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Research article

Error estimation of the finite difference and Galerkin spectral for the optimal control problem of time-fractional fourth-order partial differential equations

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Abstract: This paper proposes a numerical method with uniform temporal accuracy for solving an optimal control problem (OCP) governed by time-fractional fourth-order partial differential equations. The approach combines a high-order finite difference scheme in time and a Legendre-Galerkin spectral method in space to provide an efficient and accurate framework for the discretization and solution of the given OCP. The first-order optimal condition theorem is derived by using calculus variations. A fully discrete scheme for OCP is established by combining the first-order optimality conditions, spatial spectral discretization, and L2 numerical scheme of time-fractional derivatives, which achieves a uniform temporal convergence order $(3-\theta)$ for time-fractional derivatives order of $0 < \theta < 1$ by using block-by-block techniques to achieve high-precision time convergence order only for the first two time layers. The error estimation for the OCP is rigorously completed by using first-order optimal condition and three proposed auxiliary equations with uniform accuracy $(3-\theta)$-order in time and spatial spectral accuracy. The stability analysis for the full discrete is also provided rigorously. The numerical results confirm the error estimation theory of the presented fully discrete scheme.

Keywords: optimal condition; time-fractional fourth-order partial differential equations; error estimation; spectral method
Mathematics Subject Classification: 65L12, 65M06, 65M12

1. Introduction

Optimal control problems (OCPs) with partial differential equations as state equations are widely encountered in scientific and engineering applications. Approximation solutions of OCPs constrained by second-order partial differential equations have been extensively conducted, and various effective methods have gradually been developed, such as [1–3]. The incorporation of time-fractional biharmonic models into studies of thin film growth with hereditary characteristics is a recent research trend. In [4], the OCP of time-fractional fourth-order partial differential equations was systematically explored. There are currently several computational methods available for solving fourth-order partial differential equations, such as [5–7]; the OCP for fourth-order partial differential equations has been investigated using various computational methods including the Galerkin spectral method [8,9], the fast Krylov subspace solver [10], the finite element method [11], radial basis functions [12], the interior penalty method [13], spectral methods [14], and derivative-informed neural operators [15].

The spectral method, as a class of efficient numerical methods, has now been applied in many fields of scientific computing. To achieve efficient discretization of partial differential equations, the spectral method uses global polynomials as the core trial functions. When the functions in the OCP possess smoothness, the solution process can achieve an exponential convergence rate. This paper employs the Galerkin spectral method to solve the OCP of time-fractional fourth-order partial differential equations:

$$\min_{(v,u)} J(v,u) = \frac{1}{2}\|u(\mathbf{x},t) - u_d(\mathbf{x},t)\|^2_{L^2(\Omega_T)} + \frac{\alpha}{2}\|v(\mathbf{x},t)\|^2_{L^2(\Omega_T)}, \tag{1.1}$$

subject to

$$\begin{aligned} &{}_0D_t^\theta u(\mathbf{x},t) + \Delta^2 u(\mathbf{x},t) = v(\mathbf{x},t) + f(\mathbf{x},t), && (\mathbf{x},t) \in \Omega_T, \\ &u(\mathbf{x},t) = \frac{\partial u(\mathbf{x},t)}{\partial n} = 0, && (\mathbf{x},t) \in \Gamma_T, \\ &u(\mathbf{x},0) = 0, && \mathbf{x} \in \Omega, \end{aligned} \tag{1.2}$$

where ${}_0D_t^\theta$ is the classical left Caputo fractional derivative, $0 < \theta < 1$, $\Omega = [-1,1]^d$, $\mathbf{x} = (x_1,\cdots,x_d), d \geq 1$, $\Omega_T = \Omega \times [0,T], \Gamma_T = \partial\Omega \times (0,T]$, $L^2(\Omega_T) \doteq L^2(0,T;L^2(\Omega))$, $V = \{v(\mathbf{x},t) \in L^2(\Omega_T) | v(\mathbf{x},\cdot) \in H_0^2(\Omega), v(\cdot,t) \in C^\theta([0,T])\}$, and $Y = H_0^2(\Omega)$.

The definitions of the Riemann-Liouville ${}_0^RD_t^\theta$, ${}_t^RD_T^\theta$ and the Caputo fractional derivative ${}_0D_t^\theta$, ${}_tD_T^\theta$ have the following relationship:

$${}_0^RD_t^\theta u(\mathbf{x},t) = \frac{u(\mathbf{x},0)\, t^{-\theta}}{\Gamma(1-\theta)} + {}_0D_t^\theta u(\mathbf{x},t), \tag{1.3}$$

$${}_t^RD_T^\theta u(\mathbf{x},t) = \frac{u(\mathbf{x},T)\,(T-t)^{-\theta}}{\Gamma(1-\theta)} + {}_tD_T^\theta u(\mathbf{x},t), \tag{1.4}$$

$$\int_{\Omega_T} {}_0^RD_t^\theta u(\mathbf{x},t) \cdot \hat{v}(\mathbf{x},t)\, d\mathbf{x}dt = \int_{\Omega_T} u(\mathbf{x},t) \cdot {}_t^RD_T^\theta \hat{v}(\mathbf{x},t)\, d\mathbf{x}dt, \quad \forall\, \hat{v}(\mathbf{x},t) \in V. \tag{1.5}$$

Most of the existing related work has focused on the OCP of second-order state equations. However, the research results for higher-order OCPs are still relatively scarce. For the fourth-order partial differential OCP with integer order time derivatives, readers can refer to [16] and [17]. The numerical scheme of the existing time integer-order space fourth-order OCP provides great inspiration for our research in this paper. In [18], they constructed a spectrally accurate, fully discrete scheme for two-dimensional distributed-order time-fractional fourth-order partial differential equations by utilizing the composite Simpson formula to discretize the distributed-order integral and the L2-1_σ formula to approximate the multiterm Caputo fractional derivatives with unconditionally stable and convergent analysis. In [19], they constructed the discrete scheme of the time second layer in the L2 scheme and for the first time presented the stability theory of the $(3-\theta)$-order scheme of time-fractional partial differential equations with an L1 scheme for the first layer. In [20], they proposed a compact difference-Galerkin spectral method for dealing with fourth-order equations in multidimensional space with time-fractional derivative orders $\theta \in (1,2)$ with stable and convergent analysis. In [21], they proposed a time-space Petrov-Galerkin spectral method for distributed-order discretization time-fractional partial differential equations and verified the exponential convergence property of its scheme. In [22], they proposed a spectral discretization scheme for the time-fractional optimal control problem with control and state variables subject to integral constraints. In [23], Galerkin spectral approximation of the OCP without a penalty term was established with a posteriori error estimates of the OCP by virtue of some lemmas and auxiliary equations. In [24], they proposed a multigrid preconditioning method under intermittent Galerkin discretization to solve an elliptic OCP. In [25], they studied the OCP with an H1-norm state constraint by the Galerkin spectral method. The optimality conditions and a posteriori error estimates were presented, which was helpful for developing the adaptive strategy in the spectral method. In [26], they studied the Legendre-Galerkin spectral approximation for a constrained OCP with a priori error estimates for both the state and the control approximation. In [27], they investigated a linear-quadratic OCP governed by a two-dimensional stochastic fractional Laplacian equation, incorporating additive white noise over a disk with regularity estimates for both the white noise and the optimal control variables within a weighted Sobolev space. In [28], they addressed the OCP of a fourth-order equation with L^2-norm control-constrained. They established the Galerkin spectral approximation scheme and analyzed its convergence and numerical accuracy. In [29], the authors examined two finite element discretization schemes for fractional-order OCPs. The numerical experiments demonstrated that the adaptive method has the advantage of higher computational efficiency.

The above work for OCPs of time-fractional fourth-order partial differential equations are only solved by a low-convergence-order L1 scheme or L2-1_σ scheme in time. In contrast, the present work proposes and analyzes a Galerkin spectral method for such OCPs, in which the time-fractional derivative is approximated by a uniform-accuracy L2 scheme. A key feature of the proposed discretization is the use of a block-by-block technique, which restores high-precision temporal convergence of order $(3-\theta)$ on the initial two layers and thereby circumvents the order reduction inherent to low-order schemes in that region. Theoretically, we rigorously establish first-order optimality conditions and prove stability and convergence of the fully discrete scheme.

The framework of the paper is as follows. In Section 2, we present the first-order optimality conditions and stability analysis for the OCP. In Section 3, we derive the error estimates for the OCP. In Section 4, we verify the effectiveness of our method through numerical experiments and confirm that our approach is equally applicable to nonlinear situations. In Section 5, we summarize the achievements related to the OCP and look forward to the future development of the work.

2. The first-order optimality conditions and stability analysis of the OCP

2.1. The first-order optimality conditions of the OCP

For the convenience of readers, we list the variables and functions involved in this paper, as shown in Table 1.

Table 1. Notation used throughout the paper.

Symbol	Description	Symbol	Description
Ω	Spatial domain	$\mathbf{x}$	Spatial variable
Ω_T	Spatiotemporal domain	d	Spatial dimension
Γ_T	Spatiotemporal boundary	J	Cost functional
Y	The Sobolev space $H_0^2(\Omega)$	V	Bochner spatiotemporal space
$L^2(0,T;L^2(\Omega))$	Bochner spatiotemporal space	θ	Caputo fractional order
${}_0D_t^\theta,\ {}_tD_T^\theta$	Left and right Caputo fractional derivatives	Δ	Laplacian operator
$\mathbb{Q}_{2,N}^0$	Legendre-Galerkin projection operator	I	Identity operator
$\|\cdot\|_0$	L^2 spatial norm	$(\cdot,\cdot)$	Inner product on Ω
Y_N	Spectral finite-dimensional subspace	u_d	Desired state
α	Regularization parameter	f	linear source term
u	State variable	v	Control variable
p	Adjoint variable	τ	Time-step size
T	Terminal time	N	Polynomial degree
U_N^k	Discrete state solution	V_N^k	Discrete control solution
P_N^k	Discrete adjoint solution		

In the following theorem, we provide first-order optimality conditions of the OCP (1.1) and (1.2).

Theorem 2.1. *The pair $(u,v) \in V \times L^2(\Omega_T)$ is the solution of the optimal control problem (1.1) and (1.2), only if there exists $p \in V$ such that (u,v,p) satisfies the following optimality conditions:*

$$({}_0D_t^\theta u, \bar{v}) + (\Delta u, \Delta \bar{v}) = (v + f, \bar{v}), \quad \forall \bar{v} \in Y, \tag{2.1}$$

$$({}_tD_T^\theta p, \bar{v}) + (\Delta p, \Delta \bar{v}) = (u - u_d, \bar{v}), \quad \forall \bar{v} \in Y, \tag{2.2}$$

$$\alpha v + p = 0, \quad a.e.\ in\ \Omega_T, \tag{2.3}$$

where ${}_tD_T^\theta$ is the classical right Caputo fractional derivative.

