Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(9): 30979–31004.
DOI: 10.3934/math.20261226
Received: 10 April 2026
Revised: 24 July 2026
Accepted: 20 August 2026
Published: 21 September 2026

Research article

Three iterative methods of Darcy–Brinkman equations based on the finite element method

Peixin Sun, Lunji Song* **and Youlei Liang**

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

* **Correspondence:** Email: song@lzu.edu.cn; Tel: +8609318912483; Fax: +860931891241.

Abstract: In this paper, three iterative methods—Stokes, Newton, and Oseen iterations—are applied to solve the nonlinear coupled system resulting from the finite element discretization of Darcy–Brinkman equations. The three iterative methods reduce the nonlinear problem to a sequence of linear subproblems, enhancing computational efficiency while maintaining accuracy. Error estimates and convergence are derived for the three iterative methods within the finite element framework. Numerical experiments are conducted to evaluate and compare the convergence rates of the three iterative methods, validating the theory.

Keywords: steady Darcy–Brinkman equations; Stokes iterative method; Newton iterative method; Oseen iterative method; finite element method
Mathematics Subject Classification: 35Q35, 65N30, 76S05

1. Introduction

Steady Darcy–Brinkman equations are complex systems coupling velocity, temperature, pressure, and concentration, governing the steady double-diffusive convection phenomenon arising from combined heat and mass transfer in saturated porous media. The classical Darcy law characterizes seepage flow in porous media at low Reynolds numbers. Because it assumes a uniformly distributed velocity profile, it cannot handle scenarios involving significant velocity shear or coupling with free-flow regions. To overcome this limitation, Brinkman proposed an extended model incorporating an additional viscous stress term, known as Darcy–Brinkman equations. This model is capable of describing flow behaviors ranging from seepage in low-permeability porous media to flows in high-porosity media that more closely resemble pure viscous fluid motion.

These equations are well-suited for addressing more complex porous media flow problems encountered in industrial and natural environments. For instance, in hydrocarbon recovery from fractured reservoirs, fluid transfer must account for both the permeability of the porous matrix and the viscous resistance at fracture walls. In cases where fracture apertures are small, the traditional Darcy model can lead to significant errors, whereas Darcy–Brinkman equations provide more accurate calculations of relative permeabilities for water and gas flow [1]. In another example, Darcy–Brinkman equations have been used to investigate the relationship between heat transfer performance and jet impingement distance in open-cell copper foam [2]. Furthermore, Darcy–Brinkman equations have increasingly become a research focus in areas such as heat and mass transfer in porous electrodes of solid oxide fuel cells and in solute transport in groundwater pollution remediation [3–5].

The primary difficulty associated with steady Darcy–Brinkman equations compared to other models lies in their complex multiphysics coupling. The equations not only couple velocity, pressure, temperature, and concentration nonlearly but also acount for inertial effects, viscous resistance, and double-diffusive (thermal and solutal) effects within the porous medium. This complexity poses significant challenges for numerical solution. To address these challenges, researchers have developed various numerical methods. The traditional finite element method is a classic approach for solving these equations. It directly discretizes the governing equations and employs iterative techniques to handle the nonlinear terms. Rigorous mathematical analysis is then conducted to establish the existence and uniqueness of the numerical solution as well as to evaluate the stability and error bounds of the scheme [6]. Furthermore, mixed finite element methods reformulate the original problem as a saddle-point problem by introducing additional variables such as vorticity. This approach is particularly suitable for handling complex flow scenarios involving three-dimensional geometries or the coupling of highly permeable media with classical porous media [7]. To significantly enhance computational efficiency for problems involving complex parameters or long-term evolution, reduced-order modeling techniques, especially those combining variational multiscale methods with proper orthogonal decomposition, have been successfully applied to extract dominant system features, thereby substantially reducing computational cost while maintaining accuracy [8]. Additionally, operator splitting methods effectively decouple the original coupled problem into a sequence of relatively simpler subproblems (e.g., a flow subproblem and a transport subproblem) solved alternately, providing an effective framework for handling incompressibility, strong nonlinearities, and multiphysics coupling [9]. As problem complexity increases, other approaches such as multiscale methods (e.g., two-scale finite element methods, lattice Boltzmann methods) and data-driven machine learning techniques are also being explored for simulating flow in highly heterogeneous porous media and addressing numerical upscaling challenges.

Zhu et al. [6] studied the Newton iterative method based on finite element discretization for Darcy–Brinkman equations. They used an amplification technique to prove convergence. In this paper, we adopt a different approach to derive sharper convergence criteria for the three iterative methods. Our criteria are significantly less restrictive than those in [6] and, in certain cases, provide an exact characterization of the convergence domain. In particular, we obtain explicit spectral radius conditions that are directly amenable to numerical verification.

The remainder of this paper is organized as follows. Section 2 introduces the necessary preliminaries, including the variational formulation of Darcy–Brinkman equations and the key mathematical properties required for the subsequent analysis. In Section 3, we describe the Stokes, Newton, and Oseen iterative methods; derive error estimates for the three iterative methods; and establish their convergence. Section 4 presents numerical experiments based on an analytical solution, where we compute and compare the convergence rates of the three iterative methods. Finally, we conclude the paper with a summary of our findings and an outlook on future research directions.

2. Preliminaries

2.1. Darcy–Brinkman equations

We consider the following steady Darcy–Brinkman equations:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + D_a^{-1}\boldsymbol{u} + \nabla p = (\beta_T T + \beta_C C)\boldsymbol{g}, \boldsymbol{x}\in\Omega, \\ \nabla\cdot\boldsymbol{u} = 0, \boldsymbol{x}\in\Omega, \\ -\gamma\Delta T + (\boldsymbol{u}\cdot\nabla)T = 0, \boldsymbol{x}\in\Omega, \\ -D_c\Delta C + (\boldsymbol{u}\cdot\nabla)C = 0, \boldsymbol{x}\in\Omega, \\ \boldsymbol{u} = 0, \boldsymbol{x}\in\partial\Omega, \\ T = T_0, C = C_0, \boldsymbol{x}\in\partial\Gamma, \\ \frac{\partial T}{\partial \mathbf{n}} = \frac{\partial C}{\partial \mathbf{n}} = 0, \boldsymbol{x}\in\partial\Omega\backslash\partial\Gamma, \end{cases} \tag{2.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with Lipschitz continuous boundaries $\partial\Omega$ and $\partial\Gamma$, and $\partial\Gamma$ is a nonempty proper subset of $\partial\Omega$. Let $\boldsymbol{x} = (x_1, x_2)$. We denote the velocity vector by $\boldsymbol{u} = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))^T$, the fluid pressure by $p = p(\boldsymbol{x})$, the temperature by $T = T(\boldsymbol{x})$, and the concentration by $C = C(\boldsymbol{x})$. Let $\mathbf{n}$ be the unit outward normal, $\boldsymbol{g}$ be the gravitational acceleration vector, $\nu > 0$ be the viscosity, D_a be the Darcy number, $\gamma > 0$ be the heat diffusivity, $D_c > 0$ be the mass diffusivity, and β_T and β_C be the thermal and solutal expansion coefficients, respectively. The Eq (2.1) is written in its dimensional form.

Next, we introduce the function spaces and the commonly used norms for these function spaces employed in this paper [10]:

$$\begin{aligned} &X = H_0^1(\Omega)^2 = \{\boldsymbol{u}\in H^1(\Omega)^2 : \boldsymbol{u}|_{\partial\Omega} = 0\}, \\ &W = H^1(\Omega), \\ &W_0 = \{\phi\in H^1(\Omega) : \phi|_{\partial\Gamma} = 0\}, \\ &E = L_0^2(\Omega) = \{q\in L^2(\Omega) : \int_\Omega q\,dx = 0\}, \\ &V = \{\boldsymbol{v}\in X : \int_\Omega \varphi \mathrm{div}\boldsymbol{v}\,dx = 0, \forall\varphi\in E\}. \end{aligned}$$

Let $\|\cdot\|_0$ denote the norm induced by the inner product of $L^2(\Omega)$. Furthermore, $(\nabla\boldsymbol{u},\nabla\boldsymbol{v})$ and $\|\nabla\boldsymbol{u}\|_0$ represent the inner product and norm in $H_0^1(\Omega)$ and X, respectively. The symbol $\|\cdot\|_{k,r}$ denotes the norm of the classical Sobolev space $W^{k,r}(\Omega)$.

We further define the bilinear forms and introduce the trilinear forms as follows:

$$a(\boldsymbol{u},\boldsymbol{v}) = (\nabla\boldsymbol{u},\nabla\boldsymbol{v}), \quad \forall\boldsymbol{u},\boldsymbol{v}\in X, \tag{2.2}$$

$$\overline{a}(T,\psi) = (\nabla T,\nabla\psi), \quad \forall T,\psi\in W, \tag{2.3}$$

$$d(q,\boldsymbol{v}) = (q,\mathrm{div}\boldsymbol{v}), \quad \forall\boldsymbol{v}\in X, \forall q\in E, \tag{2.4}$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) + \frac{1}{2}((\mathrm{div}\boldsymbol{u})\boldsymbol{v},\boldsymbol{w})$$

$$= \frac{1}{2}((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) - \frac{1}{2}((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{v}), \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in X, \tag{2.5}$$

$$\overline{b}(\boldsymbol{u},T,\psi) = ((\boldsymbol{u}\cdot\nabla)T,\psi) + \frac{1}{2}((\mathrm{div}\boldsymbol{u})T,\psi)$$
$$= \frac{1}{2}((\boldsymbol{u}\cdot\nabla)T,\psi) - \frac{1}{2}((\boldsymbol{u}\cdot\nabla)\psi,T), \quad \forall \boldsymbol{u}\in X, \forall T,\psi\in W. \tag{2.6}$$

It can be verified that the bilinear forms $a(\cdot,\cdot)$ and $\overline{a}(\cdot,\cdot)$ are continuous and coercive. The bilinear form $d(\cdot,\cdot)$ is continuous, and satisfies the following property [11]: There exists a positive constant β such that

$$\sup_{\boldsymbol{v}\in X,\boldsymbol{v}\neq 0}\frac{|d(q,\boldsymbol{v})|}{\|\nabla\boldsymbol{v}\|_0} \geq \beta\|q\|_0, \quad \forall q\in E. \tag{2.7}$$

Similarly, we can also establish that these trilinear forms are continuous and satisfy the following property [12, 13]:

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}), \tag{2.8}$$
$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0, \tag{2.9}$$
$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leq N\|\nabla\boldsymbol{u}\|_0\|\nabla\boldsymbol{v}\|_0\|\nabla\boldsymbol{w}\|_0, \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in X, \tag{2.10}$$
$$\overline{b}(\boldsymbol{u},T,\psi) = -\overline{b}(\boldsymbol{u},\psi,T), \tag{2.11}$$
$$\overline{b}(\boldsymbol{u},T,T) = 0, \tag{2.12}$$
$$|\overline{b}(\boldsymbol{u},T,\psi)| \leq \overline{N}\|\nabla\boldsymbol{u}\|_0\|\nabla T\|_0\|\nabla\psi\|_0, \quad \forall \boldsymbol{u}\in X, \forall T,\psi\in W, \tag{2.13}$$

where

$$N = \sup_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in X}\frac{|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|}{\|\nabla\boldsymbol{u}\|_0\|\nabla\boldsymbol{v}\|_0\|\nabla\boldsymbol{w}\|_0}, \tag{2.14}$$
$$\overline{N} = \sup_{\boldsymbol{u}\in X,\, T,\psi\in W}\frac{|\overline{b}(\boldsymbol{u},T,\psi)|}{\|\nabla\boldsymbol{u}\|_0\|\nabla T\|_0\|\nabla\psi\|_0}. \tag{2.15}$$

Additionally, the following properties hold in Sobolev spaces:

$$\|\boldsymbol{u}\|_0 \leq \alpha\|\nabla\boldsymbol{u}\|_0, \quad \forall \boldsymbol{u}\in X(\text{or } W), \tag{2.16}$$

where α is a positive constant depending only on Ω.

We make the following assumptions regarding the domain Ω and its boundary conditions [14, 15]:

(A1) For any $\boldsymbol{f}\in H^{l-2}(\Omega)^2$ and the Stokes problem

$$-\Delta\boldsymbol{v} + \nabla q = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v} = 0 \quad in \quad \Omega, \quad \boldsymbol{v}|_{\partial\Omega} = 0, \tag{2.17}$$

the solution $(\boldsymbol{v}, q)$ satisfies the following regularity result:

$$\|\boldsymbol{v}\|_l + \|q\|_{l-1} \leq \kappa\|\boldsymbol{f}\|_{l-2}, \quad l = 1,2,3, \tag{2.18}$$

where κ is a positive constant depending on Ω.

