Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(9): 30815–30829.
DOI: 10.3934/math.20261220
Received: 05 August 2026
Revised: 14 September 2026
Accepted: 16 September 2026
Published: 21 September 2026

Research article

Composition operators from the Bloch space to α-Bergman–Morrey spaces

Xiangling Zhu[1] **and Qinghua Hu**[2,*]

[1] University of Electronic Science and Technology of China, Zhongshan Institute, Zhongshan 528402, Guangdong, China
[2] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China

* **Correspondence:** Email: hqhmath@sina.com.

Abstract: Let $1 \le p < \infty$, $0 < \lambda \le 2$, and $\alpha > 0$. This paper provides a four-case classification of the boundedness and compactness of composition operators from the Bloch space $\mathcal{B}$ into the α–Bergman–Morrey space $A^{p,\lambda,\alpha}$. Quantitative integral criteria for boundedness and compactness of such operators are also derived. As a corollary, we obtain a new integral description for the compactness of $C_\phi : \mathcal{B} \to \mathcal{B}$.

Keywords: Bloch space; composition operator; α-Bergman–Morrey space
Mathematics Subject Classification: 30H30, 47B33

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ stand for the vector space of all analytic functions on $\mathbb{D}$. Let $H^\infty = H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$. Then H^∞ is a Banach algebra with the supremum norm $\|g\|_\infty = \sup_{z\in\mathbb{D}} |g(z)|$. For $0 < p < \infty$, the Bergman space A^p consists of all $g \in H(\mathbb{D})$ satisfying

$$\|g\|_{A^p}^p = \int_{\mathbb{D}} |g(w)|^p \, dA(w) < \infty,$$

where dA represents the normalized Lebesgue area measure on $\mathbb{D}$, namely $dA = \frac{1}{\pi}\, dx\, dy$.

The Bloch space $\mathcal{B}$ consists of all $g \in H(\mathbb{D})$ such that

$$\|g\|_{\mathcal{B}} = |g(0)| + \|g\|_{\mathcal{B},*} < \infty, \qquad \|g\|_{\mathcal{B},*} = \sup_{z\in\mathbb{D}}(1 - |z|^2)|g'(z)|.$$

A well-known result from [1] states that

$$\|g\|_{\mathcal{B},*} \asymp \sup_{c\in\mathbb{D}} \left\| g \circ \sigma_c - g(c) \right\|_{A^p},$$

in which $\sigma_c(z) = \frac{c-z}{1-\bar{c}z}$ denotes the Möbius automorphism of $\mathbb{D}$.

Building on the abovementioned result, Zhu and Yang introduced [20] the analytic α-Bergman–Morrey space $A^{p,\lambda,\alpha}$ for $1 \leq p < \infty$, $0 < \lambda \leq 2$, and $\alpha > 0$. We use the following equivalent norm from [20]: An analytic function g belongs to $A^{p,\lambda,\alpha}$ if

$$\|g\|^p_{A^{p,\lambda,\alpha}} = |g(0)|^p + \|g\|^p_{A^{p,\lambda,\alpha},*} < \infty, \tag{1.1}$$

where

$$\|g\|^p_{A^{p,\lambda,\alpha},*} = \sup_{c\in\mathbb{D}}(1-|c|^2)^{2-\lambda}\int_{\mathbb{D}}|g'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\sigma_c(z)|^2)^{p-p\alpha+2}\,dA(z).$$

The space $A^{p,\lambda,\alpha}$ is a Banach space under this norm, and norm convergence implies uniform convergence on every compact subset of $\mathbb{D}$; see [20]. Setting $\alpha = 1$ recovers the Bergman–Morrey space A^p_λ, originally investigated by Yang and Liu in [17]. If $1 - (2-\lambda)/p > 0$, then the choice $\alpha = 1 - (2-\lambda)/p$ gives the classical Bloch space $\mathcal{B}$ with equivalent norms. Further properties of $A^{p,\lambda,\alpha}$ can be found in [20].

Given an analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$, the associated composition operator C_ϕ is defined by

$$C_\phi f = f \circ \phi, \qquad \forall\, f \in H(\mathbb{D}).$$

An immediate corollary of Littlewood's subordination theorem asserts that every analytic self-map of $\mathbb{D}$ induces a bounded composition operator on all Hardy spaces and Bergman spaces over $\mathbb{D}$. Another standard consequence of Littlewood's subordination theorem guarantees that each composition operator acts as a bounded linear operator on BMOA, the bounded mean oscillation analytic function space. Recall that a function $f \in H(\mathbb{D})$ lies in BMOA if and only if

$$\|g\|_* = \sup_{c\in\mathbb{D}} \|g \circ \sigma_c - g(c)\|_{H^2} < \infty,$$

where H^2 denotes the Hardy space. The compactness properties of C_ϕ : BMOA $\to$ BMOA have been extensively studied in [2, 8, 11, 14, 15]. In particular, Wulan established in [14] that C_ϕ is compact on BMOA precisely when

$$\lim_{k\to\infty} \|\phi^k\|_* = 0 \quad \text{and} \quad \lim_{|c|\to 1} \|\sigma_c \circ \phi\|_* = 0.$$

Subsequently, Wulan et al. refined this characterization in [15] by proving that the condition $\lim_{k\to\infty} \|\phi^k\|_* = 0$ alone suffices for the compactness of C_ϕ : BMOA $\to$ BMOA.

For an analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$, the Schwarz–Pick type derivative $\phi^\#$ of ϕ is given by

$$\phi^\#(z) = \frac{1-|z|^2}{1-|\phi(z)|^2}\phi'(z).$$

From the Schwarz–Pick lemma, for all $z \in \mathbb{D}$, one has $|\phi^\#(z)| \leq 1$, which implies that C_ϕ acts boundedly on the Bloch space. The compactness of composition operators on the Bloch space was first investigated in [9]. Those authors established that $C_\phi : \mathcal{B} \to \mathcal{B}$ is compact if and only if

$$\lim_{|\phi(z)|\to 1} |\phi^\#(z)| = 0.$$

Tjani provided an equivalent characterization in [12] and showed that $C_\phi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{|a|\to 1} \|\sigma_a \circ \phi\|_{\mathcal{B},*} = 0$. In [15], Wulan et al. obtained another criterion for the compactness of composition operators on the Bloch space. Their theorem states that $C_\phi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{k\to\infty} \|\phi^k\|_{\mathcal{B}} = 0$.

