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Research article

On a difference equation which is a sum of rational and linear terms

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Abstract: A third order-difference equation defined as sum of rational and linear terms is solved explicitly. The importance of periodic solutions for difference equations lies, for instance, in their ability to model seasonal cycles in biology, recurring fluctuations in economics, and oscillatory behaviors in engineering systems. For this reason, and by relying on the explicit formulas of the solutions of our equation, we propose a rigorous approach to studying the existence of prime periodic solutions. Furthermore, the obtained formulas are used to investigate the asymptotic behavior of the solutions. Finally, several numerical examples are presented to illustrate and support the theoretical results.

Keywords: difference equations; form of the solutions; periodic solutions; limiting behavior of solutions
Mathematics Subject Classification: 39A05, 39A10, 39A20, 39A30

1. Introduction and some preliminaries

Difference equations are extensively used to model real-world discrete phenomena across a wide range of modern scientific disciplines. The following are some illustrative examples of how difference equations are applied. In biology, difference equations describe population dynamics, such as predator-prey interactions and the spread of infectious diseases. In economics, they provide the evolution of stock prices and are also used to analyze how debt evolves with respect to interest rates, growth, and budget deficits. In computer science, difference equations describe the time complexity of recursive and iterative procedures. For some applications of difference equations to concrete real-world models, we refer the reader to the following references [2, 11, 20].

It is worth noting that difference equations also provide a powerful framework for analyzing continuous models governed, for example, by ordinary differential equations, partial differential equations, or partial-integro differential equations. In general, this is achieved using a variety of discretizations schemes, whether standard or nonstandard; the reader is referred to the following contributions [3, 13, 15].

The solutions of difference equations reveal, among others, stability, oscillations, convergence and periodicity, so knowing their formulas is essential for predicting their long-term behavior and understanding the dynamics of the corresponding phenomena. This fact has motivated numerous researchers to study the solvability of a wide range of specific classes of difference equations and their systems. In [6], Gumus et al. investigated the following rational difference equation:

$$x_{n+1} = \frac{a x_n x_{n-k+1}}{b x_{n-3} + x_{n-k+1} + c x_{n-k}},\ n \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, a, b, c are positive real numbers, the initial values x_{-i}, $i = 0, ..., k$ are real numbers, and k is a positive integer. In [21], Zabat et al. solved explicitly the following non-autonomous difference equation of higher order:

$$x_{n+1} = \alpha_n x_{n-k} + \frac{\beta_n}{x_n x_{n-1} ... x_{n-k+1}},\ n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, $(\alpha)_{n\in\mathbb{N}_0}$, $(\beta)_{n\in\mathbb{N}_0}$ are real sequences, and the initial values x_{-i}, $i = 0, 1, ..., k$ are real and non-zero. Kara et al. in [10], as extension of the work of Haddad et al. [7], studied the following three-dimensional system of difference equations:

$$x_{n+1} = \frac{a x_n z_{n-1}}{z_n - \beta} + \gamma,\ y_{n+1} = \frac{b y_n x_{n-1}}{x_n - \gamma} + \alpha,\ z_{n+1} = \frac{c z_n y_{n-1}}{y_n - \alpha} + \beta,\ n \in \mathbb{N}_0,$$

where the parameters a, b, c, α, β, γ and the initial values x_{-i}, y_{-i}, $i \in \{0, 1\}$ are non-zero real numbers. For other works dealing with the solvability of difference equations and their systems, we can consult the following papers [1, 8, 9]. For further contributions to the study of difference equations and their applications, we refer to [12, 14, 16], where the authors investigated the existence and qualitative properties of solutions to challenging problems involving Laplacian partial difference equations and fractional symmetric Hahn difference equations.

The evolution of some phenomenons depends on both linear accumulation and nonlinear feedback. This motivated us to consider a new class of nonlinear difference equations defined as sum of a rational and linear terms. More precisely, in this study we investigate the following third-order difference equation defined by

$$x_{n+1} = \frac{\alpha x_n x_{n-2}}{x_{n-1}} \left(\frac{x_n - a x_{n-1} - b}{x_{n-1} - a x_{n-2} - b} \right) + a x_n + b,\ n \in \mathbb{N}_0, \tag{1.1}$$

where α and the initial values x_{-2}, x_{-1}, x_0 are non-zero real numbers while the parameters a, b are real numbers.

In the present study, we are inspired by the work of Touafek et al. in [19], in which the authors studied in detail the behavior of the solutions of the following system of difference equations:

$$x_{n+1} = \frac{a x_n y_{n-1}}{y_n - \beta y_{n-1} - \gamma} + b x_n + c,\ y_{n+1} = \frac{\alpha y_n x_{n-1}}{x_n - b x_{n-1} - c} + \beta y_n + \gamma,\ n \in \mathbb{N}_0, \tag{1.2}$$

where a, α and the initial values x_{-1}, y_{-1}, x_0, and y_0 are non-zero real numbers, while b, c, β, and γ are real numbers. We observe that, by building on the ideas and techniques introduced in [19], Ghezal et al. [4] were able to derive an explicit solution to a three-dimensional system of difference equations that extends system (1.2).

In order to derive explicit formulas of the solutions of (1.1), we transform Eq (1.1) into an equivalent solvable difference equation with periodic pattern via a rational change of variable, and then reduce it to six first-order linear difference equations. This allows a systematic construction of all solutions of (1.1) and, subsequently, to characterize their periodicity and asymptotic behavior.

Remark 1.1. *For $\alpha = 0$, Eq* (1.1) *takes the following simple form:*

$$x_{n+1} = ax_n + b,\ n \in \mathbb{N}_0. \tag{1.3}$$

Eq (1.3) is a first order difference equation, and the formulas of its solutions are well known, that is,

$$x_n = a^n x_0 + b\left(\frac{a^n - 1}{a - 1}\right),\ a \neq 1,\ x_n = x_0 + bn,\ a = 1,$$

and this justifies the assumption that α is non-zero in our study.

Remark 1.2. *For $a = b = 0$, Eq* (1.1) *reduces to*

$$x_{n+1} = \frac{\alpha x_n^2 x_{n-2}}{x_{n-1}^2},\ n \in \mathbb{N}_0. \tag{1.4}$$

Thus, Eq (1.1) can be viewed as an extension of Eq (1.4), which is an example of product-type difference equations. For some studies on this type of difference equations, see [17, 18].

From (1.1), we obtain

$$\frac{x_{n+1} - ax_n - b}{x_n} = \alpha\left(\frac{\frac{x_n - ax_{n-1} - b}{x_{n-1}}}{\frac{x_{n-1} - ax_{n-2} - b}{x_{n-2}}}\right),\ n \in \mathbb{N}_0. \tag{1.5}$$

Identity (1.5) is defined, provided that the terms x_n, x_{n-1}, x_{n-2}, and $x_{n-1} - ax_{n-2} - b$ are non-zero for all $n \in \mathbb{N}_0$. In particular, this implies that x_{-2}, x_{-1}, and x_0 must be non-zero, and this justified our choice for the initial values.

Definition 1.1. *A solution $(x_n)_{n=-2}^{+\infty}$ of Eq* (1.1) *is said well-defined solution if*

$$x_{n-2} \neq 0,\ x_{n-1} - ax_{n-2} - b \neq 0,\ n \in \mathbb{N}_0.$$

Through this study, by a solution of Eq (1.1), we mean a well-defined solution. Also, we adopt the following definition regarding periodic solutions: see [5].

Definition 1.2. *A solution $(x_n)_{n=-2}^{+\infty}$ of Eq* (1.1) *is said eventually periodic of period $p \in \mathbb{N}$, or simply eventually p-periodic, if there exist $k \geq -2$ such that*

$$x_{n+p} = x_n,\ n \geq k.$$

If, in addition, $k = -2$, then the solution is said periodic of period p, or simply p-periodic.

Remark 1.3. *In Definition 1.2, if p is the smallest $j \in \mathbb{N}$ for which $x_{n+j} = x_n$, $n \geq k$, then the solution is said eventually periodic of prime period p, or simply eventually prime p-periodic. If, in addition, $k = -2$, then the solution is said periodic of prime period p or simply prime p-periodic.*

Our main contribution can be summarized as follows: First, we obtain an explicit solution of the third-order difference Eq (1.1). Based on the resulting solution formulas, we develop a rigorous approach to characterize the existence of prime periodic solutions and derive several results concerning the asymptotic behavior of the solutions.

The paper is organized as follows. Section 2 is devoted to establishing the solvability of (1.1). Sections 3–6 examine the existence of prime periodic solutions and the asymptotic behavior of the solutions. Section 7 provides numerical examples to illustrate the main theoretical results. Finally, in the conclusion, we propose the results of the present work to a system of difference equations.

2. Expressions of the solutions for Eq (1.1)

In this section, we provide explicit formulas for the solutions of Eq (1.1) by first transforming it into an equivalent equation that is easier to solve. By exploiting the periodicity of the solutions of the equivalent equation, we reduce the problem to six solvable first-order linear difference equations, whose solutions provide the explicit form of the solutions of Eq (1.1).