(A2) Let ε be a sufficiently small positive constant, and suppose that $\partial\Omega \in C^{k,j}$ with $k \geq 0$ and $j > 0$. Then, for $T_0, C_0 \in C^{k,j}(\partial\Omega)$ in Eq (2.1), there exist corresponding extensions $\tilde{T}_0, \tilde{C}_0 \in C^{k,j}(\Omega)$ satisfying

$$\begin{aligned} \|\tilde{T}_0\|_{k,r} &\leq \frac{\varepsilon}{8}, \quad k \geq 0,\ 1 \leq r \leq \infty, \\ \|\tilde{C}_0\|_{k,r} &\leq \frac{\varepsilon}{8}, \quad k \geq 0,\ 1 \leq r \leq \infty. \end{aligned} \tag{2.19}$$

2.2. *Variational formulation*

Based on the discussion in the previous subsection, the variational formulation of Eq (2.1) can be derived as follows: Find $(\boldsymbol{u}, p, T, C) \in X \times E \times W \times W$ such that for all $(\boldsymbol{v}, \varphi, \psi, s) \in X \times E \times W_0 \times W_0$, it satisfies

$$\begin{cases} \nu a(\boldsymbol{u}, \boldsymbol{v}) + D_a^{-1}(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + d(\varphi, \boldsymbol{u}) - d(p, \boldsymbol{v}) = (\beta_T T g + \beta_C C \boldsymbol{g}, \boldsymbol{v}), \\ \gamma \overline{a}(T, \psi) + \overline{b}(\boldsymbol{u}, T, \psi) = 0, \\ D_c \overline{a}(C, s) + \overline{b}(\boldsymbol{u}, C, s) = 0. \end{cases} \tag{2.20}$$

Lemma 2.1. *[6] If Assumptions (A1) and (A2) hold, and*

$$\frac{25 m \alpha^2 N \varepsilon}{3\nu^2} < 1, \quad \frac{32 m \alpha^2 \mu \overline{N} \varepsilon}{3\nu} < 1, \tag{2.21}$$

then Eq (2.20) *admits a unique solution* $(\boldsymbol{u}, p, T, C) \in X \times E \times W \times W$ *and satisfies*

$$\|\nabla \boldsymbol{u}\|_0 \leq \frac{5 m \alpha^2 \varepsilon}{3\nu}, \quad \|\nabla T\|_0 \leq \frac{\varepsilon}{2}, \quad \|\nabla C\|_0 \leq \frac{\varepsilon}{2}, \tag{2.22}$$

where $m = |\boldsymbol{g}| max\{|\beta_T|, |\beta_C|\}$, $\mu = max\{\gamma^{-1}, D_c^{-1}\}$, *and* ε *is a sufficiently small positive constant.*

To introduce iterative methods for the steady Darcy–Brinkman equation (2.1), it is necessary to discretize the equations based on the variational formulation. First, we introduce a triangulation of the aforementioned function spaces. Let the mesh size be a positive real number $0 < h < 1$. Then, we consider a collection $K_h = \{K : \bigcup_{K \subset \Omega} \overline{K} = \overline{\Omega}\}$ of nonoverlapping regular and uniform triangulations of $\overline{\Omega}$. For a fixed partition K_h, we define the following finite element subspaces X_h, E_h and W_h:

$$\begin{aligned} X_h &= \{\boldsymbol{v}_h \in X \cap C^0(\overline{\Omega})^2 : \boldsymbol{v}_{h|K} \in P_i(K)^2, \forall K \in K_h\}, \\ E_h &= \{q_h \in E \cap C^0(\overline{\Omega}) : q_{h|K} \in P_j(K), \forall K \in K_h\}, \\ W_h &= \{s_h \in W \cap C^0(\overline{\Omega}) : s_{h|K} \in P_k(K), \forall K \in K_h\}. \end{aligned}$$

Here, $P_i(K)$ denotes the set of piecewise polynomials of degree at most i on K, and i, j, k are non-negative integers. Next, we define $W_{0h} = W_h \cap W_0$. Meanwhile, we assume that the finite element spaces (X_h, E_h) satisfy the following discrete inf-sup condition: For any $q_h \in E_h$, there exists a positive constant $\beta > 0$, independent of h, such that

$$\sup_{\boldsymbol{v}_h \in X_h, \boldsymbol{v}_h \neq 0} \frac{|d(q_h, \boldsymbol{v}_h)|}{\|\nabla \boldsymbol{v}_h\|_0} \geq \beta \|q_h\|_0. \tag{2.23}$$

(A3) For any $\boldsymbol{v} \in D^{r+1}(\Omega)^2 \cap X$ and $q \in H^r(\Omega) \cap E$, there exist approximations $\pi_h \boldsymbol{v} \in X_h$ and $\rho_h q \in E_h$ satisfying

$$\begin{aligned} &\|\nabla(\boldsymbol{v} - \pi_h \boldsymbol{v})\|_0 \leq ch^r \|\boldsymbol{v}\|_{r+1}, \quad r = 1, 2, \\ &\|q - \rho_h q\|_0 \leq ch^r \|q\|_r, \quad r = 1, 2. \end{aligned} \tag{2.24}$$

With these preliminaries, the standard finite element Galerkin approximation of the variational equations is given as follows: Find $(\boldsymbol{u}_h, p_h, T_h, C_h) \in X_h \times E_h \times W_h \times W_h$ such that

$$\begin{cases} \nu a(\boldsymbol{u}_h, \boldsymbol{v}_h) - d(p_h, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{u}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + D_a^{-1}(\boldsymbol{u}_h, \boldsymbol{v}_h) & \\ = (\beta_T T_h \boldsymbol{g} + \beta_C C_h \boldsymbol{g}, \boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in X_h, \varphi_h \in E_h, \\ \gamma \overline{a}(T_h, \psi_h) + \overline{b}(\boldsymbol{u}_h, T_h, \psi_h) = 0, & \forall \psi_h \in W_{0h}, \\ D_c \overline{a}(C_h, s_h) + \overline{b}(\boldsymbol{u}_h, C_h, s_h) = 0, & \forall s_h \in W_{0h}. \end{cases} \tag{2.25}$$

We also present some primary results and error estimates for the finite element method.

Theorem 2.2. *[6] Under the assumptions of Lemma 2.1 and (A3), there exists a unique solution $(\boldsymbol{u}_h, p_h, T_h, C_h) \in X_h \times E_h \times W_h \times W_h$ to the finite element Galerkin approximation* (2.25) *provided that*

$$\|\nabla \boldsymbol{u}_h\|_0 \leq \frac{5m\alpha^2 \varepsilon}{3\nu}, \quad \|\nabla T_h\|_0 \leq \frac{\varepsilon}{2}, \quad \|\nabla C_h\|_0 \leq \frac{\varepsilon}{2}, \quad \|p_h\|_0 \leq \kappa. \tag{2.26}$$

Theorem 2.3. *[6] Under the assumptions of Theorem 2.2, suppose that $(\boldsymbol{u}, p, T, C) \in H^{r+1}(\Omega) \times H^r(\Omega) \times H^{r+1}(\Omega) \times H^{r+1}(\Omega)$ is a solution to problem* (2.20)*. Let $l = \min\{r, i, j+1, k\}$; then, the finite element solution $(\boldsymbol{u}_h, p_h, T_h, C_h) \in X_h \times E_h \times W_h \times W_h$ to Problem* (2.25) *satisfies*

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 + \|T - T_h\|_0 + \|C - C_h\|_0 \\ \leq\ & \kappa h^{l+1}(\|\boldsymbol{u}\|_{r+1,2} + \|p\|_{r,2} + \|T\|_{r+1,2} + \|C\|_{r+1,2}), \end{aligned} \tag{2.27}$$

$$\begin{aligned} &\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_0 + \|p - p_h\|_0 + \|\nabla(T - T_h)\|_0 + \|\nabla(C - C_h)\|_0 \\ \leq\ & \kappa h^{l}(\|\boldsymbol{u}\|_{r+1,2} + \|p\|_{r,2} + \|T\|_{r+1,2} + \|C\|_{r+1,2}), \quad r = 1, 2. \end{aligned} \tag{2.28}$$

3. Iterative methods based on finite element discretization

We propose three iterative methods—Stokes, Newton, and Oseen—to solve for the solution $(\boldsymbol{u}_h, p_h, T_h, C_h)$ in Eq (2.25). Each subsection introduces one method and establishes its convergence.

3.1. Stokes iterative method

For $n = 1, 2, \ldots$, given $(\boldsymbol{u}_h^{n-1}, T_h^{n-1}, C_h^{n-1})$, we find $(\boldsymbol{u}_h^n, p_h^n, T_h^n, C_h^n) \in X_h \times E_h \times W_h \times W_h$ such that for all $\boldsymbol{v}_h \in X_h$, $\varphi_h \in E_h$, $\psi_h \in W_{0h}$, $s_h \in W_{0h}$, the following holds:

$$\begin{cases} \nu a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + D_a^{-1}(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + b(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{u}_h^n) - d(p_h^n, \boldsymbol{v}_h) \\ = (\beta_T T_h^n \boldsymbol{g} + \beta_C C_h^n \boldsymbol{g}, \boldsymbol{v}_h), \\ \gamma \overline{a}(T_h^n, \psi_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, T_h^{n-1}, \psi_h) = 0, \\ D_c \overline{a}(C_h^n, s_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, C_h^{n-1}, s_h) = 0. \end{cases} \tag{3.1}$$

For computational convenience, we set the initial values as $\boldsymbol{u}_h^0 = 0$, $T_h^0 = 0$, $C_h^0 = 0$. The following boundedness theorem for the Stokes iterative method is a straightforward consequence of induction, analogous to that in [6]; we omit the proof for brevity.

Theorem 3.1. *The solution $(\boldsymbol{u}_h^n, p_h^n, T_h^n, C_h^n)$ obtained by the Stokes iterative method satisfies*

$$\|\nabla \boldsymbol{u}_h^n\|_0 \leq \frac{5m\alpha^2\varepsilon}{3\nu}, \quad \|\nabla T_h^n\|_0 \leq \frac{\varepsilon}{2}, \quad \|\nabla C_h^n\|_0 \leq \frac{\varepsilon}{2}, \quad \|p_h^n\|_0 \leq \kappa. \tag{3.2}$$

To prove the convergence of the Stokes iterative method, we define $(\boldsymbol{e}^n, \xi^n, \delta^n, \eta^n) = (\boldsymbol{u}_h - \boldsymbol{u}_h^n, T_h - T_h^n, C_h - C_h^n, p_h - p_h^n)$. In the following, we will demonstrate the convergence of the Stokes iterative method.

Theorem 3.2. *For the Stokes iterative method, under the assumptions of Lemma 2.1, let*

$$\tilde{D}_1 = \sqrt{\left(\frac{16\mu M\overline{N}}{3\nu} + \frac{20MN}{3\nu^2}\right)^2 - \frac{800\mu M^2 N\overline{N}}{9\nu^3}}. \tag{3.3}$$

If

$$0 < \varepsilon < \frac{2}{\tilde{D}_1 + \frac{16\mu M\overline{N}}{3\nu} + \frac{20MN}{3\nu^2}}, \tag{3.4}$$

the following convergence estimate holds: $\lim_{n\to\infty}(\|\nabla \boldsymbol{e}^n\|_0 + \|\nabla \xi^n\|_0 + \|\nabla \delta^n\|_0 + \|\eta^n\|_0) = 0$.

Proof. First, subtracting (2.25) from (3.1) yields the fundamental expressions. All subsequent calculations are based on these three expressions.

$$\begin{aligned} &\nu a(\boldsymbol{e}^n, \boldsymbol{v}_h) - d(\eta^n, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{e}^n) + D_a^{-1}(\boldsymbol{e}^n, \boldsymbol{v}_h) + b(\boldsymbol{e}^{n-1}, \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) \\ &+ b(\boldsymbol{u}_h, \boldsymbol{e}^{n-1}, \boldsymbol{v}_h) = (\beta_T \xi^n \boldsymbol{g} + \beta_C \delta^n \boldsymbol{g}, \boldsymbol{v}_h), \end{aligned} \tag{3.5}$$

$$\gamma \overline{a}(\xi^n, \psi_h) + \overline{b}(\boldsymbol{e}^{n-1}, T_h, \psi_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, \xi^{n-1}, \psi_h) = 0, \tag{3.6}$$

$$D_c \overline{a}(\delta^n, s_h) + \overline{b}(\boldsymbol{e}^{n-1}, C_h, s_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, \delta^{n-1}, s_h) = 0. \tag{3.7}$$

Because the expression for $\|\nabla \boldsymbol{e}^n\|_0$ involves $\|\nabla \xi^n\|_0$ and $\|\nabla \delta^n\|_0$, we first estimate $\|\nabla \xi^n\|_0$ and $\|\nabla \delta^n\|_0$ and then estimate $\|\nabla \boldsymbol{e}^n\|_0$. Finally, $\|\eta^n\|_0$ will be estimated.