Composition operators from the Bloch space into $F(p, q, s)$ and Q_K spaces were studied in [3, 7, 13, 19], while [6] concerns the products of differentiation and composition operators from the Bloch space into analytic Morrey-type spaces. The work [16] investigates structural properties and operator-theoretic applications of the Bergman–Morrey spaces corresponding to the case $\alpha = 1$. More recently, the spaces $A^{p,\lambda,\alpha}$ were introduced in [20], where their basic structure and the Volterra operators acting on them were studied. None of these results, however, provides a characterization of the composition operators $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ throughout the full range $1 \le p < \infty$, $0 < \lambda \le 2$, and $\alpha > 0$. In particular, the simultaneous dependence of the norm on the parameter λ and the parameter α produces two transition indices that do not arise in the previously studied settings. In this paper, we conduct a systematic study for the boundedness and compactness of composition operators that map the Bloch space $\mathcal{B}$ into the α-Bergman–Morrey space $A^{p,\lambda,\alpha}$.

The first main contribution of the present paper is a complete parameter classification governed by

$$\alpha_0 = 1 - \frac{2-\lambda}{p}, \qquad \alpha_1 = 1 + \frac{4-\lambda}{p}.$$

The four resulting cases exhibit essentially different operator-theoretic behavior. In the range $0 < \alpha < \alpha_0$, boundedness and compactness coincide and hold precisely when

$$\phi \in A^{p,\lambda,\alpha} \quad \text{and} \quad \|\phi\|_\infty < 1.$$

When $\alpha = \alpha_0 > 0$, the space $A^{p,\lambda,\alpha}$ agrees with the Bloch space up to equivalent norms, and compactness is governed by the classical vanishing hyperbolic-derivative condition. In the range $\alpha_0 < \alpha \le \alpha_1$, every analytic self-map of $\mathbb{D}$ induces a compact composition operator from $\mathcal{B}$ into $A^{p,\lambda,\alpha}$. Finally, when $\alpha > \alpha_1$, boundedness is possible only for constant symbols. Thus, the paper identifies both the exact phase-transition values and the corresponding changes from symbol-dependent boundedness to automatic compactness and, ultimately, to rigidity.

The second contribution is an intrinsic integral characterization of boundedness, together with a two-sided estimate for the operator norm. We also establish two equivalent compactness criteria. One expressed in terms of the Möbius test family $\sigma_a \circ \phi$, and the other through the vanishing of a weighted integral over the sets $\{z \in \mathbb{D} : |\phi(z)| > r\}$. As a consequence, a new family of integral characterizations of compact composition operators on the classical Bloch space is obtained.

The decisive technical ingredient is the following representation

$$\|g\|^p_{A^{p,\lambda,\alpha},*} = \sup_{c\in\mathbb{D}} (1-|c|^2)^{p-p\alpha+4-\lambda} \int_{\mathbb{D}} \frac{|g'(z)|^p (1-|z|^2)^p}{|1-\bar{c}z|^{2p+4-2p\alpha}}\, dA(z).$$

This representation makes the two critical indices explicit and permits uniform estimates at all relevant endpoints. Consequently, the parameter classification and the compactness results are not merely formal applications of the existing theory for $F(p, q, s)$, Q_K, or the classical Bergman–Morrey spaces.

The paper is organized as follows. Section 2 collects some auxiliary lemmas used in the proofs of our main theorems. Section 3 gives a complete parameter classification and establishes the main results and proofs in this paper.

We adopt the notation $X \lesssim Y$ whenever $X \leq CY$ for some constant that is independent of the relevant variables, and $X \asymp Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold. Unless explicitly stated otherwise, implicit constants may depend on the fixed parameters p, λ, α and the prescribed auxiliary quantities, but not on variables over which suprema or limits are taken. The supremum over an empty set is defined to be zero.

2. Preliminaries

We assemble several auxiliary lemmas below, which constitute the foundational estimates for the proofs of our main theorems. The following integral bounds are standard and may be found in [18, Lemma 3.10].

Lemma 2.1. *Let $z \in \mathbb{D}$, $t > -1$, and let c be a real constant. Define*

$$I(z) = \int_{\mathbb{D}} \frac{(1-|\xi|^2)^t}{|1-z\bar{\xi}|^{2+t+c}} dA(\xi).$$

(i) If $c < 0$, then $I(z)$ is uniformly bounded on $\mathbb{D}$;
(ii) If $c > 0$, then $I(z) \asymp \frac{1}{(1-|z|^2)^c}$ as $|z| \to 1^-$;
(iii) If $c = 0$, then $I(z) \asymp \log \frac{1}{1-|z|^2}$ as $|z| \to 1^-$.

The identity

$$1 - |\sigma_c(z)|^2 = \frac{(1-|c|^2)(1-|z|^2)}{|1-\overline{c}z|^2}$$

yields

$$\begin{aligned}
&(1-|c|^2)^{2-\lambda}(1-|z|^2)^{p\alpha-2}(1-|\sigma_c(z)|^2)^{p-p\alpha+2} \\
&= \frac{(1-|c|^2)^{2-\lambda+p-p\alpha+2}(1-|z|^2)^{p\alpha-2+p-p\alpha+2}}{|1-\overline{c}z|^{2p-2p\alpha+4}} \\
&= \frac{(1-|c|^2)^{p-p\alpha+4-\lambda}(1-|z|^2)^{p}}{|1-\overline{c}z|^{2p+4-2p\alpha}}.
\end{aligned}$$