Consider the following change of variable:

$$u_n = \frac{x_n - ax_{n-1} - b}{x_{n-1}},\ n = -1, 0, ..., \tag{2.1}$$

then from (1.5) and (2.1), we obtain

$$u_{n+1} = \frac{\alpha u_n}{u_{n-1}},\ \ n \in \mathbb{N}_0. \tag{2.2}$$

It is easy to see that every solution of (2.2) is periodic of period six and is given by

$$u_{6n} = u_0,\ u_{6n+1} = \frac{\alpha u_0}{u_{-1}},\ u_{6n+2} = \frac{\alpha^2}{u_{-1}},\ u_{6n+3} = \frac{\alpha^2}{u_0},\ u_{6n+4} = \frac{\alpha u_{-1}}{u_0},\ u_{6n+5} = u_{-1},\ n \in \mathbb{N}_0, \tag{2.3}$$

or equivalently,

$$(u_n)_{n=-1}^{+\infty} = \left(u_{-1}, u_0, \frac{\alpha u_0}{u_{-1}}, \frac{\alpha^2}{u_{-1}}, \frac{\alpha^2}{u_0}, \frac{\alpha u_{-1}}{u_0}, u_{-1}, u_0, ...,\right).$$

Using (2.1), we get

$$x_{n+1} = (a + u_{n+1})x_n + b,\ n \in \mathbb{N}_0. \tag{2.4}$$

Replacing n by $6n$, $6n + 1$,..., $6n + 5$ in (2.4), we obtain

$$x_{6n+1} = (a + u_{6n+1})x_{6n} + b,\ n \in \mathbb{N}_0, \tag{2.5}$$

$$x_{6n+2} = (a + u_{6n+2})x_{6n+1} + b,\ n \in \mathbb{N}_0, \tag{2.6}$$

$$x_{6n+3} = (a + u_{6n+3})x_{6n+2} + b,\ n \in \mathbb{N}_0, \tag{2.7}$$

$$x_{6n+4} = (a + u_{6n+4})x_{6n+3} + b,\ n \in \mathbb{N}_0, \tag{2.8}$$

$$x_{6n+5} = (a + u_{6n+5})x_{6n+4} + b,\ n \in \mathbb{N}_0, \tag{2.9}$$

$$x_{6n+6} = (a + u_{6n+6})x_{6n+5} + b,\ n \in \mathbb{N}_0. \tag{2.10}$$

Now, take into consideration the formulas of u_n given in (2.3). Eqs (2.5)–(2.10) become

$$x_{6n+1} = (a + \frac{\alpha u_0}{u_{-1}})x_{6n} + b,\ n \in \mathbb{N}_0, \tag{2.11}$$

$$x_{6n+2} = (a + \frac{\alpha^2}{u_{-1}})x_{6n+1} + b,\ n \in \mathbb{N}_0, \tag{2.12}$$

$$x_{6n+3} = (a + \frac{\alpha^2}{u_0})x_{6n+2} + b,\ n \in \mathbb{N}_0, \tag{2.13}$$

$$x_{6n+4} = (a + \frac{\alpha u_{-1}}{u_0})x_{6n+3} + b,\ n \in \mathbb{N}_0, \tag{2.14}$$

$$x_{6n+5} = (a + u_{-1})x_{6n+4} + b,\ n \in \mathbb{N}_0, \tag{2.15}$$

$$x_{6n+6} = (a + u_0)x_{6n+5} + b,\ n \in \mathbb{N}_0. \tag{2.16}$$

Define the following quantities:

$$A_i = \prod_{j=1}^{i} p_j,\ B_i = b\sum_{k=1}^{i}\left(\prod_{j=k+1}^{i} p_j\right),\ i = 1, ..., 6, \tag{2.17}$$

where

$$p_1 = a + \frac{\alpha u_0}{u_{-1}},\ p_2 = a + \frac{\alpha^2}{u_{-1}},\ p_3 = a + \frac{\alpha^2}{u_0},\ p_4 = a + \frac{\alpha u_{-1}}{u_0},\ p_5 = a + u_{-1},\ p_6 = a + u_0,$$

with the convention that

$$\prod_{l=m}^{n} q_l = 1,\ n < m.$$

Using (2.17), it follows from Eqs (2.11)–(2.16) that

$$x_{6n+1} = A_1 x_{6n} + B_1,\ n \in \mathbb{N}_0, \tag{2.18}$$

$$x_{6n+2} = A_2 x_{6n} + B_2,\ n \in \mathbb{N}_0, \tag{2.19}$$

$$x_{6n+3} = A_3 x_{6n} + B_3,\ n \in \mathbb{N}_0, \tag{2.20}$$

$$x_{6n+4} = A_4 x_{6n} + B_4,\ n \in \mathbb{N}_0, \tag{2.21}$$

$$x_{6n+5} = A_5 x_{6n} + B_5,\ n \in \mathbb{N}_0, \tag{2.22}$$

$$x_{6n+6} = A_6 x_{6n} + B_6,\ n \in \mathbb{N}_0. \tag{2.23}$$

Eq (2.23) is a first-order linear difference equation (in x_{6n}), and for all $n \in \mathbb{N}_0$, we have

$$x_{6n} = A_6^n x_0 + B_6\left(\frac{A_6^n - 1}{A_6 - 1}\right),\ \text{if}\, A_6 \neq 1,\ \text{and}\, x_{6n} = x_0 + B_6 n,\ \text{if}\, A_6 = 1. \tag{2.24}$$

Now, from (2.24) and (2.18)–(2.22), it follows that for all $n \in \mathbb{N}_0$, we have

$$x_{6n+1} = \begin{cases} A_1 A_6^n x_0 + A_1 B_6 \left(\dfrac{A_6^n - 1}{A_6 - 1} \right) + B_1, & A_6 \neq 1, \\ A_1 x_0 + A_1 B_6 n + B_1, & A_6 = 1. \end{cases} \tag{2.25}$$

$$x_{6n+2} = \begin{cases} A_2 A_6^n x_0 + A_2 B_6 \left(\dfrac{A_6^n - 1}{A_6 - 1} \right) + B_2, & A_6 \neq 1, \\ A_2 x_0 + A_2 B_6 n + B_2, & A_6 = 1. \end{cases} \tag{2.26}$$

$$x_{6n+3} = \begin{cases} A_3 A_6^n x_0 + A_3 B_6 \left(\dfrac{A_6^n - 1}{A_6 - 1} \right) + B_3, & A_6 \neq 1, \\ A_3 x_0 + A_3 B_6 n + B_3, & A_6 = 1. \end{cases} \tag{2.27}$$

$$x_{6n+4} = \begin{cases} A_4 A_6^n x_0 + A_4 B_6 \left(\dfrac{A_6^n - 1}{A_6 - 1} \right) + B_4, & A_6 \neq 1, \\ A_4 x_0 + A_4 B_6 n + B_4, & A_6 = 1. \end{cases} \tag{2.28}$$

$$x_{6n+5} = \begin{cases} A_5 A_6^n x_0 + A_5 B_6 \left(\dfrac{A_6^n - 1}{A_6 - 1} \right) + B_5, & A_6 \neq 1, \\ A_5 x_0 + A_5 B_6 n + B_5, & A_6 = 1. \end{cases} \tag{2.29}$$

In summary, we have the following result.

Theorem 2.1. *Let $(x_n)_{n=-2}^{+\infty}$ be a solution of* (1.1). *Then, for all $n \in \mathbb{N}_0$, we have*

1) If $A_6 \neq 1$,

$$x_{6n} = A_6^n x_0 + B_6 \left(\frac{A_6^n - 1}{A_6 - 1} \right),\ x_{6n+1} = A_1 A_6^n x_0 + A_1 B_6 \left(\frac{A_6^n - 1}{A_6 - 1} \right) + B_1,$$
$$x_{6n+2} = A_2 A_6^n x_0 + A_2 B_6 \left(\frac{A_6^n - 1}{A_6 - 1} \right) + B_2,\ x_{6n+3} = A_3 A_6^n x_0 + A_3 B_6 \left(\frac{A_6^n - 1}{A_6 - 1} \right) + B_3,$$
$$x_{6n+4} = A_4 A_6^n x_0 + A_4 B_6 \left(\frac{A_6^n - 1}{A_6 - 1} \right) + B_4,\ x_{6n+5} = A_5 A_6^n x_0 + A_5 B_6 \left(\frac{A_6^n - 1}{A_6 - 1} \right) + B_5.$$

2) If $A_6 = 1$,

$$x_{6n} = x_0 + B_6 n,\ x_{6n+1} = A_1 x_0 + A_1 B_6 n + B_1,\ x_{6n+2} = A_2 x_0 + A_2 B_6 n + B_2,$$
$$x_{6n+3} = A_3 x_0 + A_3 B_6 n + B_3,\ x_{6n+4} = A_4 x_0 + A_4 B_6 n + B_4,\ x_{6n+5} = A_5 x_0 + A_5 B_6 n + B_5,$$

where

$$A_i = \prod_{j=1}^{i} p_j,\ B_i = b \sum_{k=1}^{i} \left(\prod_{j=k+1}^{i} p_j \right),\ i = 1, ..., 6,$$

with

$$p_1 = a + \frac{\alpha u_0}{u_{-1}},\ p_2 = a + \frac{\alpha^2}{u_{-1}},\ p_3 = a + \frac{\alpha^2}{u_0},\ p_4 = a + \frac{\alpha u_{-1}}{u_0},\ p_5 = a + u_{-1},\ p_6 = a + u_0,$$

and

$$u_0 = \frac{x_0 - a x_{-1} - b}{x_{-1}},\ u_{-1} = \frac{x_{-1} - a x_{-2} - b}{x_{-2}}.$$

3. The case $A_6 = 1$.

Here, we discuss the form and the behavior of the solutions of Eq (1.1) when the parameter $A_6 = 1$. We begin by the following result, by distinguishing the cases where the parameter B_6 is zero or not.

Lemma 3.1. *Let* $(x_n)_{n=-2}^{+\infty}$ *be a solution of* (1.1).

- If $B_6 = 0$, then for all $n \in \mathbb{N}_0$, the solution takes the form
$$x_{6n} = x_0,\ x_{6n+1} = A_1x_0 + B_1,\ x_{6n+2} = A_2x_0 + B_2,$$
$$x_{6n+3} = A_3x_0 + B_3,\ x_{6n+4} = A_4x_0 + B_4,\ x_{6n+5} = A_5x_0 + B_5.$$
That is, the solution is eventually 6-periodic. Moreover, if
$$A_4x_0 + B_4 = x_{-2},\ A_5x_0 + B_5 = x_{-1},$$
the solution will be 6-periodic.
- If $B_6 \neq 0$, then for all $n \in \mathbb{N}_0$, the solution takes the form
$$x_{6n} = x_0 + B_6n,\ x_{6n+1} = A_1x_0 + A_1B_6n + B_1,\ x_{6n+2} = A_2x_0 + A_2B_6n + B_2,$$
$$x_{6n+3} = A_3x_0 + A_3B_6n + B_3,\ x_{6n+4} = A_4x_0 + A_4B_6n + B_4,\ x_{6n+5} = A_5x_0 + A_5B_6n + B_5,$$
from which it follows that
$$\lim_{n\to\infty} |x_{6n+i}| = +\infty,\ i = 0, 1, ..., 5.$$

Proof.