Now, we consider $n = 1$. Because $\boldsymbol{u}_h^0 = T_h^0 = C_h^0 = 0$, it follows from (3.6) and (3.7) that we obtain

$$\gamma \overline{a}(\xi^1, \psi_h) + \overline{b}(\boldsymbol{u}_h, T_h, \psi_h) = 0, \tag{3.8}$$

$$D_c \overline{a}(\delta^1, s_h) + \overline{b}(\boldsymbol{u}_h, C_h, s_h) = 0. \tag{3.9}$$

According to (2.26) and (2.13), setting $\psi_h = \xi^1$ and $s_h = \delta^1$, we obtain

$$\begin{aligned} \|\nabla \xi^1\|_0 &\leq \gamma^{-1}\overline{N}\|\nabla \boldsymbol{u}_h\|_0 \|\nabla T_h\|_0 \\ &\leq \gamma^{-1}\overline{N}\frac{5m\alpha^2\varepsilon}{3\nu}\frac{\varepsilon}{2} \leq \frac{\varepsilon}{12}, \\ \|\nabla \delta^1\|_0 &\leq D_c^{-1}\overline{N}\|\nabla \boldsymbol{u}_h\|_0 \|\nabla C_h\|_0 \end{aligned} \tag{3.10}$$

$$\leq D_c^{-1}\overline{N}\frac{5m\alpha^2\varepsilon}{3\nu}\frac{\varepsilon}{2} \leq \frac{\varepsilon}{12}. \tag{3.11}$$

For $n = 1$, Eq (3.5) follows:

$$\nu a(\boldsymbol{e}^1, \boldsymbol{v}_h) - d(\eta^1, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{e}^1) + D_a^{-1}(\boldsymbol{e}^1, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = (\beta_T\xi^1\boldsymbol{g} + \beta_C\delta^1\boldsymbol{g}, \boldsymbol{v}_h). \tag{3.12}$$

For this equation, we take $\boldsymbol{v}_h = \boldsymbol{e}^1$, $\varphi_h = \eta^1$. Then, according to (2.10), we obtain

$$\begin{aligned}
\|\nabla\boldsymbol{e}^1\|_0 &\leq \nu^{-1}N\|\nabla\boldsymbol{u}_h\|_0^2 + \nu^{-1}m\alpha^2(\|\nabla\xi^1\|_0 + \|\nabla\delta^1\|_0) \\
&\leq \nu^{-1}N(\frac{5m\alpha^2\varepsilon}{3\nu})^2 + \nu^{-1}m\alpha^2\frac{\varepsilon}{6} \leq \nu^{-1}M\varepsilon,
\end{aligned} \tag{3.13}$$

where $M = (m\alpha^2)/2$. For $n \geq 2$ with $\mu = \max\{\gamma^{-1}, D_c^{-1}\}$, it follows from (3.6) that

$$\begin{aligned}
\gamma\overline{a}(\xi^n, \xi^n) &= \gamma\|\nabla\xi^n\|_0^2 \\
&\leq |\overline{b}(\boldsymbol{e}^{n-1}, T_h, \xi^n)| + |\overline{b}(\boldsymbol{u}_h^{n-1}, \xi^{n-1}, \xi^n)|.
\end{aligned} \tag{3.14}$$

Then,

$$\begin{aligned}
\|\nabla\xi^n\|_0 &\leq \mu\overline{N}\|\nabla\boldsymbol{e}^{n-1}\|_0\|\nabla T_h\|_0 + \mu\overline{N}\|\nabla\boldsymbol{u}_h^{n-1}\|_0\|\nabla\xi^{n-1}\|_0 \\
&\leq \mu\overline{N}\|\nabla\boldsymbol{e}^{n-1}\|_0\frac{\varepsilon}{2} + \overline{N}\frac{5m\alpha^2\varepsilon}{3\nu}\mu\|\nabla\xi^{n-1}\|_0 \\
&= \frac{1}{2}\mu\varepsilon\overline{N}\|\nabla\boldsymbol{e}^{n-1}\|_0 + \frac{10}{3}\nu^{-1}\mu M\varepsilon\overline{N}\|\nabla\xi^{n-1}\|_0.
\end{aligned} \tag{3.15}$$

Similarly, from (3.7), we obtain

$$\begin{aligned}
D_c\overline{a}(\delta^n, \delta^n) &= D_c\|\nabla\delta^n\|_0^2 \\
&\leq |\overline{b}(\boldsymbol{e}^{n-1}, C_h, \delta^n)| + |\overline{b}(\boldsymbol{u}_h^{n-1}, \delta^{n-1}, \delta^n)|.
\end{aligned} \tag{3.16}$$

Then,

$$\|\nabla\delta^n\|_0 \leq \frac{1}{2}\mu\varepsilon\overline{N}\|\nabla\boldsymbol{e}^{n-1}\|_0 + \frac{10}{3}\nu^{-1}\mu M\varepsilon\overline{N}\|\nabla\delta^{n-1}\|_0. \tag{3.17}$$

Substituting $\boldsymbol{v}_h = \boldsymbol{e}^n$, $\varphi_h = \eta^n$ into (3.5) yields

$$\nu a(\boldsymbol{e}^n, \boldsymbol{e}^n) - d(\eta^n, \boldsymbol{e}^n) + d(\eta^n, \boldsymbol{e}^n) + D_a^{-1}(\boldsymbol{e}^n, \boldsymbol{e}^n) + b(\boldsymbol{e}^{n-1}, \boldsymbol{u}_h^{n-1}, \boldsymbol{e}^n) + b(\boldsymbol{u}_h, \boldsymbol{e}^{n-1}, \boldsymbol{e}^n) = (\beta_T\xi^n\boldsymbol{g} + \beta_C\delta^n\boldsymbol{g}, \boldsymbol{e}^n).$$

From (2.10), (3.15), and (3.17), we obtain

$$\begin{aligned}
\nu a(\boldsymbol{e}^n, \boldsymbol{e}^n) &= \nu\|\nabla\boldsymbol{e}^n\|_0^2 \\
&\leq |b(\boldsymbol{e}^{n-1}, \boldsymbol{u}_h^{n-1}, \boldsymbol{e}^n)| + |b(\boldsymbol{u}_h, \boldsymbol{e}^{n-1}, \boldsymbol{e}^n)| + |(\beta_T\xi^n\boldsymbol{g} + \beta_C\delta^n\boldsymbol{g}, \boldsymbol{e}^n)|.
\end{aligned} \tag{3.18}$$

Then,

$$\|\nabla\boldsymbol{e}^n\|_0 \leq \nu^{-1}N\|\nabla\boldsymbol{e}^{n-1}\|_0\|\nabla\boldsymbol{u}_h^{n-1}\|_0 + \nu^{-1}N\|\nabla\boldsymbol{u}_h\|_0\|\nabla\boldsymbol{e}^{n-1}\|_0 + \nu^{-1}m\alpha^2(\|\nabla\xi^n\|_0 + \|\nabla\delta^n\|_0)$$

$$\leq 2\nu^{-1}N\|\nabla \boldsymbol{e}^{n-1}\|_0 \frac{5m\alpha^2\varepsilon}{3\nu} + \nu^{-1}m\alpha^2[\mu\varepsilon\overline{N}\|\nabla \boldsymbol{e}^{n-1}\|_0 + \frac{10}{3}\nu^{-1}\mu M\varepsilon\overline{N}(\|\nabla\xi^{n-1}\|_0 + \|\nabla\delta^{n-1}\|_0)]$$

$$\leq 2\nu^{-1}M\varepsilon\|\nabla \boldsymbol{e}^{n-1}\|_0\left(\frac{10}{3}\nu^{-1}N + \mu\overline{N}\right) + \frac{10}{3}\mu M\varepsilon\nu^{-2}m\alpha^2\overline{N}(\|\nabla\xi^{n-1}\|_0 + \|\nabla\delta^{n-1}\|_0). \tag{3.19}$$

We set

$$a_1 = \frac{m\alpha^2\varepsilon}{2\nu}, \quad b_1 = \frac{\varepsilon}{12}, \quad c_1 = \frac{\varepsilon}{12}, \tag{3.20}$$

and

$$a_n = \|\nabla \boldsymbol{e}^n\|_0, \quad b_n = \|\nabla\xi^n\|_0, \quad c_n = \|\nabla\delta^n\|_0, \quad n > 1, \tag{3.21}$$

$$A_1 = \frac{20MN\varepsilon}{3\nu^2} + \frac{2\mu M\overline{N}\varepsilon}{\nu}, \quad A_2 = \frac{20\mu M^2\overline{N}\varepsilon}{3\nu^2},$$

$$A_3 = \frac{1}{2}\mu\overline{N}\varepsilon, \quad A_4 = \frac{10}{3}\nu^{-1}\mu\overline{N}M\varepsilon. \tag{3.22}$$

Then, we have for $n > 1$

$$\begin{cases} a_n \leq A_1 a_{n-1} + A_2(b_{n-1} + c_{n-1}), \\ b_n \leq A_3 a_{n-1} + A_4 b_{n-1}, \\ c_n \leq A_3 a_{n-1} + A_4 c_{n-1}. \end{cases} \tag{3.23}$$

Let

$$z_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} A_1 & A_2 & A_2 \\ A_3 & A_4 & 0 \\ A_3 & 0 & A_4 \end{pmatrix}. \tag{3.24}$$

Then, we obtain $z_n \leq \mathbf{A}z_{n-1} \leq \mathbf{A}^{n-1}z_1$. The vector inequality $z_n \leq \mathbf{A}^{n-1}z_1$ is understood componentwise. To prove $z_n \to 0$, it suffices to show $\mathbf{A}^{n-1}z_1 \to 0$. For a non-negative matrix $\mathbf{A}$, the condition $\mathbf{A}^{n-1} \to 0$ is equivalent to $\rho(\mathbf{A}) < 1$, where $\rho(\mathbf{A})$ denotes the spectral radius. Therefore, $z_n \to 0$ is guaranteed whenever $\rho(\mathbf{A}) < 1$. The eigenvalues of the matrix $\mathbf{A}$ are $\lambda_1, \lambda_2, \lambda_3$, where

$$\lambda_1 = A_4, \tag{3.25}$$

$$\lambda_2 = \frac{A_1 + A_4 - \sqrt{(A_1 - A_4)^2 + 8A_2A_3}}{2}, \tag{3.26}$$

$$\lambda_3 = \frac{A_1 + A_4 + \sqrt{(A_1 - A_4)^2 + 8A_2A_3}}{2}. \tag{3.27}$$

Therefore, $z_n \to 0$ is guaranteed if $\rho(\mathbf{A}) = \lambda_3 < 1$, as $z_n \leq \mathbf{A}^{n-1}z_1$. Then, from $\lambda_3 < 1$, we obtain

$$2A_2A_3 < (1 - A_1)(1 - A_4). \tag{3.28}$$

Substituting A_1, A_2, A_3, A_4 into the inequality and simplifying the resulting quadratic inequality in ε, we obtain

$$\varepsilon < \frac{2}{\tilde{D}_1 + \frac{16\mu M\overline{N}}{3\nu} + \frac{20MN}{3\nu^2}}, \tag{3.29}$$

where

$$\tilde{D_1} = \sqrt{\left(\frac{16\mu M\overline{N}}{3\nu} + \frac{20MN}{3\nu^2}\right)^2 - \frac{800\mu M^2 N\overline{N}}{9\nu^3}}. \tag{3.30}$$

Finally, we obtain that when

$$\varepsilon < \frac{2}{\tilde{D_1} + \frac{16\mu M\overline{N}}{3\nu} + \frac{20MN}{3\nu^2}}, \tag{3.31}$$

we have

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{n\to\infty} b_n = 0, \quad \lim_{n\to\infty} c_n = 0. \tag{3.32}$$

Hence,

$$\lim_{n\to\infty}(\|\nabla \boldsymbol{e}^n\|_0 + \|\nabla \xi^n\|_0 + \|\nabla \delta^n\|_0) = 0. \tag{3.33}$$

Based on the previous results, we can estimate $\|\eta^n\|_0$. Setting $\varphi_h = 0$ in (3.12), we estimate $\|\eta^1\|_0$ by

$$\beta\|\eta^1\|_0 \leq \sup_{\boldsymbol{v}\in X_h} \frac{|d(\eta^1, \boldsymbol{v}_h)|}{\|\nabla \boldsymbol{v}_h\|_0} \tag{3.34}$$

and

$$d(\eta^1, \boldsymbol{v}_h) \leq |\nu a(\boldsymbol{e}^1, \boldsymbol{v}_h)| + |D_a^{-1}(\boldsymbol{e}^1, \boldsymbol{v}_h)| + |b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h)| + |(\beta_T \xi^1 \boldsymbol{g} + \beta_C \delta^1 \boldsymbol{g}, \boldsymbol{v}_h)|. \tag{3.35}$$

Then, we have

$$\begin{aligned}
\beta\|\eta^1\|_0 \leq & \nu\|\nabla \boldsymbol{e}^1\|_0 + D_a^{-1}\alpha^2\|\nabla \boldsymbol{e}^1\|_0 + N\|\nabla \boldsymbol{u}_h\|_0^2 + m\alpha^2(\|\nabla \xi^1\|_0 + \|\nabla \delta^1\|_0) \\
\leq & M\varepsilon + D_a^{-1}\alpha^2\nu^{-1}M\varepsilon + N\left(\frac{5m\alpha^2\varepsilon}{3\nu}\right)^2 + \frac{1}{3}M\varepsilon \\
\leq & \left(1 + D_a^{-1}\alpha^2\nu^{-1} + 1\right)M\varepsilon \\
\leq & Q_1 M\varepsilon,
\end{aligned} \tag{3.36}$$

where $Q_1 = 2 + D_a^{-1}\alpha^2\nu^{-1}$. Then, $\|\eta^1\|_0 \leq \beta^{-1}Q_1 M\varepsilon$. For $n \geq 2$, setting $\varphi_h = 0$ in (3.5) yields

$$\begin{aligned}
\beta\|\eta^n\|_0 \leq & \nu\|\nabla \boldsymbol{e}^n\|_0 + D_a^{-1}\alpha^2\|\nabla \boldsymbol{e}^n\|_0 + N\|\nabla \boldsymbol{e}^{n-1}\|_0\|\nabla \boldsymbol{u}_h^{n-1}\|_0 \\
& + N\|\nabla \boldsymbol{e}^{n-1}\|_0\|\nabla \boldsymbol{u}_h\|_0 + m\alpha^2(\|\nabla \xi^n\|_0 + \|\nabla \delta^n\|_0) \\
\leq & (\nu + D_a^{-1}\alpha^2)\|\nabla \boldsymbol{e}^n\|_0 + \frac{20}{3}\frac{MN\varepsilon}{\nu}\|\nabla \boldsymbol{e}^{n-1}\|_0 + m\alpha^2(\|\nabla \xi^n\|_0 + \|\nabla \delta^n\|_0).
\end{aligned} \tag{3.37}$$

Because $\|\nabla e^n\|_0$, $\|\nabla \xi^n\|_0$, $\|\nabla \delta^n\|_0$ have already been established, it follows that $\lim_{n\to\infty} \|\eta^n\|_0 = 0$. Thus, the proof is complete. □

Lemma 3.3. *Consider the inequality system*

$$\begin{cases} a_n \leq A_1 a_{n-1} + A_2(b_{n-1} + c_{n-1}), \\ b_n \leq A_3 a_{n-1} + A_4 b_{n-1}, & n \geq 2, \\ c_n \leq A_3 a_{n-1} + A_4 c_{n-1}, \end{cases} \tag{3.38}$$

with non-negative initial data a_1, b_1, c_1. Define a new sequence $(\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)$ by taking equalities and the initial values

$$\tilde{a}_1 = a_1, \quad \tilde{b}_1 = \tilde{c}_1 = \max(b_1, c_1), \tag{3.39}$$

and for $n \geq 2$,

$$\begin{cases} \tilde{a}_n = A_1 \tilde{a}_{n-1} + A_2(\tilde{b}_{n-1} + \tilde{c}_{n-1}), \\ \tilde{b}_n = A_3 \tilde{a}_{n-1} + A_4 \tilde{b}_{n-1}, \\ \tilde{c}_n = A_3 \tilde{a}_{n-1} + A_4 \tilde{c}_{n-1}. \end{cases} \tag{3.40}$$

Then, for any original sequence (a_n, b_n, c_n) satisfying the inequalities, we have

$$a_n \leq \tilde{a}_n, \quad b_n \leq \tilde{b}_n, \quad c_n \leq \tilde{c}_n, \quad \forall n \geq 1. \tag{3.41}$$

Consequently, the original sequence converges to zero if the equality sequence $(\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)$ converges to zero.