Substituting this into (1.1), we obtain

$$\|g\|^p_{A^{p,\lambda,\alpha},*} = \sup_{c\in\mathbb{D}} (1-|c|^2)^{p-p\alpha+4-\lambda} \int_{\mathbb{D}} \frac{|g'(z)|^p(1-|z|^2)^p}{|1-\overline{c}z|^{2p+4-2p\alpha}} dA(z). \tag{2.1}$$

For simplicity of notation, we introduce two critical indices for the lemma below.

$$\alpha_0 = 1 - \frac{2-\lambda}{p}, \qquad \alpha_1 = 1 + \frac{4-\lambda}{p}. \tag{2.2}$$

Lemma 2.2. *Let $1 \leq p < \infty$, $0 < \lambda \leq 2$, and $\alpha > 0$. Assume $\alpha_0 \leq \alpha \leq \alpha_1$. Then*

$$\sup_{c\in\mathbb{D}} (1-|c|^2)^{p-p\alpha+4-\lambda} \int_{\mathbb{D}} \frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}} < \infty. \tag{2.3}$$

If additionally $\alpha > \alpha_0$, then

$$\lim_{R\to 1^-} \sup_{c\in\mathbb{D}} (1-|c|^2)^{p-p\alpha+4-\lambda} \int_{|z|>R} \frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}} = 0. \tag{2.4}$$

Proof. The condition $\alpha \leq \alpha_1$ is equivalent to

$$p - p\alpha + 4 - \lambda \geq 0.$$

First, suppose $2p + 4 - 2p\alpha > 2$. Applying Lemma 2.1,

$$\begin{aligned}(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{\mathbb{D}}\frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}} &\lesssim (1-|c|^2)^{p-p\alpha+4-\lambda+2-(2p+4-2p\alpha)}\\ &= (1-|c|^2)^{p\alpha-p+2-\lambda} = (1-|c|^2)^{p(\alpha-\alpha_0)} \leq 1.\end{aligned} \tag{2.5}$$

Next, consider the case $2p+4-2p\alpha = 2$. This equality gives $\alpha = 1+1/p$, and $p-p\alpha+4-\lambda = 3-\lambda > 0$. Therefore

$$(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{\mathbb{D}}\frac{dA(z)}{|1-\overline{c}z|^{2}} \lesssim (1-|c|^2)^{3-\lambda}\log\frac{e}{1-|c|^2}.$$

If $2p + 4 - 2p\alpha < 2$, Lemma 2.1 implies

$$(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{\mathbb{D}}\frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}} \lesssim (1-|c|^2)^{p-p\alpha+4-\lambda} \leq 1.$$

The cases above establish (2.3).

Now assume $\alpha > \alpha_0$. We first consider the case of $2p + 4 - 2p\alpha > 2$. Fix $0 < \rho < 1$ and decompose the supremum over $|c| \leq \rho$ and $|c| > \rho$. For $|c| \leq \rho$, we have $|1 - \overline{c}z| \geq 1 - \rho$ and $(1 - |c|^2)^{p-p\alpha+4-\lambda} \leq 1$. The integrals over $\{|z| > R\}$ converge to zero uniformly for $|c| \leq \rho$ as $R \to 1$. For $|c| > \rho$, the integral over $\{|z| > R\}$ is dominated by the full integral over $\mathbb{D}$. By (2.5), this bound is

$$C(1-\rho^2)^{p(\alpha-\alpha_0)},$$

which vanishes as $\rho \to 1$.

When $2p+4-2p\alpha = 2$, the contribution from $|c| \leq \rho$ again tends to zero with R, owing to the lower bound $|1 - \overline{c}z| \geq 1 - \rho$. For $|c| > \rho$, the integral over $\{|z| > R\}$ is bounded by the integral over $\mathbb{D}$. The previously derived estimate yields

$$C\sup_{|c|>\rho}(1-|c|^2)^{3-\lambda}\log\frac{e}{1-|c|^2}.$$

Since $3 - \lambda \geq 1$, this expression approaches zero as $\rho \to 1$.

For $0 < 2p+4-2p\alpha < 2$, select $t > 1$ such that $(2p+4-2p\alpha)t < 2$. Let $E \subset \mathbb{D}$ be any measurable set. The inequality $p - p\alpha + 4 - \lambda \geq 0$ ensures $(1 - |c|^2)^{p-p\alpha+4-\lambda} \leq 1$. Hölder's inequality combined with Lemma 2.1 gives

$$\begin{gathered}(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{E}\frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}}\\ \leq\left(\int_{\mathbb{D}}\frac{dA(z)}{|1-\overline{c}z|^{(2p+4-2p\alpha)t}}\right)^{1/t}A(E)^{1-1/t} \lesssim A(E)^{1-1/t}.\end{gathered}$$

Setting $E = \{|z| > R\}$ verifies (2.4). Finally, if $2p + 4 - 2p\alpha \leq 0$, then

$$\frac{1}{|1-\overline{c}z|^{2p+4-2p\alpha}} \leq 2^{-(2p+4-2p\alpha)}.$$

So

$$(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{|z|>R}\frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}}\le 2^{-(2p+4-2p\alpha)}A(|z|>R).$$

The right-hand side converges to zero uniformly in c as $R\to 1$. ◻

The following lemma can be proved similarly to Proposition 3.11 in [4], and we omit the details.

Lemma 2.3. *Let* $1\le p<\infty$, $0<\lambda\le 2$, *and* $\alpha>0$. *Suppose* $T:\mathcal{B}\to A^{p,\lambda,\alpha}$ *is a bounded operator that is either the canonical embedding or a composition operator* C_ϕ. *Then* T *is compact if and only if* $\|Tf_n\|_{A^{p,\lambda,\alpha}}\to 0$ *holds for every bounded sequence* $(f_n)\subset\mathcal{B}$ *converging uniformly to zero on all compact subsets of* $\mathbb{D}$.

Lemma 2.4. *Let* $1\le p<\infty$, $0<\lambda\le 2$, *and* $\alpha>0$. *Let* α_0 *and* α_1 *be defined by* (2.2).