Proof. We introduce the following Lagrangian functional:

$$\mathcal{L}(v,u,p) = \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega_T)} + \frac{\alpha}{2}\|v\|^2_{L^2(\Omega_T)} - \int_{\Omega_T} {}_0D_t^\theta u \cdot p\, d\mathbf{x}dt - \int_{\Omega_T} \Delta^2 u \cdot p\, d\mathbf{x}dt + \int_{\Omega_T} (v+f)\cdot p\, d\mathbf{x}dt.$$

According to [30], we have

$$\int_\Omega \Delta^2 u \cdot p d\mathbf{x} = \int_\Omega \Delta u \cdot \Delta p d\mathbf{x} = \int_\Omega u \cdot \Delta^2 p d\mathbf{x}, \text{for}\ \ u, p \in Y. \tag{2.4}$$

By means of (1.3)–(1.5) and the fractional integration-by-parts formula [31] with the conditions of $u(\mathbf{x},0) = 0$ and $p(\mathbf{x},T) = 0$, we have

$$\int_{\Omega_T} {}_0D_t^\theta u \cdot p d\mathbf{x}dt = \int_{\Omega_T} {}_0^RD_t^\theta u \cdot p d\mathbf{x}dt = \int_{\Omega_T} {}_t^RD_T^\theta p \cdot u d\mathbf{x}dt = \int_{\Omega_T} {}_tD_T^\theta p \cdot u d\mathbf{x}dt. \tag{2.5}$$

First, set a partial derivative with respect to u for $\mathcal{L}(v, u, p)$ by combining (2.4) and (2.5). Let it equal zero so that we obtain

$$\mathcal{L}_u(v, u, p) \cdot w = \int_{\Omega_T} (u - u_d) \cdot w \, d\mathbf{x}dt - \int_{\Omega_T} {}_tD_T^\theta p \cdot w \, d\mathbf{x}dt - \int_{\Omega_T} \Delta p \cdot \Delta w d\mathbf{x}dt = 0, \quad \forall w \in Y,$$

which implies that

$$\int_{\Omega_T} {}_tD_T^\theta p \cdot w + \Delta p \cdot \Delta w d\mathbf{x}dt = \int_{\Omega_T} (u - u_d) \cdot w \, d\mathbf{x}dt, \quad \forall w \in Y.$$

Set a partial derivative with respect to v for $\mathcal{L}(v, u, p)$, and let it equal zero so that we have

$$\mathcal{L}_v(v, u, p) \cdot h = \int_{\Omega_T} (\alpha v + p) \cdot h \, d\mathbf{x}dt = 0, \quad \forall h \in Y,$$

which implies that

$$\alpha v + p = 0, \quad \text{a.e. in } \Omega_T.$$

Altogether, we obtain the following optimality system:

$$\begin{aligned} ({}_0D_t^\theta u, \bar{v}) + (\Delta u, \Delta \bar{v}) &= (v + f, \bar{v}), \quad \forall \bar{v} \in Y, \\ ({}_tD_T^\theta p, \bar{v}) + (\Delta p, \Delta \bar{v}) &= (u - u_d, \bar{v}), \quad \forall \bar{v} \in Y, \\ \alpha v + p &= 0 \quad \text{a.e. in } \Omega_T. \end{aligned}$$

The proof of Theorem 2.1 has been completed. □

2.2. Temporal and spatial discretizations

For the time-fractional Caputo derivatives in Eqs (2.1) and (2.2), we employ the L2 scheme for discretization in the time. We make the following assumption: Let $t_k = k\tau$, $k = 0, 1, \cdots, K$; $\alpha_0 = \Gamma(3-\theta)\tau^\theta$, where $\tau = T/K$ represents the time-step. We can obtain the corresponding L2 discrete scheme of ${}_0D_\tau^\theta u(\mathbf{x}, t_k)$, ${}_\tau D_T^\theta p(\mathbf{x}, t_k)$ [32, 33] as follows:

$${}_0D_\tau^\theta u(\mathbf{x}, t_k) = \begin{cases} \alpha_0^{-1}\left(\bar{A}_1 u(\mathbf{x}, t_0) + \bar{B}_1 u(\mathbf{x}, t_1) + \bar{C}_1 u(\mathbf{x}, t_2)\right), \; k = 1, \\ \alpha_0^{-1}\left(\bar{A}_2 u(\mathbf{x}, t_0) + \bar{B}_2 u(\mathbf{x}, t_1) + \bar{C}_2 u(\mathbf{x}, t_2)\right), \; k = 2, \\ \alpha_0^{-1}\Big\{\bar{A}_k u(\mathbf{x}, t_0) + \bar{B}_k u(\mathbf{x}, t_1) + \bar{C}_k u(\mathbf{x}, t_2) \\ \quad + \displaystyle\sum_{m=1}^{k-1} [A_m u(\mathbf{x}, t_{k-m-1}) + B_m u(\mathbf{x}, t_{k-m}) + C_m u(\mathbf{x}, t_{k-m+1})]\Big\}, \; k \geq 3, \end{cases} \tag{2.6}$$

and

$${}_\tau D_T^\theta p(\mathbf{x}, t_k) = \begin{cases} \alpha_0^{-1}\left(\bar{E}_0 p(\mathbf{x}, t_{K-2}) + \bar{E}_1 p(\mathbf{x}, t_{K-1}) + \bar{E}_2 p(\mathbf{x}, t_K)\right), \; k = K-1, \\ \alpha_0^{-1}\left(E_0 p(\mathbf{x}, t_{K-2}) + E_1 p(\mathbf{x}, t_{K-1}) + E_2 p(\mathbf{x}, t_K)\right), \; k = K-2, \\ \alpha_0^{-1}\Big\{\bar{F}_k p(\mathbf{x}, t_{K-2}) + \bar{G}_k p(\mathbf{x}, t_{K-1}) + \bar{H}_k p(\mathbf{x}, t_K) \\ \quad + \displaystyle\sum_{m=k+1}^{K-1} [F_m p(\mathbf{x}, t_{m-1}) + G_m p(\mathbf{x}, t_m) + H_m p(\mathbf{x}, t_{m+1})]\Big\}, \; k \leq K-3. \end{cases} \tag{2.7}$$

The values of all coefficients A, B, C in (2.6) and E, F, G, H in (2.7) are as follows:

$$\bar{A}_1 = \frac{3}{2}\theta - 2, \quad \bar{B}_1 = 2 - 2\theta, \quad \bar{C}_1 = \frac{1}{2}\theta;$$
$$\bar{A}_2 = \frac{3\theta}{2^\theta} - 2^{1-\theta}, \quad \bar{B}_2 = -\frac{\theta}{2^{\theta-2}}, \quad \bar{C}_2 = \frac{\theta}{2^\theta} + 2^{1-\theta};$$
$$\bar{A}_k = \left(1 - \frac{\theta}{2}\right)(k-1)^{1-\theta} - (k-1)^{2-\theta} - \left(3 - \frac{3\theta}{2}\right)k^{1-\theta} + k^{2-\theta}, \quad k \geq 3;$$
$$\bar{B}_k = 2(k-1)^{2-\theta} - 2k^{2-\theta} + 2(2-\theta)\,k^{1-\theta}, \quad k \geq 3;$$
$$\bar{C}_k = \left(\frac{\theta}{2} - 1\right)k^{1-\theta} + \left(\frac{\theta}{2} - 1\right)(k-1)^{1-\theta} + k^{2-\theta} - (k-1)^{2-\theta}, \quad k \geq 3;$$
$$A_m = m^{2-\theta} + \left(\frac{\theta}{2} - 1\right)(m-1)^{1-\theta} + \left(\frac{\theta}{2} - 1\right)m^{1-\theta} - (m-1)^{2-\theta};$$
$$B_m = (4-2\theta)(m-1)^{1-\theta} + 2(m-1)^{2-\theta} - 2m^{2-\theta};$$
$$C_m = \left(1 - \frac{\theta}{2}\right)m^{1-\theta} + m^{2-\theta} - \left(3 - \frac{3\theta}{2}\right)(m-1)^{1-\theta} - (m-1)^{2-\theta};$$
$$\bar{E}_0 = \frac{1}{2}\theta, \quad \bar{E}_1 = 2 - 2\theta, \quad \bar{E}_2 = \frac{3}{2}\theta - 2;$$
$$E_0 = \frac{\theta}{2^\theta} + 2^{1-\theta}, \quad E_1 = -\frac{\theta}{2^{\theta-2}}, \quad E_2 = \frac{3\theta}{2^\theta} - 2^{1-\theta};$$
$$\bar{F}_k = \left(\frac{\theta}{2} - 1\right)(K-k)^{1-\theta} + (K-k)^{2-\theta} + \left(\frac{\theta}{2} - 1\right)(K-k-1)^{1-\theta} - (K-k-1)^{2-\theta}, \quad k \leq K-3;$$
$$\bar{G}_k = (4-2\theta)(K-k)^{1-\theta} - 2(K-k)^{2-\theta} + 2(K-k-1)^{2-\theta}, \quad k \leq K-3;$$
$$\bar{H}_k = \left(\frac{3}{2}\theta - 3\right)(K-k)^{1-\theta} + (K-k)^{2-\theta} + \left(1 - \frac{\theta}{2}\right)(K-k-1)^{1-\theta} - (K-k-1)^{2-\theta}, \quad k \leq K-3;$$
$$F_m = \left(\frac{3}{2}\theta - 3\right)(m-k-1)^{1-\theta} - (m-k-1)^{2-\theta} + \left(1 - \frac{\theta}{2}\right)(m-k)^{1-\theta} + (m-k)^{2-\theta};$$
$$G_m = (4-2\theta)(m-k-1)^{1-\theta} - 2(m-k)^{2-\theta} + 2(m-k-1)^{2-\theta};$$
$$H_m = \left(\frac{\theta}{2} - 1\right)(m-k)^{1-\theta} + (m-k)^{2-\theta} + \left(\frac{\theta}{2} - 1\right)(m-k-1)^{1-\theta} - (m-k-1)^{2-\theta}.$$

The local truncation error estimates for the time-discrete scheme (2.6) and (2.7) can be given by Lemma 2.1 below.

Lemma 2.1. *[32, 33] Let the functions $u(\cdot,t)$, $p(\cdot,t)$ have the property of fourth-order derivatives with respect to time t:*

$$\left|\bar{R}_\tau^k\right| = \left|{}_0D_t^\theta u(\cdot,t_k) - {}_0D_\tau^\theta u(\cdot,t_k)\right| \leq C\tau^{3-\theta}, \quad \forall 1 \leq k \leq K, 0 < \theta < 1,$$
$$\left|R_\tau^k\right| = \left|{}_tD_T^\theta p(\cdot,t_k) - {}_\tau D_T^\theta p(\cdot,t_k)\right| \leq C\tau^{3-\theta}, \quad \forall 0 \leq k \leq K-1, 0 < \theta < 1,$$

where the constant $C > 0$ is independent of τ.

For convenience in presentation without loss of generality, we consider the case of $d = 2$. Similar to [34], in the discrete space we introduce the one-dimensional $L_k(s)$, the kth degree Legendre polynomial, and

$$Y_N = \text{span}\{\psi_i(x_1)\psi_j(x_2):\ i, j = 0, 1, \cdots, N-4\},$$

where

$$\psi_k(s) = \frac{1}{\sqrt{(2k+3)^2(4k+10)}}\Big[L_k(s) - \frac{4k+10}{2k+7}L_{k+2}(s) + \frac{2k+3}{2k+7}L_{k+4}(s)\Big], k = 0, 1, \cdots, N-4.$$

We introduce the norm, which is denoted as $|||\cdot|||_2$, as follows:

$$|||u|||_2 = \left(\|u\|_0^2 + \alpha_0\beta_0^{-1}\|\Delta u\|_0^2\right)^{\frac{1}{2}}, \tag{2.8}$$

where $\beta_0 = 2 - 0.5\theta > 0, \forall\theta \in (0,1)$. Regarding Eq (2.2), the constructed fully discrete numerical scheme is as follows.

Find $P_N^k \in Y_N, k = 0, 1, \cdots, K-1$, and $\forall\bar{v}_N \in Y_N$ such that they satisfy the following equation:

$$\begin{aligned}
&(\bar{E}_0 P_N^{K-2}, \bar{v}_N) + (\bar{E}_1 P_N^{K-1}, \bar{v}_N) + (\bar{E}_2 P_N^{K}, \bar{v}_N) + \alpha_0(\Delta P_N^{K-1}, \Delta\bar{v}_N) = \alpha_0(U_N^{K-1} - u_d^{K-1}, \bar{v}_N),\ k = K-1,\\
&(E_0 P_N^{K-2}, \bar{v}_N) + (E_1 P_N^{K-1}, \bar{v}_N) + (E_2 P_N^{K}, \bar{v}_N) + \alpha_0(\Delta P_N^{K-2}, \Delta\bar{v}_N) = \alpha_0(U_N^{K-2} - u_d^{K-2}, \bar{v}_N),\ k = K-2,\\
&(P_N^k, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta P_N^k, \Delta\bar{v}_N) = \Big(\sum_{s=k+1}^{K} d_s^k P_N^s, \bar{v}_N\Big) + \alpha_0\beta_0^{-1}(U_N^k - u_d^k, \bar{v}_N),\ k \le K-3,
\end{aligned} \tag{2.9}$$

where $F_{k+1} = 2 - 0.5\theta$, according to the coefficient expression of (2.7). Therefore, $\beta_0 = F_{k+1}$. For $k = K-3$,

$$\begin{aligned}
&d_{K-2}^{K-3} = -\beta_0^{-1}(\bar{F}_{K-3} + G_{K-2} + F_{K-1}),\ \ d_{K-1}^{K-3} = -\beta_0^{-1}(H_{K-2} + G_{K-1} + \bar{G}_{K-3}),\\
&d_K^{K-3} = -\beta_0^{-1}(\bar{H}_{K-3} + H_{K-1}),
\end{aligned}$$

and for $k \le K-4$,

$$\begin{aligned}
&d_{k+1}^k = -\beta_0^{-1}(G_{k+1} + F_{k+2}), d_{k+2}^k = -\beta_0^{-1}(H_{k+1} + G_{k+2} + F_{k+3}),\\
&d_s^k = -\beta_0^{-1}(H_{s-1} + G_s + F_{s+1}), s = k+3, k+4, \cdots, K-3,\\
&d_{K-2}^k = -\beta_0^{-1}(\bar{F}_k + H_{K-3} + G_{K-2} + F_{K-1}),\\
&d_{K-1}^k = -\beta_0^{-1}(H_{K-2} + G_{K-1} + \bar{G}_k),\ \ d_K^k = -\beta_0^{-1}(\bar{H}_k + H_{K-1}).
\end{aligned}$$

Lemma 2.2 establishes the properties of the coefficient d_s^k in (2.9).

