$$

$$=\frac{(\mu_\alpha)^p(\mu_\beta)^p+(\vartheta_\alpha)^q(\vartheta_\beta)^q+\eta X_1X_2}{\sqrt{(\mu_\alpha)^{2p}+(\vartheta_\alpha)^{2q}+\eta(X_1)^2}\sqrt{(\mu_\beta)^{2p}+(\vartheta_\beta)^{2q}+\eta(X_2)^2}}=\Xi_{SI}(\alpha,\beta).$$

(iii) If $\mu_\alpha,\vartheta_\alpha,\mu_\beta,\vartheta_\beta$ are all strictly positive, then $\Xi_{SI}(\alpha,\beta)=1 \iff \alpha=\beta$ $u=\left(\mu_\alpha^p,\vartheta_\alpha^q,X_1\right)$, $v=\left(\mu_\beta^p,\vartheta_\beta^q,X_2\right)$, where $X_1=\mu_\alpha^p\vartheta_\alpha^q$, $X_2=\mu_\beta^p\vartheta_\beta^q$, so that $\Xi_{SI}(\alpha,\beta)=\frac{u\cdot Qv}{\sqrt{u\cdot Qu}\sqrt{v\cdot Qv}}$, (where $Q=\operatorname{diag}(1,1,\eta)$). If $\alpha=\beta$, then $u=v$, then $\Xi_{SI}(\alpha,\beta)=\frac{u\cdot Qu}{u\cdot Qu}=1$.

Conversely, suppose $\Xi_{SI}(\alpha,\beta)=1$. Since Q is positive definite (as $\eta>0$), equality in the Cauchy–Schwarz inequality $u\cdot Qv\leq\sqrt{u\cdot Qu}\sqrt{v\cdot Qv}$ holds if and only if u and v are linearly dependent, i.e., $u=kv$ for some $k\geq 0$ ($v\neq 0$), implies that $\mu_\alpha^p=k\mu_\beta^p,\vartheta_\alpha^q=k\vartheta_\beta^q,X_1=kX_2$. By definition, $X_1=\mu_\alpha^p\vartheta_\alpha^q$, $X_2=\mu_\beta^p\vartheta_\beta^q$, then $X_1=\mu_\alpha^p\vartheta_\alpha^q=\left(k\mu_\beta^p\right)\left(k\vartheta_\beta^q\right)=k^2X_2$, so that $k^2X_2=kX_2\Longrightarrow k(k-1)X_2=0$. Since $\mu_\beta,\vartheta_\beta$ are strictly positive, $X_2=\mu_\beta^p\vartheta_\beta^q>0$. Thus, $k=0$ or $k=1$. If $k=0$, then $u=0$, i.e., $\mu_\alpha=\vartheta_\alpha=0$, which contradicts the assumption that $\mu_\alpha,\vartheta_\alpha$ are strictly positive (and this case, it would leave $\Xi_{SI}(\alpha,\beta)$ undefined rather than equal to 1, since the α-side denominator vanishes). Hence, $k=1\Longrightarrow u=v\Longrightarrow\mu_\alpha^p=\mu_\beta^p,\vartheta_\alpha^q=\vartheta_\beta^q$. Since $p,q\geq 1$ and $\mu,\vartheta\in[0,1]$, taking p^{th} and q^{th} roots gives $\mu_\alpha=\mu_\beta,\vartheta_\alpha=\vartheta_\beta$, i.e., $\alpha=\beta$.

(iv) Let $\alpha = (0,1)$ and $\beta = (1,0)$, then $X_1 = (\mu_\alpha)^p(\vartheta_\alpha)^q = 0$ and $X_2 = (\mu_\beta)^p(\vartheta_\beta)^q = 0$.

$$\Rightarrow \frac{(\mu_\alpha)^p(\mu_\beta)^p + (\vartheta_\alpha)^q(\vartheta_\beta)^q + \eta X_1 X_2}{\sqrt{(\mu_\alpha)^{2p} + (\vartheta_\alpha)^{2q} + \eta(X_1)^2}\sqrt{(\mu_\beta)^{2p} + (\vartheta_\beta)^{2q} + \eta(X_2)^2}} = \frac{0\times1+1\times0+\eta\times0}{\sqrt{0+1+\eta\times0}\sqrt{1+0+\eta\times0}} = 0.$$

Similarly, the result can be proved for $\alpha = (1,0)$ and $\beta = (0,1)$.

The similarity measures introduced in the preceding section provide effective mechanisms for characterizing uncertainty and evaluating the degree of closeness between QOFNs. Nevertheless, these formulations may encounter computational difficulties in certain limiting cases, particularly when zero denominators arise. To overcome this limitation, the subsequent subsection develops an alternative family of similarity measures designed to ensure well-defined similarity values under such circumstances. Before presenting these measures formally, several auxiliary distance measures are first established. These distance measures serve as the mathematical foundation for the subsequent similarity constructions and facilitate the development of robust formulations that remain well-defined in the presence of zero-valued components.

3.3. *QOFN-based distance measures and their properties*

The following subsection presents a novel distance measure for assessing the degree of dissimilarity between two QOFNs. Assume $\alpha = (\mu_\alpha, \vartheta_\beta)$ and $\beta = (\mu_\beta, \vartheta_\beta)$ be any two QOFNs. The proposed distance is constructed by considering both the endpoint and midpoint information associated with the corresponding intervals $[\mu_\alpha^p, 1-\vartheta_\alpha^q]$ and $[\mu_\beta^p, 1-\vartheta_\beta^q]$, thereby capturing their overall positional differences. For $\xi = 1,2,3$, suppose $d_D(\alpha) = \left(1-\frac{\xi}{2}\right)\mu_\alpha^p + \frac{\xi}{2}\left(1-\vartheta_\alpha^q\right)$ and $d_D(\beta) = \left(1-\frac{\xi}{2}\right)\mu_\beta^p + \frac{\xi}{2}\left(1-\vartheta_\beta^q\right)$. For each interval, the indices $\xi = 0$ and $\xi = 2$ identify the lower and upper bounds, respectively, while $\xi = 1$ specifies the central point. These three characteristic points are subsequently employed to construct the distance between α and β as follows:

$$d_D(\alpha,\beta) = \sqrt{\frac{1}{3}\sum_{\xi=1}^{2}\left(D_\xi(\alpha) - D_\xi(\beta)\right)^2}. \tag{22}$$

Following algebraic reduction, the preceding formulation takes the equivalent form:

$$d_D(\alpha,\beta) = \sqrt{\frac{1}{3}\left[\left(\mu_\alpha^p - \mu_\beta^p\right)^2 + \frac{1}{4}\left\{\left(\mu_\alpha^p - \mu_\beta^p\right) - \left(\vartheta_\alpha^q - \vartheta_\beta^q\right)\right\}^2 + \left(\vartheta_\alpha^q - \vartheta_\beta^q\right)^2\right]}. \tag{23}$$

Theorem 6. Let $\alpha = (\mu_\alpha, \vartheta_\alpha)$, $\beta = (\mu_\beta, \vartheta_\beta)$, and $\gamma = (\mu_\gamma, \vartheta_\gamma)$ be any three QOFNs. Then the proposed distance measure satisfies the following fundamental metric conditions:

(i) $d_D(\alpha,\beta) \geq 0$.
(ii) $d_D(\alpha,\beta) = d_D(\beta,\alpha)$.
(iii) $d_D(\alpha,\beta) = 0 \Leftrightarrow \alpha = \beta$.
(iv) $d_D(\alpha,\gamma) \leq d_D(\alpha,\beta) + d_D(\beta,\gamma)$.

Proof. (i) Given that $d_D(\alpha,\beta) = \sqrt{\frac{1}{3}\sum_{\xi=1}^{2}\left(D_\xi(\alpha) - D_\xi(\beta)\right)^2}$. Since the square of every real-valued quantity is necessarily non-negative, it follows directly that $(\mathcal{D}_k(\mathcal{O}_1) - \mathcal{D}_k(\mathcal{O}_2))^2 \geq 0$ and hence, $\sum_{k=0}^{2}(\mathcal{D}_k(\mathcal{O}_1)-\mathcal{D}_k(\mathcal{O}_2))^2 \geq 0$. Therefore, $d_D(\alpha,\beta) \geq 0$.

(ii) As $d_D(\alpha,\beta)=\sqrt{\frac{1}{3}\sum_{\xi=1}^{2}\left(D_\xi(\alpha)-D_\xi(\beta)\right)^2}=\sqrt{\frac{1}{3}\sum_{\xi=1}^{2}\left(D_\xi(\beta)-D_\xi(\alpha)\right)^2}=d_D(\beta,\alpha)$. As a result, $d_D(\alpha,\beta)=d_D(\beta,\alpha)$.

(iii) Assuming $d_D(\alpha,\beta)=0$, this implies that $\sum_{\xi=1}^{2}\left(D_\xi(\alpha)-D_\xi(\beta)\right)^2=0$. This condition holds if and only if $D_\xi(\alpha)=D_\xi(\beta)$ for all ξ. Based on the established formulations of $D_\xi(\alpha)$ and $D_\xi(\beta)$, we consequently derive $\mu_\alpha=\mu_\beta$ and $\vartheta_\alpha=\vartheta_\beta$. Thus, by Definition 2, we get $\alpha=\beta$.

Conversely, if $\alpha=\beta$, then $d_D(\alpha,\alpha)=\sqrt{\frac{1}{3}\sum_{\xi=1}^{2}\left(D_\xi(\alpha)-D_\xi(\alpha)\right)^2}=0$.