- If $B_6 = 0$, then from Theorem 2.1, we get for all $n \in \mathbb{N}_0$
$$x_{6n} = x_0,\ x_{6n+1} = A_1x_0+B_1,\ x_{6n+2} = A_2x_0+B_2,\ x_{6n+3} = A_3x_0+B_3,\ x_{6n+4} = A_4x_0+B_4,\ x_{6n+5} = A_5x_0+B_5,$$
or equivalently,
$$x_{n+6} = x_n,\ n = 0, 1, ...,$$
that is, the solution is eventually 6-periodic. Moreover, if
$$A_4x_0 + B_4 = x_{-2},\ A_5x_0 + B_5 = x_{-1},$$
we get
$$x_{n+6} = x_n,\ n = -2, -1, ...,$$
that is, the solution is 6-periodic.
- If $B_6 \neq 0$, then from Theorem 2.1, we get for all $n \in \mathbb{N}_0$
$$x_{6n} = x_0 + B_6n,\ x_{6n+1} = A_1x_0 + A_1B_6n + B_1,\ x_{6n+2} = A_2x_0 + A_2B_6n + B_2, \tag{3.1}$$
$$x_{6n+3} = A_3x_0 + A_3B_6n + B_3,\ x_{6n+4} = A_4x_0 + A_4B_6n + B_4,\ x_{6n+5} = A_5x_0 + A_5B_6n + B_5. \tag{3.2}$$
From the fact that $A_6 = 1$, it follows that $A_i \neq 0$, $i = 1, ..., 5$. So, it follows from (3.1) and (3.2) that
$$\lim_{n\to\infty} |x_{6n+i}| = +\infty,\ i = 0, 1, ..., 5.$$

□

Based on the formulas given in Lemma 3.1, we discuss the existence of prime periodic solutions. We begin with the existence of prime 3-periodic solutions.

Theorem 3.1. *Assume that $B_6 = 0$. Then, the solution $(x_n)_{n=-2}^{+\infty}$ of Eq* (1.1) *is eventually prime* 3-*periodic if and only if*

$$A_3x_0+B_3 = x_0,\ A_4x_0+B_4 = A_1x_0+B_1,\ A_5x_0+B_5 = A_2x_0+B_2,\ (A_1x_0+B_1, A_2x_0+B_2) \neq (x_0, x_0). \tag{3.3}$$

Proof. First, assume that (3.3) holds. Since $A_6 = 1$, $B_6 = 0$, formulas given in Lemma 3.1 (part one) are valid. Now, from (3.3) and Lemma 3.1, we get for all $n \in \mathbb{N}_0$

$$x_{6n} = x_0,\ x_{6n+1} = A_1x_0 + B_1,\ x_{6n+2} = A_2x_0 + B_2,\ x_{6n+3} = A_3x_0 + B_3 = x_0,$$

$$x_{6n+4} = A_4x_0 + B_4 = A_1x_0 + B_1,\ x_{6n+5} = A_5x_0 + B_5 = A_2x_0 + B_2,$$

or equivalently,

$$x_{3n} = x_0,\ x_{3n+1} = A_1x_0 + B_1,\ x_{3n+2} = A_2x_0 + B_2,\ n \in \mathbb{N}_0. \tag{3.4}$$

We distinguish four cases:

1) The quantities x_0, $A_1x_0 + B_1$, $A_2x_0 + B_2$ are pairwise different: in this case, the solution takes the form in (3.4), or equivalently,

$$(x_n)_{n=-2}^{+\infty} = (x_{-2}, x_{-1}, \overbrace{x_0, A_1x_0 + B_1, A_2x_0 + B_2}, \overbrace{x_0, A_1x_0 + B_1, A_2x_0 + B_2}, ...,). \tag{3.5}$$

As the quantities x_0, $A_1x_0 + B_1$, $A_2x_0 + B_2$ are pairwise different, and the solution repeats with the cycle of the three terms x_0, $A_1x_0 + B_1$, $A_2x_0 + B_2$, it follows that the solution is an eventually prime 3-periodic.

2) $A_2x_0 + B_2 = A_1x_0 + B_1 \neq x_0$: in this case, the solution takes the form

$$x_{3n} = x_0,\ x_{3n+1} = A_1x_0 + B_1,\ x_{3n+2} = A_1x_0 + B_1,\ n \in \mathbb{N}_0,$$

or equivalently,

$$(x_n)_{n=-2}^{+\infty} = (x_{-2}, x_{-1}, \overbrace{x_0, A_1x_0 + B_1, A_1x_0 + B_1}, \overbrace{x_0, A_1x_0 + B_1, A_1x_0 + B_1}, ...,). \tag{3.6}$$

As the quantities x_0, $A_1x_0 + B_1$ are different, and the solution repeats with the cycle of the three terms x_0, $A_1x_0 + B_1$, $A_1x_0 + B_1$ it follows that the solution is an eventually prime 3-periodic.

3) $A_2x_0 + B_2 = x_0 \neq A_1x_0 + B_1$: in this case, the solution takes the form

$$x_{3n} = x_0,\ x_{3n+1} = A_1x_0 + B_1,\ x_{3n+2} = x_0,\ n \in \mathbb{N}_0,$$

or equivalently,

$$(x_n)_{n=-2}^{+\infty} = (x_{-2}, x_{-1}, \overbrace{x_0, A_1x_0 + B_1, x_0}, \overbrace{x_0, A_1x_0 + B_1, x_0}, ...,). \tag{3.7}$$

As the quantities x_0, $A_1x_0 + B_1$ are different, and the solution repeats with the cycle of the three terms x_0, $A_1x_0 + B_1$, x_0, it follows that the solution is an eventually prime 3-periodic.

4) $A_1x_0 + B_1 = x_0 \neq A_2x_0 + B_2$: in this case, the solution takes the form

$$x_{3n} = x_0,\ x_{3n+1} = x_0,\ x_{3n+2} = A_2x_0 + B_2,\ n \in \mathbb{N}_0,$$

or equivalently,

$$(x_n)_{n=-2}^{+\infty} = (x_{-2}, x_{-1}, \overbrace{x_0, x_0, A_2x_0 + B_2}, \overbrace{x_0, x_0, A_2x_0 + B_2}, ...,). \tag{3.8}$$

As the quantities x_0, $A_2x_0 + B_2$ are different, and the solution repeats with the cycle of the three terms x_0, x_0, $A_2x_0 + B_2$, it follows that the solution is an eventually prime 3-periodic.

Second, assume that $(x_n)_{n=-2}^{+\infty}$ is an eventually prime 3-periodic, that is,

$$\exists n_0 \geq -2 :\ x_{n+3} = x_n,\ n \geq n_0. \tag{3.9}$$

As $n_0 + 2 \geq 0$, $A_6 = 1$, $B_6 = 0$, it follows from Lemma 3.1 (part one) that

$$x_{6(n_0+2)} = x_0,\ x_{6(n_0+2)+1} = A_1x_0 + B_1,\ x_{6(n_0+2)+2} = A_2x_0 + B_2, \tag{3.10}$$

$$x_{6(n_0+2)+3} = A_3x_0 + B_3,\ x_{6(n_0+2)+4} = A_4x_0 + B_4,\ x_{6(n_0+2)+5} = A_5x_0 + B_5. \tag{3.11}$$

From (3.9), we get

$$x_{6(n_0+2)+3} = x_{6(n_0+2)},\ x_{6(n_0+2)+4} = x_{6(n_0+2)+1},\ x_{6(n_0+2)+5} = x_{6(n_0+2)+2},$$

from which, using (3.10) and (3.11), it follows that

$$A_3x_0 + B_3 = x_0,\ A_4x_0 + B_4 = A_1x_0 + B_1,\ A_5x_0 + B_5 = A_2x_0 + B_2,$$

and $(A_1x_0 + B_1, A_2x_0 + B_2)$ must be different from (x_0, x_0). In fact, if we assume that

$$A_1x_0 + B_1 = x_0 = A_2x_0 + B_2,$$

we get

$$A_1x_0 + B_1 = A_2x_0 + B_2 = A_3x_0 + B_3 = A_4x_0 + B_4 = A_5x_0 + B_5 = x_0,$$

and from Lemma 3.1 (part one), we obtain

$$x_n = x_0,\ n \in \mathbb{N}_0,$$

and this contradicts the fact that $(x)_{n=-2}^{+\infty}$ is eventually prime 3-periodic. □

The next result is devoted to the existence of eventually prime 2-periodic solutions.

Theorem 3.2. *Assume that* $B_6 = 0$. *Then, the* $(x_n)_{n=-2}^{+\infty}$ *of Eq* (1.1) *is eventually prime* 2*-periodic if and only if*

$$A_5x_0 + B_5 = A_3x_0 + B_3 = A_1x_0 + B_1,\ A_4x_0 + B_4 = A_2x_0 + B_2 = x_0,\ A_1x_0 + B_1 \neq x_0. \tag{3.12}$$

Proof. First, assume that (3.12) holds. Since $A_6 = 1,\ B_6 = 0$, we have the formulas in Lemma 3.1 (part one). Now, from (3.12) and Lemma 3.1, we get for all $n \in \mathbb{N}_0$

$$x_{6n} = x_0,\ x_{6n+1} = A_1x_0 + B_1,\ x_{6n+2} = A_2x_0 + B_2 = x_0,\ x_{6n+3} = A_3x_0 + B_3 = A_1x_0 + B_1,$$

$$x_{6n+4} = A_4x_0 + B_4 = x_0,\ x_{6n+5} = A_5x_0 + B_5 = A_1x_0 + B_1,$$

or equivalently,

$$x_{2n} = x_0,\ x_{2n+1} = A_1x_0 + B_1,\ n \in \mathbb{N}_0, \tag{3.13}$$

that is,

$$(x_n)_{n=-2}^{+\infty} = (x_{-2}, x_{-1}, \overbrace{x_0, A_1x_0 + B_1}, \overbrace{x_0, A_1x_0 + B_1}, ...,). \tag{3.14}$$

As $A_1x_0 + B_1 \neq x_0$, the solution repeats with the cycle of the two terms x_0, $A_1x_0 + B_1$, and it follows that the solution is eventually prime 2-periodic.