Lemma 2.2. *[33] For the given $0 < \theta < 1$, to satisfy $k \le K-3$, the coefficients in the fully discrete numerical scheme (2.9) of the adjoint equation must satisfy the following conditions:*

(1) $\beta_0 = F_{k+1} = 2 - 0.5\theta \in (1.5, 2)$;

(2) $\sum_{s=k+1}^{K} d_s^k \equiv 1$;

(3) $d_s^k > 0, s = K, K-1, \cdots, k+4, k+3$;

(4) $0 < d_{k+1}^k < \frac{4}{3}$;

(5) *The symbol for d_{k+2}^k cannot be determined;*

(6) $d_{k+2}^k + \left(d_{k+1}^k\right)^2/4 > 0$.

From Lemma 2.2, it follows that the sign of the coefficient d^k_{k+2} does not have a definite sign for $\forall\theta \in (0, 1)$. Therefore, we expect to adopt a new approach to ensure that all coefficients in the plan are positive and that an unconditionally stable numerical scheme can be derived for $\forall\theta \in (0, 1)$. Based on the properties of the coefficients of (2.9) in Lemma 2.2, we construct

$$g = \frac{1}{2} d^k_{k+1}.$$

Based on the definition of g, the last line of (2.9) can be re-expressed as

$$\begin{aligned}
&(P^k_N - gP^{k+1}_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta P^k_N, \Delta\bar{v}_N)\\
=&g(P^{k+1}_N - gP^{k+2}_N, \bar{v}_N) + (g^2 + d^k_{k+2})(P^{k+2}_N - gP^{k+3}_N, \bar{v}_N)\\
&+ (g^3 + gd^k_{k+2} + d^k_{k+3})(P^{k+3}_N - gP^{k+4}_N, \bar{v}_N) + \cdots\\
&+ (g^{K-k-2} + g^{K-k-4}d^k_{k+2} + \cdots + gd^k_{K-3} + d^k_{K-2})(P^{K-2}_N - gP^{K-1}_N, \bar{v}_N)\\
&+ (g^{K-k-1} + g^{K-k-3}d^k_{k+2} + \cdots + gd^k_{K-2} + d^k_{K-1})(P^{K-1}_N - gP^K_N, \bar{v}_N)\\
&+ (g^{K-k} + g^{K-k-2}d^k_{k+2} + \cdots + gd^k_{K-1} + d^k_K)(P^K_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(U^k_N - u^k_d, \bar{v}_N).
\end{aligned}$$

Next, we denote

$$\begin{aligned}
\bar{d}^k_s &= g^{s-k} + \sum_{m=k+2}^{s} g^{s-m}d^k_m, \quad s = K, K-1, K-2, \cdots, k+2,\\
\bar{P}^s_N &= P^s_N - gP^{s+1}_N, \quad s = K-1, K-2, \cdots, k.
\end{aligned} \tag{2.10}$$

As $k = K - 3$, we obtain

$$\begin{aligned}
&(P^{K-3}_N - gP^{K-2}_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta P^{K-3}_N, \Delta\bar{v}_N)\\
=&(d^{K-3}_{K-2} - g)(P^{K-2}_N - gP^{K-1}_N) + [d^{K-3}_{K-1} + g(d^{K-3}_{K-2} - g)](P^{K-1}_N - gP^K_N)\\
&+ [d^{K-3}_K + g(d^{K-3}_{K-1} + g(d^{K-3}_{K-2} - g))](P^K_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(U^{K-3}_N - u^{K-3}_d, \bar{v}_N),
\end{aligned}$$

and we denote

$$\bar{d}^{K-3}_{K-2} = d^{K-3}_{K-2} - g, \bar{d}^{K-3}_{K-1} = d^{K-3}_{K-1} + g\bar{d}^{K-3}_{K-2}, \bar{d}^{K-3}_K = d^{K-3}_K + g\bar{d}^{K-3}_{K-1}.$$

As $k = K - 3$, we get

$$(\bar{P}^{K-3}_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta P^{K-3}_N, \Delta\bar{v}_N) = (\sum_{s=K-2}^{K} \bar{d}^{K-3}_s \bar{P}^s_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(U^{K-3}_N - u^{K-3}_d, \bar{v}_N).$$

When $k \leq K - 4$, we have

$$(\bar{P}^k_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta P^k_N, \Delta\bar{v}_N) = (g\bar{P}^{k+1}_N + \sum_{s=k+2}^{K-1} \bar{d}^k_s \bar{P}^s_N + \bar{d}^k_K P^K_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(U^k_N - u^k_d, \bar{v}_N).$$

Therefore, (2.9) has the following equivalent expression:

$$\begin{aligned}
&(\bar{E}_0 P^{K-2}_N, \bar{v}_N) + (\bar{E}_1 P^{K-1}_N, \bar{v}_N) + (\bar{E}_2 P^K_N, \bar{v}_N) + \alpha_0(\Delta P^{K-1}_N, \Delta\bar{v}_N) = \alpha_0(U^{K-1}_N - u^{K-1}_d, \bar{v}_N),\ k = K-1,\\
&(E_0 P^{K-2}_N, \bar{v}_N) + (E_1 P^{K-1}_N, \bar{v}_N) + (E_2 P^K_N, \bar{v}_N) + \alpha_0(\Delta P^{K-2}_N, \Delta\bar{v}_N) = \alpha_0(U^{K-2}_N - u^{K-2}_d, \bar{v}_N),\ k = K-2,\\
&(\bar{P}^{K-3}_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta P^{K-3}_N, \Delta\bar{v}_N) = (\sum_{s=K-2}^{K} \bar{d}^{K-3}_s \bar{P}^s_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(U^{K-3}_N - u^{K-3}_d, \bar{v}_N),\ k = K-3,\\
&(\bar{P}^k_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta P^k_N, \Delta\bar{v}_N) = (g\bar{P}^{k+1}_N + \sum_{s=k+2}^{K-1} \bar{d}^k_s \bar{P}^s_N + \bar{d}^k_K P^K_N, \bar{v}_N) + \alpha_0\beta_0^{-1}(U^k_N - u^k_d, \bar{v}_N),\ k \leq K-4,
\end{aligned} \tag{2.11}$$

for $P_N^k \in Y_N$, $k = 0, 1, \cdots, K-1$, and $\forall \bar{v}_N \in Y_N$.

Before analyzing the stability of the scheme (2.11), Lemma 2.3 will elaborate on some excellent properties of the coefficients in detail.

Lemma 2.3. *[33] For all* $0 < \theta < 1, k \leq K-3$*, the coefficients of (2.11) must satisfy the following conditions:*

(1) $0 < g < 2/3$;

(2) $\bar{d}_s^k > 0, \quad s = K, K-1, K-2, \cdots, k+2$;

(3) $g + \sum_{s=k+2}^{K-1} \bar{d}_s^k + \bar{d}_K^k \leq 1$;

(4) $1/\bar{d}_K^k < 1/d_K^k < 3(K-k)^\theta/[2(1-\theta)(2-\theta)]$;

(5) $\bar{d}_{K-i}^{K-3} > 0, i = 0, 1, 2$;

(6) $\bar{d}_{K-2}^{K-3} + \bar{d}_{K-1}^{K-3} + \bar{d}_K^{K-3} \leq 1$;

(7) $\bar{d}_{K-2}^{K-3} - g \leq 0$.

2.3. Stability and convergence analysis of the adjoint equation

Compared with (2.9), in which the sign of the coefficient d_{k+2}^k cannot be guaranteed, the modified formulation (2.11) possesses more favorable properties, which facilitates the subsequent stability analysis.

Lemma 2.4. *Let* $\{P_N^k\}_{k=0}^{K-1}$ *be the numerical solution corresponding to Eq (2.9). This numerical solution must satisfy the following result:*

$$\left\|P_N^k\right\|_0 + \sqrt{\alpha_0 \beta_0^{-1}} \|\Delta P_N^k\|_0 \leq 4\Big(S_2 C_u [\Gamma(1-\theta)]^2 T^{2\theta} \max_{0 \leq i \leq K-1} \left\|U_N^i - u_d^i\right\|_0^2\Big)^{\frac{1}{2}}, \quad 0 \leq k \leq K-1,$$

where the constant $C_u > 0$ *is independent of* N *and* τ*, and*

$$S_2 = \max\{2B_1, 2B_2 + 2g^2 B_1\}, \tag{2.12}$$

and B_1, B_2 *are defined as follows:*

$$B_1 = \Big\{(2-\theta)^2[0.5 + \frac{2^{2\theta-3}(1-\theta)}{\theta+2}]\Big\}, \; B_2 = \Big\{\frac{8(1-\theta)(2-\theta)^2}{2+\theta} + \frac{2^{2\theta+1}(1-\theta)^2(2-\theta)^2}{(2+\theta)^2}\Big\}.$$

Proof. The detailed proof process can be found in Appendix A. □

The equation of state is similar to (2.9):

$$\begin{aligned}
&(\bar{A}_1 U_N^0, \bar{v}_N) + (\bar{B}_1 U_N^1, \bar{v}_N) + (\bar{C}_1 U_N^2, \bar{v}_N) + \alpha_0(\Delta U_N^1, \Delta \bar{v}_N) = \alpha_0(f^1 + V_N^1, \bar{v}_N), \; k = 1, \\
&(\bar{A}_2 U_N^0, \bar{v}_N) + (\bar{B}_2 U_N^1, \bar{v}_N) + (\bar{C}_2 U_N^2, \bar{v}_N) + \alpha_0(\Delta U_N^2, \Delta \bar{v}_N) = \alpha_0(f^2 + V_N^2, \bar{v}_N), \; k = 2, \\
&(U_N^k, \bar{v}_N) + \alpha_0 \beta_0^{-1}(\Delta U_N^k, \Delta \bar{v}_N) = \Big(\sum_{s=1}^{K} D_{k-s}^k U_N^{k-s}, \bar{v}_N\Big) + \alpha_0 \beta_0^{-1}(f^k + V_N^k, \bar{v}_N), \; k \geq 3,
\end{aligned} \tag{2.13}$$

where D_{k-s}^k are the same definition as in [32].

Lemma 2.5. *[32] Let $\{U_N^k\}_{k=1}^K$ be the numerical solution corresponding to Eq (2.13). This numerical solution must satisfy the following result:*

$$\left\|U_N^k\right\|_0 + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta U_N^k\|_0 \le 4\Big(S_2 C_{f,v}[\Gamma(1-\theta)]^2 T^{2\theta} \max_{1\le i\le K}\left\|f^i + V_N^i\right\|_0^2\Big)^{\frac{1}{2}}, \quad 1 \le k \le K,$$

where the constant $C_{f,v} > 0$ is independent of N, τ and S_2 in Lemma 2.4.

Lemma 2.6. *[18] Let the projection operator $\mathbb{Q}_{2,N}^0 : Y \mapsto Y_N$ be defined. For $\forall\phi \in Y$, there is*

$$(\Delta(\phi - \mathbb{Q}_{2,N}^0\phi), \Delta\bar{v}_N) = 0, \quad \forall\bar{v}_N \in Y_N.$$

Thus, for $\forall\phi \in H^l(\Omega) \cap Y$ such that $l \ge 2$, the following holds:

$$\|\phi - \mathbb{Q}_{2,N}^0\phi\|_m \le CN^{m-l}\|\phi\|_l, \ m = 0, 1, 2.$$

Theorem 2.2. *Let $p(\mathbf{x}, t_k)$ be the exact solution of Eq (2.2), and let $P_N^k, k = 0, 1, \cdots, K-1$ be the approximate solution of Eq (2.9), with $p(\mathbf{x}, T) = 0$. Under the assumption that $\partial_t^4 p \in L^\infty((0, T]; H^l(\Omega)\cap H_0^2(\Omega)), l \ge 2$. The following error estimation holds:*

$$|||P_N^k - p(\mathbf{x}, t_k)|||_2 \le C(\tau^{3-\theta} + N^{2-l}) + C\max_{0\le i\le K-1}\|U_N^k - u(x, t_k)\|_0, \quad k = 0, 1, \cdots, K-1,$$

where the constant C has no relation to N and τ.