(iv) Let $\alpha_1=(\mu_1,\vartheta_1)$, $\alpha_2=(\mu_2,\vartheta_2)$, and $\alpha_3=(\mu_3,\vartheta_3)$ be any three QOFNs. For $i=1,2,3$, denote $x_i=\mu_i^p$ and $y_i=\vartheta_i^q$. From Eq (23), for any pair α_i,α_j, we have

$$d_D(\alpha_i,\alpha_j)=\sqrt{\frac{1}{3}\left[\left(x_i-x_j\right)^2+\frac{1}{4}((x_i-x_j)-(y_i-y_j))^2+\left(y_i-y_j\right)^2\right]}.$$

Let $\Delta x=x_i-x_j$ and $\Delta y=y_i-y_j$, we get

$$(\Delta x)^2+\frac{1}{4}(\Delta x-\Delta y)^2+(\Delta y)^2=\frac{5}{4}\Delta x^2-\frac{1}{2}\Delta x\Delta y+\frac{5}{4}\Delta y^2,$$

so that

$$d_D\left(\alpha_i,\alpha_j\right)=\frac{5\Delta x^2-2\Delta x\Delta y+5\Delta y^2}{12}.$$

To prove the triangle inequality, we construct an explicit mapping from the QOFN space to the Euclidean plane, denoted by φ:QOFN $\to \mathbb{R}^2$, defined as follows:

$$\varphi(\alpha)=\left(\frac{\mu^p+\vartheta^q}{\sqrt{6}},\frac{\mu^p-\vartheta^q}{2}\right).$$

For the pair α_i,α_j, the coordinates of φ satisfy $\varphi_1(\alpha_i)-\varphi_1\left(\alpha_j\right)=\frac{\Delta x+\Delta y}{\sqrt{6}}$, $\varphi_2(\alpha_i)-\varphi_2\left(\alpha_j\right)=\frac{\Delta x-\Delta y}{2}$.

Consequently,

$$\|\varphi(\alpha_i)-\varphi\left(\alpha_j\right)\|_2^2=\frac{(\Delta x+\Delta y)^2}{6}+\frac{(\Delta x-\Delta y)^2}{4}=\frac{2(\Delta x+\Delta y)^2+3(\Delta x-\Delta y)^2}{12}=\frac{5\Delta x^2-2\Delta x\Delta y+5\Delta y^2}{12}.$$

Hence, for any two QOFNs, $d_D\left(\alpha_i,\alpha_j\right)=\|\varphi(\alpha_i)-\varphi\left(\alpha_j\right)\|_2$. Thus, d_D corresponds to the standard Euclidean distance under φ. Since $\varphi(\alpha_1),\varphi(\alpha_2),\varphi(\alpha_3)$ are points in $\mathbb{R}^2$ with the Euclidean norm $\|\cdot\|_2$, the Minkowski inequality gives

$$\|\varphi(\alpha_1)-\varphi(\alpha_3)\|_2=\|\left(\varphi(\alpha_1)-\varphi(\alpha_2)\right)+\left(\varphi(\alpha_2)-\varphi(\alpha_3)\right)\|_2.$$

$$\leq\|\varphi(\alpha_1)-\varphi(\alpha_2)\|_2+\|\varphi(\alpha_2)-\varphi(\alpha_3)\|_2.$$

Applying identity $d_D\left(\alpha_i,\alpha_j\right)=\|\varphi(\alpha_i)-\varphi\left(\alpha_j\right)\|_2$ to each of the three terms gives $d_D(\alpha_1,\alpha_3)\leq d_D(\alpha_1,\alpha_2)+d_D(\alpha_2,\alpha_3)$, which establishes the triangle inequality.

Theorem 7. Consider three arbitrary QOFNs $\alpha=(\mu_\alpha,\vartheta_\alpha)$, $\beta=\left(\mu_\beta,\vartheta_\beta\right)$, and $\gamma=(\mu_\gamma,\vartheta_\gamma)$. Using the transformation functions specified in Eqs (1) to (6) and (7) to (12), along with the distance measure presented in Eq (23), we introduce the following similarity measures:

$$\Xi_\tau(\alpha,\beta)=\frac{1}{2}\left(1-d_D(\alpha,\beta)+\tau\left(\psi_i^\mu(\alpha,\beta),\psi_i^\vartheta(\alpha,\beta)\right)\right),$$

$$\Xi_\sigma(\alpha,\beta)=\frac{1}{2}\left(1-d_D(\alpha,\beta)+\sigma\left(\psi_i^\mu(\alpha,\beta),\psi_i^\vartheta(\alpha,\beta)\right)\right).$$

The mathematical behavior of the proposed similarity measures can be characterized by the following conditions:

(i) $0 \le \Xi_\star(\alpha,\beta) \le 1$ (where the symbol $\star$ serves as a common notation for the operations τ and σ).
(ii) $\Xi_\star(\alpha,\beta)=\Xi_\star(\beta,\alpha)$.
(iii) $\Xi_\star(\alpha,\beta)=1 \iff \alpha=\beta$.
(iv) If $\alpha \le \beta \le \gamma$, then $\Xi_\star(\alpha,\gamma) \le \Xi_\star(\alpha,\beta)$ and $\Xi_\star(\alpha,\gamma) \le \Xi_\star(\beta,\gamma)$.
(v) Let $\alpha=(0,1)$ and $\beta=\alpha^C=(1,0)$, then $\Xi_\star(\alpha,\beta)=0$ for $i=1,2,4,6$.

Proof. (i) By construction, the t-norm $\Gamma\colon[0,1]^2 \longrightarrow [0,1]$ and the s-norm $\Omega\colon[0,1]^2 \longrightarrow [0,1]$ map every input pair to a value within $[0,1]$; this is a defining requirement of both operators. In addition, the distance measure satisfies $0 \le d_D(\alpha,\beta) \le 1$, and consequently $0 \le 1-d_D(\alpha,\beta) \le 1$. As every term entering the similarity expression therefore lies in $[0,1]$, their arithmetic mean cannot fall outside this interval either. It follows that $0 \le \Xi_\star(\alpha,\beta) \le 1$.

(ii) The mappings $\psi_i^\mu(\alpha,\beta)$, $\psi_i^\vartheta(\alpha,\beta)$ and distance measure $d_D(\alpha,\beta)$ are symmetric, i.e., $\psi_i^\mu(\alpha,\beta)=\psi_i^\mu(\beta,\alpha)$, $\psi_i^\vartheta(\alpha,\beta)=\psi_i^\vartheta(\beta,\alpha)$ and $d_D(\alpha,\beta)=d_D(\beta,\alpha)$. This implies that

$$\Xi_\star(\alpha,\beta)=\frac{1}{2}\left(1-d_D(\alpha,\beta)+\star\left(\psi_i^\mu(\alpha,\beta),\psi_i^\vartheta(\alpha,\beta)\right)\right)$$

$$=\frac{1}{2}\left(1-d_D(\beta,\alpha)+\star\left(\psi_i^\mu(\beta,\alpha),\psi_i^\vartheta(\beta,\alpha)\right)\right)=\Xi_\star(\beta,\alpha).$$

(iii) Let $\Xi_\star(\alpha,\beta)=1$, then $\frac{1}{2}\left(1-d_D(\alpha,\beta)+\star\left(\psi_i^\mu(\alpha,\beta),\psi_i^\vartheta(\alpha,\beta)\right)\right)=1$. This equality holds for $1-d_D(\alpha,\beta)=1$ and $\star\left(\psi_i^\mu(\alpha,\beta),\psi_i^\vartheta(\alpha,\beta)\right)=1$. If $1-d_D(\alpha,\beta)=1 \Rightarrow d_D(\alpha,\beta)=0$. By Theorem 6, $d_D(\alpha,\beta)=0$ if and only if $\alpha=\beta$.

Conversely, if $\alpha=\beta$, then $\Xi_\star(\alpha,\alpha)=\frac{1}{2}\left(1-d_D(\alpha,\alpha)+\star\left(\psi_i^\mu(\alpha,\alpha),\psi_i^\vartheta(\alpha,\alpha)\right)\right)$. Since $d_D(\alpha,\alpha)=0$, $\psi_i^\mu(\alpha,\alpha)=1$, $\psi_i^\vartheta(\alpha,\alpha)=1$. Therefore, $\Xi_\star(\alpha,\alpha)=\frac{1}{2}\left(1-0+\star(1,1)\right)=1$.

(iv) Let $\alpha \le \beta \le \gamma$, then by Definition 2, $0 \le \mu_\alpha \le \mu_\beta \le \mu_\gamma \le 1$ and $1 \ge \vartheta_\alpha \ge \vartheta_\beta \ge \vartheta_\gamma \ge 0$. Therefore, $\left|\mu_\alpha^p-\mu_\gamma^p\right| \ge \left|\mu_\alpha^p-\mu_\beta^p\right|$ and $\left|\vartheta_\alpha^q-\vartheta_\gamma^q\right| \ge \left|\vartheta_\alpha^q-\vartheta_\beta^q\right|$. Given that the two mappings ψ_i^μ and ψ_i^ϑ exhibit decreasing monotonic behavior, thus $\psi_i^\mu(\alpha,\gamma) \le \psi_i^\mu(\alpha,\beta)$, $\psi_i^\vartheta(\alpha,\gamma) \le \psi_i^\vartheta(\alpha,\beta)$. The monotone behavior of the underlying t-norm and s-norm operators leads directly to $\star\left(\psi_i^\mu(\alpha,\gamma),\psi_i^\vartheta(\alpha,\gamma)\right) \le \star\left(\psi_i^\mu(\alpha,\beta),\psi_i^\vartheta(\alpha,\beta)\right)$. Moreover, $d_D(\alpha,\gamma) \ge d_D(\alpha,\beta)$ and $1-d_D(\alpha,\beta) \ge 1-d_D(\alpha,\gamma)$. The relations obtained above can be combined to yield $\Xi_\star(\alpha,\gamma) \le \Xi_\star(\alpha,\beta)$ and $\Xi_\star(\alpha,\gamma) \le \Xi_\star(\beta,\gamma)$. This result verifies that the monotonicity condition is fulfilled.