Proof. The system (3.38) can be written in the matrix form as

$$z_n \leq \mathbf{O} z_{n-1}, \quad n \geq 2, \tag{3.42}$$

where the vector inequality is understood componentwise, and for $n \geq 1$,

$$z_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \mathbf{O} = \begin{pmatrix} A_1 & A_2 & A_2 \\ A_3 & A_4 & 0 \\ A_3 & 0 & A_4 \end{pmatrix}. \tag{3.43}$$

Similarly, we define the corresponding equality sequence (3.40) by

$$\begin{aligned} &\tilde{z}_n = \mathbf{O} \tilde{z}_{n-1}, \quad n \geq 2, \\ &\tilde{z}_n = (\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)^T, \quad n \geq 1. \end{aligned} \tag{3.44}$$

Because all entries of $\mathbf{O}$ are non-negative, the map $\mathcal{T}_1(z_n) = \mathbf{O} z_n$ is monotone with respect to the componentwise partial order. Consequently, $\boldsymbol{X} \leq \boldsymbol{Y}$ implies $\mathcal{T}_1(\boldsymbol{X}) \leq \mathcal{T}_1(\boldsymbol{Y})$.

Suppose $\tilde{z}_n \to 0$. From the definition of the initial values, we have $z_1 \leq \tilde{z}_1$. We then have

$$z_2 \leq \mathcal{T}_1(z_1) \leq \mathcal{T}_1(\tilde{z}_1) = \tilde{z}_2. \tag{3.45}$$

Assume inductively that $z_{n-1} \leq \tilde{z}_{n-1}$. Then,

$$z_n \leq \mathcal{T}_1(z_{n-1}) \leq \mathcal{T}_1(\tilde{z}_{n-1}) = \tilde{z}_n. \tag{3.46}$$

Then, $z_n \leq \tilde{z}_n$ for all n and consequently $z_n \to 0$. Therefore, the inequality sequence converges to zero if the associated equality sequence converges to zero. □

3.2. Newton iterative method

For $n = 1, 2, \ldots$, given $(\boldsymbol{u}_h^{n-1}, T_h^{n-1}, C_h^{n-1})$, we find $(\boldsymbol{u}_h^n, p_h^n, T_h^n, C_h^n) \in X_h \times E_h \times W_h \times W_h$ such that for all $\boldsymbol{v}_h \in X_h$, $\varphi_h \in E_h$, $\psi_h \in W_{0h}$, $s_h \in W_{0h}$,

$$\begin{cases} \nu a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + D_a^{-1}(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + b(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) + b(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{u}_h^n) - d(p_h^n, \boldsymbol{v}_h) \\ = b(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) + (\beta_T T_h^n \boldsymbol{g} + \beta_C C_h^n \boldsymbol{g}, \boldsymbol{v}_h), \\ \gamma \overline{a}(T_h^n, \psi_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, T_h^n, \psi_h) + \overline{b}(\boldsymbol{u}_h^n, T_h^{n-1}, \psi_h) = \overline{b}(\boldsymbol{u}_h^{n-1}, T_h^{n-1}, \psi_h), \\ D_c \overline{a}(C_h^n, s_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, C_h^n, s_h) + \overline{b}(\boldsymbol{u}_h^n, C_h^{n-1}, s_h) = \overline{b}(\boldsymbol{u}_h^{n-1}, C_h^{n-1}, s_h). \end{cases} \tag{3.47}$$

For computational convenience, we also set the initial values as $\boldsymbol{u}_h^0 = 0$, $T_h^0 = 0$, $C_h^0 = 0$. Define $(\boldsymbol{e}^n, \xi^n, \delta^n, \eta^n) := (\boldsymbol{u}_h - \boldsymbol{u}_h^n, T_h - T_h^n, C_h - C_h^n, p_h - p_h^n)$.

Theorem 3.4. *[6] The solution $(\boldsymbol{u}_h^n, p_h^n, T_h^n, C_h^n)$ obtained by the Newton iterative method satisfies*

$$\|\nabla \boldsymbol{u}_h^n\|_0 \le \frac{5m\alpha^2\varepsilon}{3\nu}, \quad \|\nabla T_h^n\|_0 \le \frac{\varepsilon}{2}, \quad \|\nabla C_h^n\|_0 \le \frac{\varepsilon}{2}, \quad \|p_h^n\|_0 \le \kappa. \tag{3.48}$$

Lemma 3.5. *Consider the inequality system*

$$\begin{cases} a_n \le A_5 a_{n-1}^2 + A_6 a_{n-1}(b_{n-1} + c_{n-1}), \\ b_n \le A_7 a_n + A_8 a_{n-1} b_{n-1}, \\ c_n \le A_7 a_n + A_8 a_{n-1} c_{n-1}, \end{cases} \qquad n \ge 2, \tag{3.49}$$

with non-negative initial data a_1, b_1, c_1. Define a new sequence $(\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)$, for $n \ge 1$ by taking equalities and the initial values

$$\tilde{a}_1 = a_1, \quad \tilde{b}_1 = \tilde{c}_1 = \max(b_1, c_1), \tag{3.50}$$

and for $n \ge 2$,

$$\tilde{a}_n = A_5 \tilde{a}_{n-1}^2 + A_6 \tilde{a}_{n-1}(\tilde{b}_{n-1} + \tilde{c}_{n-1}), \tag{3.51}$$

$$\tilde{b}_n = A_7 \tilde{a}_n + A_8 \tilde{a}_{n-1} \tilde{b}_{n-1}, \tag{3.52}$$

$$\tilde{c}_n = A_7 \tilde{a}_n + A_8 \tilde{a}_{n-1} \tilde{c}_{n-1}. \tag{3.53}$$

Then, for any original sequence (a_n, b_n, c_n) satisfying the inequalities, we have

$$a_n \le \tilde{a}_n, \quad b_n \le \tilde{b}_n, \quad c_n \le \tilde{c}_n, \quad n \ge 1. \tag{3.54}$$

Consequently, the original sequence converges to zero if the equality sequence $(\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)$ converges to zero.

Proof. The system (3.49) can be written separately in the following matrix quadratic forms:

$$a_n \le \begin{pmatrix} a_{n-1} & b_{n-1} & c_{n-1} \end{pmatrix} \begin{pmatrix} A_5 & \frac{A_6}{2} & \frac{A_6}{2} \\ \frac{A_6}{2} & 0 & 0 \\ \frac{A_6}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{pmatrix}, \tag{3.55}$$

$$b_n \leq A_7 a_n + \begin{pmatrix} a_{n-1} & b_{n-1} & c_{n-1} \end{pmatrix} \begin{pmatrix} 0 & \frac{A_8}{2} & 0 \\ \frac{A_8}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{pmatrix}, \tag{3.56}$$

$$c_n \leq A_7 a_n + \begin{pmatrix} a_{n-1} & b_{n-1} & c_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{A_8}{2} \\ 0 & 0 & 0 \\ \frac{A_8}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{pmatrix}, \tag{3.57}$$

where $n \geq 2$. Write the inequalities in the compact matrix form

$$\mathbf{P}\boldsymbol{X}_n \leq R(\boldsymbol{X}_{n-1}), \quad n \geq 2, \tag{3.58}$$

where the vector inequality is understood componentwise, and

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ -A_7 & 1 & 0 \\ -A_7 & 0 & 1 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} A_5 & \frac{A_6}{2} & \frac{A_6}{2} \\ \frac{A_6}{2} & 0 & 0 \\ \frac{A_6}{2} & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & \frac{A_8}{2} & 0 \\ \frac{A_8}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} 0 & 0 & \frac{A_8}{2} \\ 0 & 0 & 0 \\ \frac{A_8}{2} & 0 & 0 \end{pmatrix}, \tag{3.59}$$

$$\boldsymbol{X}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad n \geq 1, \quad R(\boldsymbol{X}_{n-1}) = \begin{pmatrix} \boldsymbol{X}_{n-1}^T \mathbf{B}_1 \boldsymbol{X}_{n-1} \\ \boldsymbol{X}_{n-1}^T \mathbf{B}_2 \boldsymbol{X}_{n-1} \\ \boldsymbol{X}_{n-1}^T \mathbf{B}_3 \boldsymbol{X}_{n-1} \end{pmatrix}, \quad n \geq 2. \tag{3.60}$$

Similarly, Eqs (3.51)–(3.53) satisfy

$$\mathbf{P}\tilde{\boldsymbol{X}}_n = R(\tilde{\boldsymbol{X}}_{n-1}), \quad n \geq 2, \quad \tilde{\boldsymbol{X}}_n = (\tilde{a}_n, \tilde{b}_n, \tilde{c}_n)^T, \quad n \geq 1. \tag{3.61}$$

Because all entries of $\mathbf{P}^{-1}$ are non-negative, multiplying both sides of the vector inequality by $\mathbf{P}^{-1}$ preserves the componentwise order. Hence,

$$\boldsymbol{X}_n \leq \mathbf{P}^{-1} R(\boldsymbol{X}_{n-1}) =: \mathcal{T}_2(\boldsymbol{X}_{n-1}). \tag{3.62}$$

Similarly, Eqs (3.51)–(3.53) satisfy

$$\tilde{\boldsymbol{X}}_n = \mathcal{T}_2(\tilde{\boldsymbol{X}}_{n-1}). \tag{3.63}$$

Because $\mathbf{P}^{-1}$ has non-negative entries, and $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3$ are symmetric, non-negative matrices, the quadratic forms $\boldsymbol{X}^T \mathbf{B}_i \boldsymbol{X}$ are nondecreasing with respect to the componentwise order (i.e., if $0 \leq \boldsymbol{X} \leq \boldsymbol{Y}$, then $\boldsymbol{X}^T \mathbf{B}_i \boldsymbol{X} \leq \boldsymbol{Y}^T \mathbf{B}_i \boldsymbol{Y}$). Hence, the map $\mathcal{T}_2(\boldsymbol{X})$ is also componentwise nondecreasing.

From the definition of the initial values, we have $\boldsymbol{X}_1 \leq \tilde{\boldsymbol{X}}_1$ componentwise. The original sequence satisfies $\boldsymbol{X}_n \leq \mathcal{T}_2(\boldsymbol{X}_{n-1})$, and the equality sequence satisfies $\tilde{\boldsymbol{X}}_n = \mathcal{T}_2(\tilde{\boldsymbol{X}}_{n-1})$.

We prove the assertion. Suppose $\tilde{\boldsymbol{X}}_n \to 0$ componentwise. Because $\boldsymbol{X}_1 \leq \tilde{\boldsymbol{X}}_1$ and $\mathcal{T}_2$ is monotone, we have

$$\boldsymbol{X}_2 \leq \mathcal{T}_2(\boldsymbol{X}_1) \leq \mathcal{T}_2(\tilde{\boldsymbol{X}}_1) = \tilde{\boldsymbol{X}}_2. \tag{3.64}$$

Assume inductively that $\boldsymbol{X}_{n-1} \leq \tilde{\boldsymbol{X}}_{n-1}$. Then,

$$\boldsymbol{X}_n \leq \mathcal{T}_2(\boldsymbol{X}_{n-1}) \leq \mathcal{T}_2(\tilde{\boldsymbol{X}}_{n-1}) = \tilde{\boldsymbol{X}}_n. \tag{3.65}$$

Thus, $\boldsymbol{X}_n \leq \tilde{\boldsymbol{X}}_n$ for all n and consequently $\boldsymbol{X}_n \to 0$.

We conclude that the inequality sequence converges to zero if the associated equality sequence converges to zero. □

Theorem 3.6. *Set*

$$\varepsilon_1 = \frac{\mu\overline{N} + \frac{3N}{2\nu} - \sqrt{\left(\sqrt{13}\mu\overline{N} - \frac{3N}{2\nu}\right)^2 + \frac{(3\sqrt{13}-3)\mu\overline{N}N}{\nu}}}{\frac{3m\alpha^2\mu\overline{N}}{\nu}\left(\frac{N}{2\nu} - \mu\overline{N}\right)}, \tag{3.66}$$

$$\varepsilon_2 = \frac{\mu\overline{N} + \frac{3N}{2\nu} + \sqrt{\left(\sqrt{13}\mu\overline{N} - \frac{3N}{2\nu}\right)^2 + \frac{(3\sqrt{13}-3)\mu\overline{N}N}{\nu}}}{\frac{3m\alpha^2\mu\overline{N}}{\nu}\left(\frac{N}{2\nu} - \mu\overline{N}\right)}, \tag{3.67}$$

$$F_2 = -\frac{m\alpha^2}{2\nu}\left(\mu\overline{N} + \frac{3N}{2\nu}\right). \tag{3.68}$$

If $\varepsilon > 0$ satisfies

- $0 < \varepsilon < -1/F_2 = \nu^2/(m\alpha^2 N)$ *when* $N/(2\nu) = \mu\overline{N}$,
- $0 < \varepsilon < \varepsilon_1$ *when* $N/(2\nu) > \mu\overline{N}$,
- $0 < \varepsilon < \varepsilon_1$ *when* $N/(2\nu) < \mu\overline{N}$,

then the Newton iterative method satisfies the convergence estimate $\lim_{n\to\infty}(\|\nabla \boldsymbol{e}^n\|_0 + \|\nabla\xi^n\|_0 + \|\nabla\delta^n\|_0 + \|\eta^n\|_0) = 0$.