Second, assume that $(x_n)_{n=-2}^{+\infty}$ is eventually prime 2-periodic, that is,

$$\exists n_0 \geq -2 :\ x_{n+2} = x_n,\ n \geq n_0. \tag{3.15}$$

As $n_0 + 2 \geq 0,\ A_6 = 1,\ B_6 = 0$, it follows from Lemma 3.1 (part one) that

$$x_{6(n_0+2)} = x_0,\ x_{6(n_0+2)+1} = A_1x_0 + B_1,\ x_{6(n_0+2)+2} = A_2x_0 + B_2, \tag{3.16}$$

$$x_{6(n_0+2)+3} = A_3x_0 + B_3,\ x_{6(n_0+2)+4} = A_4x_0 + B_4,\ x_{6(n_0+2)+5} = A_5x_0 + B_5. \tag{3.17}$$

From (3.15), we get

$$x_{6(n_0+2)+2} = x_{6(n_0+2)},\ x_{6(n_0+2)+3} = x_{6(n_0+2)+1},\ x_{6(n_0+2)+4} = x_{6(n_0+2)+2},\ x_{6(n_0+2)+5} = x_{6(n_0+2)+3},$$

from which, using (3.16) and (3.17), it follows that

$$A_2x_0 + B_2 = x_0,\ A_3x_0 + B_3 = A_1x_0 + B_1,\ A_4x_0 + B_4 = A_2x_0 + B_2,\ A_5x_0 + B_5 = A_3x_0 + B_3,$$

and $A_1x_0 + B_1$ must be different from x_0. In fact, if we assume that $A_1x_0 + B_1 = x_0$, we get

$$A_1x_0 + B_1 = A_2x_0 + B_2 = A_3x_0 + B_3 = A_4x_0 + B_4 = A_5x_0 + B_5 = x_0,$$

and from Lemma 3.1 (part one), we obtain

$$x_n = x_0,\ n \in \mathbb{N}_0,$$

and this contradicts the fact that $(x)_{n=-2}^{+\infty}$ is eventually prime 2-periodic. □

The following result is devoted to the existence of eventually prime 1-periodic solutions.

Theorem 3.3. *Assume that $B_6 = 0$. Then, the solution $(x_n)_{n=-2}^{+\infty}$ of Eq* (1.1) *is eventually prime* 1-*periodic if and only if*

$$A_5x_0 + B_5 = A_4x_0 + B_4 = A_3x_0 + B_3 = A_2x_0 + B_2 = A_1x_0 + B_1 = x_0. \tag{3.18}$$

Proof. First, assume that (3.18) holds. Since $A_6 = 1$, $B_6 = 0$, we have the formulas in Lemma (3.1) (part one). Now, from (3.18) and Lemma 3.1, we get for all $n \in \mathbb{N}_0$

$$x_{6n} = x_0,\ x_{6n+1} = A_1x_0 + B_1 = x_0,\ x_{6n+2} = A_2x_0 + B_2 = x_0,$$

$$x_{6n+3} = A_3x_0 + B_3 = x_0,\ x_{6n+4} = A_4x_0 + B_4 = x_0,\ x_{6n+5} = A_5x_0 + B_5 = x_0,$$

or equivalently,

$$x_n = x_0,\ n \in \mathbb{N}_0,$$

that is,

$$(x_n)_{n=-2}^{+\infty} = (x_{-2}, x_{-1}, \overbrace{x_0}, \overbrace{x_0}, ...,).$$

The solution repeats with the cycle of one term x_0, and it follows that the solution is eventually prime 1-periodic.

Second, assume that $(x_n)_{n=-2}^{+\infty}$ is eventually prime 1-periodic, that is,

$$\exists n_0 \geq -2 :\ x_{n+1} = x_n,\ n \geq n_0. \tag{3.19}$$

As $n_0 + 2 \geq 0$, $A_6 = 1$, $B_6 = 0$, it follows from Lemma 3.1 (part one) that

$$x_{6(n_0+2)} = x_0,\ x_{6(n_0+2)+1} = A_1x_0 + B_1,\ x_{6(n_0+2)+2} = A_2x_0 + B_2, \tag{3.20}$$

$$x_{6(n_0+2)+3} = A_3x_0 + B_3,\ x_{6(n_0+2)+4} = A_4x_0 + B_4,\ x_{6(n_0+2)+5} = A_5x_0 + B_5. \tag{3.21}$$

From (3.19), we get

$$x_{6(n_0+2)+5} = x_{6(n_0+2)+4} = x_{6(n_0+2)+3} = x_{6(n_0+2)+2} = x_{6(n_0+2)+1} = x_{6(n_0+2)},$$

from which, using (3.20) and (3.21), it follows that

$$A_5x_0 + B_5 = A_4x_0 + B_4 = A_3x_0 + B_3 = A_2x_0 + B_2 = A_1x_0 + B_1 = x_0.$$

□

The following result is a direct consequence of Theorem 3.1, Theorem 3.2, and Theorem 3.3.

Theorem 3.4. *Assume that $B_6 = 0$. Then, the solution $(x_n)_{n=-2}^{+\infty}$ of Eq* (1.1) *is eventually prime* 6-*periodic if and only if the quantities x_0, $A_1x_0 + B_1$, $A_2x_0 + B_2$, $A_3x_0 + B_3$, $A_4x_0 + B_4$, and $A_5x_0 + B_5$ do not satisfy* (3.3) *nor* (3.12) *nor* (3.18).

4. The case $A_6 = -1$

As in the previous section, we use the form of the solutions when $A_6 = -1$ to study the existence of prime periodic solutions for Eq (1.1).

Lemma 4.1. *Let $(x_n)_{n=-2}^{+\infty}$ be a solution of* (1.1)*. Then, for all $n \in \mathbb{N}_0$, we have*

$$x_{12n} = x_0,\ x_{12n+1} = A_1x_0 + B_1,\ x_{12n+2} = A_2x_0 + B_2,$$

$$x_{12n+3} = A_3x_0 + B_3,,\ x_{12n+4} = A_4x_0 + B_4,\ x_{12n+5} = A_5x_0 + B_5,$$

$$x_{12n+6} = -x_0 + B_6,\ x_{12n+7} = -A_1x_0 + A_1B_6 + B_1,\ x_{12n+8} = -A_2x_0 + A_2B_6 + B_2,$$

$$x_{12n+9} = -A_3x_0 + A_3B_6 + B_3,\ x_{12n+10} = -A_4x_0 + A_4B_6 + B_4,\ x_{12n+11} = -A_5x_0 + A_5B_6 + B_5,$$

and the solution will be eventually 12-periodic.

Proof. Taking $A_6 = -1$ in the formulas of the solutions of Eq (1.1) given in Theorem 2.1, we get

$$x_{6n} = (-1)^n x_0 + B_6\left(\frac{(-1)^n - 1}{-2}\right),\ x_{6n+1} = A_1(-1)^n x_0 + A_1 B_6\left(\frac{(-1)^n - 1}{-2}\right) + B_1,$$

$$x_{6n+2} = A_2(-1)^n x_0 + A_2 B_6\left(\frac{(-1)^n - 1}{-2}\right) + B_2,\ x_{6n+3} = A_3(-1)^n x_0 + A_3 B_6\left(\frac{(-1)^n - 1}{-2}\right) + B_3,$$

$$x_{6n+4} = A_4(-1)^n x_0 + A_4 B_6\left(\frac{(-1)^n - 1}{-2}\right) + B_4,\ x_{6n+5} = A_5(-1)^n x_0 + A_5 B_6\left(\frac{(-1)^n - 1}{-2}\right) + B_5,$$

Depending on the parity of n, we get

$$x_{12n} = x_0,\ x_{12n+1} = A_1 x_0 + B_1,\ x_{12n+2} = A_2 x_0 + B_2,$$

$$x_{12n+3} = A_3 x_0 + B_3,,\ x_{12n+4} = A_4 x_0 + B_4,\ x_{12n+5} = A_5 x_0 + B_5,$$

$$x_{12n+6} = -x_0 + B_6,\ x_{12n+7} = -A_1 x_0 + A_1 B_6 + B_1,\ x_{12n+8} = -A_2 x_0 + A_2 B_6 + B_2,$$

$$x_{12n+9} = -A_3 x_0 + A_3 B_6 + B_3,\ x_{12n+10} = -A_4 x_0 + A_4 B_6 + B_4,\ x_{12n+11} = -A_5 x_0 + A_5 B_6 + B_5.$$

□

Theorem 4.1. *Assume that $B_6 \neq 0$, and let $(x_n)_{n=-2}^{+\infty}$ be a solution of Eq* (1.1)*. Then, $(x_n)_{n=-2}^{+\infty}$ is eventually prime* 6*-periodic if and only if*

$$B_6 = 2x_0,\ (A_1 x_0 + B_1, A_2 x_0 + B_2, A_3 x_0 + B_3, A_4 x_0 + B_4, A_5 x_0 + B_5) \neq (x_0, x_0, x_0, x_0, x_0). \tag{4.1}$$

Proof. First, assume that $(x_n)_{n=-2}^{+\infty}$ is eventually prime 6-periodic, that is,

$$\exists n_0 \geq -2 : x_{n+6} = x_n,\ n \geq n_0. \tag{4.2}$$

So, as $n_0 + 2 \geq 0$, it follows from (4.2), that

$$x_{12(n_0+2)+6} = x_{12(n_0+2)}, \tag{4.3}$$

$$x_{12(n_0+2)+7} = x_{12(n_0+2)+1}, \tag{4.4}$$

$$x_{12(n_0+2)+8} = x_{12(n_0+2)+2}, \tag{4.5}$$

$$x_{12(n_0+2)+9} = x_{12(n_0+2)+3}, \tag{4.6}$$

$$x_{12(n_0+2)+10} = x_{12(n_0+2)+4}, \tag{4.7}$$

$$x_{12(n_0+2)+11} = x_{12(n_0+2)+5}. \tag{4.8}$$

From (4.3)–(4.8) and Lemma 4.1, we obtain

$$-x_0 + B_6 = x_0, \tag{4.9}$$

$$-A_1 x_0 + A_1 B_6 + B_1 = A_1 x_0 + B_1, \tag{4.10}$$

$$-A_2 x_0 + A_2 B_6 + B_2 = A_2 x_0 + B_2, \tag{4.11}$$

$$-A_3x_0 + A_3B_6 + B_3 = A_3x_0 + B_3, \tag{4.12}$$

$$-A_4x_0 + A_4B_6 + B_4 = A_4x_0 + B_4, \tag{4.13}$$

$$-A_5x_0 + A_5B_6 + B_5 = A_5x_0 + B_5. \tag{4.14}$$

We note that if (4.9) is satisfied, that is, $B_6 = 2x_0$, all other equalities in (4.10)–(4.14) are automatically satisfied. Using (4.9) in Lemma 4.1, we get for all $n = 0, 1, ...,$

$$x_{6n} = x_0,\ x_{6n+i} = A_i x_0 + B_i,\ i = 1, ..., 5, \tag{4.15}$$

and we must have

$$(A_1x_0 + B_1, A_2x_0 + B_2, A_3x_0 + B_3, A_4x_0 + B_4, A_5x_0 + B_5) \neq (x_0, x_0, x_0, x_0, x_0).$$

In fact, if $A_i x_0 + B_i = x_0$, $i = 1, ..., 5$, from (4.15), we obtain

$$x_n = x_0,\ n \in \mathbb{N}_0,$$

and this contradicts the fact that the solution is eventually prime 6-periodic.