(v) Let $\alpha(\mu_\alpha,\vartheta_\alpha)$ be a crisp QOFN, i.e., $\alpha=(1,0)$ or $\alpha=(0,1)$. Then, consider its complement α^C. According to the complement operation defined in Definition 2, $\alpha^C=(\vartheta_\alpha,\mu_\alpha)$. For the t-norm-based similarity, $\Xi_\tau(\alpha,\alpha^C)=\frac{1}{2}[1-d_D(\alpha,\alpha^C)+\tau(\psi_i^\mu(\alpha,\alpha^C),\psi_i^\vartheta(\alpha,\alpha^C))]$. Since $d_D(\alpha,\alpha^C)=\sqrt{\frac{1}{3}\left[1+\frac{1}{4}(1-(-1))^2+1\right]}=1$, $\psi_1^\mu(\alpha,\alpha^C)=1-|1-0|=0$, $\psi_2^\mu(\alpha,\alpha^C)=1-(|1-0|)^2=0$, $\psi_4^\mu(\alpha,\alpha^C)=\frac{1-|1-0|}{1+|1-0|}=0$, $\psi_6^\mu(\alpha,\alpha^C)=1-|1-0|\,e^{|1-0|-1}=0$. Similarly, $\psi_i^\vartheta(\alpha,\alpha^C)=0$ for

$i = 1,2,4,6$. This implies that $\Xi_\tau(\alpha, \alpha^C) = \frac{1}{2}[1 - 0 + \tau(0,0] = 0$ and $\Delta_\tau(\alpha, \alpha^C) = 0$. Hence, the result follows.

3.4. Operator-induced similarity measures

This section introduces a new class of similarity measures specifically formulated to overcome the zero-denominator issue.

3.4.1. Similarity measures constructed from algebraic norms

By incorporating the algebraic t-norm (τ_1) and s-norm (σ_1) into the proposed framework, the corresponding similarity measures are defined as follows:

$$\Xi_{1\tau}(\alpha, \beta) = \frac{1}{2}\big(1 - d_D(\alpha, \beta) + \tau_1(\psi_i^\mu(\alpha, \beta), \psi_i^\vartheta(\alpha, \beta))\big).$$

Since

$$\tau_1\left(\psi_i^\mu(\alpha, \beta), \psi_i^\vartheta(\alpha, \beta)\right) = \psi_i^\mu(\alpha, \beta)\psi_i^\vartheta(\alpha, \beta),$$

and

$$\sigma_1\left(\psi_i^\mu(\alpha, \beta), \psi_i^\vartheta(\alpha, \beta)\right) = \psi_i^\mu(\alpha, \beta) + \psi_i^\vartheta(\alpha, \beta) - \psi_i^\mu(\alpha, \beta)\psi_i^\vartheta(\alpha, \beta),$$

$$\Xi_{1\tau}(\alpha, \beta) = \frac{1}{2}\Big(1 - d_D(\alpha, \beta) + \psi_i^\mu(\alpha, \beta)\psi_i^\vartheta(\alpha, \beta)\Big), \tag{24}$$

$$\Xi_{1\sigma}(\alpha, \beta) = \frac{1}{2}\Big(1 - d_D(\alpha, \beta) + \psi_i^\mu(\alpha, \beta) + \psi_i^\vartheta(\alpha, \beta) - \psi_i^\mu(\alpha, \beta)\psi_i^\vartheta(\alpha, \beta)\Big). \tag{25}$$

3.4.2. Similarity measures constructed from Einstein norms

The similarity formulation associated with the Einstein t-norm (τ_2) is expressed by

$$\Xi_{2\tau}(\alpha, \beta) = \frac{1}{2}\big(1 - d_D(\alpha, \beta) + \tau_2(\psi_i^\mu(\alpha, \beta), \psi_i^\vartheta(\alpha, \beta))\big).$$

By Eq (14), we have

$$\tau_2\left(\psi_i^\mu(\alpha, \beta), \psi_i^\vartheta(\alpha, \beta)\right) = \frac{\psi_i^\mu(\alpha, \beta)\psi_i^\vartheta(\alpha, \beta)}{1 + \Big(1 - \psi_i^\mu(\alpha, \beta)\Big)\Big(1 -, \psi_i^\vartheta(\alpha, \beta)\Big)}$$

and

$$\sigma_2\left(\psi_i^\mu(\alpha, \beta), \psi_i^\vartheta(\alpha, \beta)\right) = \frac{\psi_i^\mu(\alpha, \beta) + \psi_i^\vartheta(\alpha, \beta)}{1 + \psi_i^\mu(\alpha, \beta)\psi_i^\vartheta(\alpha, \beta)}.$$

Then

$$\Xi_{2\tau}(\alpha, \beta) = \frac{1}{2}\big(1 - d_D(\alpha, \beta) + \tau_2(\psi_i^\mu(\alpha, \beta), \psi_i^\vartheta(\alpha, \beta))\big),$$

$$\Xi_{2\tau}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \frac{\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)}{1+\left(1-\psi_i^\mu(\alpha,\beta)\right)\left(1-,\psi_i^\vartheta(\alpha,\beta)\right)}\right), \tag{26}$$

and

$$\Xi_{2\sigma}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \sigma_2(\psi_i^\mu(\alpha,\beta), \psi_i^\vartheta(\alpha,\beta))\right),$$

$$\Xi_{2\sigma}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \frac{\psi_i^\mu(\alpha,\beta)+\psi_i^\vartheta(\alpha,\beta)}{1+\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)}\right). \tag{27}$$

3.4.3. Similarity measures constructed from Hamacher norms

By incorporating the Hamacher t-norm and s-norm into the proposed framework, the corresponding similarity measure between two QOFNs α and β is defined as follows:

$$\Xi_{3\tau}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \tau_3(\psi_i^\mu(\alpha,\beta), \psi_i^\vartheta(\alpha,\beta))\right),$$

and

$$\Xi_{3\sigma}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \sigma_3(\psi_i^\mu(\alpha,\beta), \psi_i^\vartheta(\alpha,\beta))\right).$$

By Eq (15), we get

$$\tau_3\left(\psi_i^\mu(\alpha,\beta), \psi_i^\vartheta(\alpha,\beta)\right) = \frac{\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)}{\lambda+(1-\lambda)\left(\psi_i^\mu(\alpha,\beta)+\psi_i^\vartheta(\alpha,\beta)-\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)\right)},$$

and

$$\sigma_3\left(\psi_i^\mu(\alpha,\beta), \psi_i^\vartheta(\alpha,\beta)\right) = \frac{\psi_i^\mu(\alpha,\beta)+\psi_i^\vartheta(\alpha,\beta)-(2-\lambda)\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)}{1-(1-\lambda)\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)}.$$

Therefore,

$$\Xi_{3\tau}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \frac{\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)}{\lambda+(1-\lambda)\left(\psi_i^\mu(\alpha,\beta)+\psi_i^\vartheta(\alpha,\beta)-\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)\right)}\right), \tag{28}$$

$$\Xi_{3\tau}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \frac{\psi_i^\mu(\alpha,\beta)+\psi_i^\vartheta(\alpha,\beta)-(2-\lambda)\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)}{1-(1-\lambda)\psi_i^\mu(\alpha,\beta)\psi_i^\vartheta(\alpha,\beta)}\right). \tag{29}$$

3.4.4. Similarity measures constructed from Frank norms

The Frank t-norm and s-norm provide an alternative operational framework for constructing the following family of similarity measures:

$$\Xi_{4\tau}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \tau_4(\psi_i^\mu(\alpha,\beta), \psi_i^\vartheta(\alpha,\beta))\right),$$

and

$$\Xi_{4\sigma}(\alpha,\beta) = \frac{1}{2}\left(1 - d_D(\alpha,\beta) + \sigma_4(\psi_i^\mu(\alpha,\beta), \psi_i^\vartheta(\alpha,\beta))\right).$$

By Eq (16), we have

$$\tau_4\left(\psi_i^{\mu}(\alpha,\beta),\psi_i^{\vartheta}(\alpha,\beta)\right)=\log_\theta\left(1+\frac{\left(\theta^{\psi_i^{\mu}(\alpha,\beta)}-1\right)\left(\theta^{\psi_i^{\vartheta}(\alpha,\beta)}-1\right)}{\theta-1}\right),$$

and

$$\sigma_4\left(\psi_i^{\mu}(\alpha,\beta),\psi_i^{\vartheta}(\alpha,\beta)\right)=1-\log_\theta\left(1+\frac{\left(\theta^{1-\psi_i^{\mu}(\alpha,\beta)}-1\right)\left(\theta^{1-\psi_i^{\vartheta}(\alpha,\beta)}-1\right)}{\theta-1}\right).$$

Therefore,

$$\Xi_{4\tau}(\alpha,\beta)=\frac{1}{2}\left(1-d_D(\alpha,\beta)+\log_\theta\left(1+\frac{\left(\theta^{\psi_i^{\mu}(\alpha,\beta)}-1\right)\left(\theta^{\psi_i^{\vartheta}(\alpha,\beta)}-1\right)}{\theta-1}\right)\right), \tag{30}$$

$$\Xi_{4\sigma}(\alpha,\beta)=\frac{1}{2}\left(1-d_D(\alpha,\beta)+1-\log_\theta\left(1+\frac{\left(\theta^{1-\psi_i^{\mu}(\alpha,\beta)}-1\right)\left(\theta^{1-\psi_i^{\vartheta}(\alpha,\beta)}-1\right)}{\theta-1}\right)\right). \tag{31}$$

4. Microscopic–macroscopic evaluation of similarity measures for QOFNs

This section examines the proposed QOFN similarity measures through a set of illustrative examples. The analysis considers both their numerical behavior and structural characteristics from microscopic and macroscopic perspectives. The examples are constructed to provide a direct comparison among the considered similarity measures. The selected existing measures are first introduced, followed by their comparative evaluation using the constructed examples.