Proof. Subtracting Eq (2.25) from Eq (3.47), we obtain

$$\begin{aligned}
&\nu a(\boldsymbol{e}^n, \boldsymbol{v}_h) + D_a^{-1}(\boldsymbol{e}^n, \boldsymbol{v}_h) - d(\eta^n, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{e}^n) + b(\boldsymbol{e}^n, \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) \\
&\quad + b(\boldsymbol{u}_h^{n-1}, \boldsymbol{e}^n, \boldsymbol{v}_h) + b(\boldsymbol{e}^{n-1}, \boldsymbol{e}^{n-1}, \boldsymbol{v}_h) = (\beta_T\xi^n\boldsymbol{g} + \beta_C\delta^n\boldsymbol{g}, \boldsymbol{v}_h), \qquad (3.69)\\
&\gamma\overline{a}(\xi^n, \psi_h) + \overline{b}(\boldsymbol{e}^n, T_h^{n-1}, \psi_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, \xi^n, \psi_h) + \overline{b}(\boldsymbol{e}^{n-1}, \xi^{n-1}, \psi_h) = 0, \qquad (3.70)\\
&D_c\overline{a}(\delta^n, s_h) + \overline{b}(\boldsymbol{e}^n, C_h^{n-1}, s_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, \delta^n, s_h) + \overline{b}(\boldsymbol{e}^{n-1}, \delta^{n-1}, s_h) = 0. \qquad (3.71)
\end{aligned}$$

We first consider the case $n = 1$. Because $\boldsymbol{u}_h^0 = T_h^0 = C_h^0 = 0$, Eqs (3.70) and (3.71) reduce to

$$\gamma\overline{a}(\xi^1, \psi_h) + \overline{b}(\boldsymbol{u}_h, T_h, \psi_h) = 0, \tag{3.72}$$

$$D_c\overline{a}(\delta^1, s_h) + \overline{b}(\boldsymbol{u}_h, C_h, s_h) = 0. \tag{3.73}$$

Using (2.10) and (2.26), we have

$$\|\nabla\xi^1\|_0 \le \gamma^{-1}\overline{N}\|\nabla\boldsymbol{u}_h\|_0\|\nabla T_h\|_0 \le \mu\overline{N}\frac{\varepsilon}{2}\frac{5m\alpha^2\varepsilon}{3\nu} \le \frac{\varepsilon}{12}, \tag{3.74}$$

$$\|\nabla\delta^1\|_0 \le D_c^{-1}\overline{N}\|\nabla\boldsymbol{u}_h\|_0\|\nabla C_h\|_0 \le \mu\overline{N}\frac{\varepsilon}{2}\frac{5m\alpha^2\varepsilon}{3\nu} \le \frac{\varepsilon}{12}. \tag{3.75}$$

For (3.69), we have

$$\nu a(\boldsymbol{e}^1, \boldsymbol{v}_h) + D_a^{-1}(\boldsymbol{e}^1, \boldsymbol{v}_h) - d(\eta^1, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{e}^1) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = (\beta_T\xi^1\boldsymbol{g} + \beta_C\delta^1\boldsymbol{g}, \boldsymbol{v}_h). \tag{3.76}$$

Taking $\boldsymbol{v}_h = \boldsymbol{e}^1, \varphi_h = \eta^1$ in (3.76) and using (2.10), we have

$$\|\nabla\boldsymbol{e}^1\|_0 \le \nu^{-1}N\|\nabla\boldsymbol{u}_h\|_0^2 + \nu^{-1}m\alpha^2(\|\nabla\xi^1\|_0 + \|\nabla\delta^1\|_0) \le \nu^{-1}N\frac{25m^2\alpha^4\varepsilon^2}{9\nu^2} + \frac{m\alpha^2\varepsilon}{6\nu} \le \nu^{-1}M\varepsilon, \tag{3.77}$$

where $M = (m\alpha^2)/2$. Taking $\varphi_h = 0$ in (3.76), by (2.26), we obtain

$$\begin{aligned}\|\eta^1\|_0 &\le \beta^{-1}\nu\|\nabla \boldsymbol{e}^1\|_0 + \beta^{-1}D_a^{-1}\alpha^2\|\nabla \boldsymbol{e}^1\|_0 + \beta^{-1}N\|\nabla \boldsymbol{u}_h\|_0^2 + \beta^{-1}m\alpha^2(\|\nabla\xi^1\|_0 + \|\nabla\delta^1\|_0)\\ &\le \beta^{-1}M\varepsilon + \beta^{-1}D_a^{-1}\alpha^2\nu^{-1}M\varepsilon + \frac{25m^2\alpha^4\varepsilon^2 N}{9\nu^2\beta} + \beta^{-1}m\alpha^2(\|\nabla\xi^1\|_0 + \|\nabla\delta^1\|_0)\\ &\le \beta^{-1}(2 + D_a^{-1}\alpha^2\nu^{-1})M\varepsilon. \end{aligned}\tag{3.78}$$

Taking $\psi_h = \xi^n$ and $s_h = \delta^n$ in (3.70) and (3.71), we obtain

$$\gamma\overline{a}(\xi^n, \xi^n) + \overline{b}(\boldsymbol{e}^n, T_h^{n-1}, \xi^n) + \overline{b}(\boldsymbol{e}^{n-1}, \xi^{n-1}, \xi^n) = 0, \tag{3.79}$$

$$D_c\overline{a}(\delta^n, \delta^n) + \overline{b}(\boldsymbol{e}^n, C_h^{n-1}, \delta^n) + \overline{b}(\boldsymbol{e}^{n-1}, \delta^{n-1}, \delta^n) = 0. \tag{3.80}$$

It follows from (2.13) that

$$\|\nabla\xi^n\|_0 \le \frac{\varepsilon}{2}\mu\overline{N}\|\nabla\boldsymbol{e}^n\|_0 + \mu\overline{N}\|\nabla\boldsymbol{e}^{n-1}\|_0\|\nabla\xi^{n-1}\|_0, \tag{3.81}$$

$$\|\nabla\delta^n\|_0 \le \frac{\varepsilon}{2}\mu\overline{N}\|\nabla\boldsymbol{e}^n\|_0 + \mu\overline{N}\|\nabla\boldsymbol{e}^{n-1}\|_0\|\nabla\delta^{n-1}\|_0. \tag{3.82}$$

Taking $\boldsymbol{v}_h = \boldsymbol{e}^n$, $\varphi_h = \eta^n$ in (3.69) leads to

$$\nu a(\boldsymbol{e}^n, \boldsymbol{e}^n) + D_a^{-1}(\boldsymbol{e}^n, \boldsymbol{e}^n) + b(\boldsymbol{e}^n, \boldsymbol{u}_h^{n-1}, \boldsymbol{e}^n) + b(\boldsymbol{e}^{n-1}, \boldsymbol{e}^{n-1}, \boldsymbol{e}^n) = (\beta_T\xi^n\boldsymbol{g} + \beta_C\delta^n\boldsymbol{g}, \boldsymbol{e}^n). \tag{3.83}$$

We then deduce from (2.10), (3.81), and (3.82) that

$$(\nu - N\|\nabla\boldsymbol{u}_h^{n-1}\|_0 - m\alpha^2\mu\overline{N}\varepsilon)\|\nabla\boldsymbol{e}^n\|_0 \le N\|\nabla\boldsymbol{e}^{n-1}\|_0^2 + m\alpha^2\mu\overline{N}\|\nabla\boldsymbol{e}^{n-1}\|_0(\|\nabla\xi^{n-1}\|_0 + \|\nabla\delta^{n-1}\|_0). \tag{3.84}$$

Using (3.48) and (2.21), we have

$$\|\nabla\boldsymbol{e}^n\|_0 \le \frac{3}{2}\nu^{-1}N\|\nabla\boldsymbol{e}^{n-1}\|_0^2 + \frac{3}{2}\nu^{-1}m\alpha^2\mu\overline{N}\|\nabla\boldsymbol{e}^{n-1}\|_0(\|\nabla\xi^{n-1}\|_0 + \|\nabla\delta^{n-1}\|_0). \tag{3.85}$$

Taking $\varphi_h = 0$ in (3.69), by (2.10) and (3.48), we obtain

$$\begin{aligned}\|\eta^n\|_0 &\le \beta^{-1}(\nu + D_a^{-1}\alpha^2 + 2N\|\nabla\boldsymbol{u}_h^{n-1}\|_0)\|\nabla\boldsymbol{e}^n\|_0 + \beta^{-1}N\|\nabla\boldsymbol{e}^{n-1}\|_0^2 + \beta^{-1}m\alpha^2(\|\nabla\xi^n\|_0 + \|\nabla\delta^n\|_0)\\ &\le \beta^{-1}(\nu + D_a^{-1}\alpha^2 + \frac{2}{5}\nu)\|\nabla\boldsymbol{e}^n\|_0 + \beta^{-1}N\|\nabla\boldsymbol{e}^{n-1}\|_0^2 + \beta^{-1}m\alpha^2(\|\nabla\xi^n\|_0 + \|\nabla\delta^n\|_0). \end{aligned}\tag{3.86}$$

We set

$$a_n = \|\nabla\boldsymbol{e}^n\|_0, \quad b_n = \|\nabla\xi^n\|_0, \quad c_n = \|\nabla\delta^n\|_0, \quad n \ge 1, \tag{3.87}$$

$$A_5 = \frac{3}{2}\nu^{-1}N, \quad A_6 = \frac{3}{2}\nu^{-1}m\alpha^2\mu\overline{N}, \quad A_7 = \frac{\varepsilon}{2}\mu\overline{N}, \quad A_8 = \mu\overline{N}. \tag{3.88}$$

Using (3.85), (3.81), and (3.82), we have

$$\begin{cases} a_n \le A_5 a_{n-1}^2 + A_6 a_{n-1}(b_{n-1} + c_{n-1}), \\ b_n \le A_7 a_n + A_8 a_{n-1} b_{n-1}, \\ c_n \le A_7 a_n + A_8 a_{n-1} c_{n-1}, \end{cases} \qquad n \ge 2, \tag{3.89}$$

which can be written in matrix form as

$$\mathbf{P}_2 \boldsymbol{X}_n \le \begin{pmatrix} \boldsymbol{X}_{n-1}^T \mathbf{B}_4 \boldsymbol{X}_{n-1} \\ \boldsymbol{X}_{n-1}^T \mathbf{B}_5 \boldsymbol{X}_{n-1} \\ \boldsymbol{X}_{n-1}^T \mathbf{B}_6 \boldsymbol{X}_{n-1} \end{pmatrix}, \quad n \ge 2, \tag{3.90}$$

where $\boldsymbol{X}_n = (a_n, b_n, c_n)^T, n \ge 1$, and

$$\boldsymbol{X}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ -A_7 & 1 & 0 \\ -A_7 & 0 & 1 \end{pmatrix},$$
$$\mathbf{B}_4 = \begin{pmatrix} A_5 & \frac{A_6}{2} & \frac{A_6}{2} \\ \frac{A_6}{2} & 0 & 0 \\ \frac{A_6}{2} & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_5 = \begin{pmatrix} 0 & \frac{A_8}{2} & 0 \\ \frac{A_8}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B}_6 = \begin{pmatrix} 0 & 0 & \frac{A_8}{2} \\ 0 & 0 & 0 \\ \frac{A_8}{2} & 0 & 0 \end{pmatrix}. \tag{3.91}$$

By Lemma 3.5, it is sufficient to analyze the associated equality sequence with the majorizing initial values. Due to the identity $\tilde{b}_n = \tilde{c}_n$ for all n, we omit the tildes, set $b_n = c_n =: u_n$, and work directly with the equalities.

Because a_1, b_1, c_1 are non-negative, and all entries of $\mathbf{P}_2^{-1}, \mathbf{B}_4, \mathbf{B}_5, \mathbf{B}_6$ are non-negative, by induction using (3.89), we have $a_n, b_n, c_n > 0$ for all n; that is, $\boldsymbol{X}_n > 0$. Let $u_n = b_n = c_n$, and set $r_n = u_n/a_n$. Then, we obtain

$$a_n = A_5 a_{n-1}^2 + 2A_6 a_{n-1} u_{n-1} = a_{n-1}^2 (A_5 + 2A_6 r_{n-1}) \tag{3.92}$$

and

$$\begin{aligned} u_n &= A_7 a_n + A_8 a_{n-1} u_{n-1} \\ &= A_7 a_{n-1}^2 (A_5 + 2A_6 r_{n-1}) + A_8 a_{n-1} u_{n-1} \\ &= a_{n-1}^2 (A_5 A_7 + 2A_6 A_7 r_{n-1} + A_8 r_{n-1}). \end{aligned} \tag{3.93}$$

We then have

$$r_n = \frac{u_n}{a_n} = \frac{A_5 A_7 + (2A_6 A_7 + A_8) r_{n-1}}{A_5 + 2A_6 r_{n-1}} = f(r_{n-1}). \tag{3.94}$$

We now study the fixed point of f. First, we show that f is strictly increasing for $r > 0$ by computing its derivative:

$$\begin{aligned} f'(r) &= \frac{(2A_6 A_7 + A_8)(A_5 + 2A_6 r_{n-1}) - 2A_6 [A_5 A_7 + (2A_6 A_7 + A_8) r_{n-1}]}{(A_5 + 2A_6 r_{n-1})^2} \\ &= \frac{A_5 A_8}{(A_5 + 2A_6 r_{n-1})^2} > 0, \end{aligned} \tag{3.95}$$

which means that f is strictly increasing on $(0, \infty)$.