(a) If $0<\alpha<\alpha_0$, *then* $A^{p,\lambda,\alpha}\hookrightarrow H^\infty$ *continuously.*
(b) If $\alpha_0>0$ *and* $\alpha=\alpha_0$, *then* $A^{p,\lambda,\alpha}=\mathcal{B}$ *with equivalent norms.*
(c) If $\alpha_0<\alpha\le\alpha_1$, *then the embedding* $\mathcal{B}\hookrightarrow A^{p,\lambda,\alpha}$ *is compact.*
(d) If $\alpha>\alpha_1$, *then* $A^{p,\lambda,\alpha}$ *contains only constant functions.*

Proof. By the result in [20], for all $g\in A^{p,\lambda,\alpha}$ and every $z\in\mathbb{D}$, we have

$$|g'(z)|\lesssim\|g\|_{A^{p,\lambda,\alpha}}(1-|z|^2)^{\alpha_0-\alpha-1}.$$

Then

$$|g(z)-g(0)|\lesssim\|g\|_{A_{p,\lambda,\alpha}}\int_0^{|z|}(1-t^2)^{\alpha_0-\alpha-1}dt\lesssim\|g\|_{A_{p,\lambda,\alpha}}.$$

Hence, $A^{p,\lambda,\alpha}\hookrightarrow H^\infty$ continuously whenever $0<\alpha<\alpha_0$.

Statement (b) is established in [20].

Now assume $\alpha_0<\alpha\le\alpha_1$. For any $g\in\mathcal{B}$, the estimate

$$|g'(z)|^p(1-|z|^2)^p\le\|g\|^p_{\mathcal{B},*}$$

remains valid. Substituting this bound into (2.1) and applying (2.3) yields

$$\|g\|^p_{A^{p,\lambda,\alpha}}\lesssim|g(0)|^p+\|g\|^p_{\mathcal{B},*}\lesssim\|g\|^p_{\mathcal{B}}.$$

Hence, the embedding in (c) is bounded.

We next verify compactness for case (c). Take a sequence (g_n) bounded in $\mathcal{B}$ and converging uniformly to zero on compact subsets of $\mathbb{D}$. Fix $0<R<1$. From (2.1), we have

$$\begin{aligned}&\sup_{c\in\mathbb{D}}(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{|z|\le R}\frac{|g_n'(z)|^p(1-|z|^2)^p}{|1-\overline{c}z|^{2p+4-2p\alpha}}\,dA(z)\\ &\le\sup_{|z|\le R}|g_n'(z)|^p\sup_{c\in\mathbb{D}}(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{\mathbb{D}}\frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}},\end{aligned}$$

where we use $(1-|z|^2)^p \le 1$. The second supremum is finite by (2.3), while Cauchy's integral formula implies $\sup_{|z|\le R}|g_n'(z)| \to 0$ as $n \to \infty$. Accordingly, the right-hand side vanishes as $n \to \infty$. By the definition of Bloch space, we have

$$\sup_{c\in\mathbb{D}}(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{|z|>R}\frac{|g_n'(z)|^p(1-|z|^2)^p}{|1-\overline{c}z|^{2p+4-2p\alpha}}\,dA(z)$$
$$\le \sup_n \|g_n\|_{\mathcal{B},*}^p \sup_{c\in\mathbb{D}}(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{|z|>R}\frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}}.$$

Lemma 2.2 ensures this term can be made uniformly small by selecting R sufficiently close to 1. Since additionally $g_n(0) \to 0$ as $n \to \infty$, Lemma 2.3 completes the proof of (c).

Finally, suppose $\alpha > \alpha_1$. In this case, $p - p\alpha + 4 - \lambda < 0$. If g were nonconstant, there is a closed disk $E \subset \mathbb{D}$ satisfying

$$\int_E |g'(z)|^p(1-|z|^2)^p\,dA(z) > 0.$$

For $z \in E$ and $c \in \mathbb{D}$, the expression $\frac{1}{|1-\overline{c}z|^{2p+4-2p\alpha}}$ is bounded below by a positive constant independent of c. We therefore obtain

$$\|g\|_{A^{p,\lambda,\alpha},*}^p \gtrsim \sup_{c\in\mathbb{D}}(1-|c|^2)^{p-p\alpha+4-\lambda} = \infty.$$

This contradiction establishes (d). □

3. Main results and proofs

Theorem 3.1. *Let $1 \le p < \infty$, $0 < \lambda \le 2$, and $\alpha > 0$. Suppose ϕ is an analytic self-map of $\mathbb{D}$, and let α_0, α_1 be defined via* (2.2).

(a) If $0 < \alpha < \alpha_0$, the following assertions are equivalent:
  (i) $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ is bounded;
  (ii) $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ is compact;
  (iii) $\phi \in A^{p,\lambda,\alpha}$ and $\|\phi\|_\infty < 1$.

(b) If $\alpha_0 > 0$ and $\alpha = \alpha_0$, then $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ is always bounded. The operator C_ϕ is compact if and only if

$$\lim_{r\to 1^-}\sup_{|\phi(z)|>r}|\phi^{\#}(z)| = 0.$$

(c) If $\alpha_0 < \alpha \le \alpha_1$, then $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ is compact for every analytic self-map ϕ of $\mathbb{D}$.

(d) If $\alpha > \alpha_1$, then $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ is bounded if and only if it is compact, and these conditions hold if and only if ϕ is constant.