Proof. Denote

$$P_N^k - p(\mathbf{x}, t_k) = P_N^k - \mathbb{Q}_{2,N}^0 p(\mathbf{x}, t_k) + \mathbb{Q}_{2,N}^0 p(\mathbf{x}, t_k) - p(\mathbf{x}, t_k) \doteq \lambda_{P_N}^k + \varepsilon_p^k,$$

where $\lambda_{P_N}^k = P_N^k - \mathbb{Q}_{2,N}^0 p(\mathbf{x}, t_k), \varepsilon_p^k = \mathbb{Q}_{2,N}^0 p(\mathbf{x}, t_k) - p(\mathbf{x}, t_k)$ for $k = 0, 1, \cdots, K-1$.

When $k = K-1, K-2$ in (2.9), from the definition of R_τ^k and Lemma 2.1, $\forall\bar{v}_N \in Y_N$, we have

$$\sum_{i=0}^{2}\bar{E}_i(\lambda_{P_N}^{K-2+i} + \varepsilon_p^{K-2+i}, \bar{v}_N) + \alpha_0(\Delta\lambda_{P_N}^{K-1}, \Delta\bar{v}_N) = \alpha_0(R_\tau^{K-1} + U_N^{K-1} - u(x, t_{K-1}), \bar{v}_N); \quad (2.14)$$

$$\sum_{i=0}^{2}E_i(\lambda_{P_N}^{K-2+i} + \varepsilon_p^{K-2+i}, \bar{v}_N) + \alpha_0(\Delta\lambda_{P_N}^{K-2}, \Delta\bar{v}_N) = \alpha_0(R_\tau^{K-2} + U_N^{K-2} - u(x, t_{K-2}), \bar{v}_N). \quad (2.15)$$

Let $\bar{v}_N = -E_1\lambda_{P_N}^{K-1}$ in (2.14) and $\bar{v}_N = \bar{E}_0\lambda_{P_N}^{K-2}$ in (2.15) hold. Add the two together and apply Young's inequality [35]: $ab \le a^p/p + b^q/q, 1/p + 1/q = 1, \ p, q > 1$. Substitute a with $\sqrt{\kappa}a$ and b with $b\sqrt{\kappa}$ so that we have $\sqrt{\kappa}a \cdot (b/\sqrt{\kappa}) \le (\sqrt{\kappa}a)^p/p + (b/\sqrt{\kappa})^q/q$. Take $p = q = 2$ so that $ab \le \kappa a^2/2 + b^2/(2\kappa)$. We then have

$$\begin{aligned}
&-\bar{E}_1E_1\left\|\lambda_{P_N}^{K-1}\right\|_0^2 + \bar{E}_0E_0\left\|\lambda_{P_N}^{K-2}\right\|_0^2 - \alpha_0E_1\left\|\Delta\lambda_{P_N}^{K-1}\right\|_0^2 + \alpha_0\bar{E}_0\left\|\Delta\lambda_{P_N}^{K-2}\right\|_0^2\\
=&-\alpha_0E_1(R_\tau^{K-1} + U_N^{K-1} - u(x, t_{K-1}), \lambda_{P_N}^{K-1}) + \alpha_0\bar{E}_0(R_\tau^{K-2} + U_N^{K-2} - u(x, t_{K-2}), \lambda_{P_N}^{K-2}) + \bar{E}_0E_1(\varepsilon_p^{K-2}, \lambda_{P_N}^{K-1})\\
&-E_1\bar{E}_0(\varepsilon_p^{K-1}, \lambda_{P_N}^{K-2}) + \bar{E}_1E_1(\varepsilon_p^{K-1}, \lambda_{P_N}^{K-1}) - \bar{E}_0E_0(\varepsilon_p^{K-2}, \lambda_{P_N}^{K-2}) + \bar{E}_2E_1(\lambda_{P_N}^K, \lambda_{P_N}^{K-1})\\
&-E_2\bar{E}_0(\lambda_{P_N}^K, \lambda_{P_N}^{K-2}) + E_1\bar{E}_2(\varepsilon_p^K, \lambda_{P_N}^{K-1}) - E_2\bar{E}_0(\varepsilon_p^K, \lambda_{P_N}^{K-2})
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(-\alpha_0 E_1)^2}{2\epsilon_1}\|R_\tau^{K-1}+U_N^{K-1}-u(x,t_{K-1})\|_0^2+\frac{(\alpha_0\bar{E}_0)^2}{2\epsilon_2}\|R_\tau^{K-2}+U_N^{K-2}-u(x,t_{K-2})\|_0^2\\
&\quad+\frac{5\epsilon_1}{2}\|\lambda_{P_N}^{K-1}\|_0^2+\frac{5\epsilon_2}{2}\|\lambda_{P_N}^{K-2}\|_0^2+\Big[\frac{(\bar{E}_1E_1)^2}{2\epsilon_1}+\frac{(-E_1\bar{E}_0)^2}{2\epsilon_2}\Big]\|\varepsilon_p^{K-1}\|_0^2+\Big[\frac{(\bar{E}_0E_1)^2}{2\epsilon_1}+\frac{(-E_0\bar{E}_0)^2}{2\epsilon_2}\Big]\|\varepsilon_p^{K-2}\|_0^2\\
&\quad+\Big[\frac{(\bar{E}_2E_1)^2}{2\epsilon_1}+\frac{(-E_2\bar{E}_0)^2}{2\epsilon_2}\Big](\|\lambda_{P_N}^K\|_0^2+\|\varepsilon_p^K\|_0^2).
\end{aligned}\tag{2.16}
$$

Let $\epsilon_1=-\bar{E}_1E_1/5, \epsilon_2=\bar{E}_0E_0/5$ in (2.16), and we get

$$
\begin{aligned}
&-\bar{E}_1E_1\|\lambda_{P_N}^{K-1}\|_0^2+\bar{E}_0E_0\|\lambda_{P_N}^{K-2}\|_0^2-2\alpha_0E_1\|\Delta\lambda_{P_N}^{K-1}\|_0^2+2\alpha_0\bar{E}_0\|\Delta\lambda_{P_N}^{K-2}\|_0^2\\
&\leq\frac{5(-\alpha_0E_1)^2}{-\bar{E}_1E_1}\|R_\tau^{K-1}+U_N^{K-1}-u(x,t_{K-1})\|_0^2+\frac{5(\alpha_0\bar{E}_0)^2}{E_0\bar{E}_0}\|R_\tau^{K-2}+U_N^{K-2}-u(x,t_{K-2})\|_0^2\\
&\quad+\frac{5(E_2^2\bar{E}_0\bar{E}_1-\bar{E}_2^2E_1E_0)}{E_0\bar{E}_1}(\|\lambda_{P_N}^K\|_0^2+\|\varepsilon_p^K\|_0^2)+\frac{5(E_1^2\bar{E}_0-\bar{E}_1E_0E_1)}{E_0}\|\varepsilon_p^{K-1}\|_0^2\\
&\quad+\frac{5(E_0\bar{E}_0\bar{E}_1-\bar{E}_0^2E_1)}{\bar{E}_1}\|\varepsilon_p^{K-2}\|_0^2.
\end{aligned}\tag{2.17}
$$

According to Lemmas 2.1 and 2.6, the inequality (2.17) becomes

$$
\begin{aligned}
&\|\lambda_{P_N}^{K-1}\|_0^2+\alpha_0\beta_0^{-1}\|\Delta\lambda_{P_N}^{K-1}\|_0^2<\|\lambda_{P_N}^{K-1}\|_0^2+\frac{2\alpha_0}{\bar{E}_1}\|\Delta\lambda_{P_N}^{K-1}\|_0^2\\
&\leq\Big(\frac{5\alpha_0^2}{\bar{E}_1^2}-\frac{5\alpha_0^2\bar{E}_0}{E_0\bar{E}_1E_1}\Big)\max_{i=K-1,K-2}\|R_\tau^i+U_N^i-u(x,t_i)\|_0^2+\frac{5(E_2^2\bar{E}_0\bar{E}_1-\bar{E}_2^2E_1E_0)}{-E_1E_0\bar{E}_1^2}(\|\lambda_{P_N}^K\|_0^2+\|\varepsilon_p^K\|_0^2)\\
&\quad+\frac{5(E_1\bar{E}_0-\bar{E}_1E_0)}{-\bar{E}_1E_0}\|\varepsilon_p^{K-1}\|_0^2+\frac{5(E_0\bar{E}_0\bar{E}_1-\bar{E}_0^2E_1)}{-\bar{E}_1^2E_1}\|\varepsilon_p^{K-2}\|_0^2\\
&=\frac{5}{2}B_1[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\|R_\tau^i+U_N^i-u(x,t_i)\|_0^2+\frac{5}{2}A_1(\|\varepsilon_p^K\|_0^2+\|\lambda_{P_N}^K\|_0^2)+D_1\|\varepsilon_p^{K-1}\|_0^2+D_2\|\varepsilon_p^{K-2}\|_0^2\\
&\leq 5B_1[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}(\|R_\tau^i\|_0^2+\|U_N^i-u(x,t_i)\|_0^2)+\frac{5}{2}A_1(\|\varepsilon_p^K\|_0^2+\|\lambda_{P_N}^K\|_0^2)+D_1\|\varepsilon_p^{K-1}\|_0^2+D_2\|\varepsilon_p^{K-2}\|_0^2\\
&\leq C(\tau^{3-\theta}+N^{-l})^2+C\max_{i=K-1,K-2}\|U_N^i-u(x,t_i)\|_0^2\\
&\leq C(\tau^{3-\theta}+N^{-l}+\max_{i=K-1,K-2}\|U_N^i-u(x,t_i)\|_0)^2,
\end{aligned}\tag{2.18}
$$

where A_1,B_1 are given in (A.6), and D_1,D_2 are defined as follows:

$$
D_1=\frac{10-5\theta}{-\theta^2-\theta+2},\quad D_2=\frac{10-5\theta}{16(1-\theta)^2}.\tag{2.19}
$$

Based on the norm $|||\cdot|||_2$ in (2.8), (2.18), and Lemma 2.6, we have

$$
\begin{aligned}
|||P_N^{K-1}-p(\mathbf{x},t_{K-1})|||_2 &\leq|||\lambda_{P_N}^{K-1}|||_2+|||\varepsilon_p^{K-1}|||_2\leq\|\lambda_{P_N}^{K-1}\|_2+\|\varepsilon_p^{K-1}\|_2\\
&\leq C(\tau^{3-\theta}+N^{-l})+CN^{2-l}+C\max_{i=K-1,K-2}\|U_N^i-u(x,t_i)\|_0\\
&\leq C(\tau^{3-\theta}+N^{2-l})+C\max_{i=K-1,K-2}\|U_N^i-u(x,t_i)\|_0.
\end{aligned}
$$

Similarly, according to (2.17), we get

$$
\begin{aligned}
&\|\lambda_{P_N}^{K-2}\|_0^2+\alpha_0\beta_0^{-1}\|\Delta\lambda_{P_N}^{K-2}\|_0^2<\|\lambda_{P_N}^{K-2}\|_0^2+\frac{2\alpha_0}{E_0}\|\Delta\lambda_{P_N}^{K-2}\|_0^2\\
\le&\Big(\frac{5\alpha_0^2}{E_0^2}-\frac{5\alpha_0^2E_1}{E_0\bar{E}_1\bar{E}_0}\Big)\max_{i=K-1,K-2}\|R_\tau^i+U_N^i-u(x,t_i)\|_0^2+\frac{5(E_2^2\bar{E}_0\bar{E}_1-\bar{E}_2^2E_1E_0)}{E_0^2\bar{E}_1\bar{E}_0}(\|\lambda_{P_N}^K\|_0^2+\|\varepsilon_p^K\|_0^2)\\
&+\frac{5(E_1^2\bar{E}_0-\bar{E}_1E_0E_1)}{\bar{E}_0E_0^2}\|\varepsilon_p^{K-1}\|_0^2+\frac{5(E_0\bar{E}_1-\bar{E}_0E_1)}{\bar{E}_1E_0}\|\varepsilon_p^{K-2}\|_0^2\\
=&\frac{5}{2}B_2[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\|R_\tau^i+U_N^i-u(x,t_i)\|_0^2+\frac{5}{2}A_2(\|\varepsilon_p^K\|_0^2+\|\lambda_{P_N}^K\|_0^2)+D_3\|\varepsilon_p^{K-1}\|_0^2+D_1\|\varepsilon_p^{K-2}\|_0^2\\
\le&5B_2[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}(\|R_\tau^i\|_0^2+\|U_N^i-u(x,t_i)\|_0^2)+\frac{5}{2}A_2(\|\varepsilon_p^K\|_0^2+\|\lambda_{P_N}^K\|_0^2)+D_3\|\varepsilon_p^{K-1}\|_0^2+D_1\|\varepsilon_p^{K-2}\|_0^2\\
\le&C(\tau^{3-\theta}+N^{-l})^2+C\max_{i=K-1,K-2}\|U_N^i-u(x,t_i)\|_0^2\\
\le&C(\tau^{3-\theta}+N^{-l}+\max_{i=K-1,K-2}\|U_N^i-u(x,t_i)\|_0)^2,
\end{aligned}
$$

where A_2,B_2 are defined in (A.10), D_1 is defined in (2.19) and $D_3=5(32-16\theta)/(\theta+2)^2$, so we get

$$
\begin{aligned}
|||P_N^{K-2}-p(\mathbf{x},t_{K-2})|||_2&\le|||\lambda_{P_N}^{K-2}|||_2+|||\varepsilon_p^{K-2}|||_2\le\|\lambda_{P_N}^{K-2}\|_2+\|\varepsilon_p^{K-2}\|_2\\
&\le C(\tau^{3-\theta}+N^{-l})+C\max_{i=K-1,K-2}\|U_N^i-u(x,t_i)\|_0+CN^{2-l}\\
&\le C(\tau^{3-\theta}+N^{2-l})+C\max_{i=K-1,K-2}\|U_N^i-u(x,t_i)\|_0.
\end{aligned}
$$