Define $g(r) = f(r) - r$. Then, g is continuous on $(0, \infty)$, and

$$g(r) = \frac{A_5 A_7 + (2A_6 A_7 + A_8) r}{A_5 + 2A_6 r} - r$$

$$= \frac{-2A_6r^2 + (2A_6A_7 + A_8 - A_5)r + A_5A_7}{A_5 + 2A_6r}. \tag{3.96}$$

Because the denominator $A_5 + 2A_6r$ is positive, the sign of $g(r)$ is determined by its numerator

$$g_1(r) = -2A_6r^2 + (2A_6A_7 + A_8 - A_5)r + A_5A_7. \tag{3.97}$$

Observe that $g_1(r)$ is a quadratic function opening downward. Moreover, $g_1(0) = A_5A_7 > 0$ because $A_5, A_7 > 0$. Consequently, $g_1(r)$ has exactly one positive zero, denoted by r^*, and $g_1(r) > 0$ for $0 < r < r^*$, and $g_1(r) < 0$ for $r > r^*$. Hence, the same sign properties hold for $g(r)$. The unique positive root r^* is given by

$$r^* = \frac{2A_6A_7 + A_8 - A_5 + \sqrt{(2A_6A_7 + A_8 - A_5)^2 + 8A_5A_6A_7}}{4A_6} > 0. \tag{3.98}$$

Now, consider the sequence $\{r_n\}$ defined by $r_n = f(r_{n-1})$ with $r_1 = u_1/a_1 > 0$. Recall that f is strictly increasing on $(0, \infty)$ and $r^* = f(r^*)$. We examine the convergence of r_n to r^* according to the initial value r_1.

- If $r_1 = r^*$, then trivially, $r_n = r^*$ for all n.
- If $0 < r_1 < r^*$, from the sign properties of $g(r)$, we have $r < f(r) < f(r^*) = r^*$ for $0 < r < r^*$. Applying this to r_1 yields $r_1 < r_2 = f(r_1) < f(r^*) = r^*$. Assuming $0 < r_{n-1} < r^*$, the same reasoning gives $r_{n-1} < r_n = f(r_{n-1}) < f(r^*) = r^*$. Hence, by induction, $\{r_n\}$ is strictly increasing and bounded above by r^* and therefore converges to a limit, which must equal r^* by continuity and uniqueness of the fixed point.
- If $r_1 > r^*$, then $f(r) < r$ for $r > r^*$; also, $f(r) > r^*$ because f is increasing, and $f(r^*) = r^*$. Thus, $r_2 = f(r_1) < r_1$, and $r_2 = f(r_1) > f(r^*) = r^*$. By induction, we obtain $r^* < r_n < r_{n-1}$ for all n, so $\{r_n\}$ is strictly decreasing and bounded below by r^*, converging to r^*.

In all cases, regardless of the initial value of $r_1 > 0$, we have $\lim_{n\to\infty} r_n = r^*$.

From the recurrence $a_n = a_{n-1}^2(A_5 + 2A_6r_{n-1})$ and the fact that $r_n \to r^*$, we have

$$\begin{aligned}\lambda^* &= \lim_{n\to\infty}(A_5 + 2A_6r_{n-1}) = A_5 + 2A_6r^* \\ &= \frac{A_5 + 2A_6A_7 + A_8 + \sqrt{(A_5 + 2A_6A_7 + A_8)^2 - 4A_5A_8}}{2}.\end{aligned} \tag{3.99}$$

Because $r^* > 0$, it follows that $\lambda^* > A_5$. Iterating the recurrence yields

$$a_n \sim \lambda^* a_{n-1}^2 = \frac{(\lambda^* a_1)^{2^{n-1}}}{\lambda^*} \quad \text{as } n \to \infty. \tag{3.100}$$

Consequently, if $\lambda^* a_1 < 1$, then $a_n \to 0$ exponentially fast. Otherwise, the sequence does not converge to 0: It either diverges to ∞ (if $\lambda^* a_1 > 1$) or tends to the positive constant $1/\lambda^*$ (if $\lambda^* a_1 = 1$).

Recall that

$$\lambda^* = \frac{A_5 + 2A_6A_7 + A_8 + \sqrt{(A_5 + 2A_6A_7 + A_8)^2 - 4A_5A_8}}{2}, \quad a_1 = \frac{m\alpha^2\varepsilon}{2\nu}. \tag{3.101}$$

We have

$$a_1^2 A_5 A_8 - a_1(A_5 + 2A_6A_7 + A_8) + 1 > 0. \tag{3.102}$$

Using (3.88), we obtain

$$\frac{3m^2\alpha^4\mu\overline{N}}{4\nu^2}\left(\frac{N}{2\nu} - \mu\overline{N}\right)\varepsilon^2 - \frac{m\alpha^2}{2\nu}\left(\mu\overline{N} + \frac{3N}{2\nu}\right)\varepsilon + 1 > 0. \tag{3.103}$$

Define the coefficients by

$$F_1 = \frac{3m^2\alpha^4\mu\overline{N}}{4\nu^2}\left(\frac{N}{2\nu} - \mu\overline{N}\right), \quad F_2 = -\frac{m\alpha^2}{2\nu}\left(\mu\overline{N} + \frac{3N}{2\nu}\right) < 0. \tag{3.104}$$

Set

$$G(\varepsilon) := F_1\varepsilon^2 + F_2\varepsilon + 1. \tag{3.105}$$

If $F_1 = 0$ (i.e., $N/(2\nu) = \mu\overline{N}$), (3.103) reduces to $F_2\varepsilon + 1 > 0$, so we have $0 < \varepsilon < -1/F_2 = \nu^2/(m\alpha^2 N)$. If $F_1 \neq 0$, the discriminant satisfies

$$\begin{aligned}
\Delta = F_2^2 - 4F_1 &= \frac{m^2\alpha^4}{4\nu^2}\left(\mu\overline{N} + \frac{3N}{2\nu}\right)^2 - \frac{3m^2\alpha^4\mu\overline{N}}{\nu^2}\left(\frac{N}{2\nu} - \mu\overline{N}\right) \\
&= \frac{m^2\alpha^4}{4\nu^2}\left[\left(\mu\overline{N} + \frac{3N}{2\nu}\right)^2 - 12\mu\overline{N}\left(\frac{N}{2\nu} - \mu\overline{N}\right)\right] \\
&= \frac{m^2\alpha^4}{4\nu^2}\left[\left(\sqrt{13}\mu\overline{N} - \frac{3N}{2\nu}\right)^2 + \frac{(3\sqrt{13} - 3)\mu\overline{N}N}{\nu}\right] \\
&> 0.
\end{aligned} \tag{3.106}$$

The equation $F_1\varepsilon^2 + F_2\varepsilon + 1 = 0$ has two distinct real roots, denoted by $\varepsilon_1, \varepsilon_2$, where

$$\begin{aligned}
\varepsilon_1 &= \frac{-F_2 - \sqrt{\Delta}}{2F_1} \\
&= \frac{\mu\overline{N} + \frac{3N}{2\nu} - \sqrt{\left(\sqrt{13}\mu\overline{N} - \frac{3N}{2\nu}\right)^2 + \frac{(3\sqrt{13}-3)\mu\overline{N}N}{\nu}}}{\frac{3m\alpha^2\mu\overline{N}}{\nu}\left(\frac{N}{2\nu} - \mu\overline{N}\right)},
\end{aligned} \tag{3.107}$$

$$\varepsilon_2 = \frac{\mu\overline{N} + \frac{3N}{2\nu} + \sqrt{\left(\sqrt{13}\mu\overline{N} - \frac{3N}{2\nu}\right)^2 + \frac{(3\sqrt{13}-3)\mu\overline{N}N}{\nu}}}{\frac{3m\alpha^2\mu\overline{N}}{\nu}\left(\frac{N}{2\nu} - \mu\overline{N}\right)}. \tag{3.108}$$

If $F_1 > 0$ (i.e., $N/(2\nu) > \mu\overline{N}$), the smaller root ε_1 is positive, and the inequality holds for $0 < \varepsilon < \varepsilon_1$. If $F_1 < 0$ (i.e., $N/(2\nu) < \mu\overline{N}$), the roots $\varepsilon_1 > 0$ and $\varepsilon_2 < 0$ hold. Then, the inequatlity holds for $0 < \varepsilon < \varepsilon_1$.

Summarizing the above cases, the convergence condition can be expressed uniformly as $0 < \varepsilon < \min\{-1/F_2, \varepsilon_1\}$. For such ε, we have $\lambda^* a_1 < 1$, which implies $a_n \to 0$. Moreover, because $b_n = c_n =$

$a_n r_n$ and $r_n \to r^*$, it follows that $b_n \to 0$, and $c_n \to 0$. By Lemma 3.5, the original inequality sequence converges to zero.

From the preceding analysis, we have established that for all $0 < \varepsilon < \min\{-1/F_2, \varepsilon_1\}$, the sequences a_n, b_n, c_n converge to zero:

$$a_n = \|\nabla \boldsymbol{e}^n\|_0 \to 0, \qquad b_n = \|\nabla \xi^n\|_0 \to 0, \qquad c_n = \|\nabla \delta^n\|_0 \to 0. \tag{3.109}$$

It remains to prove the convergence of $\|\eta^n\|_0$. Using the estimate (3.86), the coefficients on the right-hand side are non-negative constants independent of n. Therefore, substituting a_n, b_n, c_n into the right-hand side and using their convergence to zero, we obtain $\|\eta^n\|_0 \to 0$ as $n \to \infty$.

Thus, all error components converge to zero. This completes the proof. □

3.3. *Oseen iterative method*

For $n = 1, 2, \ldots$, given $(\boldsymbol{u}_h^{n-1}, T_h^{n-1}, C_h^{n-1})$, find $(\boldsymbol{u}_h^n, p_h^n, T_h^n, C_h^n) \in X_h \times E_h \times W_h \times W_h$ such that for all $\boldsymbol{v}_h \in X_h$, $\varphi_h \in E_h$, $\psi_h \in W_{0h}$, $s_h \in W_{0h}$,

$$\begin{cases} \nu a(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + D_a^{-1}(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + b(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h) - d(p_h^n, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{u}_h^n) \\ = (\beta_T T_h^n \boldsymbol{g} + \beta_C C_h^n \boldsymbol{g}, \boldsymbol{v}_h), \\ \gamma \overline{a}(T_h^n, \psi_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, T_h^n, \psi_h) = 0, \\ D_c \overline{a}(C_h^n, s_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, C_h^n, s_h) = 0. \end{cases} \tag{3.110}$$

Set the initial values as $\boldsymbol{u}_h^0 = 0$, $T_h^0 = 0$, $C_h^0 = 0$. Let $(\boldsymbol{e}^n, \xi^n, \delta^n, \eta^n) = (\boldsymbol{u}_h - \boldsymbol{u}_h^n, T_h - T_h^n, C_h - C_h^n, p_h - p_h^n)$. The following boundedness theorem for the Oseen iterative method is a straightforward consequence of induction, analogous to [6]; we omit the proof for brevity.

Theorem 3.7. *The solution $(\boldsymbol{u}_h^n, p_h^n, T_h^n, C_h^n)$ obtained by the Oseen iterative method satisfies*

$$\|\nabla \boldsymbol{u}_h^n\|_0 \le \frac{5m\alpha^2\varepsilon}{3\nu}, \quad \|\nabla T_h^n\|_0 \le \frac{\varepsilon}{2}, \quad \|\nabla C_h^n\|_0 \le \frac{\varepsilon}{2}, \quad \|p_h^n\|_0 \le \kappa. \tag{3.111}$$

Theorem 3.8. *For the Oseen iterative method, the following estimate holds:*

$$\|\nabla \boldsymbol{e}^n\|_0 \le \begin{cases} \nu^{-1} M\varepsilon, \quad n = 1, \\ \nu^{-1} M\varepsilon \left(\frac{25MN\varepsilon}{6\nu^2} + \frac{15\mu\overline{N}M\varepsilon}{2(3\nu - 10M\varepsilon\mu\overline{N})} \right)^{n-1}, \quad n \ge 2, \end{cases} \tag{3.112}$$

$$\|\nabla \xi^n\|_0 \le \begin{cases} \frac{\varepsilon}{12}, \quad n = 1, \\ \frac{3\mu\overline{N}M\varepsilon^2}{2(3\nu - 10M\varepsilon\mu\overline{N})} \left(\frac{25MN\varepsilon}{6\nu^2} + \frac{15\mu\overline{N}M\varepsilon}{2(3\nu - 10M\varepsilon\mu\overline{N})} \right)^{n-2}, \quad n \ge 2, \end{cases} \tag{3.113}$$

$$\|\nabla \delta^n\|_0 \le \begin{cases} \frac{\varepsilon}{12}, \quad n = 1, \\ \frac{3\mu\overline{N}M\varepsilon^2}{2(3\nu - 10M\varepsilon\mu\overline{N})} \left(\frac{25MN\varepsilon}{6\nu^2} + \frac{15\mu\overline{N}M\varepsilon}{2(3\nu - 10M\varepsilon\mu\overline{N})} \right)^{n-2}, \quad n \ge 2, \end{cases} \tag{3.114}$$

$$\|\eta^n\|_0 \le \beta^{-1} Q_3 M\varepsilon \left(\frac{25MN\varepsilon}{6\nu^2} + \frac{15\mu\overline{N}M\varepsilon}{2(3\nu - 10M\varepsilon\mu\overline{N})} \right)^{n-1}, \quad n \ge 1, \tag{3.115}$$

where $M = (m\alpha^2)/2$, $Q_3 = 2 + D_a^{-1}\alpha^2\nu^{-1}$, $\mu = max\{\gamma^{-1}, D_c^{-1}\}$, and the definitions of N and $\overline{N}$ are given in (2.14) *and* (2.15).