Proof. We first consider the case $0 < \alpha < \alpha_0$. Assume that C_ϕ is bounded. Applying this composition operator to the function $f(z) = z$ yields

$$\phi = C_\phi f \in A^{p,\lambda,\alpha}.$$

By Lemma 2.4, we get

$$\|f\circ\phi\|_\infty \lesssim \|C_\phi f\|_{A^{p,\lambda,\alpha}} \lesssim \|f\|_{\mathcal{B}}. \tag{3.1}$$

We proceed by contradiction and suppose $\|\phi\|_\infty = 1$. Select a sequence $\{z_n\}$ such that

$$w_n = \phi(z_n), \qquad |w_n| \to 1,$$

and define $\xi_n = w_n/|w_n|$ for all sufficiently large indices n. Set

$$f_n(w) = \log \frac{1}{1 - \overline{\xi_n} w}.$$

The estimate

$$(1 - |w|^2)|f_n'(w)| = \frac{1 - |w|^2}{|1 - \overline{\xi_n} w|} \leq \frac{(1 - |w|)(1 + |w|)}{1 - |w|} \leq 2$$

shows that $\{f_n\}$ remains bounded within $\mathcal{B}$. Meanwhile, we have

$$|f_n(w_n)| = \log \frac{1}{1 - |w_n|} \to \infty, \quad (n \to \infty),$$

which contradicts Inequality (3.1). Consequently, $\|\phi\|_\infty < 1$.

Conversely, let $R = \|\phi\|_\infty < 1$ and assume $\phi \in A^{p,\lambda,\alpha}$. For an arbitrary $f \in \mathcal{B}$, we have

$$\sup_{|w| \leq R} |f'(w)| \leq \frac{\|f\|_{\mathcal{B},*}}{1 - R^2},$$

and

$$|f(\phi(0))| \leq |f(0)| + \frac{1}{2}\|f\|_{\mathcal{B},*} \log \frac{1 + R}{1 - R} \lesssim \|f\|_{\mathcal{B}}.$$

Substituting these bounds into (1.1) gives

$$\begin{aligned}
\|C_\phi f\|^p_{A^{p,\lambda,\alpha}} &\leq |f(\phi(0))|^p + \left(\sup_{|w| \leq R} |f'(w)|\right)^p \|\phi\|^p_{A^{p,\lambda,\alpha},*} \\
&\lesssim (1 + \|\phi\|^p_{A^{p,\lambda,\alpha},*})\|f\|^p_{\mathcal{B}}.
\end{aligned}$$

This establishes the boundedness of C_ϕ.

Now let $\{f_n\}$ be a bounded sequence in $\mathcal{B}$ converging to zero locally uniformly on $\mathbb{D}$. Then

$$f_n(\phi(0)) \to 0, \quad \sup_{|w| \leq R} |f_n'(w)| \to 0$$

as $n \to \infty$. The convergence of derivatives follows from Cauchy's integral formula applied on any disk whose radius lies strictly between R and 1. Explicitly, we have

$$\|C_\phi f_n\|^p_{A^{p,\lambda,\alpha}} \leq |f_n(\phi(0))|^p + \left(\sup_{|w| \leq R} |f_n'(w)|\right)^p \|\phi\|^p_{A^{p,\lambda,\alpha},*},$$

and both terms on the right-hand side vanish in the limit. Therefore, $\|C_\phi f_n\|_{A^{p,\lambda,\alpha}} \to 0$ as $n \to \infty$. By Lemma 2.3, C_ϕ is compact. This completes the proof of (a).

If $\alpha = \alpha_0 > 0$, Lemma 2.4(b) identifies $A^{p,\lambda,\alpha}$ with $\mathcal{B}$ up to equivalent norms. Boundedness then follows from the Schwarz–Pick lemma, while the asserted compactness criterion is precisely the theorem of Madigan and Matheson [9]. This proves (b).

We next consider the range $\alpha_0 < \alpha \le \alpha_1$. For every analytic self-map ϕ of $\mathbb{D}$, it is well known that C_ϕ acts as a bounded operator on $\mathcal{B}$. Composing C_ϕ with the compact embedding stated in Lemma 2.4(c) yields (c).

Finally, suppose $\alpha > \alpha_1$. Take a bounded operator C_ϕ and apply it to $f(z) = z$. Then $C_\phi f = \phi \in A^{p,\lambda,\alpha}$, and Lemma 2.4(d) forces ϕ to be constant. Conversely, constant symbols induce rank-one operators and hence compact operators. This verifies (d). □

To estimate the norm of a composition operator $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$, we use

$$\|C_\phi\| = \sup_{\|f\|_{\mathcal{B}}\le 1} \|C_\phi f\|_{A^{p,\lambda,\alpha}}.$$

Theorem 3.2. *Let $1 \le p < \infty$, $0 < \lambda \le 2$, and $\alpha > 0$. Let ϕ be an analytic self-map of $\mathbb{D}$. The composition operator $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ is bounded if and only if*

$$\sup_{c\in\mathbb{D}} (1-|c|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{|\phi'(z)|^p (1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p} (1-|\sigma_c(z)|^2)^{p-p\alpha+2}\, dA(z) < \infty. \tag{3.2}$$

In addition, whenever $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ is bounded, we have

$$\begin{aligned}\|C_\phi\|^p \asymp &\left(1 + \log\frac{1}{1-|\phi(0)|^2}\right)^p + \\ &\sup_{c\in\mathbb{D}}(1-|c|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{|\phi'(z)|^p (1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p} (1-|\sigma_c(z)|^2)^{p-p\alpha+2}\, dA(z),\end{aligned} \tag{3.3}$$

where the implicit constants in the equivalence relation depend only on the parameters p, λ and α.

Proof. The proof of boundedness follows a similar framework as that in [7], yet we supply the full argument here to derive the corresponding norm estimate. Taking an arbitrary function $f \in \mathcal{B}$, we get

$$|(f\circ\phi)'(z)|^p = |f'(\phi(z))|^p |\phi'(z)|^p \le \|f\|^p_{\mathcal{B},*} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^p}.$$

We also have the pointwise bound

$$|f(\phi(0))| \lesssim \left(1 + \log\frac{1}{1-|\phi(0)|^2}\right)\|f\|_{\mathcal{B}}.$$

Substituting these two inequalities into (1.1), we obtain

$$\begin{aligned}\|C_\phi f\|^p_{A^{p,\lambda,\alpha}} \lesssim &\left(1 + \log\frac{1}{1-|\phi(0)|^2}\right)^p \|f\|^p_{\mathcal{B}} \\ &+ \sup_{c\in\mathbb{D}}(1-|c|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{|\phi'(z)|^p (1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p} (1-|\sigma_c(z)|^2)^{p-p\alpha+2}\, dA(z) \|f\|^p_{\mathcal{B}}.\end{aligned} \tag{3.4}$$

This establishes the sufficiency direction and the upper bound appearing in (3.3).