Secondly, assume that (4.1) holds. Since $A_6 = -1$, from Lemma 4.1 we have for all $n \in \mathbb{N}_0$

$$x_{12n} = x_0,\ x_{12n+6} = x_0,$$

$$x_{12n+1} = A_1x_0 + B_1,\ x_{12n+7} = A_1x_0 + B_1,$$

$$x_{12n+2} = A_2x_0 + B_2,\ x_{12n+8} = A_2x_0 + B_2,$$

$$x_{12n+3} = A_3x_0 + B_3,\ x_{12n+9} == A_3x_0 + b_3,$$

$$x_{12n+4} = A_4x_0 + B_4,\ x_{12n+10} = A_4x_0 + B_4,$$

$$x_{12n+5} = A_5x_0 + B_5,\ x_{12n+11} = A_5x_0 + B_5.$$

Or equivalently,

$$x_{6n} = x_0,\ x_{6n+1} = A_1x_0 + B_1,\ x_{6n+2} = A_2x_0 + B_2,$$

$$x_{6n+3} = A_3x_0 + B_3,\ x_{6n+4} = A_4x_0 + B_4,\ x_{6n+5} = A_5x_0 + B_5,$$

that is,

$$x_{n+6} = x_n,\ n \in \mathbb{N}_0,$$

which means that the solution is eventually 6-periodic, and as

$$(A_1x_0 + B_1, A_2x_0 + B_2, A_3x_0 + B_3, A_4x_0 + B_4, A_5x_0 + B_5) \neq (x_0, x_0, x_0, x_0, x_0),$$

the solution will be eventually prime 6-periodic. □

In the next theorem, we discus the existence of eventually prime 4-periodic solutions for Eq (1.1).

Theorem 4.2. *Let $(x_n)_{n=-2}^{+\infty}$ be a solution of Eq* (1.1)*. Then, $(x_n)_{n=-2}^{+\infty}$ is eventually prime* 4*-periodic if and only if*

$$x_0 = A_4x_0 + B_4 = -A_2x_0 + A_2B_6 + B_2, \tag{4.16}$$

$$A_1x_0 + B_1 = A_5x_0 + B_5 = -A_3x_0 + A_3B_6 + B_3, \tag{4.17}$$

$$A_2x_0 + B_2 = -x_0 + B_6 = -A_4x_0 + A_4B_6 + B_4, \tag{4.18}$$

$$A_3x_0 + B_3 = -A_1x_0 + A_1B_6 + B_1 = -A_5x_0 + A_5B_6 + B_5, \tag{4.19}$$

$$(A_1x_0 + B_1, A_2x_0 + B_2, A_3x_0 + B_3) \neq (x_0, x_0, x_0). \tag{4.20}$$

Proof. First, assume that $(x_n)_{n=-2}^{+\infty}$ is eventually prime 4-periodic, that is,

$$\exists n_0 \geq -2 : x_{n+4} = x_n,\ n \geq n_0. \tag{4.21}$$

So, as $n_0 + 2 \geq 0$, it follows from (4.21), that

$$x_{12(n_0+2)+8} = x_{12(n_0+2)+4} = x_{12(n_0+2)}, \tag{4.22}$$

$$x_{12(n_0+2)+9} = x_{12(n_0+2)+5} = x_{12(n_0+2)+1}, \tag{4.23}$$

$$x_{12(n_0+2)+10} = x_{12(n_0+2)+6} = x_{12(n_0+2)+2}, \tag{4.24}$$

$$x_{12(n_0+2)+11} = x_{12(n_0+2)+7} = x_{12(n_0+2)+3}. \tag{4.25}$$

From (4.22)–(4.25) and Lemma 4.1, we obtain

$$x_0 = A_4x_0 + B_4 = -A_2x_0 + A_2B_6 + B_2, \tag{4.26}$$

$$A_1x_0 + B_1 = A_5x_0 + B_5 = -A_3x_0 + A_3B_6 + B_3, \tag{4.27}$$

$$A_2x_0 + B_2 = -x_0 + B_6 = -A_4x_0 + A_4B_6 + B_4, \tag{4.28}$$

$$A_3x_0 + B_3 = -A_1x_0 + A_1B_6 + B_1 = -A_5x_0 + A_5B_6 + B_5. \tag{4.29}$$

Using (4.26)–(4.29) in Lemma 4.1, we get

$$x_0 = x_4 = x_8 = x_{12} = x_{16} = x_{20} = ...,$$

$$A_1x_0 + B_1 = x_1 = x_5 = x_9 = x_{13} = x_{17} = x_{21} = ...,$$

$$A_2x_0 + B_2 = x_2 = x_6 = x_{10} = x_{14} = x_{18} = x_{22} = ...,$$

$$A_3x_0 + B_3 = x_3 = x_7 = x_{11} = x_{15} = x_{19} = x_{23} = ...,$$

that is, for all $n \in \mathbb{N}_0$, we have

$$x_{4n} = x_0,\ x_{4n+i} = A_ix_0 + B_i,\ i = 1, 2, 3, \tag{4.30}$$

and we must have

$$(A_1x_0 + B_1, A_2x_0 + B_2, A_3x_0 + B_3) \neq (x_0, x_0, x_0).$$

In fact, assume that $A_i x_0 + B_i = x_0,\ i = 1, 2, 3$, and it follows from (4.30) that

$$x_n = x_0,\ n \in \mathbb{N}_0,$$

and this contradicts the fact that the solution is eventually prime 4-periodic.

Second, assume that (4.16)–(4.20) hold. From Lemma 4.1, we have

$$x_{12n} = x_0,\ x_{12n+4} = x_0,\ x_{12n+8} = x_0,$$

$$x_{12n+1} = A_1 x_0 + B_1,\ x_{12n+5} = A_1 x_0 + B_1,\ x_{12n+9} = A_1 x_0 + B_1,$$

$$x_{12n+2} = A_2 x_0 + B_2,\ x_{12n+6} = A_2 x_0 + B_2,\ x_{12n+10} = A_2 x_0 + B_2,$$

$$x_{12n+3} = A_3 x_0 + B_3,\ x_{12n+7} = A_3 x_0 + B_3,\ x_{12n+11} = A_3 x_0 + B_3.$$

Or equivalently,

$$x_{4n} = x_0,\ x_{4n+1} = A_1 x_0 + B_1,\ x_{4n+2} = A_2 x_0 + B_2,\ x_{4n+3} = A_3 x_0 + B_3,$$

that is,

$$x_{n+4} = x_n,\ n \in \mathbb{N}_0,$$

which means that the solution is eventually 4-periodic, and as

$$(A_1 x_0 + B_1, A_2 x_0 + B_2, A_3 x_0 + B_3) \neq (x_0, x_0, x_0),$$

the solution will be eventually prime 4-periodic. □

The following theorem is devoted to the existence of eventually prime 3-periodic.

Theorem 4.3. *Assume that $B_6 \neq 0$, and let $(x_n)_{n=-2}^{+\infty}$ be a solution of Eq* (1.1). *Then, $(x_n)_{n=-2}^{+\infty}$ is eventually prime* 3*-periodic if and only if*

$$x_0 = A_3 x_0 + B_3 = -x_0 + B_6 = -A_3 x_0 + A_3 B_6 + B_3, \tag{4.31}$$

$$A_1 x_0 + B_1 = A_4 x_0 + B_4 = -A_1 x_0 + A_1 B_6 + B_1 = -A_4 x_0 + A_4 B_6 + B_4, \tag{4.32}$$

$$A_2 x_0 + B_2 = A_5 x_0 + B_5 = -A_2 x_0 + A_2 B_6 + B_2 = -A_5 x_0 + A_5 B_6 + B_5, \tag{4.33}$$

$$(A_1 x_0 + B_1, A_2 x_0 + B_2) \neq (x_0, x_0). \tag{4.34}$$

Proof. First, assume that $(x_n)_{n=-2}^{+\infty}$ is eventually prime 3-periodic, that is,

$$\exists n_0 \geq -2 : x_{n+3} = x_n,\ n \geq n_0. \tag{4.35}$$

So, as $n_0 + 2 \geq 0$, it follows from (4.35), that

$$x_{12(n_0+2)+9} = x_{12(n_0+2)+6} = x_{12(n_0+2)+3} = x_{12(n_0+2)}, \tag{4.36}$$

$$x_{12(n_0+2)+10} = x_{12(n_0+2)+7} = x_{12(n_0+2)+4} = x_{12(n_0+2)+1}, \tag{4.37}$$

$$x_{12(n_0+2)+11} = x_{12(n_0+2)+8} = x_{12(n_0+2)+5} = x_{12(n_0+2)+2}. \tag{4.38}$$

From (4.36)–(4.38) and Lemma 4.1, we obtain

$$x_0 = A_3x_0 + B_3 = -x_0 + B_6 = -A_3x_0 + A_3B_6 + B_3, \tag{4.39}$$

$$A_1x_0 + B_1 = A_4x_0 + B_4 = -A_1x_0 + A_1B_6 + B_1 = -A_4x_0 + A_4B_6 + B_4, \tag{4.40}$$

$$A_2x_0 + B_2 = A_5x_0 + B_5 = -A_2x_0 + A_2B_6 + B_2 = -A_5x_0 + A_5B_6 + B_5. \tag{4.41}$$

Using (4.39)–(4.41) in Lemma 4.1, we get for all $n \in \mathbb{N}_0$

$$x_{3n} = x_0,\ x_{3n+i} = A_ix_0 + B_i,\ i = 1, 2, \tag{4.42}$$

and we must have

$$(A_1x_0 + B_1, A_2x_0 + B_2) \neq (x_0, x_0).$$

In fact, assume that $A_ix_0 + B_i = x_0,\ i = 1, 2$, and it follows from (4.42) that

$$x_n = x_0,\ n \in \mathbb{N}_0,$$

and this contradicts the fact that the solution is eventually prime 3-periodic.