Similarly in (2.14) and (2.15), for $k\le K-3$ in (2.9), we obtain

$$
(\lambda_{P_N}^k+\varepsilon_p^k,\bar{v}_N)+\alpha_0\beta_0^{-1}(\Delta\lambda_{P_N}^k,\Delta\bar{v}_N)=\sum_{s=k+1}^{K}d_s^k(\lambda_{P_N}^s+\varepsilon_p^s,\bar{v}_N)+\alpha_0\beta_0^{-1}(R_\tau^k+U_N^k-u(x,t_k),\bar{v}_N),\ \forall\bar{v}_N\in Y_N. \quad (2.20)
$$

Based on Lemma 2.1 and (2.20), the following conclusion can be drawn:

$$
\begin{aligned}
&(\lambda_{P_N}^k,\bar{v}_N)+\alpha_0\beta_0^{-1}(\Delta\lambda_{P_N}^k,\Delta\bar{v}_N)\\
=&\sum_{s=k+1}^{K}d_s^k(\lambda_{P_N}^s,\bar{v}_N)-\Big[(\varepsilon_p^k,\bar{v}_N)-\sum_{s=k+1}^{K}d_s^k(\varepsilon_p^s,\bar{v}_N)-\alpha_0\beta_0^{-1}(R_\tau^k+U_N^k-u(x,t_k),\bar{v}_N)\Big]\\
=&\sum_{s=k+1}^{K}d_s^k(\lambda_{P_N}^s,\bar{v}_N)-(\mathbb{Q}_{2,N}^0p(\mathbf{x},t_k)-p(\mathbf{x},t_k),\bar{v}_N)\\
&+\sum_{s=k+1}^{K}d_s^k(\mathbb{Q}_{2,N}^0p(\mathbf{x},t_s)-p(\mathbf{x},t_s),\bar{v}_N)+\alpha_0\beta_0^{-1}(R_\tau^k+U_N^k-u(x,t_k),\bar{v}_N)\\
=&\sum_{s=k+1}^{K}d_s^k(\lambda_{P_N}^s,\bar{v}_N)+(I-\mathbb{Q}_{2,N}^0)(p(\mathbf{x},t_k),\bar{v}_N)\\
&-(I-\mathbb{Q}_{2,N}^0)\sum_{s=k+1}^{K}d_s^k(p(\mathbf{x},t_s),\bar{v}_N)+\alpha_0\beta_0^{-1}(R_\tau^k+U_N^k-u(x,t_k),\bar{v}_N)
\end{aligned}
\quad (2.21)
$$

$$= \sum_{s=k+1}^{K} d_s^k(\lambda_{P_N}^s, \bar{v}_N) + (I - \mathbb{Q}_{2,N}^0)(p(\mathbf{x}, t_k) - \sum_{s=k+1}^{K} d_s^k p(\mathbf{x}, t_s), \bar{v}_N) + \alpha_0\beta_0^{-1}(R_\tau^k + U_N^k - u(x, t_k), \bar{v}_N)$$
$$= \sum_{s=k+1}^{K} d_s^k(\lambda_{P_N}^s, \bar{v}_N) + (I - \mathbb{Q}_{2,N}^0)(\alpha_0\beta_0^{-1}({}_tD_T^\theta p(\mathbf{x}, t_k) - R_\tau^k), \bar{v}_N) + \alpha_0\beta_0^{-1}(R_\tau^k + U_N^k - u(x, t_k), \bar{v}_N)$$
$$= \sum_{s=k+1}^{K} d_s^k(\lambda_{P_N}^s, \bar{v}_N) + \alpha_0\beta_0^{-1}((I - \mathbb{Q}_{2,N}^0)({}_tD_T^\theta p(\mathbf{x}, t_k) - R_\tau^k) + R_\tau^k + U_N^k - u(x, t_k), \bar{v}_N)$$
$$= \sum_{s=k+1}^{K} d_s^k(\lambda_{P_N}^s, \bar{v}_N) + \alpha_0\beta_0^{-1}(\zeta_N^k p, \bar{v}_N), \quad \forall \bar{v}_N \in Y_N,$$

where $\zeta_N^k p = (I - \mathbb{Q}_{2,N}^0)({}_tD_T^\theta p(\mathbf{x}, t_k) - R_\tau^k) + R_\tau^k + U_N^k - u(x, t_k)$, I is the identity operator, and R_τ^k is given in Lemma 2.1.

Using the triangle inequality leads to

$$\|\zeta_N^k p\|_0 \le \|(I - \mathbb{Q}_{2,N}^0){}_tD_T^\theta p(\mathbf{x}, t_k)\|_0 + \|(I - \mathbb{Q}_{2,N}^0)R_\tau^k\|_0 + \|R_\tau^k\|_0 + \|U_N^k - u(x, t_k)\|_0.$$

According to Lemma 2.1, we can draw a conclusion:

$$\|R_\tau^k\|_0 \le C\tau^{3-\theta},$$

and according to Lemma 2.6,

$$\|(I - \mathbb{Q}_{2,N}^0)\, {}_tD_T^\theta p(\mathbf{x}, t_k)\|_0 \le CN^{-l}.$$

Combining Lemmas 2.1 and 2.6, we have

$$\|(I - \mathbb{Q}_{2,N}^0)R_\tau^k\|_0 \le CN^{-l}\tau^{3-\theta}.$$

Thus, we have

$$\|\zeta_N^k p\|_0 \le C(\tau^{3-\theta} + N^{-l} + \tau^{3-\theta}N^{-l}) + \|U_N^k - u(x, t_k)\|_0.$$

Let $\bar{\lambda}_{P_N}^i = \lambda_{P_N}^i - g\lambda_{P_N}^{i+1}$, $i = 0, 1, \cdots, K-1$, and the expression (2.21) can then be rewritten as

$$(\bar{\lambda}_{P_N}^k, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta\lambda_{P_N}^k, \Delta\bar{v}_N) = g(\bar{\lambda}_{P_N}^{k+1}, \bar{v}_N) + \sum_{s=k+2}^{K-1} \bar{d}_s^k(\bar{\lambda}_{P_N}^s, \bar{v}_N) + \bar{d}_K^k(\bar{\lambda}_{P_N}^K, \bar{v}_N) + \alpha_0\beta_0^{-1}(\zeta_N^k p, \bar{v}_N), k \le K-4. \quad (2.22)$$

According to (7) in Lemma 2.3, it can be deduced from (2.22) that the situation of $k = K-3$ also holds:

$$(\bar{\lambda}_{P_N}^{K-3}, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta\lambda_{P_N}^{K-3}, \Delta\bar{v}_N) = (\sum_{s=K-2}^{K} \bar{d}_s^{K-3}\bar{\lambda}_{P_N}^s, \bar{v}_N) + \alpha_0\beta_0^{-1}(\zeta_N^{K-3}p, \bar{v}_N).$$

By following a similar approach to Lemma 2.4,

$$\begin{aligned}
&|||\lambda_{P_N}^k|||_2^2 = \|\lambda_{P_N}^k\|_0^2 + \alpha_0\beta_0^{-1}\|\Delta\lambda_{P_N}^k\|_0^2 \\
&\le 16(S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\le i\le K-1}\|\zeta_N^i p\|_0^2) \\
&\le C(\tau^{3-\theta} + N^{-l} + \tau^{3-\theta}N^{-l} + \max_{0\le i\le K-1}\|U_N^k - u(x, t_k)\|_0)^2.
\end{aligned}$$

Based on the above estimation and the norm $|||\cdot|||_2$, we have

$$
\begin{aligned}
&|||P_N^k - p(\mathbf{x}, t_k)|||_2 \le |||\lambda_{P_N}^k|||_2 + |||\varepsilon_p^k|||_2 \le \left\|\lambda_{P_N}^k\right\|_2 + \left\|\varepsilon_p^k\right\|_2 \\
\le& C(\tau^{3-\theta} + N^{-l} + \tau^{3-\theta}N^{-l}) + CN^{2-l} + C\max_{0\le i\le K-1}\|U_N^k - u(x, t_k)\|_0 \\
\le& C(\tau^{3-\theta} + N^{2-l}) + C\max_{0\le i\le K-1}\|U_N^k - u(x, t_k)\|_0,
\end{aligned}
$$

where the constant C has no relation to N and τ.

Therefore, the proof of Theorem 2.2 has been completed. □

Following the analysis of Theorem 2.2, we present the following lemma on the convergence of the state equation.

Lemma 2.7. *[32] Let $u(\mathbf{x}, t_k)$ be the exact solution of Eq (2.1), and let $U_N^k, k = 1, 2, \cdots, K$ be the approximate solution of Eq (2.13), with $u(\mathbf{x}, 0) = 0$. Under the assumption that $\partial_t^4 u \in L^\infty((0, T]; H^l(\Omega) \cap H_0^2(\Omega)), l \ge 2$, the following error estimation holds:*

$$
|||U_N^k - u(\mathbf{x}, t_k)|||_2 \le C(\tau^{3-\theta} + N^{2-l}) + C\max_{1\le i\le K}\|V_N^k - v(\mathbf{x}, t_k)\|_0, \quad k = 1, 2, \cdots, K,
$$

where the constant C has no relation to N and τ.

2.4. Stability analysis of the OCP

In this subsection, we present the fully discrete scheme for the OCP and provide the corresponding stability analysis. Regarding the discretized form of the cost function, its definition is

$$
\begin{aligned}
J_{N,\tau}(\boldsymbol{V}_N, \boldsymbol{U}_N) =& \frac{2\tau}{6}\Big(\|U_N^0 - u_d^0\|_0^2 + 4\sum_{m=0}^{K-1}\|U_N^{2m+1} - u_d^{2m+1}\|_0^2 \\
&+ 2\sum_{m=0}^{K-1}\|U_N^{2m} - u_d^{2m}\|_0^2 + \|U_N^{2N} - u_d^{2N}\|_0^2 + \alpha\|V_N^0\|_0^2 + 4\alpha\sum_{m=0}^{K-1}\|V_N^{2m+1}\|_0^2 \\
&+ 2\alpha\sum_{m=0}^{K-1}\|V_N^{2m}\|_0^2 + \alpha\|V_N^{2K}\|_0^2\Big) \doteq (\boldsymbol{V}_N, \boldsymbol{U}_N)_{\Omega_T,\tau},
\end{aligned}
$$

where the Simpson's rule is employed to convert the time integral of the cost function into a discrete form. $\boldsymbol{U}_N = (U_N^1, \cdots, U_N^K)$, $\boldsymbol{V}_N = (V_N^1, \cdots, V_N^K)$. The complete discretization forms can be obtained by finding $(\boldsymbol{U}_N, \boldsymbol{V}_N) \in Y_N \times Y_N$ such that

$$
\min_{\boldsymbol{v}_N \in Y_N} J_{N,\tau}(\boldsymbol{U}_N, \boldsymbol{V}_N),
$$

subject to

$$
\begin{cases}
({}_0D_\tau^\theta U_N^k, \bar{v}_N) + (\Delta U_N^k, \Delta\bar{v}_N) = (f^k + V_N^k, \bar{v}_N), \forall \bar{v}_N \in Y_N, \\
U_N^0 = 0, \quad k = 1, 2, \cdots, K.
\end{cases}
$$

For the sake of concision, we rewrite (2.9) in the following form:

$$
\begin{cases}
({}_\tau D_T^\theta P_N^k, \bar{v}_N) + (\Delta P_N^k, \Delta\bar{v}_N) = (U_N^k - u_d^k, \bar{v}_N), \forall \bar{v}_N \in Y_N, \\
p_N^K = 0, \ k = 0, \cdots, K - 1.
\end{cases}
$$

Therefore, the fully discrete scheme for the OCP (1.1) and (1.2) can be expressed as

$$\begin{aligned}
&({}_0D_\tau^\theta U_N^k, \bar{v}_N) + (\Delta U_N^k, \Delta \bar{v}_N) = (f^k + V_N^k, \bar{v}_N), \forall \bar{v}_N \in Y_N,\ 1 \le k \le K,\\
&({}_\tau D_T^\theta P_N^k, \bar{v}_N) + (\Delta P_N^k, \Delta \bar{v}_N) = (U_N^k - u_d^k, \bar{v}_N), \forall \bar{v}_N \in Y_N,\ 0 \le k \le K-1,\\
&(\alpha V_N^k + P_N^k, \bar{v}_N) = 0, \forall \bar{v}_N \in Y_N,\ 0 \le k \le K,\\
&U_N^0 = 0, P_N^K = 0,
\end{aligned} \tag{2.23}$$

where ${}_0D_\tau^\theta$ and ${}_\tau D_T^\theta$ converge with high uniform accuracy of order $3-\theta$, as known from Lemma 2.1. Next, we will prove the stability and convergence of the numerical scheme obtained by the time-finite difference discretization and the spatial spectral discretization methods.