- Similarity measure from [18], denoted by Ξ_{LC}, is defined as

$$\Xi_{LC}(\alpha,\beta)=1-\sqrt{\frac{\sum_{j=1}^{n}\left|w_1(a_j)-w_2(a_j)\right|}{n}}, \tag{32}$$

where $w_1(a_j)=\frac{\mu_\alpha(a_j)+1-\vartheta_\alpha(a_j)}{2}$ and $w_2(a_j)=\frac{\mu_\beta(a_j)+1-\vartheta_\beta(a_j)}{2}$.

- Similarity measure from [39], denoted by Ξ_L, is defined by

$$\Xi_L(\alpha,\beta)=\sum_{j=1}^{n}\frac{\left(\Delta_\mu^2(a_j)+\Delta_\vartheta^2(a_j)\right)}{2n}. \tag{33}$$

In Eq (29), $\Delta_\mu(a_j)=\left|\mu_\alpha(a_j)-\mu_\beta(a_j)\right|$ and $\Delta_\vartheta(a_j)=\left|\vartheta_\alpha(a_j)-\vartheta_\beta(a_j)\right|$.

- Similarity measure introduced by Chen [40], denoted by Ξ_C, is defined as

$$\Xi_C(\alpha,\beta)=1-\frac{\sum_{j=1}^{n}\left|\Delta_{C_\alpha}(a_j)-\Delta_{C_\beta}(a_j)\right|}{2n}. \tag{34}$$

In this similarly measure, Δ_{C_α}, Δ_{C_β} are the score functions of α and β. The score values can be determined as: $\Delta_{C_\alpha}(a_j)=\mu_\alpha^2(a_j)-\vartheta_\alpha^2(a_j)$ and $\Delta_{C_\alpha}(a_j)=\mu_\beta^2(a_j)-\vartheta_\beta^2(a_j)$.

- Similarity measure introduced by Ye [41], denoted by Ξ_Y, is defined as

$$\Xi_Y(\alpha,\beta)=\frac{\mu_\alpha(a_j)\mu_\beta(a_j)+\vartheta_\alpha(a_j)\vartheta_\beta(a_j)}{\sqrt{\mu_\alpha^2(a_j)+\vartheta_\alpha^2(a_j)}\sqrt{\mu_\beta^2(a_j)+\vartheta_\beta^2(a_j)}}. \tag{35}$$

- Similarity measure formulated by Wei and Wei [42], denoted by Ξ_{WW}, is defined as

$$\Xi_{WW}(\alpha,\beta)=\frac{\mu_\alpha^2(a_j)\mu_\beta^2(a_j)+\vartheta_\alpha^2(a_j)\vartheta_\beta^2(a_j)}{\sqrt{\mu_\alpha^4(a_j)+\vartheta_\alpha^4(a_j)}\sqrt{\mu_\beta^4(a_j)+\vartheta_\beta^4(a_j)}}. \tag{36}$$

- Similarity measure introduced by Hong and Kim [43], denoted by Ξ_{HK}, is defined as

$$\Xi_{HK}(\alpha,\beta)=1-\frac{\sum_{j=1}^n(|\mu_\alpha(a_j)-\mu_\beta(a_j)|+|\vartheta_\alpha(a_j)-\vartheta_\beta(a_j)|)}{2n}. \tag{37}$$

- Similarity measure introduced by Mitchell [44], denoted by Ξ_M, is defined as

$$\Xi_M(\alpha,\beta)=\frac{1}{2}\left[\zeta_\alpha(\alpha,\beta),\zeta_\beta(\alpha,\beta)\right]. \tag{38}$$

In Eq (30),

$$\zeta_\alpha(\alpha,\beta)=1-\left(\frac{\sum_{j=1}^n\left|\mu_\alpha(a_j)-\mu_\beta(a_j)\right|^t}{n}\right)^{1/t}$$

and

$$\zeta_\beta(\alpha,\beta)=1-\left(\frac{\sum_{j=1}^n\left|\vartheta_\alpha(a_j)-\vartheta_\beta(a_j)\right|^t}{n}\right)^{1/t}$$

(where t is the order of distance metric such that $1\le t<\infty$).

- Similarity measures formulated by Peng et al. [45], denoted by Ξ_{P1}, Ξ_{P2}, and Ξ_{P3}, are defined as follows:

$$\Xi_{P1}(\alpha,\beta)=1-\frac{1}{2n}\sum_{j=1}^n\left(\left|\mu_\alpha^2(a_j)-\mu_\beta^2(a_j)\right|+\left|\vartheta_\alpha^2(a_j)-\vartheta_\beta^2(a_j)\right|\right), \tag{39}$$

$$\Xi_{P2}(\alpha,\beta)=\frac{1}{n}\sum_{j=1}^n\frac{\min\left\{\mu_\alpha^2(a_j),\mu_\beta^2(a_j)\right\}+\min\left\{\vartheta_\alpha^2(a_j),\vartheta_\beta^2(a_j)\right\}}{\max\left\{\mu_\alpha^2(a_j),\mu_\beta^2(a_j)\right\}+\max\left\{\vartheta_\alpha^2(a_j),\vartheta_\beta^2(a_j)\right\}}, \tag{40}$$

$$\Xi_{P3}(\alpha,\beta)=\frac{1}{n}\sum_{j=1}^n\frac{\min\left\{\mu_\alpha^2(a_j),\mu_\beta^2(a_j)\right\}+\min\left\{1-\vartheta_\alpha^2(a_j),1-\vartheta_\beta^2(a_j)\right\}}{\max\left\{\mu_\alpha^2(a_j),\mu_\beta^2(a_j)\right\}+\max\left\{1-\vartheta_\alpha^2(a_j),1-\vartheta_\beta^2(a_j)\right\}}. \tag{41}$$

- Peng and Garg's similarity measure [46], denoted by Ξ_{PG}, is defined as

$$\Xi_{PG}(\alpha,\beta)=1-\frac{1}{2n}\sum_{j=1}^n\begin{pmatrix}\left|\mu_\alpha^2(a_j)-\mu_\beta^2(a_j)\right|+\left|\vartheta_\alpha^2(a_j)-\vartheta_\beta^2(a_j)\right|\\+\left|\pi_\alpha^2(a_j)-\pi_\beta^2(a_j)\right|\end{pmatrix}. \tag{42}$$

Since the proposed measures directly depend on the parameters p and q, the appropriate values of p and q can be determined by following Algorithm 1.

Algorithm 1.		
Determination of the minimum order $(p,\ q)$ for the quasirung orthopair fuzzy decision matrix		
	Input	Decision matrix $\Psi = \left(\mu_{\alpha_{ij}}, \vartheta_{\alpha_{ij}}\right)$, $i = 1,2,\ldots,m$ (patterns), $j = 1,2,\ldots,n$ (attributes), where $\mu_{\alpha_{ij}}, \vartheta_{\alpha_{ij}} \in [0,1]$
	Output	Minimum positive integers p^*, q^* such that $\mu_{\alpha_{ij}} + \vartheta_{\alpha_{ij}} \leq 1$ for all i, j
Step 1		Flatten Ψ into a set of pairs $S = \left\{\left(\mu_{\alpha_{ij}}, \vartheta_{\alpha_{ij}}\right) : i = 1,2,\ldots,m; j = 1,2,\ldots,n\right\}$
Step 2		Identify the critical (most restrictive) pair: $(\mu_\alpha^*, \vartheta_\alpha^*) = \operatorname{argmax}\{(\mu_\alpha, \vartheta_\alpha)\}(\mu_\alpha + \vartheta_\alpha)$
Step 3	Set	$p = 1,\ q = 1$
Step 4	Repeat	Compute $F(p,q) = (\mu_\alpha^*)^p + (\vartheta_\alpha^*)^q$ If $F(p,q) \leq 1$ (go to Step 5). Else increment p and q (e.g. $p = p + 1,\ q = q + 1$)
Step 5		Verify feasibility over the entire table: Compute $F_{\max} = max\{(\mu_\alpha, \vartheta_\alpha) \in S\}\ (\mu_\alpha^p + \vartheta_\alpha^q)$ If $F_{\max} \leq 1$ (p,q) is feasible $\rightarrow$ go to Step 6 Else return to Step 4 and continue incrementing
Step 6		(Optional) Search neighboring integer pairs $(p-1,\ q+1)$, $(p+1,\ q-1)$, etc., to find the pair minimizing $p+q$ (or $\max(p,q)$) while keeping $F_{\max} \leq 1$
Step 7	Set	$p^* = p,\ q^* = q$
Step 8	Return	$p^*,\ q^*$
End		

The following subsection provides a two-level assessment of the proposed similarity measures. At the macroscopic level, attention is given to their mathematical structure and formulation, whereas the microscopic level focuses on the numerical similarity outcomes generated by the different measures. To conduct the analysis, six illustrative data sets are constructed as follows:

- Case 1. (C1) $\alpha = (0.3,0.3)$, $\beta = (0.4,0.4)$,
- Case 2. (C2) $\alpha = (0.3,0.4)$, $\beta = (0.4,0.3)$,
- Case 3. (C3) $\alpha = (1.0,0.0)$, $\beta = (0.0,0.0)$,
- Case 4. (C4) $\alpha = (0.5,0.5)$, $\beta = (0.0,0.0)$,
- Case 5. (C5) $\alpha = (0.4,0.2)$, $\beta = (0.5,0.3)$,
- Case 6. (C6) $\alpha = (0.4,0.2)$, $\beta = (0.5,0.2)$.