Second, assume that (4.31)–(4.34) hold. From Lemma 4.1 we get

$$x_{12n} = x_{12n+3} = x_{12n+6} = x_{12n+9} = x_0,$$

$$x_{12n+1} = x_{12n+4} = x_{12n+7} = x_{12n+10} = A_1x_0 + B_1,$$

$$x_{12n+2} = x_{12n+5} = x_{12n+8} = x_{12n+11} = A_2x_0 + B_2.$$

Or equivalently,

$$x_{3n} = x_0,\ x_{3n+1} = A_1x_0 + B_1,\ x_{3n+2} = A_2x_0 + B_2,$$

that is,

$$x_{n+3} = x_n,\ n \in \mathbb{N}_0,$$

which means that the solution is eventually 3-periodic, and as

$$(A_1x_0 + B_1, A_2x_0 + B_2) \neq (x_0, x_0),$$

the solution will be eventually prime 3-periodic. □

The following theorem is devoted to the existence of eventually prime 2-periodic solutions.

Theorem 4.4. *Assume that $B_6 \neq 0$, and let $(x_n)_{n=-2}^{+\infty}$ be a solution of Eq (1.1). Then, $(x_n)_{n=-2}^{+\infty}$ is eventually prime 2-periodic if and only if*

$$x_0 = A_2x_0 + B_2 = A_4x_0 + B_4 = -x_0 + B_6 = -A_2x_0 + A_2B_6 + B_2 = -A_4x_0 + A_4B_6 + B_4, \tag{4.43}$$

$$A_1x_0+B_1 = A_3x_0+B_3 = A_5x_0+B_5 = -A_1x_0+A_1B_6+B_1 = -A_3x_0+A_3B_6+B_3 = -A_5x_0+A_5B_6+B_5, \tag{4.44}$$

$$A_1x_0 + B_1 \neq x_0. \tag{4.45}$$

Proof. First, assume that $(x_n)_{n=-2}^{+\infty}$ is eventually prime 2-periodic, that is,

$$\exists n_0 \geq -2 : x_{n+2} = x_n,\ n \geq n_0. \tag{4.46}$$

So, as $n_0 + 2 \geq 0$, it follows from (4.46), that

$$x_{12(n_0+2)+10} = x_{12(n_0+2)+8} = x_{12(n_0+2)+6} = x_{12(n_0+2)+4} = x_{12(n_0+2)+2} = x_{12(n_0+2)}, \tag{4.47}$$

$$x_{12(n_0+2)+11} = x_{12(n_0+2)+9} = x_{12(n_0+2)+7} = x_{12(n_0+2)+5} = x_{12(n_0+2)+3} = x_{12(n_0+2)+1}. \tag{4.48}$$

From (4.47), (4.48), and Lemma 4.1, we obtain

$$x_0 = A_2x_0 + B_2 = A_4x_0 + B_4 = -x_0 + B_6 = -A_2x_0 + A_2B_6 + B_2 = -A_4x_0 + A_4B_6 + B_4, \tag{4.49}$$

$$A_1x_0+B_1 = A_3x_0+B_3 = A_5x_0+B_5 = -A_1x_0+A_1B_6+B_1 = -A_3x_0+A_3B_6+B_3 = -A_5x_0+A_5B_6+B_5. \tag{4.50}$$

Using (4.49) and (4.50) in Lemma 4.1, we get for all $n \in \mathbb{N}_0$

$$x_{2n} = x_0,\ x_{2n+1} = A_1x_0 + B_1, \tag{4.51}$$

and we must have

$$A_1x_0 + B_1 \neq x_0.$$

In fact, assume that $A_1x_0 + B_1 = x_0$, and it follows from (4.51) that

$$x_n = x_0,\ n \in \mathbb{N}_0,$$

and this contradicts the fact that the solution is eventually prime 2-periodic.

Second, assume that (4.43)–(4.45) hold. From Lemma 4.1, we get

$$x_{12n} = x_{12n+2} = x_{12n+4} = x_{12n+6} = x_{12n+8} = x_{12n+10} = x_0,$$

$$x_{12n+1} = x_{12n+3} = x_{12n+5} = x_{12n+7} = x_{12n+9} = x_{12n+11} = A_1x_0 + B_1.$$

Or equivalently,

$$x_{2n} = x_0,\ x_{2n+1} = A_1x_0 + B_1,$$

that is,

$$x_{n+2} = x_n,\ n \in \mathbb{N}_0,$$

which means that the solution is eventually 2-periodic, and as

$$A_1x_0 + B_1 \neq x_0,$$

the solution will be eventually prime 2-periodic. □

In the following theorem, we discuss the existence of eventually prime 1-periodic solutions.

Theorem 4.5. *Assume that $B_6 \neq 0$, and let $(x_n)_{n=-2}^{+\infty}$ be a solution of Eq* (1.1)*. Then, $(x_n)_{n=-2}^{+\infty}$ is eventually prime* 1*-periodic if and only if*

$$x_0 = -x_0 + B_6 = A_ix_0 + B_i = -A_ix_0 + A_iB_6 + B_i,\ i = 1, ..., 5. \tag{4.52}$$

Proof. First, assume that $(x_n)_{n=-2}^{+\infty}$ is eventually prime 1-periodic, that is,

$$\exists n_0 \geq -2 : x_{n+1} = x_n,\ n \geq n_0. \tag{4.53}$$

So, as $n_0 + 2 \geq 0$, it follows from (4.53) that

$$x_{12(n_0+2)+11} = x_{12(n_0+2)+10} = x_{12(n_0+2)+9} = ... = x_{12(n_0+2)+1} = x_{12(n_0+2)}. \tag{4.54}$$

From (4.54) and Lemma 4.1, we obtain that all the quantities x_0, $A_1x_0+B_1$, $A_2x_0+B_2$, $A_3x_0+B_3$ $A_4x_0+B_4$, $A_5x_0 + B_5$, $-x_0 + B_6$, $-A_1x_0 + A_1B_6 + B_1$, $-A_2x_0 + A_2B_6 + B_2$, $-A_3x_0 + A_3B_6 + B_3$, $-A_4x_0 + A_4B_6 + B_4$, and $-A_5x_0 + A_5B_6 + B_5$ are equal.

Second, assume that (4.52) holds. From Lemma 4.1, we get

$$x_{12n+11} = x_{12n+10} = x_{12n+9} = ... = x_{12n+1} = x_{12n+1} = x_0,$$

Or equivalently,

$$x_n = x_0,$$

that is,

$$x_{n+1} = x_n,\ n \in \mathbb{N}_0,$$

which means that the solution is eventually prime 1-periodic. □

Remark 4.1. *From Theorem 4.1, Theorem 4.3, Theorem 4.4, and Theorem 4.5, the existence of 6-periodic, 3-periodic, 2-periodic, and 1-periodic solutions requires that $x_0 = -x_0 + B_6$. If $B_6 = 0$, this leads to $x_0 = 0$, contradicting the assumption x_0 is non-zero. Hence, condition $B_6 \neq 0$ is essential in these theorems.*

5. The case $A_6 = 0$

This section concerns the form and the periodicity of the solutions of (1.1) in the case where the parameter $A_6 = 0$.

Lemma 5.1. *Assume that $A_6 = 0$, and let $(x_n)_{n=-2}^{+\infty}$ be a solution of* (1.1). *Then for all $n \in \mathbb{N}_0$, we have*

$$x_{6n} = x_0,\ x_{6n+1} = A_1x_0 + B_1,\ x_{6n+2} = A_2x_0 + B_2,$$

$$x_{6n+3} = A_3x_0 + B_3,\ x_{6n+4} = A_4x_0 + B_4,\ x_{6n+5} = A_5x_0 + B_5,$$

that is, the solution is eventually 6-periodic. Moreover, if $x_{-2} = A_4x_0 + B_4$ and $x_{-1} = A_5x_0 + B_5$, the solution will be 6-periodic.

Proof. As $A_6 = 0$, It follows from Theorem 2.1 that for all $n = 0, 1, ...,$

$$x_{6n} = B_6,\ x_{6n+i} = A_iB_6 + B_i,\ i = 1, ..., 5.$$

As the formula of x_{6n} is valid for $n = 0$, it follows that $B_6 = x_0 \neq 0$. So,

$$x_{6n} = x_0,\ x_{6n+i} = A_ix_0 + B_i,\ i = 1, ..., 5,$$

and the solution is is eventually 6-periodic. To get the solution 6-periodic, it is enough to take

$$A_4x_0 + B_4 = x_{-2},\ A_5x_0 + B_5 = x_{-1}.$$

□

Remark 5.1. *From Lemma 3.1, when $A_6 = 1$, $B_6 = 0$, and Lemma 5.1, when $A_6 = 0$, $B_6 = x_0$, the solutions of* (1.1) *take the same form. As a consequence, the following result holds, and its proof follows line by line those of Theorems 3.1–3.3.*

Theorem 5.1. *Assume that $A_6 = 0$, and let $(x_n)_{n=-2}^{+\infty}$ be a solution of* (1.1). *Then,*

The solution $(x_n)_{n=-2}^{+\infty}$ will be eventually prime 3-periodic if and only if condition (3.3) is satisfied.

The solution $(x_n)_{n=-2}^{+\infty}$ will be eventually prime 2-periodic if and only if condition (3.12) is satisfied.

The solution $(x_n)_{n=-2}^{+\infty}$ will be eventually prime 1-periodic if and only if condition (3.18) is satisfied.

The following result is a consequence of Theorem 5.1.