By combining Lemmas 2.4 and 2.5, the stability of the OCP is given as follows.

Theorem 2.3. *Let $\{U_N^{k+1}\}_{k=0}^{K-1}$,$\{P_N^k\}_{k=0}^{K-1}$ be the numerical solution corresponding to Eq (2.23). This numerical solution must satisfy the following result:*

$$\begin{aligned}
&\left\|U_N^{k+1}\right\|_0 + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta U_N^{k+1}\|_0 + \left\|P_N^k\right\|_0 + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta P_N^k\|_0\\
\le&\Big[16S_2C_{f,v,u}[\Gamma(1-\theta)]^2T^{2\theta} + 4\Big(S_2C_{f,v}[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\Big]\max_{0\le i\le K-1}(\left\|f^{i+1}\right\|_0 + C_v\left\|v^{i+1}\right\|_0 + \left\|u_d^i\right\|_0),
\end{aligned}$$

where the constant $C_{f,v,u} > 0$ is independent of N, τ and S_2 in Lemma 2.4.

Proof. Applying the triangle inequality to the inequality in Lemma 2.5, we obtain

$$\begin{aligned}
&\left\|U_N^{k+1}\right\|_0 + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta U_N^{k+1}\|_0 \le 4\Big(S_2C_{f,v}[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\le i\le K-1}\left\|f^{i+1} + V_N^{i+1}\right\|_0^2\Big)^{\frac{1}{2}}\\
\le&4\Big(S_2C_{f,v}[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\max_{0\le i\le K-1}\left\|f^{i+1} + V_N^{i+1}\right\|_0\\
\le&4\Big(S_2C_{f,v}[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\max_{0\le i\le K-1}(\left\|f^{i+1}\right\|_0 + \left\|V_N^{i+1}\right\|_0).
\end{aligned} \tag{2.24}$$

Because V_N^{i+1} is the spectral solution of v^{i+1}, we can obtain $\|V_N^{i+1}\|_0 \le C_v\|v^{i+1}\|_0$, where C_v is a constant and independent of N and τ. Therefore, (2.24) can be rewritten as

$$\left\|U_N^{k+1}\right\|_0 + \sqrt{\alpha_0\beta_0^{-1}}\|\Delta U_N^{k+1}\|_0 \le 4\Big(S_2C_{f,v}[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\max_{0\le i\le K-1}(\left\|f^{i+1}\right\|_0 + C_v\left\|v^{i+1}\right\|_0). \tag{2.25}$$

Similar to (2.24), by using the inequality in Lemma 2.4, we have

$$
\begin{aligned}
&\left\|P_N^k\right\|_0+\sqrt{\alpha_0\beta_0^{-1}}\|\Delta P_N^k\|_0\\
\leq&4\Big(S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}\\
\leq&4\Big(S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0\\
\leq&4\Big(S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\max_{0\leq i\leq K-1}(\left\|U_N^i\right\|_0+\left\|u_d^i\right\|_0)\\
\leq&4\Big(S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\max_{0\leq i\leq K-1}(\left\|U_N^i\right\|_0+\sqrt{\alpha_0\beta_0^{-1}}\left\|\Delta U_N^i\right\|_0+\left\|u_d^i\right\|_0)\\
\leq&16S_2C_{f,v,u}[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}(\left\|f^{i+1}\right\|_0+C_v\left\|v^{i+1}\right\|_0)+4\Big(S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\max_{0\leq i\leq K-1}\left\|u_d^i\right\|_0\\
\leq&16S_2C_{f,v,u}[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}(\left\|f^{i+1}\right\|_0+C_v\left\|v^{i+1}\right\|_0+\left\|u_d^i\right\|_0).
\end{aligned}
\tag{2.26}
$$

Combining (2.25) and (2.26) and simplifying, we obtain

$$
\begin{aligned}
&\left\|U_N^{k+1}\right\|_0+\sqrt{\alpha_0\beta_0^{-1}}\|\Delta U_N^{k+1}\|_0+\left\|P_N^k\right\|_0+\sqrt{\alpha_0\beta_0^{-1}}\|\Delta P_N^k\|_0\\
\leq&\Big[16S_2C_{f,v,u}[\Gamma(1-\theta)]^2T^{2\theta}+4\Big(S_2C_{f,v}[\Gamma(1-\theta)]^2T^{2\theta}\Big)^{\frac{1}{2}}\Big]\max_{0\leq i\leq K-1}(\left\|f^{i+1}\right\|_0+C_v\left\|v^{i+1}\right\|_0+\left\|u_d^i\right\|_0).
\end{aligned}
$$

Therefore, the proof of Theorem 2.3 has been completed. □

3. Error estimation for OCPs

In this section, we will establish a rigorous theoretical analysis of errors of $\|u-U_N\|_{\hat{L}^2(\Omega_T)}$, $\|v-V_N\|_{\hat{L}^2(\Omega_T)}$ and $\|p-P_N\|_{\hat{L}^2(\Omega_T)}$, where $\|\Phi(\mathbf{x},t)\|_{\hat{L}^2(\Omega_T)}=(\sum_{k=1}^K\tau\|\Phi(\mathbf{x},t_k)\|_0^2)^{1/2}, \forall\Phi(\mathbf{x},t)\in L^2(\Omega_T)$. To isolate this estimated value, we first decompose $p-P_N$ and $u-U_N$ into the following forms:

$$p-P_N=(p-P_N(u))+(P_N(u)-P_N),\ \ u-U_N=(u-U_N(v))+(U_N(v)-U_N),$$

where $U_N(v)$ and $P_N(u)$ are auxiliary variables defined by

$$({}_0D_\tau^\theta U_N^k(v),\bar{v}_N)+(\Delta U_N^k(v),\Delta\bar{v}_N)=(f^k+v^k,\bar{v}_N),k=1,2,\cdots,K.\ \forall\bar{v}_N\in Y_N,\tag{3.1}$$
$$({}_\tau D_T^\theta P_N^k(u),\bar{v}_N)+(\Delta P_N^k(u),\Delta\bar{v}_N)=(u^k-u_d^k,\bar{v}_N),k=K-1,K-2,\cdots,0.\ \forall\bar{v}_N\in Y_N.\tag{3.2}$$

As the right-hand sides of the two auxiliary equations take exact solutions as inputs, no contributions arise from the discretization errors $\|U_N^k-u(\mathbf{x},t_k)\|_0$ and $\|V_N^k-v(\mathbf{x},t_k)\|_0$. Following the arguments of Theorem 2.2, we derive the following lemmas.

Lemma 3.1. *Let $p(\mathbf{x},t)$ be the exact solution of Eq (2.2) and $P_N^k(u),k=0,1,\cdots,K-1$ be the approximate solution of Eq (3.2). Under the assumption that $\partial_t^4p\in L^\infty((0,T];H^l(\Omega)\cap H_0^2(\Omega)),l\geq2$, the following error estimation holds:*

$$|||P_N^k(u)-p(\mathbf{x},t_k)|||_2\leq C(\tau^{3-\theta}+N^{2-l}),\quad k=0,1,\cdots,K-1,$$

where the constant C has no relation to N and τ.

Proof. The detailed proof process can be found in Appendix B. □

Lemma 3.2. *Let $u(\mathbf{x},t)$ be the exact solution of Eq (1.2) and $U_N^k(v), k=1,2,\cdots,K$ be the approximate solution of Eq (3.1). Under the assumption that $\partial_t^4 u \in L^\infty((0,T];H^l(\Omega)\cap H_0^2(\Omega)), l\geq 2$, the following error estimation holds:*

$$|||U_N^k(v)-u(\mathbf{x},t_k)|||_2 \leq C(\tau^{3-\theta}+N^{2-l}), \quad k=1,2,\cdots,K,$$

where the constant C has no relation to N and τ.

Proof. See Appendix C for the detailed proof. □

Next, based on the aforementioned approach and the relevant results, we will conduct an error analysis for the fully discrete numerical scheme.

Theorem 3.1. *Let (u,v,p) be the exact solution of Eqs (1.2) and (2.2), and the approximate solution be $(U_N^k,V_N^k,P_N^k), k=1,2,\cdots,K$ of (2.23). The error estimation of $\|u-U_N\|_{\hat{L}^2(\Omega_T)}, \|v-V_N\|_{\hat{L}^2(\Omega_T)}, \|p-P_N\|_{\hat{L}^2(\Omega_T)}$ satisfies the following inequality:*

$$\|u-U_N\|_{\hat{L}^2(\Omega_T)}+\|v-V_N\|_{\hat{L}^2(\Omega_T)}+\|p-P_N\|_{\hat{L}^2(\Omega_T)} \leq C(\tau^{3-\theta}+N^{2-l}),$$

where the constant C has no relation to N and τ.

Proof. Let $e_N^{u,k}=U_N^k(v)-U_N^k$ by subtracting the first line of (2.23) from (3.1). $e_N^{u,k}$ satisfies the following equation:

$$({}_0D_\tau^\theta e_N^{u,k},\bar{v}_N)+(\Delta e_N^{u,k},\Delta\bar{v}_N)=(v^k-V_N^k,\bar{v}_N). \tag{3.3}$$

Based on the idea of Theorem 3.1 in [32], $\|e^{u,k}\|_0^2 \leq C\tau\|v^k-V_N^k\|_0^2$. Multiply both sides by τ, then sum from $k=1$ to K, to draw the following conclusion:

$$\sum_{k=1}^K \tau\|e^{u,k}\|_0^2 \leq C\sum_{k=1}^K \tau\|v^k-V_N^k\|_0^2. \tag{3.4}$$

Now, we estimate $\|v^k-V_N^k\|_0^2$ by introducing an auxiliary equation,

$$({}_\tau D_T^\theta P_N^k(U_N^k(v)),\bar{v}_N)+(\Delta P_N^k(U_N^k(v)),\Delta\bar{v}_N)=(U_N^k(v)-u_d^k,\bar{v}_N). \tag{3.5}$$

Given the conditions $\alpha v_N^k+p_N^k=0$ and $\alpha V_N^k+P_N^k=0$, we have

$$\begin{aligned}
&\alpha\sum_{k=1}^K \tau\|v^k-V_N^k\|_0^2=\alpha\sum_{k=1}^K \tau(v^k-V_N^k,v^k-V_N^k)=\sum_{k=1}^K \tau(-p^k+P_N^k,v^k-V_N^k)\\
&=\sum_{k=1}^K \tau\left(P_N^k-P_N^k(U_N^k(v))+P_N^k(U_N^k(v))-p^k,v^k-V_N^k\right)\\
&=\sum_{k=1}^K \tau\left(P_N^k-P_N^k(U_N^k(v)),v^k-V_N^k\right)+\sum_{k=1}^K \tau\left(P_N^k(U_N^k(v))-p^k,v^k-V_N^k\right).
\end{aligned} \tag{3.6}$$

Multiply both sides of (3.1) by τ, then sum from $k = 1$ to K. The following equation is obtained:

$$\sum_{k=1}^{K} \tau \left({}_0D_\tau^\theta U_N^k(v), \bar{v}_N \right) + \sum_{k=1}^{K} \tau \left(\Delta U_N^k(v), \Delta \bar{v}_N \right) = \sum_{k=1}^{K} \tau \left(f^k + v^k, \bar{v}_N \right). \tag{3.7}$$