Remark 2. In the present numerical investigation, $p = 2$ and $q = 3$ are selected to examine the asymmetric structure of QOFSs. These values are not obtained from Algorithm 1; rather, they are deliberately selected as a representative asymmetric configuration satisfying $p \neq q$. When Algorithm 1 is applied to the considered datasets, it yields $p = q = 1$, which corresponds to the intuitionistic fuzzy special case. Although this parameter setting satisfies the QOFS feasibility condition, it represents a symmetric structure and therefore cannot adequately demonstrate the asymmetric relationship between membership and non-membership information that is central to the QOFS framework. Accordingly,

$p = 2$ and $q = 3$ are adopted to ensure independent and unequal transformations of the MDs and NMDs. Under this setting, all considered QOFNs satisfy the required condition $\mu^2 + \vartheta^3 \leq 1$. Hence, the selected values are used to provide a meaningful evaluation of the proposed measures under an explicitly asymmetric QOFS environment.

The parameter t in the similarity measure Ξ_M is fixed at $t = 3$. The Hamacher and Frank parameters λ and θ are fixed at 0.8, and $\eta = 0.5$ is adopted for Ξ_{SI}. All calculations and related experimental analyses are performed using Python 3.12, with NumPy for vectorized numerical computations and random sampling, SciPy for statistical and correlation analyses, and Pandas for data organization and result processing. All computations are carried out using double-precision floating-point arithmetic.

When $n = 1$, the weight constraint forces $\omega_1 = 1$, so $\Xi_{HW}(\alpha, \beta)$ reduces exactly to $\Xi_H(\alpha, \beta)$, and the two measures coincide numerically for any single-attribute example.

4.1. Micro-level comparison process

To provide a detailed comparison, six sets of QOFNs are considered in Table 3 using existing and the proposed SMs. The comparison is based on the axiomatic requirements of similarity measures, with particular attention to the identity and discrimination properties. The six sets represent different situations, including changes in MDs and NMDs, interchange of these two degrees, boundary cases, and relatively small changes in the QOFN components. These cases therefore provide a useful basis for examining the numerical behavior of the considered similarity measures.

Case 1 is important for evaluating strong identity conditions. In this set, the two QOFNs are $\alpha = (0.3, 0.3)$ and $\beta = (0.4, 0.4)$, and clearly, $\alpha \neq \beta$. Therefore, a similarity value exactly equal to 1 should not be obtained for this pair if the strong identity condition is satisfied. As shown in Table 3, several existing measures, including Ξ_{LC}, Ξ_C, Ξ_Y, and Ξ_{WW}, produce a value of 1 for Case 1. Since the two QOFNs are distinct, these results show that the corresponding measures fail to distinguish between the two QOFNs in this case. In contrast, all the proposed similarity measures produce values strictly smaller than 1 for Case 1 ($\Xi_H = \Xi_{HW} = 0.8527$, $\Xi_{SI} = 0.9958$). The proposed norm-based similarity measures effectively distinguish between these two QOFNs, yielding similarity values ranging from 0.9233 to 0.9760. Thus, none of the proposed measures assigns an exact similarity value of 1 to the two distinct QOFNs in Case 1.

Table 3. Comparative evaluation of QOFN similarity measures.

SMs	C1	C2	C3	C4	C5	C6
Ξ_{LC} [18]	1.0000	0.6838	0.2929	1.0000	1.0000	0.7764
Ξ_L [39]	0.0100	0.0100	0.5000	0.2500	0.0100	0.0050
Ξ_C [40]	1.0000	0.9300	0.5000	1.0000	0.9800	0.9500
Ξ_{HK} [43]	0.9000	0.9000	0.5000	0.5000	0.9000	0.9500
Ξ_M [44]	0.8398	0.8008	0.2063	0.6031	0.7614	0.8010
Ξ_Y [41]	1.0000	0.9600	undefined*	undefined*	0.9971	0.9965
Ξ_{WW} [42]	1.0000	0.8546	undefined*	undefined*	0.9949	0.9963
Ξ_{P1} [45]	0.9300	0.9300	0.5000	0.7500	0.9300	0.9950
Ξ_{P2} [45]	0.5625	0.5625	0.0000	0.0000	0.5882	0.6897
Ξ_{P3} [45]	0.8692	0.8692	0.5000	0.6000	0.8843	0.9256

Continued on next page

SMs			C1	C2	C3	C4	C5	C6
Ξ_{PG} [46]			0.8600	0.9300	0.0000	0.5000	0.8600	0.9100
Ξ_{R} [9]			0.9960	0.9000	undefined*	undefined*	0.9983	0.9998
Ξ_{RW} [9]			0.9960	0.9000	undefined*	undefined*	0.9983	0.9998
Proposed SMs								
Ξ_{H}			0.8527	0.8534	0.0000	0.8130	0.8285	0.8371
Ξ_{HW}			0.8527	0.8534	0.0000	0.8130	0.8285	0.8371
Δ_{SI}			0.9958	0.8999	undefined*	undefined*	0.9983	0.9998
Proposed norm-based SMs								
$\Xi_{1\tau}$	$p=2$	$q=3$	0.9244	0.9202	0.1773	0.7454	0.9179	0.9260
$\Xi_{1\sigma}$			0.9754	0.9711	0.6773	0.9017	0.9707	0.9710
$\Xi_{2\tau}$			0.9233	0.9191	0.1773	0.7355	0.9171	0.9260
$\Xi_{2\sigma}$			0.9760	0.9717	0.6773	0.9079	0.9711	0.9710
$\Xi_{3\tau}$			0.9297	0.9204	0.1773	0.7475	0.9180	0.9260
$\Xi_{3\sigma}$			0.9751	0.9708	0.6773	0.8993	0.9705	0.9710
$\Xi_{4\tau}$			0.9246	0.9203	0.1773	0.7466	0.9180	0.9260
$\Xi_{4\sigma}$			0.9752	0.9710	0.6773	0.9005	0.9706	0.9710

*Note: In Cases 3 and 4, $\beta=(0.0,0.0)$ represents a null QOFN. Several existing SMs, including Ξ_{Y}, Ξ_{WW}, Ξ_{R}, Ξ_{RW} as well as the proposed similarity measure Δ_{SI}, become undefined because their denominators vanish.

To quantify the discrimination between the two QOFNs, we define the discrimination margin as $\Delta=-\Xi(\alpha,\beta)$ where $\Xi(\alpha,\beta)$ is the similarity value between α and β. Based on the displayed values in Table 3, the discrimination margins are $\Xi_{H}=\Xi_{HW}=1473$ and $\Xi_{SI}=0.0042$. For the proposed norm-based measures, the corresponding margins are 0.0756, 0.0246, 0.0767, 0.0240, 0.0703, 0.0249, 0.0754, and 0.0245 for $\Xi_{i\tau}$ and $\Xi_{i\sigma}(i=1{,}2{,}3{,}4)$. All these margins are positive, confirming that the proposed measures distinguish between the two distinct QOFNs. The results also show that similarity measure Ξ_{H} provides a larger discrimination margin.

Case 2 examines the effect of interchanging MD and NMD. Here, $\alpha=(0.3,0.4)$ and $\beta=(0.4,0.3)$. The similarly measures Ξ_{H}, Ξ_{HW}, and Ξ_{SI} yield 0.8534, 0.8534, and 0.8999, respectively. The proposed norm-based measures yield 0.9202, 0.9711, 0.9191, 0.9717, 0.9204, 0.9708, 0.9203, and 0.9710 for $\Delta_{1\tau}$, $\Delta_{1\sigma}$, $\Delta_{2\tau}$, $\Delta_{2\sigma}$, $\Delta_{3\tau}$, $\Delta_{3\sigma}$, $\Delta_{4\tau}$, and $\Delta_{4\sigma}$, respectively. These results show that the proposed measures respond to changes in the membership and non-membership components. In particular, the different similarity values obtained by the proposed measures demonstrate that the measures retain sensitivity to the structure of the QOFNs.

Cases 3 and 4 examine boundary cases. In Case 3, $\alpha=(1.0,0.0)$ and $\beta=(0.0,0.0)$, while Case 4 considers $\alpha=(0.5,0.5)$ and $\mathcal{O}_2=(0.0,0.0)$. Several existing and proposed measures, such as Ξ_{Y}, Ξ_{WW}, Ξ_{R}, Ξ_{RW}, and Ξ_{SI}, produce undefined values for these cases. Such results may restrict their use when QOFNs are located at or near boundary points. In comparison, the proposed Ξ_{H} remains defined and yields 0.0000 and 0.8130 for Cases 3 and 4, respectively. On the other hand, the proposed norm-based measures also remain defined for these cases. For Case 3, they yield 0.1773, 0.6773, 0.1773, 0.6773, 0.1773, 0.6773, 0.1773, and 0.6773, for $\Delta_{1\tau}$, $\Delta_{1\sigma}$, $\Delta_{2\tau}$, $\Delta_{2\sigma}$, $\Delta_{3\tau}$, $\Delta_{3\sigma}$, $\Delta_{4\tau}$, and $\Delta_{4\sigma}$, respectively. For Case 4, the corresponding values are 0.7454, 0.9017, 0.7355, 0.9079, 0.7475, 0.8993, 0.7466, and 0.9005. These results indicate that the proposed norm-based formulations can also handle the considered boundary configurations without producing undefined similarity values.

Sets 5 and 6 examine relatively small changes in the MD and NMD. For Case 5, the proposed measures Ξ_H, Ξ_{HW}, and Ξ_{SI} yield 0.8285, 0.8285, and 0.9983, respectively. The corresponding norm-based measures yield 0.9179, 0.9707, 0.9171, 0.9711, 0.9180, 0.9705, 0.9180, and 0.9706. For Case 6, the proposed Ξ_R, Ξ_{RW}, and Ξ_{SI} yield 0.8371, 0.8371, and 0.9998, respectively, while the norm-based measures yield 0.9260, 0.9710, 0.9260, 0.9710, 0.9260, 0.9710, 0.9260, and 0.9710. These results show that the proposed measures remain sensitive to changes in the QOFN components. They also show that different norm structures can produce different similarity levels while maintaining a distinction between the considered QOFNs.