Proof. Subtracting Eq (2.25) from Eq (3.110) yields the following fundamental expressions:

$$\nu a(\boldsymbol{e}^n, \boldsymbol{v}_h) + D_a^{-1}(\boldsymbol{e}^n, \boldsymbol{v}_h) + b(\boldsymbol{e}^{n-1}, \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{u}_h^{n-1}, \boldsymbol{e}^n, \boldsymbol{v}_h) - d(\eta^n, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{e}^n)$$
$$= (\beta_T \xi^n \boldsymbol{g} + \beta_C \delta^n \boldsymbol{g}, \boldsymbol{v}_h), \tag{3.116}$$
$$\gamma \overline{a}(\xi^n, \psi_h) + \overline{b}(\boldsymbol{e}^{n-1}, T_h, \psi_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, \xi^n, \psi_h) = 0, \tag{3.117}$$
$$D_c \overline{a}(\delta^n, s_h) + \overline{b}(\boldsymbol{e}^{n-1}, C_h, s_h) + \overline{b}(\boldsymbol{u}_h^{n-1}, \delta^n, s_h) = 0. \tag{3.118}$$

We first consider the case $n = 1$. From Eqs (3.116)–(3.118), we obtain

$$\nu a(\boldsymbol{e}^1, \boldsymbol{v}_h) + D_a^{-1}(\boldsymbol{e}^1, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - d(\eta^1, \boldsymbol{v}_h) + d(\varphi_h, \boldsymbol{e}^1) = (\beta_T \xi^1 \boldsymbol{g} + \beta_C \delta^1 \boldsymbol{g}, \boldsymbol{v}_h), \tag{3.119}$$
$$\gamma \overline{a}(\xi^1, \psi_h) + \overline{b}(\boldsymbol{u}_h, T_h, \psi_h) = 0, \tag{3.120}$$
$$D_c \overline{a}(\delta^1, s_h) + \overline{b}(\boldsymbol{u}_h, C_h, s_h) = 0. \tag{3.121}$$

Then, we obtain

$$||\nabla \boldsymbol{e}^1||_0 \le \nu^{-1} N ||\nabla \boldsymbol{u}_h||_0^2 + \nu^{-1} m\alpha^2 (||\nabla \xi^1||_0 + ||\nabla \delta^1||_0) \le \nu^{-1} M\varepsilon, \tag{3.122}$$
$$||\nabla \xi^1||_0 \le \mu \overline{N} ||\nabla \boldsymbol{u}_h||_0 ||\nabla T_h||_0 \le \frac{\varepsilon}{12}, \tag{3.123}$$
$$||\nabla \delta^1||_0 \le \mu \overline{N} ||\nabla \boldsymbol{u}_h||_0 ||\nabla C||_0 \le \frac{\varepsilon}{12}, \tag{3.124}$$
$$||\eta^1||_0 \le \beta^{-1}(\nu + D_a^{-1}\alpha^2) ||\nabla \boldsymbol{e}^1||_0 + \beta^{-1} N ||\nabla \boldsymbol{u}_h||_0^2 + \beta^{-1} m\alpha^2 (||\nabla \xi^1||_0 + ||\nabla \delta^1||_0)$$
$$\le \beta^{-1} Q_3 M\varepsilon, \tag{3.125}$$

where $Q_3 = 2 + D_a^{-1}\alpha^2\nu^{-1}$. For $n \ge 2$, from Eqs (3.116)–(3.118), we obtain

$$||\nabla \boldsymbol{e}^n||_0 \le \left(\nu - N ||\nabla \boldsymbol{u}_h^{n-1}||_0\right)^{-1} \left(N ||\nabla \boldsymbol{u}_h||_0 ||\nabla \boldsymbol{e}^{n-1}||_0 + m\alpha^2 (||\nabla \xi^n||_0 + ||\nabla \delta^n||_0)\right)$$
$$\le \left(\frac{25MN\varepsilon}{6\nu^2} + \frac{15\mu M\overline{N}\varepsilon}{2(3\nu - 10\mu M\overline{N}\varepsilon)}\right) ||\nabla \boldsymbol{e}^{n-1}||_0, \tag{3.126}$$
$$||\nabla \xi^n||_0 \le \left(1 - \mu \overline{N} ||\nabla \boldsymbol{u}_h^{n-1}||_0\right)^{-1} \mu \overline{N} ||\nabla \boldsymbol{e}^{n-1}||_0 ||\nabla T_h||_0$$
$$\le \frac{3\mu\nu\overline{N}\varepsilon}{2(3\nu - 10\mu M\overline{N}\varepsilon)} ||\nabla \boldsymbol{e}^{n-1}||_0, \tag{3.127}$$
$$||\nabla \delta^n||_0 \le \left(1 - \mu \overline{N} ||\nabla \boldsymbol{u}_h^{n-1}||_0\right)^{-1} \mu \overline{N} ||\nabla \boldsymbol{e}^{n-1}||_0 ||\nabla C_h||_0$$
$$\le \frac{3\mu\nu\overline{N}\varepsilon}{2(3\nu - 10\mu M\overline{N}\varepsilon)} ||\nabla \boldsymbol{e}^{n-1}||_0, \tag{3.128}$$
$$\beta ||\eta^n||_0 \le (\nu + D_a^{-1}\alpha^2 + N ||\nabla \boldsymbol{u}_h^{n-1}||_0) ||\nabla \boldsymbol{e}^n||_0 + N ||\nabla \boldsymbol{u}_h||_0 ||\nabla \boldsymbol{e}^{n-1}||_0 + m\alpha^2 (||\nabla \xi^n||_0 + ||\nabla \delta^n||_0)$$
$$\le \left(\nu + D_a^{-1}\alpha^2 + \frac{10MN\varepsilon}{3\nu}\right) ||\nabla \boldsymbol{e}^n||_0 + \frac{10MN\varepsilon}{3\nu} ||\nabla \boldsymbol{e}^{n-1}||_0 + m\alpha^2 (||\nabla \xi^n||_0 + ||\nabla \delta^n||_0). \tag{3.129}$$

Then, we can obtain

$$||\nabla \boldsymbol{e}^n||_0 \le q^{n-1} ||\nabla \boldsymbol{e}^1||_0$$

$$
\begin{aligned}
&= \nu^{-1}M\varepsilon q^{n-1}, && (3.130)\\
\|\nabla\xi^n\|_0 &\le \frac{3\mu\nu\overline{N}\varepsilon}{2(3\nu-10\mu M\overline{N}\varepsilon)}q^{n-2}\|\nabla e^1\|_0\\
&= Q_a q^{n-2}, && (3.131)\\
\|\nabla\delta^n\|_0 &\le \frac{3\mu\nu\overline{N}\varepsilon}{2(3\nu-10\mu M\overline{N}\varepsilon)}q^{n-2}\|\nabla e^1\|_0\\
&= Q_a q^{n-2}, && (3.132)\\
\|\eta^n\|_0 &\le \beta^{-1}\left(\nu + D_a^{-1}\alpha^2 + \frac{10MN\varepsilon}{3\nu}\right)q^{n-1}\|\nabla \boldsymbol{e}^1\|_0 + \beta^{-1}\frac{10MN\varepsilon}{3\nu}q^{n-2}\|\nabla \boldsymbol{e}^1\|_0 + \beta^{-1}2m\alpha^2 Q_a q^{n-2}\\
&\le \beta^{-1}\left(\nu + D_a^{-1}\alpha^2 + \frac{1}{5}\nu\right)\|\nabla e^1\|_0 q^{n-1} + \beta^{-1}\left(\frac{10MN\varepsilon}{3\nu} + \frac{6\mu\nu M\overline{N}\varepsilon}{3\nu-10\mu M\overline{N}\varepsilon}\right)\|\nabla e^1\|_0 q^{n-2}\\
&\le \beta^{-1}(2 + D_a^{-1}\alpha^2\nu^{-1})M\varepsilon q^{n-1}. && (3.133)
\end{aligned}
$$

Here,

$$
q = \frac{25MN\varepsilon}{6\nu^2} + \frac{15\mu M\overline{N}\varepsilon}{2(3\nu-10\mu M\overline{N}\varepsilon)}, \quad Q_a = \frac{3\mu M\overline{N}\varepsilon^2}{2(3\nu-10\mu M\overline{N}\varepsilon)}. \tag{3.134}
$$

□

Corollary 3.1. *Based on the results of Theorem 3.8 and Lemma 2.1, it holds that*

$$
\lim_{n\to\infty}(\|\nabla \boldsymbol{e}^n\|_0 + \|\nabla\xi^n\|_0 + \|\nabla\delta^n\|_0 + \|\eta^n\|_0) = 0.
$$

4. Numerical experiments

In this section, numerical experiments are conducted using the software FreeFem++ to validate the iterative methods proposed in the previous section. First, we examine the convergence behavior of these three methods and compare their convergence rates by computing the errors and convergence orders of the iterative methods. The stopping criterion for all three iterative methods is based on the L^2 norm of the difference between successive iterates, with a tolerance of 1e-6. Assume the domain $\Omega = (0,1)\times(0,1)$, and select the exact solution as

$$
\begin{aligned}
u_1(x,y) &= -\sin^2(\pi x)\sin(\pi y)\cos(\pi y),\\
u_2(x,y) &= \sin(\pi x)\cos(\pi x)\sin^2(\pi y),\\
p(x,y) &= \sin(\pi x)\cos(\pi y),\\
T(x,y) &= 1 - x + \sin(\pi x)\cos(\pi y),\\
C(x,y) &= 1 - x - \sin(\pi x)\cos(\pi y),
\end{aligned} \tag{4.1}
$$

with the coefficients $\nu = 1, \gamma = 1, D_a = 1, D_c = 1, \beta_T = 1, \beta_C = 1$. We take $\boldsymbol{g} = (0,-1)$. We solve the equations using the three iterative methods with P_2-P_1-P_2-P_2 elements on uniform triangular meshes. The resulting errors and convergence orders are presented in Tables 1–6.

Tables 1–3 present the L^2 errors and convergence orders for the Stokes iteration, Newton iteration, and Oseen iterative methods, respectively, using P_2-P_1-P_2-P_2 elements. For all three iterative methods, the convergence orders in the L^2 norm for velocity, temperature, and concentration reach up to 3, and the convergence order for pressure is 2. Tables 4–6 show the H^1 errors and convergence orders for the Stokes iteration, Newton iteration, and Oseen iterative methods, respectively, under the same finite element discretization. In this case, their convergence orders for velocity, temperature, and concentration become 2. A close inspection shows that the numerical values in these tables for different methods are not identical; the first discrepancies occur in the tenth decimal place, which is at the level of round-off errors. This does not affect the reported convergence orders. The results indicate that all three iterative methods converge, achieving optimal convergence rates consistent with the theory.

Table 1. L^2 errors and convergence orders of the Stokes iterative method.

1/h	$\|\boldsymbol{u}-\boldsymbol{u}_h^n\|_0$	Order	$\|p-p_h^n\|_0$	Order	$\|T-T_h^n\|_0$	Order	$\|C-C_h^n\|_0$	Order	Iterations
9	1.04e-03		5.94e-03		3.36e-04		3.37e-04		5
18	1.31e-04	2.99	1.30e-03	2.19	4.19e-05	3.01	4.19e-05	3.01	5
27	3.88e-05	2.99	5.68e-04	2.04	1.24e-05	3.00	1.24e-05	3.00	5
36	1.64e-05	3.00	3.18e-04	2.01	5.23e-06	3.00	5.24e-06	3.00	5
45	8.40e-06	3.00	2.04e-04	2.01	2.68e-06	3.00	2.68e-06	3.00	5
54	4.86e-06	3.00	1.41e-04	2.00	1.55e-06	3.00	1.55e-06	3.00	5

Table 2. L^2 errors and convergence orders of the Newton iterative method.

1/h	$\|\boldsymbol{u}-\boldsymbol{u}_h^n\|_0$	Order	$\|p-p_h^n\|_0$	Order	$\|T-T_h^n\|_0$	Order	$\|C-C_h^n\|_0$	Order	Iterations
9	1.04e-03		5.94e-03		3.37e-04		3.39e-04		4
18	1.31e-04	2.99	1.30e-03	2.19	4.19e-05	3.01	4.20e-05	3.01	4
27	3.88e-05	2.99	5.68e-04	2.04	1.24e-05	3.00	1.24e-05	3.00	4
36	1.64e-05	3.00	3.18e-04	2.01	5.24e-06	3.00	5.24e-06	3.00	4
45	8.40e-06	3.00	2.03e-04	2.01	2.68e-06	3.00	2.68e-06	3.00	4
54	4.86e-06	3.00	1.41e-04	2.00	1.55e-06	3.00	1.55e-06	3.00	4

Table 3. L^2 errors and convergence orders of the Oseen iterative method.