Similarly, the first equation of (2.23) can be rewritten as

$$\sum_{k=1}^{K} \tau \left({}_0D_\tau^\theta U_N^k, \bar{v}_N \right) + \sum_{k=1}^{K} \tau \left(\Delta U_N^k, \Delta \bar{v}_N \right) = \sum_{k=1}^{K} \tau \left(f^k + V_N^k, \bar{v}_N \right). \tag{3.8}$$

Using (3.7)–(3.8), and setting $\bar{v}_N = P_N^k - P_N^k(U_N^k(v))$, we have

$$\begin{aligned} &\sum_{k=1}^{K} \tau \left({}_0D_\tau^\theta \left(U_N^k(v) - U_N^k \right), P_N^k - P_N^k(U_N^k(v)) \right) + \sum_{k=1}^{K} \tau \left(\Delta \left(U_N^k(v) - U_N^k \right), \Delta (P_N^k - P_N^k(U_N^k(v))) \right) \\ &= \sum_{k=1}^{K} \tau \left(v^k - V_N^k, P_N^k - P_N^k(U_N^k(v)) \right). \end{aligned} \tag{3.9}$$

For the left side of Eq (3.9), if the integration method is applied in the time direction, with the conditions of $U_N^0(v) - U_N^0 = 0$ and $P_N^K - P_N^K(U_N^K(v)) = 0$ [31], we can obtain

$$\begin{aligned} &\sum_{k=1}^{K} \tau \left({}_0D_\tau^\theta \left(U_N^k(v) - U_N^k \right), P_N^k - P_N^k(U_N^k(v)) \right) + \sum_{k=1}^{K} \tau \left(\Delta \left(U_N^k(v) - U_N^k \right), \Delta (P_N^k - P_N^k(U_N^k(v))) \right) \\ &= \sum_{k=1}^{K} \tau \left(U_N^k(v) - U_N^k, {}_\tau D_T^\theta \left(P_N^k - P_N^k(U_N^k(v)) \right) \right) + \sum_{k=1}^{K} \tau \left(\Delta \left(U_N^k(v) - U_N^k \right), \Delta (P_N^k - P_N^k(U_N^k(v))) \right). \end{aligned} \tag{3.10}$$

Multiply both sides of (3.5) by τ, then sum from $k = 1$ to K:

$$\sum_{k=1}^{K} \tau \left({}_\tau D_T^\theta P_N^k(U_N^k(v)), \bar{v}_N \right) + \sum_{k=1}^{K} \tau \left(\Delta P_N^k(U_N^k(v)), \Delta \bar{v}_N \right) = \sum_{k=1}^{K} \tau \left(U_N^k(v) - u_d^k, \bar{v}_N \right). \tag{3.11}$$

For the second equation in (2.23), multiply both sides by τ and sum from $k = 1$ to K:

$$\sum_{k=1}^{k} \tau \left({}_\tau D_T^\theta P_N^k, \bar{v}_N \right) + \sum_{k=1}^{K} \tau \left(\Delta P_N^k, \Delta \bar{v}_N \right) = \sum_{k=1}^{K} \tau \left(U_N^k - u_d^k, \bar{v}_N \right). \tag{3.12}$$

Set $\bar{v}_N = U_N^k(v) - U_N^k$. Then, by (3.12) minus (3.11), we obtain

$$\begin{aligned} &\sum_{k=1}^{K} \tau \left({}_\tau D_T^\theta \left(P_N^k - P_N^k(U_N^k(v)) \right), U_N^k(v) - U_N^k \right) + \sum_{k=1}^{K} \tau \left(\Delta \left(P_N^k - P_N^k(U_N^k(v)) \right), \Delta \left(U_N^k(v) - U_N^k \right) \right) \\ &= \sum_{k=1}^{K} \tau \left(U_N^k - U_N^k(v), U_N^k(v) - U_N^k \right). \end{aligned} \tag{3.13}$$

By (3.13), (3.10) can be rewritten as

$$\begin{aligned}&\sum_{k=1}^{K}\tau\left({}_0D_\tau^\theta\left(U_N^k(v)-U_N^k\right),P_N^k-P_N^k(U_N^k(v))\right)+\sum_{k=1}^{K}\tau\left(\Delta\left(U_N^k(v)-U_N^k\right),\Delta(P_N^k-P_N^k(U_N^k(v)))\right)\\&=\sum_{k=1}^{K}\tau\left(U_N^k-U_N^k(v),U_N^k(v)-U_N^k\right)\le 0.\end{aligned}\tag{3.14}$$

Combining (3.9) and (3.14), we can conclude that

$$\begin{aligned}\sum_{k=1}^{K}\tau\left(v^k-V_N^k,P_N^k-P_N^k(U_N^k(v))\right)&=\sum_{k=1}^{K}\tau\left({}_0D_\tau^\theta\left(U_N^k(v)-U_N^k\right),P_N^k-P_N^k(U_N^k(v))\right)\\&+\sum_{k=1}^{K}\tau\left(\Delta\left(U_N^k(v)-U_N^k\right),\Delta(P_N^k-P_N^k(U_N^k(v)))\right)\le 0.\end{aligned}\tag{3.15}$$

Then, using (3.6), (3.15), and the Cauchy-Schwarz inequality, we have

$$\|v-V_N\|_{\hat{L}^2(\Omega_T)}\le\frac{1}{\alpha}\|P_N(U_N(v))-p\|_{\hat{L}^2(\Omega_T)}\le\frac{1}{\alpha}\|P_N(U_N(v))-P_N(u)\|_{\hat{L}^2(\Omega_T)}+\frac{1}{\alpha}\|P_N(u)-p\|_{\hat{L}^2(\Omega_T)}.\tag{3.16}$$

Next, we estimate $\|P_N(u)-p\|_{\hat{L}^2(\Omega_T)}$. Let $e_N^{p,k}=p(\mathbf{x},t_k)-P_N^k(u)$. According to Lemma 3.1, we obtain

$$\sum_{k=1}^{K}\tau\|e^{p,k}\|_0^2\le C\left(\tau^{3-\theta}+N^{2-l}\right)^2.\tag{3.17}$$

Using (3.5) minus (3.2), multiply both sides by τ and sum from $k=1$ to K. The equation can be rewritten as

$$\sum_{k=1}^{K}\tau\left({}_\tau D_T^\theta\left(P_N^k(U_N^k(v))-P_N^k(u)\right),\bar{v}_N\right)+\sum_{k=1}^{K}\tau\left(\Delta\left(P_N^k(U_N^k(v))-P_N^k(u)\right),\Delta\bar{v}_N\right)=\sum_{k=1}^{K}\tau\left(U_N^k(v)-u^k,\bar{v}_N\right).$$

Based on the idea of Lemma 2.4, we obtain that

$$\|P_N(U_N(v))-P_N(u)\|_{\hat{L}^2(\Omega_T)}\le C\|U_N(v)-u\|_{\hat{L}^2(\Omega_T)}.\tag{3.18}$$

Next, we estimate $\|U_N(v)-u\|_{\hat{L}^2(\Omega_T)}$. Let $\bar{e}_N^{u,k}=u^k-U_N^k(v)$, $\bar{e}_N^k=v(\mathbf{x},t_k)+f(\mathbf{x},t_k)-f^k-v^k$. Subtracting Eq (3.1) from the Galerkin weak form of Eq (1.2), we obtain

$$\left({}_0D_\tau^\theta\bar{e}^{u,k},\bar{v}_N\right)+\left(\Delta\bar{e}^{u,k},\Delta\bar{v}_N\right)=(\bar{e}_N^k,\bar{v}_N).\tag{3.19}$$

According to Lemma 3.2, we have

$$\sum_{k=1}^{K}\tau\|\bar{e}^{u,k}\|_0^2\le C(\tau^{3-\theta}+N^{2-l})^2.$$

By (3.16)–(3.18), we find the following result:

$$\sum_{k=1}^{K}\tau\|v^k - V_N^k\|_0^2 \le C\sum_{k=1}^{K}\tau\|\bar{e}^{u,k}\|_0^2 + C\sum_{k=1}^{K}\tau\|e^{p,k}\|_0^2 \le C(\tau^{3-\theta} + N^{2-l})^2. \tag{3.20}$$

By (3.3), (3.4), and (3.19), we can conclude the following:

$$\begin{aligned}&\sum_{k=1}^{K}\tau\|u^k - U_N^k\|_0^2 \le \sum_{k=1}^{K}\tau\|u^k - U_N^k(v)\|_0^2 + \sum_{k=1}^{K}\tau\|U_N^k(v) - U_N^k\|_0^2\\ \le& C(\tau^{3-\theta} + N^{2-l})^2 + \sum_{k=1}^{K}\tau\|v^k - V_N^k\|_0^2 \le C(\tau^{3-\theta} + N^{2-l})^2.\end{aligned} \tag{3.21}$$

Using the conditions $\alpha v_N^k + p_N^k = 0$ and $\alpha V_N^k + P_N^k = 0$, we have

$$\sum_{k=1}^{K}\tau\|p^k - P_N^k\|_0^2 = \alpha^2\sum_{k=1}^{K}\tau\|v^k - V_N^k\|_0^2 \le C(\tau^{3-\theta} + N^{2-l})^2. \tag{3.22}$$

Summarizing (3.20)–(3.22), we obtain

$$\sum_{k=1}^{K}\tau\|p^k - P_N^k\|_0^2 + \sum_{k=1}^{K}\tau\|v^k - V_N^k\|_0^2 + \sum_{k=1}^{K}\tau\|u^k - U_N^k\|_0^2 \le C(\tau^{3-\theta} + N^{2-l})^2.$$

Thus, we have

$$\|u - U_N\|_{\hat{L}^2(\Omega_T)} + \|p - P_N\|_{\hat{L}^2(\Omega_T)} + \|v - V_N\|_{\hat{L}^2(\Omega_T)} \le C(\tau^{3-\theta} + N^{2-l}).$$

Therefore, the proof of Theorem 3.1 has been completed. □

4. Numerical examples

We now present numerical results for a series of test problems designed to validate the convergence order in space and time of the proposed algorithms. By constructing 1D and 2D numerical examples with analytical solutions, we verify the spatial spectral accuracy convergence and temporally uniform high convergence order of the algorithm. In this section, numerical examples with exact solutions are presented to investigate the accuracy of the proposed method. Define

$$e_u(\tau) = (\sum_{k=1}^{K}\tau\|u(\mathbf{x}, t_k) - U_N^k\|_0^2)^{1/2}, e_v(\tau) = (\sum_{k=1}^{K}\tau\|v(\mathbf{x}, t_k) - V_N^k\|_0^2)^{1/2},$$

and

$$Order_u = \log_2(e_u(2\tau)/e_u(\tau)), Order_v = \log_2(e_v(2\tau)/e_v(\tau)),$$

for sufficiently large N.

Example 4.1. *We choose* $u_d(\mathbf{x}, t)$ *and* $f(\mathbf{x}, t)$ *in (1.1) and (1.2) with* $T = 1, \Omega = [-1, 1]$. *The initial values and boundary conditions required to satisfy (1.2) are as follows:*

$$u_d(\mathbf{x}, t) = t^4 \cos^2\left(\frac{\pi x}{2}\right) - \left[\frac{24}{\Gamma(5-\theta)}(1-t)^{4-\theta}\cos^2\left(\frac{\pi x}{2}\right) + \frac{1}{2}\pi^4(1-t)^4\cos(\pi x)\right],$$
$$f(\mathbf{x}, t) = \frac{24}{\Gamma(5-\theta)}t^{4-\theta}\cos^2\left(\frac{\pi x}{2}\right) + \frac{1}{2}\pi^4 t^4\cos(\pi x) + (1-t)^4\cos^2\left(\frac{\pi x}{2}\right).$$

Then, $u(\mathbf{x}, t)$ *and* $v(\mathbf{x}, t)$ *are determined by*

$$u(\mathbf{x}, t) = t^4\cos^2\left(\frac{\pi x}{2}\right), \quad v(\mathbf{x}, t) = -(1-t)^4\cos^2\left(\frac{\pi x}{2}\right).$$

In this example, we choose $\tau = 2^{-l}, l = 4, 5, 6, 7,$ *and* $N = 120$ *for* $\theta = 0.3, 0.5, 0.7$, *respectively. We immediately observe that the estimated order of convergence agrees with the theoretical value* $3 - \theta$ *exceptionally well for all the choices of* θ *from Table 2.*

Table 2. Temporal errors of e_u and e_v for the proposed L2 scheme in (2.23) with different values of θ in Example 4.1.