We now turn to necessity. Ramey and Ullrich constructed in [10, Proposition 5.4] two functions $F_1, F_2 \in \mathcal{B}$ and a positive constant C such that

$$\frac{1}{1-|w|} \le C\big(|F_1'(w)| + |F_2'(w)|\big), \qquad w \in \mathbb{D}.$$

Since $1-|w| \le 1-|w|^2$, we obtain

$$\frac{1}{1-|w|^2} \le \frac{1}{1-|w|} \le C(|F_1'(w)| + |F_2'(w)|), \qquad w \in \mathbb{D}.$$

For $p \ge 1$, the elementary inequality

$$(x+y)^p \le 2^{p-1}(x^p + y^p) \qquad (x, y \ge 0),$$

implies

$$\frac{1}{(1-|\phi(z)|^2)^p} \lesssim |F_1'(\phi(z))|^p + |F_2'(\phi(z))|^p.$$

Multiply both sides by the weight term

$$|\phi'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\sigma_c(z)|^2)^{p-p\alpha+2},$$

integrate over the unit disk $\mathbb{D}$, multiply the resulting integral by $(1-|c|^2)^{2-\lambda}$, and finally take the supremum over all $c \in \mathbb{D}$. This computation yields

$$\begin{aligned}
&\sup_{c\in\mathbb{D}}(1-|c|^2)^{2-\lambda}\int_{\mathbb{D}} \frac{|\phi'(z)|^p(1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p}(1-|\sigma_c(z)|^2)^{p-p\alpha+2}\,dA(z)\\
&\lesssim \sum_{j=1}^{2} \|C_\phi F_j\|^p_{A^{p,\lambda,\alpha},*} \le \|C_\phi\|^p \sum_{j=1}^{2} \|F_j\|^p_{\mathcal{B}} \lesssim \|C_\phi\|^p.
\end{aligned}$$

Therefore, the boundedness of C_ϕ implies the condition (3.2) and provides the lower bound for the integral term.

The constant function 1 immediately gives $\|C_\phi\| \ge 1$. Suppose $|\phi(0)| > 1/2$. Define

$$\xi = \frac{\phi(0)}{|\phi(0)|}, \qquad f_\xi(w) = \log\frac{1}{1-\bar{\xi}w}.$$

It is well known that $\|f_\xi\|_{\mathcal{B}} \le 2$, while

$$|C_\phi f_\xi(0)| = \log\frac{1}{1-|\phi(0)|} \asymp 1 + \log\frac{1}{1-|\phi(0)|^2}.$$

We then have

$$1 + \log\frac{1}{1-|\phi(0)|^2} \lesssim \frac{\|C_\phi f_\xi\|_{A^{p,\lambda,\alpha}}}{\|f_\xi\|_{\mathcal{B}}} \le \|C_\phi\|$$

under the assumption $|\phi(0)| > 1/2$. If instead $|\phi(0)| \le 1/2$, the quantity $1 + \log(1/(1-|\phi(0)|^2))$ is comparable to 1. In both cases, we have

$$\left(1 + \log\frac{1}{1-|\phi(0)|^2}\right)^p \lesssim \|C_\phi\|^p.$$

Combining this estimate with the lower bound for the integral term and the upper bound (3.4) completes the verification of (3.3). □

Theorem 3.3. *Let $1 \leq p < \infty$, $0 < \lambda \leq 2$, and $\alpha > 0$. Suppose that ϕ is an analytic self-map of $\mathbb{D}$ such that the composition operator $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ is bounded. The following statements are equivalent.*

(i) $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ *is compact;*

(ii)

$$\lim_{|a|\to 1^-} \|\sigma_a \circ \phi\|_{A^{p,\lambda,\alpha},*} = 0;$$

(iii)

$$\lim_{r\to 1^-} \sup_{c\in\mathbb{D}} (1-|c|^2)^{2-\lambda} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p (1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p} (1-|\sigma_c(z)|^2)^{p-p\alpha+2}\, dA(z) = 0. \tag{3.5}$$

Proof. First, assume that C_ϕ is compact. Define the function family $\{h_a\}_{a\in\mathbb{D}}$ by $h_a = \sigma_a - a$, which satisfies uniform boundedness in the Bloch space. As $|a| \to 1^-$,

$$h_a(z) = -\frac{(1-|a|^2)z}{1-\bar{a}z}$$

converges uniformly to zero on every compact subset of $\mathbb{D}$. Additionally, the relations

$$h_a(0) = 0, \qquad (1-|z|^2)|h_a'(z)| = 1 - |\sigma_a(z)|^2 \leq 1$$

confirm the uniform bound on the Bloch seminorm. Applying Lemma 2.3 yields

$$\|\sigma_a \circ \phi\|_{A^{p,\lambda,\alpha},*} = \|C_\phi h_a\|_{A^{p,\lambda,\alpha},*} \to 0$$

as $|a| \to 1^-$, so (i) implies (ii).

Next, suppose that (ii) holds. For the ranges $0 < \alpha < \alpha_0$, $\alpha_0 < \alpha \leq \alpha_1$, and $\alpha > \alpha_1$, the boundedness of C_ϕ implies its compactness by Theorem 3.1. It remains to consider $\alpha = \alpha_0 > 0$, where $\mathcal{B} = A^{p,\lambda,\alpha}$ with equivalent norms by Lemma 2.4(b). The implication $(ii) \implies (i)$ follows directly from [12].

We now show $(iii) \implies (i)$. By Theorem 3.2, the supremum in (3.2) is finite. Let $\{f_n\}$ be a bounded sequence in $\mathcal{B}$ converging to zero locally uniformly. Decompose the integral for $\|C_\phi f_n\|^p_{A^{p,\lambda,\alpha},*}$ into two parts over the subdomains $\{|\phi| \leq R\}$ and $\{|\phi| > R\}$.