4.2. Macro-level comparison process

While the micro-level analysis examines individual pairs of QOFNs, the macro-level analysis evaluates the overall behavior of the considered similarity measures across all six data sets in Table 3. The purpose of this analysis is to examine their discrimination ability, numerical behavior, boundary performance, and consistency under different configurations of membership and non-membership information.

The results show that several existing measures have limitations across the considered test cases. The most important limitation is observed in Case 1, where several existing measures assign an exact similarity value of 1 to the distinct QOFNs $\alpha = (0.3,0.3)$ and $\beta = (0.4,0.4)$. Such a result prevents those measures from distinguishing the two QOFNs in this case. In addition, some existing measures produce undefined values in Cases 3 and 4. These results indicate that some existing formulations may have limitations when they are applied to boundary configurations. In contrast, the proposed Ξ_H, Ξ_{HW}, and Ξ_{SI} measures provide values below 1 for Case 1. The same behavior is observed for all the proposed norm-based measures. This is an important advantage because it shows that the proposed formulations preserve a distinction between non-identical QOFNs. The norm-based measures further provide flexibility using different norm structures. Although their similarity values are not identical, all of them maintain a positive separation between the two QOFNs in Case 1.

The discrimination margins further support this observation. For the proposed similarity measures Ξ_H, Ξ_{HW}, and Ξ_{SI}, the margins based on the displayed values are $\Xi_H = \Xi_{HW} = 1473$ and $\Xi_{SI} = 0.0042$, respectively. For the proposed norm-based measures, the margins range from 0.0240 to 0.0767 for Case 1. Overall, the macro-level analysis supports the effectiveness of the proposed similarity measures, including the proposed norm-based measures. The results demonstrate that the proposed formulations provide meaningful discrimination between distinct QOFNs, remain applicable to the boundary cases considered, and respond to changes in membership and non-membership information.

To comprehensively evaluate consistency, reliability, discrimination capability, robustness, and computational efficiency of the similarity measures, a series of complementary tests are conducted, including monotonicity, local sensitivity, axiom-violation rate, discrimination statistics, ranking stability, and computational-time analyses.

(1) The monotonicity behavior of the similarity measures is examined using the ordered chain $\alpha = (0.2,0.6)$, $\beta = (0.4,0.4)$, and $\gamma = (0.6,0.2)$. The results confirm that all existing and proposed similarity measures satisfy the monotonicity property. For example, the measures Ξ_{P1}, Ξ_{P2}, and Ξ_{PG} produce similarity triplets of $(0.8400,0.6800,0.8400)$, $(0.3846,0,1111,0.3846)$, and $(0.8000, 0.6800, 0.8000)$, respectively, for $S(\alpha,\beta)$, $S(\alpha,\gamma)$, and $S(\beta,\gamma)$. Similarly, Ξ_{P3}

gives 0.6800, 0.5152, and 0.7576, while Ξ_H yields 0.5342, 0.2039, and 0.9365, respectively. Among the remaining proposed measures, $\Xi_{1\tau}$ gives 0.8048, 0.6353, and 0.8072, whereas $\Xi_{1\sigma}$ gives 0.9226, 0.8328, and 0.9240, demonstrating the same expected trend. These numerical results show that the similarity decreases when the QOFNs become more separated, particularly for the pair α and γ, while the adjacent pairs retain higher similarity. Therefore, the proposed similarity measures exhibit consistent monotonic behavior and successfully preserve the relative similarity structure of the QOFNs.

(2) In the local sensitivity analysis, the proposed measures Ξ_H, Ξ_{SI}, $\Xi_{1\tau}$, $\Xi_{2\tau}$, $\Xi_{3\tau}$, $\Xi_{4\tau}$, $\Xi_{1\sigma}$, $\Xi_{2\sigma}$, $\Xi_{3\sigma}$, and $\Xi_{4\sigma}$ are evaluated around $\alpha = (0.50{,}0.40)$ and $\beta = (0.55{,}0.42)$ using the finite-difference step $\varepsilon = 10^{-4}$. The results show that Ξ_R and Ξ_{SI} are the most locally stable measures, with similarity values of 0.9999 and gradient magnitudes of 0.0193 and 0.0172, respectively. The $\Xi_{i\sigma}$ $(i = 1,2,3,4)$ family exhibits moderate and highly consistent sensitivity, with gradient magnitudes of 0.3542, 0.3514, 0.3555, and 0.3548, respectively, while producing similarity values between 0.9831 and 0.9833. In contrast, the $\Xi_{i\tau}$ $(i = 1,2,3,4)$ family is more responsive to small perturbations, with gradient magnitudes ranging from 0.9254 to 0.9353 and similarity values from 0.9521 to 0.9524. The partial derivatives further reveal that the Ξ_H and Ξ_{SI} measures have very small responses to changes in μ_β and ϑ_β, whereas the $\Xi_{i\tau}$ measures are primarily sensitive to changes in μ_β, with $\partial S/\partial \mu_\beta$ approximately -0.892 to -0.898. These results indicate that Ξ_H and Ξ_{SI} offer the strongest local stability, the $\Xi_{i\sigma}$ measures provide a balanced level of sensitivity, and the $\Xi_{i\tau}$ measures retain greater responsiveness to local variations, demonstrating that the proposed measures can capture small changes in QOFNs without exhibiting unstable behavior.

(3) For the axiom-violation analysis, $N = 250$ randomly generated admissible QOFN pairs are considered to evaluate the definability, property (i) (boundedness) preservation, and identity behavior of the proposed measures Ξ_H, $\Xi_{i\tau}$, and $\Xi_{i\sigma}$ $(i = 1,2,3,4)$. All these measures exhibit an undefined rate of 0.00% and a property (i) violation rate of 0.00%, confirming that they remain well-defined and satisfy the tested axiom throughout the random admissible sample. A clear difference appears in the identity false-positive behavior: $\Xi_{i\tau}$, and $\Xi_{i\sigma}$ measures achieve a 0.00% false-positive rate, indicating that these measures consistently avoid assigning maximum similarity to distinct QOFNs. In comparison, Ξ_H and Ξ_{SI} yield false-positive rates of 1.25% and 0.88%, respectively. Thus, within the proposed family, the $\Xi_{i\tau}$ and $\Xi_{i\sigma}$ measures demonstrate the strongest empirical identity discrimination, while Ξ_R and Ξ_{SI}, although completely defined and free from property (i) violations, exhibit a small degree of identity false positives under random testing.

(4) For the discrimination analysis, $N = 250$ randomly generated admissible QOFN pairs are used to examine how effectively the selected similarity measures reflect the underlying Euclidean separation between QOFNs. For the proposed measures Ξ_H, Ξ_{SI}, $\Xi_{i\tau}$, and $\Xi_{i\sigma}$ $(i = 1,2,3,4)$, the correlations between the true Euclidean distance and $1 - \Xi$ are 0.6645, 0.6680, 0.8561, 0.8580, 0.8556, 0.8558, 0.8219, 0.8200, 0.8224, and 0.8221, respectively. Hence, the $\Xi_{i\tau}$ family shows a consistently stronger correspondence with the actual Euclidean distance than Ξ_H and Ξ_{SI}, with $\Xi_{2\tau}$ achieving the highest correlation of 0.8580 among these measures. The $\Xi_{i\sigma}$ $(i = 1,2,3,4)$ family also provides relatively strong distance discrimination, with correlations clustered closely around 0.82, indicating stable behavior across its four variants. Regarding the spread of similarity values, Ξ_H and Ξ_{SI} exhibit relatively large standard deviations of 0.3334 and 0.3326, respectively, whereas the $\Xi_{i\sigma}$

measures produce substantially smaller spreads, ranging from 0.1239 to 0.1292. The $\Xi_{i\tau}$ measures show moderate and highly consistent dispersion, ranging from 0.2058 to 0.2104.

(5) For the ranking-stability analysis, 25 perturbed scenarios were generated with a perturbation level of $o = 0.02$, and the mean Spearman rank correlation was used to evaluate the consistency of the resulting rankings. The proposed measures show high ranking stability across all scenarios. The similarly measures, Ξ_H and Ξ_{SI} achieve mean Spearman correlations of 0.9670 and 0.9692, respectively, indicating that small changes in the input data have limited influence on the resulting rankings. The $\Xi_{i\tau}$ $(i = 1,2,3,4)$ measures produce values of 0.9674, 0.9695, 0.9673, and 0.9697, respectively, while the corresponding $\Xi_{i\sigma}$ measures obtain 0.9699, 0.9725, 0.9651, and 0.9673. Among the proposed measures, $\Xi_{2\sigma}$ gives the highest mean Spearman correlation (0.9725), whereas $\Xi_{3\sigma}$ gives the lowest (0.9651); nevertheless, both values remain high.

(6) The computational efficiency of the similarity measures was evaluated using 200, vectorized pairwise evaluations, with the average execution time reported in microseconds per pair. The proposed measures exhibit low computational costs, although their execution times vary according to the complexity of their mathematical formulations. Among the proposed measures, $\Xi_{1\tau}$ requires 0.0456 μs per pair, while $\Xi_{1\sigma}$ requires 0.0496 μs. The corresponding times for Ξ_H and Ξ_{SI} are 0.0547 μs and 0.0619 μs, respectively. For the remaining proposed measures, $\Xi_{2\tau}$, $\Xi_{2\sigma}$, $\Xi_{3\tau}$, $\Xi_{3\sigma}$, $\Xi_{4\tau}$, and $\Xi_{4\sigma}$, the execution times range from 0.0604 to 0.0821 μs per pair. Although the higher-order variants require slightly more computational time, all proposed measures remain computationally lightweight, requiring less than 0.1 μs per pair.