Theorem 5.2. *Assume that $A_6 = 0$, and let $(x_n)_{n=-2}^{+\infty}$ be a solution of* (1.1). *Then, $(x_n)_{n=-2}^{+\infty}$ will be eventually prime* 6*-periodic if and only if the quantities x_0, $A_1x_0 + B_1$, $A_2x_0 + B_2$, $A_3x_0 + B_3$, $A_4x_0 + B_4$, and $A_5x_0 + B_5$ do not satisfy* (3.3) *nor* (3.12) *nor* (3.18).

6. The case $A_6 \neq -1, 0, 1$

This section investigates the behavior of solutions of (1.1), when the parameter $A_6 \neq -1, 0, 1$. The following result is a direct consequence of Theorem 2.1, therefore its proof will be omitted.

Theorem 6.1. *Let $(x_n)_{n=-2}^{+\infty}$ be a solution of* (1.1).

- Assume that $|A_6| < 1$, then

$$\lim_{n\to+\infty} x_{6n} = \frac{B_6}{1-A_6}, \lim_{n\to+\infty} x_{6n+i} = \frac{A_iB_6}{1-A_6} + B_i, \ i = 1, ..., 5,$$

that is, $(x_n)_{n=-2}^{+\infty}$ converges to an eventually 6-periodic solution.

- Assume that $|A_6| > 1$, then for every $i = 0, 1, ..., 5$, we get

$$\lim_{n\to+\infty} |x_{6n+i}| = +\infty,$$

if $x_0 + \frac{A_iB_6}{1-A_6} \neq 0$, and

$$\lim_{n\to+\infty} x_{6n+i} = -x_0 + B_i, \ i = 0, 1, ..., 5,$$

if $x_0 + \frac{A_iB_6}{1-A_6} = 0$, where $A_0 = 1$, $B_0 = 0$.

7. Numerical examples

In this section, in order to illustrate and support our theoretical findings, we present some numerical examples with different parameters and initial values. These examples confirm the correctness of our results.

Example 7.1. *Let $a = 1$, $b = 1$, and $\alpha = -2$. For this choice, Eq* (1.1) *takes the form*

$$x_{n+1} = \frac{-2x_nx_{n-2}}{x_{n-1}}\left(\frac{x_n - x_{n-1} - 1}{x_{n-1} - x_{n-2} - 1}\right) + x_n + 1. \tag{7.1}$$

If we take,

$$x_{-2} = 2, \ x_{-1} = -1, \ x_0 = 2,$$

we get

$$A_6 = 1, \ B_6 = 0,$$

$$A_5 x_0 + B_5 = A_3 x_0 + B_3 = A_1 x_0 + B_1 = -1,$$

$$A_4 x_0 + B_4 = A_2 x_0 + B_2 = x_0 = 2,$$

and for all $n = 0, 1, ...,$

$$x_{2n} = 2 = x_{-2}, \ x_{2n+1} = -1 = x_{-1}.$$

It follows that

$$x_{n+2} = x_n, \ n = -2, 0, ...,$$

that is, the solution is prime 2-periodic, and this confirms the result of Theorem 3.2. See Figure 1.

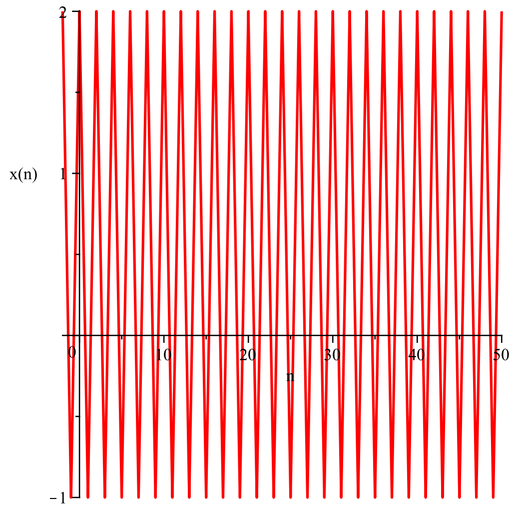

Figure 1. The graphic shows the 2-periodicity of the solution of Eq (7.1) given in Example 7.1.

Example 7.2. *Let* $a = 1$, $b = 0$, *and* $\alpha = 1$. *For this choice, Eq* (1.1) *takes the form*

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}} \left(\frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}} \right) + x_n. \tag{7.2}$$

If we take

$$x_{-2} = 2, \ x_{-1} = \frac{1}{2}, \ x_0 = 2,$$

we get

$$A_6 = 1, \ B_6 = 0,$$

$$A_5 x_0 + B_5 = \frac{1}{2}, \ A_4 x_0 + B_4 = 2, \ A_3 x_0 + B_3 = \frac{8}{3}, \ A_2 x_0 + B_2 = 2, \ A_1 x_0 + B_1 = -6.$$

Clearly, the quantities $A_i x_0 + B_i$, $i = 1, 2, ..., 5$ do not satisfy the conditions (3.3), (3.12), and (3.18). Also, for all $n = 0, 1, ...$, we have

$$x_{6n} = 2 = x_0,\ x_{6n+1} = -6,\ x_{6n+2} = 2,$$

$$x_{6n+3} = \frac{8}{3},\ x_{6n+4} = 2 = x_{-2},\ x_{6n+5} = \frac{1}{2} = x_{-1}.$$

It follows that

$$x_{n+6} = x_n,\ n = -2, 0, ...,$$

that is, the solution is prime 6-periodic, and this confirms the result of Lemma 3.1 and Theorem 3.4. See Figure 2.

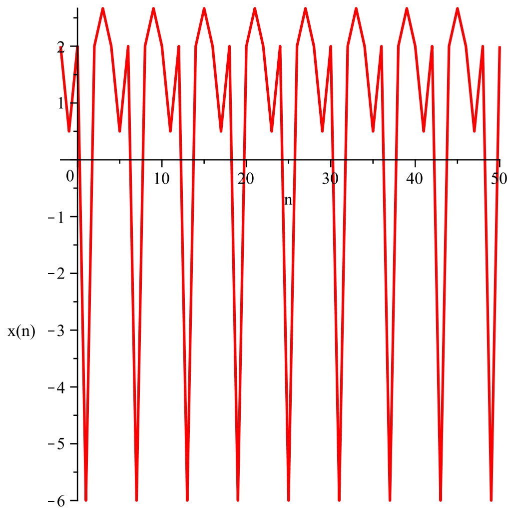

Figure 2. The graphic shows the 6-periodicity of the solution of Eq (7.2) given in Example 7.2.

Example 7.3. *Let* $a = -1$, $b = 1$, *and* $\alpha = 1$, *then Eq* (1.1) *will be*

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}} \left(\frac{x_n + x_{n-1} - 1}{x_{n-1} - x_{n-2} - 2} \right) - x_n + 1. \tag{7.3}$$

Assume that

$$x_{-2} = -\frac{1}{36}(188 + 36\sqrt{93})^{1/3} + \frac{11}{9(188 + 36\sqrt{93})^{1/3}} + \frac{5}{18},\ x_{-1} = \frac{1}{2},$$

$$x_0 = \frac{1}{18}\left(\frac{23(188 + 36\sqrt{93})^{1/3} + 3(188 + 36\sqrt{93})^{1/3}\sqrt{93} + 13(188 + 36\sqrt{93})^{2/3} + 130 - 6\sqrt{93}}{(188 + 36\sqrt{93})^{2/3}} \right).$$

For this choice of the parameters and initial values, we get

$$A_6 = -1,$$

$$B_6 = -\frac{23}{396}\left(188 + 36\sqrt{93}\right)^{\frac{1}{3}} + \frac{1}{132}\left(188 + 36\sqrt{93}\right)^{\frac{1}{3}}\sqrt{93}$$

$$
\begin{aligned}
&+\frac{65}{8712}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}+\frac{1}{2904}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}\sqrt{93}+\frac{13}{9}\\
&=2x_0,
\end{aligned}
$$

and for all $n = 0, 1, ...,$

$$
\begin{aligned}
x_{6n} &= \frac{1}{18}\left(\frac{23(188+36\sqrt{93})^{1/3}+3(188+36\sqrt{93})^{1/3}\sqrt{93}+13(188+36\sqrt{93})^{2/3}+130-6\sqrt{93}}{(188+36\sqrt{93})^{2/3}}\right)\\
&= x_0,
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+1} =& \frac{7}{54}+\frac{13}{594}\left(188+36\sqrt{93}\right)^{\frac{1}{3}}-\frac{1}{99}\left(188+36\sqrt{93}\right)^{\frac{1}{3}}\sqrt{93}\\
&+\frac{5}{8712}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}\sqrt{93}-\frac{401}{26136}\left(188+36\sqrt{93}\right)^{\frac{2}{3}},
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+2} =& -\frac{1}{484}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}\sqrt{93}+\frac{215}{8712}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}+\frac{23}{18}\\
&+\frac{1}{88}\left(188+36\sqrt{93}\right)^{\frac{1}{3}}\sqrt{93}+\frac{41}{792}\left(188+36\sqrt{93}\right)^{\frac{1}{3}},
\end{aligned}
$$

$$
\begin{aligned}
x_{6n+3} =& -\frac{11}{18}-\frac{1}{396}\left(188+36\sqrt{93}\right)^{\frac{1}{3}}+\frac{1}{132}\left(188+36\sqrt{93}\right)^{\frac{1}{3}}\sqrt{93}\\
&+\frac{14}{1089}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}-\frac{1}{1452}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}\sqrt{93},
\end{aligned}
$$

$$
x_{6n+4} = -\frac{1}{36}\left(188+36\sqrt{93}\right)^{\frac{1}{3}}-\frac{47}{17424}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}+\frac{1}{1936}\left(188+36\sqrt{93}\right)^{\frac{2}{3}}\sqrt{93}+\frac{5}{18},
$$

$$
x_{6n+5} = \frac{1}{2} = x_{-1}.
$$

It follows that

$$
x_{n+6} = x_n,\ n = -1, 0, ...,
$$

that is, the solution is eventually prime 6-periodic. This confirms the result of Theorem 4.1. See Figure 3.