1/h	$\|\boldsymbol{u}-\boldsymbol{u}_h^n\|_0$	Order	$\|p-p_h^n\|_0$	Order	$\|T-T_h^n\|_0$	Order	$\|C-C_h^n\|_0$	Order	Iterations
9	1.04e-03		5.94e-03		3.36e-04		3.37e-04		4
18	1.31e-04	2.99	1.30e-03	2.19	4.19e-05	3.01	4.19e-05	3.01	4
27	3.88e-05	2.99	5.68e-04	2.04	1.24e-05	3.00	1.24e-05	3.00	4
36	1.64e-05	3.00	3.18e-04	2.01	5.23e-06	3.00	5.24e-06	3.00	4
45	8.40e-06	3.00	2.04e-04	2.01	2.68e-06	3.00	2.68e-06	3.00	4
54	4.86e-06	3.00	1.41e-04	2.00	1.55e-06	3.00	1.55e-06	3.00	4

Table 4. H^1 errors and convergence orders of the Stokes iterative method.

1/h	$\|\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h^n)\|\|_0$	Order	$\|\|\nabla(T-T_h^n)\|\|_0$	Order	$\|\|\nabla(C-C_h^n)\|\|_0$	Order	Iterations
9	7.82e-02		2.65e-02		2.65e-02		5
18	2.00e-02	1.97	6.66e-03	1.99	6.66e-03	1.99	5
27	8.93e-03	1.99	2.96e-03	2.00	2.96e-03	2.00	5
36	5.03e-03	1.99	1.67e-03	2.00	1.67e-03	2.00	5
45	3.22e-03	2.00	1.07e-03	2.00	1.07e-03	2.00	5
54	2.24e-03	2.00	7.41e-04	2.00	7.41e-04	2.00	5

Table 5. H^1 errors and convergence orders of the Newton iterative method.

1/h	$\|\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h^n)\|\|_0$	Order	$\|\|\nabla(T-T_h^n)\|\|_0$	Order	$\|\|\nabla(C-C_h^n)\|\|_0$	Order	Iterations
9	7.82e-02		2.65e-02		2.65e-02		4
18	2.00e-02	1.97	6.66e-03	1.99	6.66e-03	1.99	4
27	8.93e-03	1.99	2.96e-03	2.00	2.96e-03	2.00	4
36	5.03e-03	1.99	1.67e-03	2.00	1.67e-03	2.00	4
45	3.22e-03	2.00	1.07e-03	2.00	1.07e-03	2.00	4
54	2.24e-03	2.00	7.41e-04	2.00	7.41e-04	2.00	4

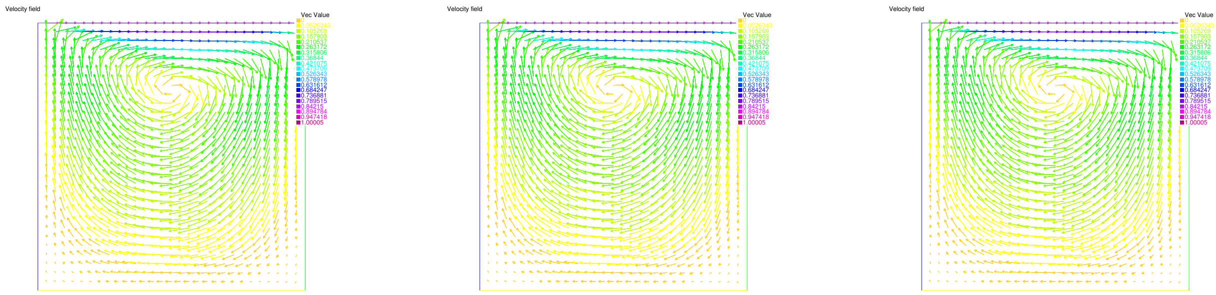
Table 6. H^1 errors and convergence orders of the Oseen iterative method.

1/h	$\|\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h^n)\|\|_0$	Order	$\|\|\nabla(T-T_h^n)\|\|_0$	Order	$\|\|\nabla(C-C_h^n)\|\|_0$	Order	Iterations
9	7.82e-02		2.65e-02		2.65e-02		4
18	2.00e-02	1.97	6.66e-03	1.99	6.66e-03	1.99	4
27	8.93e-03	1.99	2.96e-03	2.00	2.96e-03	2.00	4
36	5.03e-03	1.99	1.67e-03	2.00	1.67e-03	2.00	4
45	3.22e-03	2.00	1.07e-03	2.00	1.07e-03	2.00	4
54	2.24e-03	2.00	7.41e-04	2.00	7.41e-04	2.00	4

Second, we consider the cavity problem on the unit square $\Omega = (0,1)\times(0,1)$ with P_{1b}-P_1-P_2-P_2 elements on a uniform triangular mesh of size $h = 1/30$, where the physical parameters are fixed as $D_a = 1$, $\gamma = 1$, $D_c = 1$, $\beta_T = 1$, $\beta_C = 1$, and $\boldsymbol{g} = (0,-1)$. Following the classical benchmark setup in [14], the boundary conditions are prescribed as follows:

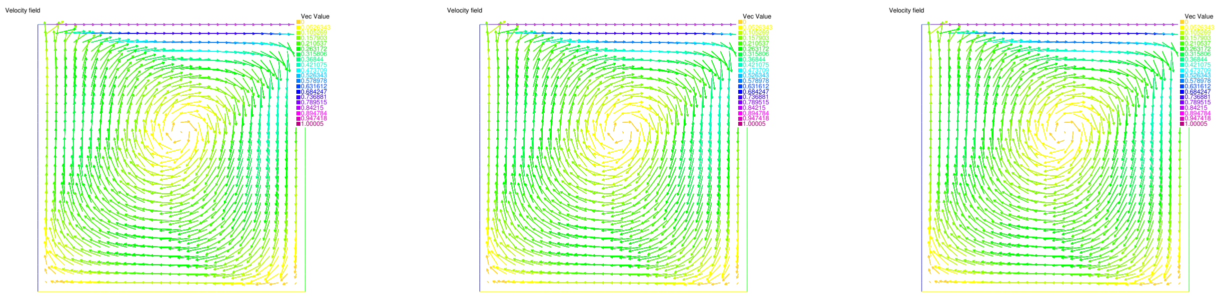
$$
\begin{aligned}
&T = 0, \quad C = 0, \quad u_1 = u_2 = 0 \quad at \quad x = 0,\\
&T = 1, \quad C = 1, \quad u_1 = u_2 = 0 \quad at \quad x = 1,\\
&\frac{\partial T}{\partial \mathbf{n}} = 0, \quad \frac{\partial C}{\partial \mathbf{n}} = 0, \quad u_1 = u_2 = 0 \quad at \quad y = 0,\\
&\frac{\partial T}{\partial \mathbf{n}} = 0, \quad \frac{\partial C}{\partial \mathbf{n}} = 0, \quad u_1 = 1, \quad u_2 = 0 \quad at \quad y = 1.
\end{aligned}
$$

Figures 1–3 present the velocity vector plots for the three iterative methods (Stokes, Newton, and Oseen) at viscosities $\nu = 1, 10^{-2}, 10^{-3}$. For $\nu = 1$, all methods produce almost identical solutions. For $\nu = 10^{-2}$, the flow becomes more complex with sharper gradients. For $\nu = 10^{-3}$, the flow is strongly convection-dominated, exhibiting a large primary vortex near the center and intensified shear layers along the moving lid and side walls, The Stokes iteration, on the other hand, diverges at the lowest viscosity due to violation of the stability condition and thus fails to produce a valid vector plot.



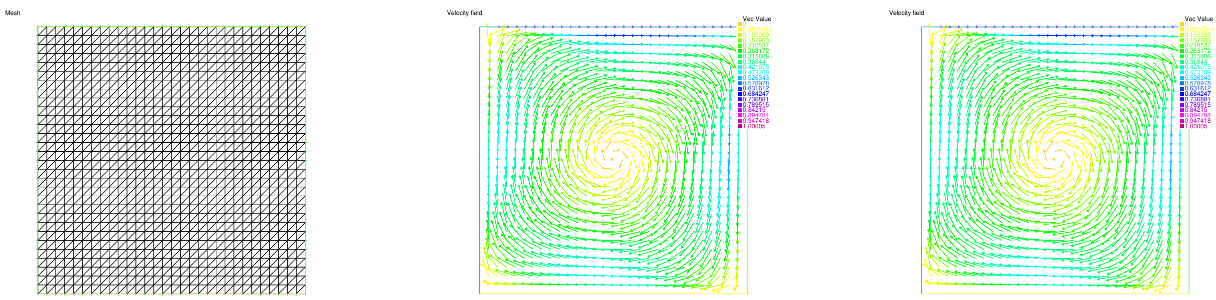
(a) Velocity vector plot for the Stokes iterative method **(b)** Velocity vector plot for the Newton iterative method **(c)** Velocity vector plot for the Oseen iterative method

Figure 1. Stokes, Newton, and Oseen methods for $\nu = 1$.



(a) Velocity vector plot for the Stokes iterative method **(b)** Velocity vector plot for the Newton iterative method **(c)** Velocity vector plot for the Oseen iterative method

Figure 2. Stokes, Newton, and Oseen methods with $\nu = 10^{-2}$.



(a) Velocity vector plot for the Stokes iterative method **(b)** Velocity vector plot for the Newton iterative method **(c)** Velocity vector plot for the Oseen iterative method

Figure 3. Stokes, Newton, and Oseen methods with $\nu = 10^{-3}$.

5. Conclusions

In this paper, we apply the Stokes, Newton, and Oseen iterative methods to the finite element discretization of steady Darcy–Brinkman equations to solve the discretized systems. We establish the convergence of all three methods. Numerical experiments confirm the theoretical convergence results and optimal convergence rates. In our future work, we plan to extend the iterative methods to (time-dependent) Darcy–Brinkman equations with variable coefficients, while also exploring strategies to prevent divergence and testing the methods on more complex benchmark problems.

Author contributions

Peixin Sun: Conceptualization, methodology, software, formal analysis, investigation, writing – original draft, writing - review & editing; Lunji Song: Conceptualization, methodology, supervision; Youlei Liang: Conceptualization, methodology, software. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Prof. Lunji Song is an editorial board member for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article.

The authors declare that they have no conflict of interest.

References

1. Q. Li, B. Teng, W. Luo, Q. Wang, Y. Yang, X. Zhang, An analytical formulation for correcting the relative permeability of Gas-water flow in propped fractures considering the effect of Brinkman flow, *Water Resour. Res.*, **60** (2024), e2023WR036625. https://doi.org/10.1029/2023WR036625
2. İ. Solmuş, Numerical investigation of heat transfer and fluid flow behaviors of a block type graphite foam heat sink inserted in a rectangular channel, *Appl. Therm. Eng.*, **78** (2015), 605–615. https://doi.org/10.1016/j.applthermaleng.2014.11.066
3. S. Bansal, A. K. Tyagi, A. K. Sharma, Modeling of planar high temperature solid oxide fuel cells, *AIP Conf. Proc.*, **2352** (2021), 030023. https://doi.org/10.1063/5.0052404
4. X. Jiang, R. Li, Z. Chen, A discontinuous Galerkin method for the Brinkman-Darcy-transport problem, *J. Comput. Appl. Math.*, **453** (2025), 116155. https://doi.org/10.1016/j.cam.2024.116155
5. M. S. Jamal, A. A. Awotunde, Darcy's model with optimized permeability distribution for the simulation of Stokes flow and contaminant transport in karst aquifers, *Hydrogeol. J.*, **28** (2020), 1249–1267. https://doi.org/10.1007/s10040-020-02124-y

6. W. Zhu, P. Huang, K. Wang, Newton iterative method based on finite element discretization for the stationary Darcy-Brinkman equations, *Comput. Math. Appl.*, **80** (2020), 3098–3122. https://doi.org/10.1016/j.camwa.2020.10.020
7. M. Alvarez, G. N. Gatica, R. Ruiz-Baier, A vorticity-based fully-mixed formulation for the 3D Brinkman-Darcy problem, *Comput. Methods Appl. Mech. Eng.*, **307** (2016), 68–95. http://doi.org/10.1016/j.cma.2016.04.017
8. F. G. Eroglu, S. Kaya, L. G. Rebholz, POD-ROM for the Darcy-Brinkman equations with double-diffusive convection, *J. Numer. Math.*, **27** (2019), 123–139. http://doi.org/10.1515/jnma-2017-0122
9. P. Lenarda, M. Paggi, R. R. Baier, Partitioned coupling of advection-diffusion-reaction systems and Brinkman flows, *J. Comput. Phys.*, **344** (2017), 281–302. http://doi.org/10.1016/j.jcp.2017.05.011
10. R. A. Adams, *Sobolev spaces*, New York: Academic Press, 1975.
11. V. Girault, P. A. Raviart, *Finite element methods for Navier-Stokes equations: Theory and agorithms*, Berlin: Springer, 1986. https://doi.org/10.1007/978-3-642-61623-5
12. J. G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization, *SIAM J. Numer. Anal.*, **19** (1982), 275–311. https://doi.org/10.1137/0719018
13. A. Labovsky, W. J. Layton, C. C. Manica, M. Neda, L. G. Rebholz , The stabilized extrapolated trapezoidal finite-element method for the Navier-Stokes equations, *Comput. Methods Appl. Mech. Eng.*, **198** (2009), 958–974. http://doi.org/10.1016/j.cma.2008.11.004
14. Y. He, J. Li, Convergence of three iterative methods based on the finite element discretization for the stationary Navier-Stokes equations, *Comput. Methods Appl. Mech. Eng.*, **198** (2009), 1351–1359. http://doi.org/10.1016/j.cma.2008.12.001
15. Y. He, A. Wang, A simplified two-level method for the steady Navier-Stokes equations, *Comput. Methods Appl. Mech. Eng.*, **197** (2008), 1569–1576. http://doi.org/10.1016/j.cma.2007.11.032