For the region $\{|\phi(z)| > R\}$, we have

$$\begin{aligned}
&\sup_{c\in\mathbb{D}} (1-|c|^2)^{2-\lambda} \int_{|\phi(z)|>R} |(f_n\circ\phi)'(z)|^p (1-|z|^2)^{p\alpha-2} (1-|\sigma_c(z)|^2)^{p-p\alpha+2}\, dA(z) \\
&\leq \sup_n \|f_n\|^p_{\mathcal{B},*} \sup_{c\in\mathbb{D}} (1-|c|^2)^{2-\lambda} \int_{|\phi(z)|>R} \frac{|\phi'(z)|^p (1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p} (1-|\sigma_c(z)|^2)^{p-p\alpha+2}\, dA(z).
\end{aligned}$$

For $\{|\phi(z)| \leq R\}$, using $(1-|\phi(z)|^2)^{-p} \geq 1$, we obtain

$$\begin{aligned}
&\sup_{c\in\mathbb{D}} (1-|c|^2)^{2-\lambda} \int_{|\phi(z)|\leq R} |(f_n\circ\phi)'(z)|^p (1-|z|^2)^{p\alpha-2} (1-|\sigma_c(z)|^2)^{p-p\alpha+2}\, dA(z) \\
&\leq \sup_{|w|\leq R} |f_n'(w)|^p \sup_{c\in\mathbb{D}} (1-|c|^2)^{2-\lambda} \int_{\mathbb{D}} \frac{|\phi'(z)|^p (1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p} (1-|\sigma_c(z)|^2)^{p-p\alpha+2}\, dA(z).
\end{aligned}$$

Given arbitrary $\varepsilon > 0$, Condition (*iii*) allows choosing R sufficiently close to 1 such that the integral over $\{|\phi| > R\}$ is bounded by ε. For this fixed R, local uniform convergence combined with the Cauchy integral formula implies

$$\sup_{|w|\leq R} |f_n'(w)| \to 0$$

as $n \to \infty$, so the integral over $\{|\phi| \leq R\}$ tends to zero. Since $f_n(\phi(0)) \to 0$ as $n \to \infty$, Lemma 2.3 confirms the compactness of C_ϕ.

Finally, we prove (*i*) $\Longrightarrow$ (*iii*). For $0 < \alpha < \alpha_0$, Theorem 3.1(a) gives $\|\phi\|_\infty < 1$, so $\{|\phi| > r\}$ is empty for any r sufficiently close to 1, making the integrand in (3.5) identically zero.

For $\alpha_0 < \alpha \leq \alpha_1$, take any $r > |\phi(0)|$. By the Schwarz–Pick lemma (see [5, Lemma 1.2]) and the lower bound for pseudohyperbolic distance (see [5, Lemma 1.4]), we have

$$|\sigma_{\phi(0)}(\phi(z))| \leq |z|, \qquad |\sigma_{\phi(0)}(\phi(z))| \geq \frac{|\phi(z)| - |\phi(0)|}{1 - |\phi(0)||\phi(z)|}.$$

Thus

$$\{|\phi| > r\} \subset \left\{|z| > \frac{r - |\phi(0)|}{1 - r|\phi(0)|}\right\},$$

where the annulus on the right recedes to the boundary as $r \to 1^-$. Using the identity

$$\begin{aligned}
&(1-|c|^2)^{2-\lambda}\frac{|\phi'(z)|^p(1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p}(1-|\sigma_c(z)|^2)^{p-p\alpha+2}\\
&=\left(\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2}\right)^p\frac{(1-|c|^2)^{p-p\alpha+4-\lambda}}{|1-\overline{c}z|^{2p+4-2p\alpha}}
\end{aligned}$$

and the Schwarz–Pick lemma, we obtain

$$\begin{aligned}
&\sup_{c\in\mathbb{D}}(1-|c|^2)^{2-\lambda}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p(1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p}(1-|\sigma_c(z)|^2)^{p-p\alpha+2}\,dA(z)\\
&\leq \sup_{c\in\mathbb{D}}(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{\{|z|>(r-|\phi(0)|)/(1-r|\phi(0)|)\}}\frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}}.
\end{aligned}$$

Lemma 2.2 ensures this term tends to zero as $r \to 1^-$.

If $\alpha > \alpha_1$, compactness (indeed, boundedness) forces ϕ to be constant by Theorem 3.1(d). Hence, $\phi' = 0$, and the expression in (3.5) vanishes identically.

For $\alpha = \alpha_0 > 0$, compactness of $C_\phi : \mathcal{B} \to A^{p,\lambda,\alpha}$ implies (see [9])

$$\lim_{r\to1^-}\sup_{|\phi(z)|>r}|\phi^\#(z)| = 0. \tag{3.6}$$

Combining (3.6) with (2.3), we get

$$\begin{aligned}
&\lim_{r\to1^-}\sup_{c\in\mathbb{D}}(1-|c|^2)^{2-\lambda}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p(1-|z|^2)^{p\alpha-2}}{(1-|\phi(z)|^2)^p}(1-|\sigma_c(z)|^2)^{p-p\alpha+2}\,dA(z)\\
&\leq \lim_{r\to1^-}\left(\sup_{|\phi(z)|>r}|\phi^\#(z)|\right)^p\sup_{c\in\mathbb{D}}(1-|c|^2)^{p-p\alpha+4-\lambda}\int_{\mathbb{D}}\frac{dA(z)}{|1-\overline{c}z|^{2p+4-2p\alpha}} = 0.
\end{aligned}$$

This verifies (*iii*) and completes the proof. □

Taking $\alpha = 1 - (2 - \lambda)/p$ in Theorem 3.3 gives Condition (v) below, which can be seen as a new characterization for the compactness of C_ϕ on the space $\mathcal{B}$. Together with the results in [9, 12, 15], this yields the following corollary.