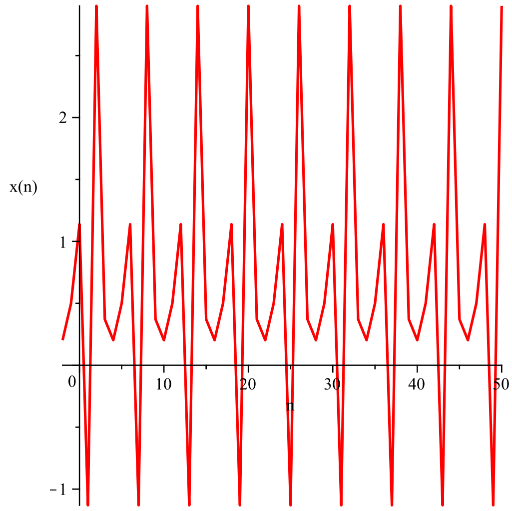

Figure 3. The graphic shows the 6-periodicity of the solution of Eq (7.3) given in Example 7.3.

Example 7.4. *Let* $a = 1$*,* $b = 2$*, and* $\alpha = -1$*. For this choice, Eq* (1.1) *takes the form*

$$x_{n+1} = \frac{-2x_n x_{n-2}}{x_{n-1}}\left(\frac{x_n - x_{n-1} - 2}{x_{n-1} - x_{n-2} - 2}\right) + x_n + 2. \tag{7.4}$$

If we take

$$x_{-2} = 1,\ x_{-1} = -2,\ x_0 = \frac{19 + \sqrt{41}}{4},$$

we get

$$A_6 = -1, B_6 = \frac{-46}{5} + \frac{3\sqrt{41}}{10},$$

and for all $n = 0, 1, ...,$

$$x_{12n} = \frac{19 + \sqrt{41}}{4} = x_0,\ x_{12n+1} = \frac{339 + \sqrt{41}}{80},\ x_{12n+2} = \frac{539 + \sqrt{41}}{100},$$

$$x_{12n+3} = \frac{121}{25} + \frac{7\sqrt{41}}{4},\ x_{12n+4} = -\frac{63}{16} + \frac{33\sqrt{41}}{80},\ x_{12n+5} = \frac{71}{4} - \frac{23\sqrt{41}}{20},$$

$$x_{12n+6} = \frac{-279 + \sqrt{41}}{20},\ x_{12n+7} = \frac{-43 + 3\sqrt{41}}{8},\ x_{12n+8} = \frac{-23 + 3\sqrt{41}}{10},$$

$$x_{12n+9} = \frac{11 + \sqrt{41}}{10},\ x_{12n+10} = 1 = x_{-2},\ x_{12n+11} = -2 = x_{-1}.$$

It follows that the solution is 12-periodic, and this confirms the result of Lemma 4.1. As

$$x_{n+12} = x_n,\ n = -2, -1, ...,$$

the solution is prime 12-periodic. See Figure 4.

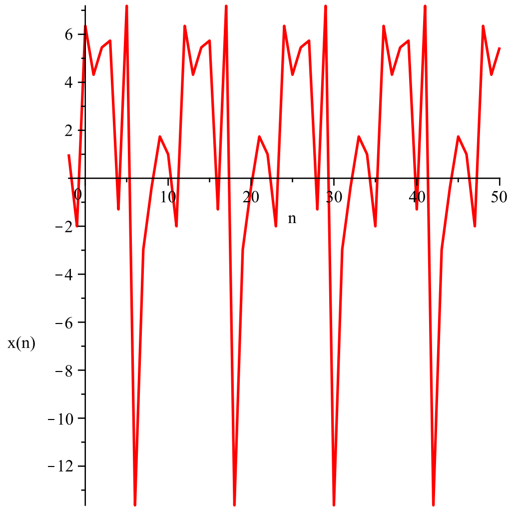

Figure 4. The graphic shows the 12-periodicity of the solution of Eq (7.4) given in Example 7.4.

Example 7.5. *Let* $a = 1$*,* $b = 1$*, and* $\alpha = 2$*. For this choice, Eq* (1.1) *takes the form*

$$x_{n+1} = \frac{2x_n x_{n-2}}{x_{n-1}} \left(\frac{x_n - x_{n-1} - 1}{x_{n-1} - x_{n-2} - 1} \right) + x_n + 1. \tag{7.5}$$

If we take

$$x_{-2} = 1,\ x_{-1} = 1,\ x_0 = 3,$$

we get

$$A_6 = 0, B_6 = 3,$$

$$A_1 x_0 + B_1 = -2,\ A_2 x_0 + B_2 = 7,\ A_3 x_0 + B_3 = 36,\ A_4 x_0 + B_4 = -35,\ A_5 x_0 + B_5 = 1,$$

and for all $n = 0, 1, ...,$

$$x_{6n} = 3 = x_0,\ x_{6n+1} = -2,\ x_{6n+2} = 7,\ x_{12n+3} = 36,\ x_{6n+4} = -5,\ x_{6n+5} = 1 = x_{-1}.$$

It follows that

$$x_{n+6} = x_n,\ n = -1, 0, ...,$$

that is, the solution is eventually prime 6-periodic, and this confirms the results of Lemma 5.1 and Theorem 5.2. See Figure 5.

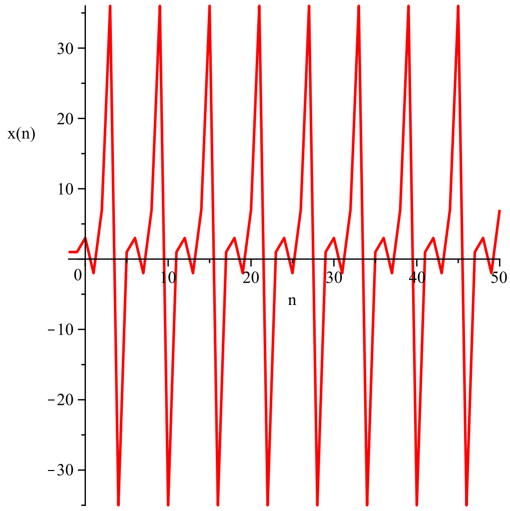

Figure 5. The graphic shows the 6-periodicity of the solution of Eq (7.5) given in Example 7.5.

Example 7.6. *Let* $a = 2$*,* $b = 1$*, and* $\alpha = -1$*. For this choice, Eq* (1.1) *takes the form*

$$x_{n+1} = -\frac{x_n x_{n-2}}{x_{n-1}}\left(\frac{x_n - 2x_{n-1} - 1}{x_{n-1} - 2x_{n-2} - 1}\right) + 2x_n + 1. \tag{7.6}$$

If we take

$$x_{-2} = 7,\ x_{-1} = -1,\ x_0 = 2,$$

we get

$$A_6 = 0.633262472 < 1,$$

$$\frac{B_6}{1 - A_6} = 5.864141463,\ \frac{A_1 B_6}{1 - A_6} + B_1 = 5.031597256,\ \frac{A_2 B_6}{1 - A_6} + B_2 = 8.861870712,$$

$$\frac{A_3 B_6}{1 - A_6} + B_3 = 15.76978452,\ \frac{A_4 B_6}{1 - A_6} + B_4 = 20.52449512,\ \frac{A_5 B_6}{1 - A_6} + B_5 = -4.864141463.$$

So, it follows that $(x_n)_{n=-2}^{+\infty}$ converges to a 6-periodic solution as follows:

$$\lim_{n\to+\infty} x_{6n} = \frac{B_6}{1 - A_6},\ \lim_{n\to+\infty} x_{6n+i} = \frac{A_i B_6}{1 - A_6} + B_i,\ i = 1, ..., 5,$$

and this confirms the result of Theorem 6.1. See Table 1.

Table 1. The table shows that the solution of Eq (7.6) converges to a 6-periodic solution.

n	x_{6n}	x_{6n+1}	x_{6n+2}	x_{6n+3}	x_{6n+4}	x_{6n+5}
0	2	2.375000000	4.710937500	8.851562500	11.95907738	-2.41687925
10	5.83873430	5.014129777	8.834577769	15.72429630	20.46817646	-4.84805033
20	5.86397450	5.031482380	8.861691216	15.76948539	20.52412491	-4.86403565
30	5.86414101	5.031596872	8.861870111	15.76978355	20.52449405	-4.86414120
40	5.86414137	5.031597100	8.861870462	15.76978412	20.52449475	-4.86414122
50	5.86414133	5.031597089	8.861870449	15.76978411	20.52449475	-4.86414124
60	5.86414137	5.031597102	8.861870461	15.76978412	20.52449475	-4.86414122
70	5.86414133	5.031597089	8.861870449	15.76978411	20.52449475	-4.86414124
80	5.86414137	5.031597102	8.861870461	15.76978412	20.52449475	-4.86414122
90	5.86414133	5.031597089	8.861870449	15.76978411	20.52449475	-4.86414124

8. Conclusions

In this work, we derived explicit formulas for the solutions of a new difference equation. Based on these formulas, we conducted a detailed analysis characterizing both the existence of periodic solutions and their asymptotic behavior. Several numerical examples were also presented to illustrate and support the theoretical results. The approach proposed in this work is original and provides a rigorous and systematic method for identifying prime periodic solutions. By relying on explicit solution formulas, it leads to clear and verifiable conditions for periodicity. Moreover, the proposed approach can potentially be extended to other nonlinear difference equations of similar forms. It can also be used to complement and extend the results obtained in [19] concerning prime periodic solutions.

Open problem: As a natural extension of the present study, we consider the following system of difference equations, to which we believe our approach can be successfully applied:

$$x_{n+1} = \frac{cx_n y_{n-2}}{y_{n-1}}\left(\frac{y_n - \alpha y_{n-1} - \beta}{y_{n-1} - \alpha y_{n-2} - \beta}\right) + ax_n + b,\ y_{n+1} = \frac{\gamma y_n x_{n-2}}{x_{n-1}}\left(\frac{x_n - ax_{n-1} - b}{x_{n-1} - ax_{n-2} - b}\right) + \alpha y_n + \beta,\ n \in \mathbb{N}_0, \quad (8.1)$$

where c, γ, and the initial values x_{-2}, y_{-2}, x_{-1}, y_{-1}, x_0, and y_0 are non-zero real numbers, while a, α, b, and β are real numbers.

Author contributions

Turki D. Alharbi: Investigation, software, Writing–review and editing, project administration; Nouressadat Touafek: Methodology, investigation, software, writing–review and editing, visualization, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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