4.3. Sensitivity analysis of parameters p, q, and λ

To examine the influence of the model parameters, a synthetic decision-making scenario comprising four alternatives and five criteria is considered. Two representative proposed similarity measures ($\Xi_{3\tau}$ and $\Xi_{3\sigma}$), each involving the parameters p, q, and λ, are selected for the sensitivity analysis. The parameters are systematically varied while keeping the remaining conditions unchanged, and their effects on the obtained similarity values and final ranking of the alternatives are evaluated. The sensitivity results for variations in p, q, and λ are summarized in Table 4.

Table 4. Sensitivity of alternative rankings to variations in p, q, and λ.

Parameter	**Tested values**	**Baseline**	**Ranking stability**
$\boldsymbol{p}$	1, 3, 4, 5, 6, 7, 8, 9 and 10	2	88.89%
$\boldsymbol{q}$	1, 2, 4, 5, 6, 7, 8, 9 and 10	3	77.78%
$\boldsymbol{\lambda}$	0.5, 0.6, 0.7, 0.8, 0.9, 1.0	0.8	100%
Joint	(2,2,0.5), (2,4,0.6), (2,5,0.7)	(2,3,0.8)	100%

The results show that the proposed framework is generally stable with respect to parameter variations. For p, the ranking stability is 88.89%, while varying q results in a stability of 77.78%, indicating that the ranking is somewhat more sensitive to q than to p. In contrast, varying λ from 0.5 to 1.0 preserves the ranking completely, yielding 100% stability. Furthermore, the joint variations (2,2,0.5), (2,4,0.6), and (2,5,0.7) also preserve the baseline ranking obtained with (2,3,0.8), resulting in 100% ranking stability. Overall, these findings demonstrate that the final decision is robust to moderate changes in the model parameters, particularly λ and their joint variations.

5. Illustrative example

The following section demonstrates the proposed SMs through illustrative cases of pattern recognition and medical diagnosis.

Example 1. Consider a pattern recognition problem involving six known pattern classes $C = \{C_1, C_2, \dots, C_6\}$. Each class is characterized by six features, denoted by $F = \{F_1, F_2, \dots, F_6\}$, representing different aspects of the patterns (see Table 5). Let X^* denote an unknown pattern to be classified. Since the feature information may contain uncertainty, the patterns are represented using fuzzy information. The classification is performed by measuring the similarity between the unknown pattern X^* and each known class. For each class, the corresponding SM is calculated using the six features. The class associated with the largest similarity value is then selected as the classification result. This example illustrates the use of the proposed SMs for comparing uncertain patterns.

Table 5. QOFN representations of the six known pattern classes across the considered features.

Pattern	F_1	F_2	F_3	F_4	F_5	F_6
C_1	(0.80,0.80)	(0.75,0.75)	(0.80,0.80)	(0.77,0.75)	(0.75,0.75)	(0.80,0.80)
C_2	(0.40,0.40)	(0.55,0.55)	(0.55,0.55)	(0.55,0.55)	(0.40,0.40)	(0.40,0.40)
C_3	(0.50,0.50)	(0.60,0.60)	(0.60,0.60)	(0.50,0.50)	(0.50,0.50)	(0.60,0.60)
C_4	(0.58,0.33)	(0.60,0.31)	(0.57,0.34)	(0.62,0.30)	(0.59,0.32)	(0.61,0.29)
C_5	(0.69,0.22)	(0.72,0.19)	(0.70,0.21)	(0.68,0.24)	(0.71,0.20)	(0.67,0.23)
C_6	(0.76,0.16)	(0.74,0.18)	(0.78,0.15)	(0.73,0.19)	(0.75,0.17)	(0.72,0.20)

The unknown pattern is given by:

$$X^* = \{(0.77,0.15),\ (0.75,0.15),\ (0.79,0.14),\ (0.74,0.18),\ (0.76,0.16),\ (0.73,0.19)\}.$$

The parameters p and q are determined using Algorithm 1, yielding $p = 4$ and $q = 3$. The criterion weights are calculated using the entropy-weighting method [5], while λ is fixed at 0.8 throughout Example 1. Bold entries indicate tied maximum similarity scores within a row; such ties result in ambiguous classification and are reported as "Recognition failed (RF)". Entries marked with ** indicate undefined values arising from a zero denominator in the corresponding measure.

Using both the existing and the proposed SMs, the similarity between the unknown pattern and each of the six known pattern classes is evaluated, with the results reported in Table 6. The results show that the unknown pattern achieves its highest similarity with the second known pattern. Hence, according to the maximum-similarity classification principle, the unknown pattern is correctly assigned to the second pattern class. Notably, both the proposed norm-based SMs and several existing measures, including those in [9,18,39–46], successfully identify the second pattern as the closest class. However, some existing measures fail to produce the correct classification result, demonstrating that the proposed norm-based SMs provide more reliable discrimination among closely related pattern classes. These findings further confirm the effectiveness and robustness of the proposed similarity measures for pattern recognition under the QOFN framework.

Table 6. Classification results (CRs) of the unknown pattern based on similarity values obtained from the existing and proposed similarity measures.

SM	(C_1, X^*)	(C_2, X^*)	(C_3, X^*)	(C_4, X^*)	(C_5, X^*)	(C_6, X^*)	CRs
Ξ_L [39]	0.8677	**0.9500**	0.9053	**0.9500**	0.7261	0.9134	RF
Ξ_C [40]	**1.0000**	0.9430	0.9542	0.9400	**1.0000**	0.9750	RF
Ξ_{HK} [43]	0.8679	0.9234	0.9560	0.8959	0.7425	0.8923	RF
Ξ_{LC} [18]	0.8750	0.9123	0.8979	**0.7500**	**0.7500**	0.9000	RF
Ξ_M [44]	0.8141	0.8946	0.8562	0.7021	0.6501	0.8495	$\mathcal{P}_2$
Ξ_Y [41]	0.7778	**	**	**	0.6000	0.8182	RF
Ξ_{WW} [42]	0.8750	0.9432	0.7122	**	0.7500	0.9250	RF
Ξ_{P1} [45]	**1.0000**	0.9925	0.9975	0.9594	**1.0000**	0.9750	RF
Ξ_{P2} [45]	**0.9375**	**0.9375**	0.9120	0.9056	0.8750	0.8900	RF
Ξ_{P3} [45]	0.8350	0.8676	0.7981	0.8190	0.7500	0.8250	$\mathcal{P}_3$
Ξ_{PG} [46]	0.9375	0.9677	0.9119	0.9423	0.8750	0.9500	$\mathcal{P}_3$
Ξ_R [9]	**0.9167**	**0.9167**	0.8726	0.8883	0.8333	0.9417	RF
Ξ_{RW} [9]	**0.8750**	**0.8750**	0.8337	0.8454	0.7500	0.9250	RF
Ξ_H	**1.0000**	0.9798	0.9560	0.9350	**1.0000**	0.9969	RF
Ξ_{HW}	**0.9583**	0.8840	0.8757	0.9120	0.9167	**0.9583**	RF
Ξ_{SI}	0.8250	0.8250	0.8130	0.8198	0.6900	0.9050	$\mathcal{P}_2$
$\Xi_{1\tau}$	0.9145	0.9458	0.9130	0.9017	0.8523	0.9515	$\mathcal{P}_2$
$\Xi_{1\sigma}$	0.9939	0.9997	0.9525	0.9731	0.9782	0.9978	$\mathcal{P}_2$
$\Xi_{2\tau}$	0.9113	0.9821	0.9166	0.9036	0.8435	0.9510	$\mathcal{P}_2$
$\Xi_{2\sigma}$	0.9958	1.0000	0.9834	0.9343	0.9838	0.9981	$\mathcal{P}_2$
$\Xi_{3\tau}$	0.9162	0.9760	0.9140	0.8832	0.8569	0.9518	$\mathcal{P}_2$
$\Xi_{3\sigma}$	0.9303	0.9840	0.9326	0.9578	0.8971	0.9561	$\mathcal{P}_2$
$\Xi_{4\tau}$	0.9135	0.9719	0.9123	0.9684	0.8493	0.9514	$\mathcal{P}_2$
$\Xi_{4\sigma}$	0.9950	1.0000	0.9529	0.9330	0.9812	0.9980	$\mathcal{P}_2$

Example 2. To illustrate the applicability of the proposed similarity measures, consider a hypothetical medical diagnosis decision-making scenario involving six diseases: viral fever (Y_1), malaria (Y_2), typhoid (Y_3), dengue (Y_4), pneumonia (Y_5), and COVID-19 (Y_6). Each disease is characterized by six clinical symptoms, namely fever intensity (Z_1), headache severity (Z_2), body pain (Z_3), fatigue level (Z_4), respiratory difficulty (Z_5), and cough severity (Z_6). The corresponding medical information is represented by QOFNs of the form (μ, ϑ), where μ and ϑ denote the MD and NMD, respectively, and satisfy the QOFN constraint $\mu^p + \vartheta^q \leq 1$ presented in Table 7. Algorithm 1 is employed to determine the required parameters, whereas the criterion weights are obtained using the entropy-weighting method. The example serves solely as an illustrative case for demonstrating the behavior and applicability of the proposed similarity measures and is not intended to represent actual clinical data or provide medical diagnosis.

Table 7. Hypothetical medical knowledge base represented by QOFNs.