Corollary 3.4. *Let ϕ be an analytic self-map of $\mathbb{D}$. Let $1 \le p < \infty$ and $0 < \lambda \le 2$ such that $p > 2 - \lambda$. The following statements are equivalent.*

(i) C_ϕ *is compact on* $\mathcal{B}$.
(ii) $\lim_{r\to 1^-} \sup_{|\phi(z)|>r} |\phi^\#(z)| = 0$.
(iii) $\lim_{|a|\to 1^-} \|\sigma_a \circ \phi\|_{\mathcal{B},*} = 0$.
(iv) $\lim_{k\to\infty} \|\phi^k\|_{\mathcal{B}} = 0$.
(v)

$$\lim_{r\to 1^-} \sup_{c\in\mathbb{D}} (1 - |c|^2)^{2-\lambda} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p (1 - |z|^2)^{p-4+\lambda}}{(1 - |\phi(z)|^2)^p} (1 - |\sigma_c(z)|^2)^{4-\lambda}\, dA(z) = 0.$$

4. Conclusions

In this paper, we characterized the boundedness and compactness of composition operators from the Bloch space $\mathcal{B}$ into the α-Bergman–Morrey spaces $A^{p,\lambda,\alpha}$ for the full range $1 \le p < \infty$, $0 < \lambda \le 2$, and $\alpha > 0$. The results reveal four distinct regimes determined by the critical indices α_0 and α_1. We also obtained an integral characterization of boundedness, a two-sided operator norm estimate, and equivalent integral criteria for compactness. As a consequence, a new integral characterization of compact composition operators on the Bloch space is obtained.

Author contributions

All authors contributed equally to formal analysis, conceptualization, methodology, writing of the original draft, and review and editing of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This paper is supported by the Natural Science Foundation of Shandong Province (No. ZR2026 MS0057).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. S. Axler, The Bergman space, the Bloch space, and commutators of multiplication operators, *Duke Math. J.*, **53** (1986), 315–332. http://dx.doi.org/10.1215/S0012-7094-86-05320-2
2. P. S. Bourdon, J. A. Cima, A. L. Matheson, Compact composition operators on BMOA, *Trans. Amer. Math. Soc.*, **351** (1999), 2183–2196. http://dx.doi.org/10.1090/s0002-9947-99-02387-9
3. M. D. Contreras, S. Díaz-Madrigal, D. Vukotić, Compact and weakly compact composition operators from the Bloch space into Möbius invariant spaces, *J. Math. Anal. Appl.*, **415** (2014), 713–735. http://dx.doi.org/10.1016/j.jmaa.2014.02.005
4. C. C. Cowen, B. D. MacCluer, *Composition operators on spaces of analytic functions*, Boca Raton, FL: CRC Press, 1995. http://dx.doi.org/10.1201/9781315139920
5. J. B. Garnett, *Bounded analytic functions*, Vol. 236, Graduate Texts in Mathematics, New York: Springer, 2007. http://dx.doi.org/10.1007/0-387-49763-3
6. Q. Hu, S. Li, Products of differentiation and composition operators from the Bloch space and weighted Dirichlet spaces to Morrey type spaces, *J. Korean Math. Soc.*, **54** (2017), 627–645. http://dx.doi.org/10.4134/JKMS.j160199
7. L. Jiang, Y. He, Composition operators from $\mathcal{B}^\alpha$ to $F(p, q, s)$, *Acta Math. Sci.*, **23** (2003), 252–260. http://dx.doi.org/10.1016/S0252-9602(17)30229-1
8. J. Laitila, P. J. Nieminen, E. Saksman, H. O. Tylli, Compact and weakly compact composition operators on BMOA, *Complex Anal. Oper. Theory*, **7** (2013), 163–181. http://dx.doi.org/10.1007/s11785-011-0130-9
9. K. Madigan, A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.*, **347** (1995), 2679–2687. http://dx.doi.org/10.1090/S0002-9947-1995-1273508-x
10. W. Ramey, D. Ullrich, Bounded mean oscillation of Bloch pull-backs, *Math. Ann.*, **291** (1991), 591–606. http://dx.doi.org/10.1007/BF01445229
11. W. Smith, Compactness of composition operators on BMOA, *Proc. Amer. Math. Soc.*, **127** (1999), 2715–2725. http://dx.doi.org/10.1090/S0002-9939-99-04856-X
12. M. Tjani, *Compact composition operators on some Möbius invariant Banach spaces*, PhD dissertation, Michigan State University, 1996.
13. H. Wulan, Compactness of composition operators from the Bloch space $\mathcal{B}$ to Q_K spaces, *Acta Math. Sin. (Engl. Ser.)*, **21** (2005), 1415–1424. http://dx.doi.org/10.1007/s10114-005-0584-7
14. H. Wulan, Compactness of composition operators on BMOA and VMOA, *Sci. China Ser. A*, **50** (2007), 997–1004. http://dx.doi.org/10.1007/s11425-007-0061-0
15. H. Wulan, D. Zheng, K. Zhu, Compact composition operators on BMOA and the Bloch space, *Proc. Amer. Math. Soc.*, **137** (2009), 3861–3868.
16. H. Xie, J. Liu, S. Ponnusamy, Properties of Bergman–Morrey space and applications, *J. Geom. Anal.*, **35** (2025), 341. http://dx.doi.org/10.1007/s12220-025-02174-2
17. Y. Yang, J. Liu, Integral operators on Bergman-Morrey spaces, *J. Geom. Anal.*, **32** (2022), 181. http://dx.doi.org/10.1007/s12220-022-00919-x

18. K. Zhu, *Operator theory in function spaces: second edition*, Mathematical Surveys and Monographs, Vol. 138, Providence, RI: American Mathematical Society, 2007. http://dx.doi.org/10.1090/surv/138

19. X. Zhu, Composition operators from Bloch type spaces to $F(p,q,s)$ spaces, *Filomat*, **21** (2007), 11–20. http://dx.doi.org/10.2298/FIL0702011Z

20. X. Zhu, R. Yang, α-Bergman–Morrey spaces and Volterra integral operators, *Mediterr. J. Math.*, **22** (2025), 189. http://dx.doi.org/10.1007/s00009-025-02959-3