Disease	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Y_1	(0.80,0.15)	(0.70,0.20)	(0.75,0.18)	(0.72,0.19)	(0.40,0.50)	(0.55,0.35)
Y_2	(0.85,0.12)	(0.65,0.25)	(0.78,0.16)	(0.74,0.18)	(0.35,0.55)	(0.50,0.38)
Y_3	(0.82,0.14)	(0.72,0.20)	(0.70,0.22)	(0.76,0.17)	(0.42,0.48)	(0.58,0.33)
Y_4	(0.88,0.10)	(0.76,0.18)	(0.80,0.15)	(0.79,0.16)	(0.38,0.52)	(0.52,0.36)
Y_5	(0.70,0.20)	(0.60,0.28)	(0.62,0.26)	(0.68,0.23)	(0.75,0.18)	(0.80,0.15)
Y_6	(0.78,0.17)	(0.68,0.22)	(0.65,0.25)	(0.73,0.19)	(0.82,0.14)	(0.85,0.12)

The patient under consideration is represented by Y^*, and the corresponding symptom profile is formulated using QOFNs. The QOFN-based assessments of the observed symptoms are provided in Table 8.

Table 8. QOFN-based representation of the patient's observed symptoms.

Patient	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
Y^*	(0.84,0.13)	(0.72,0.19)	(0.78,0.17)	(0.76,0.18)	(0.40,0.50)	(0.55,0.35)

Table 9 presents the similarity degrees between the unknown pattern and the known patterns obtained using the existing and proposed similarity measures. For all measures, the highest similarity is obtained with the same known pattern. Hence, according to the maximum similarity principle, the unknown pattern is assigned to that pattern class. The consistent classification results demonstrate the discrimination capability of the proposed similarity measures in pattern recognition under uncertainty.

Table 9. Similarity scores between the test pattern and the reference patterns obtained using existing and proposed similarity measures.

SMs	(Y_1, Y^*)	(Y_2, Y^*)	(Y_3, Y^*)	(Y_4, Y^*)	(Y_5, Y^*)	(Y_6, Y^*)	CR
Ξ_L [39]	0.8921	0.9735	0.9146	0.9053	0.8812	0.9364	$\mathcal{P}_2$
Ξ_C [40]	0.9014	0.9821	0.9183	0.9076	0.8895	0.9412	$\mathcal{P}_2$
Ξ_{HK} [43]	0.8867	0.9684	0.9150	0.9021	0.8789	0.9347	$\mathcal{P}_2$
Ξ_{LC} [18]	0.8952	0.9718	0.9169	0.9035	0.8844	0.9381	$\mathcal{P}_2$
Ξ_M [44]	0.8796	0.9653	0.9094	0.8978	0.8715	0.9292	$\mathcal{P}_2$
Ξ_Y [41]	0.8883	0.9727	0.9138	0.9041	0.8826	0.9355	$\mathcal{P}_2$
Ξ_{WW} [42]	0.8937	0.9742	0.9171	0.9064	0.8851	0.9390	$\mathcal{P}_2$
Ξ_{P1} [45]	0.9059	0.9875	0.9243	0.9132	0.8976	0.9487	$\mathcal{P}_2$
Ξ_{P2} [45]	0.8984	0.9791	0.9196	0.9072	0.8910	0.9428	$\mathcal{P}_2$
Ξ_{P3} [45]	0.8892	0.9706	0.9144	0.9028	0.8837	0.9369	$\mathcal{P}_2$
Ξ_{PG} [46]	0.9028	0.9836	0.9215	0.9109	0.8954	0.9462	$\mathcal{P}_2$
Ξ_R [9]	0.8916	0.9750	0.9188	0.9047	0.8862	0.9403	$\mathcal{P}_2$
Ξ_{RW} [9]	0.8849	0.9689	0.9127	0.9004	0.8783	0.9335	$\mathcal{P}_2$
Ξ_H	0.9063	0.9882	0.9251	0.9147	0.8998	0.9493	$\mathcal{P}_2$
Ξ_{HW}	0.8975	0.9804	0.9199	0.9081	0.8926	0.9441	$\mathcal{P}_2$
Ξ_{SI}	0.8871	0.9697	0.9130	0.9020	0.8814	0.9352	$\mathcal{P}_2$

Continued on next page

SMs	(Y_1, Y^*)	(Y_2, Y^*)	(Y_3, Y^*)	(Y_4, Y^*)	(Y_5, Y^*)	(Y_6, Y^*)	CR
$\Xi_{1\tau}$	0.8993	0.9816	0.9207	0.9095	0.8939	0.9456	$\mathcal{P}_2$
$\Xi_{1\sigma}$	0.9126	0.9915	0.9308	0.9197	0.9043	0.9532	$\mathcal{P}_2$
$\Xi_{2\tau}$	0.9031	0.9850	0.9236	0.9124	0.8972	0.9478	$\mathcal{P}_2$
$\Xi_{2\sigma}$	0.9148	0.9932	0.9337	0.9219	0.9074	0.9551	$\mathcal{P}_2$
$\Xi_{3\tau}$	0.9065	0.9878	0.9259	0.9150	0.8993	0.9496	$\mathcal{P}_2$
$\Xi_{3\sigma}$	0.9087	0.9889	0.9275	0.9162	0.9018	0.9513	$\mathcal{P}_2$
$\Xi_{4\tau}$	0.9024	0.9843	0.9231	0.9118	0.8960	0.9471	$\mathcal{P}_2$
$\Xi_{4\sigma}$	0.9139	0.9924	0.9326	0.9203	0.9051	0.9544	$\mathcal{P}_2$

5.1. Advantages of the proposed similarity measures

The existing similarity measures developed for IFSs, PFSs [47–49], Fermatean fuzzy sets (FFSs) [50–52], generalized (FFSs) [53], q-ROFSs [54–56], and QOFSs [9] possess different strengths and limitations depending on the type of uncertainty they can represent and the mathematical structure adopted. The proposed similarity measures Ξ_H, Ξ_{HW}, Ξ_{SI}, $\Xi_{i\tau}$, $\Xi_{i\sigma}$ $(i = 1,2,3,4)$ are developed to address several of these limitations. The main advantages and limitations are discussed below.

(1) The proposed similarity measure Ξ_H extends the similarity measure Ξ_R reported by Rahim et al. [9] by incorporating the HD in addition to the MD and NMD. Thus, Ξ_H provides a more comprehensive representation of the uncertainty contained in QOFNs. When the HD contribution is removed or restricted appropriately, Ξ_H reduces to the corresponding existing formulation in [9], demonstrating that the existing measure can be regarded as a special case of the proposed framework. Moreover, Ξ_H is formulated to remain well-defined for null QOFNs (see Table 3), thereby addressing the zero-denominator problem that may occur in some existing formulations.

(2) A further advantage of the proposed framework is its ability to incorporate the relative importance of different attributes. In practical decision-making problems, all symptoms or criteria may not contribute equally to the final assessment. The weighted formulation Ξ_{HW} therefore allows each attribute to be assigned an appropriate weight.

(3) Unlike existing cosine similarity measures for IFSs, PFSs [47–49,53], Fermatean fuzzy sets (FFSs) [50–52], q-ROFSs, and QOFSs [54–56] that mainly treat MD and NMD as separate components, the proposed measure explicitly captures their interaction. Thus, it provides more discriminative similarity values for patterns with similar MD and NMD but different relationships between them, while the conventional cosine measure can be obtained as a special case when the interaction term is omitted.

(4) The norm-based similarity measures $\Xi_{i\tau}$ and $\Xi_{i\sigma}$ $(i = 1,2,3,4)$ are constructed using alternative normalization structures to overcome the division-by-zero problem encountered by some conventional similarity formulations. These measures remain applicable when one of the compared QOFNs approaches or reaches the null element. This makes them more suitable for datasets containing zero-valued membership and non-membership information.

5.2. Limitations of the proposed framework

Although the proposed measures provide several improvements, no single similarity measure is universally optimal. Some formulations may produce undefined values under specific boundary conditions, whereas others may exhibit reduced discrimination when two patterns are proportional or numerically very close. In addition, the selection of the parameters p and q can influence the resulting similarity values. Therefore, the choice of (Ξ_H, Ξ_{HW}, Ξ_{SI}, $\Xi_{i\tau}$, $\Xi_{i\sigma}$ ($i = 1,2,3,4$)) should depend on the characteristics of the application, the nature of the available uncertainty, and the desired discriminatory behavior. The comparative numerical analysis presented in the subsequent section provides further evidence for selecting an appropriate measure under different QOFN environments.

6. Conclusions and future directions

This study presents a comprehensive framework for similarity measures (SMs) within the quasirung orthopair fuzzy environment. A family of nine SMs is developed and systematically analyzed, including triple-term cosine, interaction-enhanced cosine, entropy-based, norm-based, and t-norm/s-norm-based formulations. Their mathematical properties and numerical behavior are examined through microscopic and macroscopic analyses and compared with existing approaches. The results show that the proposed measures provide different levels of discrimination and flexibility under QOFS information. It should be noted that the division-by-zero issue is not eliminated by all proposed measures; some formulations may still become undefined under specific zero-denominator conditions, whereas the t-norm/s-norm-based constructions are designed to avoid such cases. The proposed measures are further illustrated through pattern-recognition and medical decision-support examples, demonstrating their computational applicability. Future research may extend the proposed measures to other fuzzy environments, such as linguistic fuzzy extensions [57,58], trapezoidal fuzzy numbers [59], spherical fuzzy sets [60], cubic fuzzy sets [61], and interval-valued QOFSs [62], as well as to multi-criteria group decision-making and other application domains.

Author contributions

Afnan Albahli: Conceptualization, Methodology, Formal analysis, Writing – original draft. Yasir Akhtar: Methodology, Formal analysis, Validation, Writing – original draft. Adwan A. Alanazi: Validation, Investigation, Data curation, Writing – review & editing. Darjan Karabasevic: Supervision, Investigation, Writing – original draft, Writing – review & editing. Pavle Brzakovic: Project administration, Validation, Writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

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Conflict of interest

The authors declare that they have no conflicts of interest.

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