	τ	e_u	$Order_u$	e_v	$Order_v$	$CPUtime(s)$
$\theta = 0.3$	1/16	2.375018×10^{-5}	–	2.203068×10^{-5}	–	0.974762
	1/32	3.930807×10^{-6}	2.595040	3.793109×10^{-6}	2.538061	1.203633
	1/64	6.362550×10^{-7}	2.627148	6.252945×10^{-7}	2.600773	1.677442
	1/128	1.015821×10^{-7}	2.646957	1.007123×10^{-7}	2.634294	2.517050
$\theta = 0.5$	1/16	7.499625×10^{-5}	–	6.973736×10^{-5}	–	1.080558
	1/32	1.389753×10^{-5}	2.431989	1.342832×10^{-5}	2.376652	1.199230
	1/64	2.527434×10^{-6}	2.459083	2.486111×10^{-6}	2.433316	1.703495
	1/128	4.547886×10^{-7}	2.474405	4.512366×10^{-7}	2.461934	2.795172
$\theta = 0.7$	1/16	2.032177×10^{-4}	–	1.892023×10^{-4}	–	0.873905
	1/32	4.266821×10^{-5}	2.251792	4.125605×10^{-5}	2.197252	1.123199
	1/64	8.819673×10^{-6}	2.274364	8.680233×10^{-6}	2.248799	1.594182
	1/128	1.808336×10^{-6}	2.286062	1.795158×10^{-6}	2.273622	2.429788

To further verify the efficiency and accuracy of the proposed method, we compare it with the scheme where the Legendre-Galerkin spectral method is used for the spatial discretization, and the L1 finite difference scheme is applied for the Caputo time derivative approximation. The numerical results under the same test case and computational parameters are presented in Table 3. The CPU times in Tables 2 and 3 show only slight differences. It can be observed from the comparison between Tables 2 and 3 that the proposed L2 scheme in (2.23) yields smaller errors with convergence order $3-\theta$, *whereas the L1 scheme has the convergence order* $2 - \theta$. *The results confirm the higher temporal accuracy and efficiency of the proposed discretization.*

Table 4 presents the temporal errors for different fractional orders θ *and time-step sizes. Here,* e_u *and* e_v *denote the errors obtained by the proposed L2 temporal scheme in (2.23), as* e_{uu} *and* e_{vv}

represent the corresponding errors computed by the high-order L2 scheme in [19]. It can be observed that the proposed L2 temporal scheme in (2.23) achieves comparable accuracy to the high-order L2 scheme in [19]. Moreover, the proposed L2 temporal scheme in (2.23) yields slightly smaller errors for all tested cases, demonstrating that the proposed L2 temporal scheme in (2.23) preserves the accuracy of the original high-order L2 discretization.

Table 3. Temporal errors of e_u and e_v computed by the L1 discretization with different values of θ in Example 4.1.

	τ	e_u	$Order_u$	e_v	$Order_v$	$CPUtime(s)$
$\theta = 0.3$	1/16	3.210271×10^{-4}	–	2.791953×10^{-4}	–	0.658165
	1/32	1.072971×10^{-4}	1.581083	1.002748×10^{-4}	1.477314	0.797126
	1/64	3.508096×10^{-5}	1.612851	3.392761×10^{-5}	1.563428	1.302184
	1/128	1.130080×10^{-5}	1.634262	1.111310×10^{-5}	1.610198	2.574216
$\theta = 0.5$	1/16	9.873023×10^{-4}	–	8.606748×10^{-4}	–	0.641268
	1/32	3.681629×10^{-4}	1.423147	3.444664×10^{-4}	1.321104	0.848637
	1/64	1.347354×10^{-4}	1.450214	1.303938×10^{-4}	1.401487	1.308580
	1/128	4.873971×10^{-5}	1.466960	4.795443×10^{-5}	1.443139	2.656249
$\theta = 0.7$	1/16	2.573160×10^{-3}	–	2.245648×10^{-3}	–	0.697200
	1/32	1.082520×10^{-3}	1.249146	1.013333×10^{-3}	1.148023	0.907862
	1/64	4.486021×10^{-4}	1.270885	4.342759×10^{-4}	1.222425	1.543885
	1/128	1.843362×10^{-4}	1.283096	1.814129×10^{-4}	1.259334	2.712370

Table 4. Comparison of temporal errors obtained by the proposed L2 scheme in (2.25) and the high-order L2 scheme in [19] for different values of θ in Example 4.1.

	τ	e_u	e_v	e_{uu} [19]	e_{vv} [19]
$\theta = 0.3$	1/16	$2.37501845 \times 10^{-5}$	$2.20306837 \times 10^{-5}$	$2.37663026 \times 10^{-5}$	$2.05899773 \times 10^{-5}$
	1/32	$3.93080783 \times 10^{-6}$	$3.79310958 \times 10^{-6}$	$3.93205425 \times 10^{-6}$	$3.79559564 \times 10^{-6}$
	1/64	$6.36255065 \times 10^{-7}$	$6.25294527 \times 10^{-7}$	$6.36362429 \times 10^{-7}$	$6.25491268 \times 10^{-7}$
	1/128	$1.01582193 \times 10^{-7}$	$1.00712392 \times 10^{-7}$	$1.01593156 \times 10^{-7}$	$1.00727638 \times 10^{-7}$
$\theta = 0.5$	1/16	$7.49962538 \times 10^{-5}$	$6.97373637 \times 10^{-5}$	$7.50375302 \times 10^{-5}$	$6.98437046 \times 10^{-5}$
	1/32	$1.38975355 \times 10^{-5}$	$1.34283273 \times 10^{-5}$	$1.39012898 \times 10^{-5}$	$1.34375173 \times 10^{-5}$
	1/64	$2.52743477 \times 10^{-6}$	$2.48611140 \times 10^{-6}$	$2.52787759 \times 10^{-6}$	$2.48688043 \times 10^{-6}$
	1/128	$4.54788662 \times 10^{-7}$	$4.51236685 \times 10^{-7}$	$4.54855464 \times 10^{-7}$	$4.51301884 \times 10^{-7}$
$\theta = 0.7$	1/16	$2.03217787 \times 10^{-4}$	$1.89202310 \times 10^{-4}$	$2.03285702 \times 10^{-4}$	$1.89499122 \times 10^{-4}$
	1/32	$4.26682172 \times 10^{-5}$	$4.12560502 \times 10^{-5}$	$4.26754511 \times 10^{-5}$	$4.12829808 \times 10^{-5}$
	1/64	$8.81967375 \times 10^{-6}$	$8.68023382 \times 10^{-6}$	$8.82091326 \times 10^{-6}$	$8.68263578 \times 10^{-6}$
	1/128	$1.80833649 \times 10^{-6}$	$1.79515866 \times 10^{-6}$	$1.80857834 \times 10^{-6}$	$1.79537179 \times 10^{-6}$

In Figure 1, we will verify the spectral convergence accuracy of the space by using a semi-log plot. For $\theta = 0.3, 0.5,$ and 0.7, we set $K = 2^{12}$ and $N = 11, 13, \cdots, 21$. Figure 1 shows that the behavior *of discretization errors reveals a remarkably linear relationship with the spatial polynomial order N. It means that the magnitude of the numerical error decreases exponentially when the degree of the spatial polynomial basis increases. In Figure 2, we plot the corresponding numerical solutions for the state variables and control variables under the conditions of $N = 120$ and $\tau = 1/32$.*

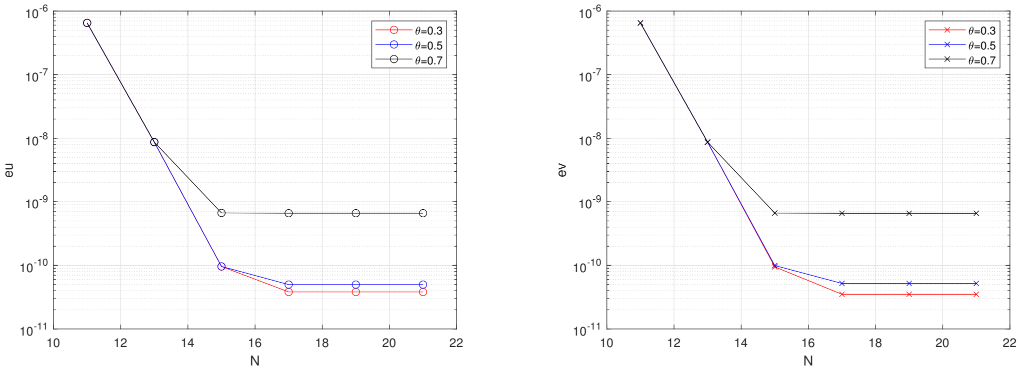

Figure 1. The deviation of the state variable (left) and the control variable (right) changes in response to the spatial polynomial's approximation order N corresponding to the parameter θ in Example 4.1.

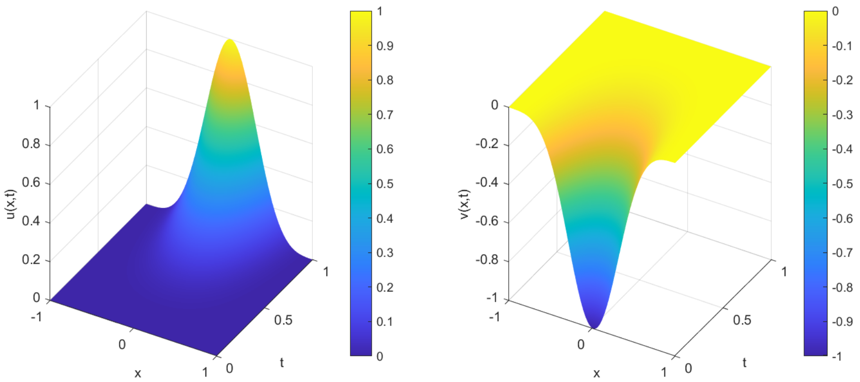

Figure 2. The numerical solution of the state variable (left) and the control variable (right) when $\theta = 0.5$ in Example 4.1.

Example 4.2. *We choose $u_d(\mathbf{x}, t)$ and $f(\mathbf{x}, t)$ in (1.1) and (1.2) with $T = 1, \Omega = [-1, 1]$. The initial values and boundary conditions required to satisfy (1.2) are as follows:*

$$u_d(\mathbf{x}, t) = t^r \cos^2\left(\frac{\pi x}{2}\right) - \left[\frac{\Gamma(r+1)}{\Gamma(r+1-\theta)}(1-t)^{r-\theta}\cos^2\left(\frac{\pi x}{2}\right) + \frac{1}{2}\pi^4(1-t)^r\cos(\pi x)\right],$$

$$f(\mathbf{x}, t) = \frac{\Gamma(r+1)}{\Gamma(r+1-\theta)}t^{r-\theta}\cos^2\left(\frac{\pi x}{2}\right) + \frac{1}{2}\pi^4 t^r\cos(\pi x) + (1-t)^r\cos^2\left(\frac{\pi x}{2}\right).$$

Then the nonsmooth exact solutions $u(\mathbf{x}, t)$ and $v(\mathbf{x}, t)$ are determined by

$$u(\mathbf{x}, t) = t^r\cos^2\left(\frac{\pi x}{2}\right), \quad v(\mathbf{x}, t) = -(1-t)^r\cos^2\left(\frac{\pi x}{2}\right).$$

Table 5 presents the temporal errors and convergence orders for the singular solution with $r = 1+\theta$. For different fractional orders $\theta = 0.4, 0.6, 0.8$, the practical convergence orders are around $1+\theta$, which *is obviously lower than the theoretical optimal rate $3-\theta$ derived under high-regularity assumptions. This numerical phenomenon confirms that the initial weak singularity of the solution degrades the temporal convergence accuracy of the proposed scheme.*

Table 5. Temporal errors of e_u and e_v for the proposed scheme with different values of θ and $r = 1+\theta$.

	τ	e_u	$Order_u$	e_v	$Order_v$
$\theta = 0.4$	1/16	5.413781×10^{-5}	–	5.408424×10^{-5}	–
	1/32	2.647513×10^{-5}	1.025924	2.646695×10^{-5}	1.022339
	1/64	1.286573×10^{-5}	1.031999	1.286443×10^{-5}	1.030162
	1/128	6.201009×10^{-6}	1.041105	6.200801×10^{-6}	1.040804
$\theta = 0.6$	1/16	8.554007×10^{-5}	–	8.538218×10^{-5}	–
	1/32	4.091882×10^{-5}	1.043032	4.089265×10^{-5}	1.037340
	1/64	1.912673×10^{-5}	1.063836	1.912234×10^{-5}	1.062094
	1/128	8.653526×10^{-6}	1.097174	8.652801×10^{-6}	1.096582
$\theta = 0.8$	1/16	8.154326×10^{-5}	–	8.127600×10^{-5}	–
	1/32	3.780615×10^{-5}	1.050115	3.775909×10^{-5}	1.041507
	1/64	1.633882×10^{-5}	1.108945	1.633086×10^{-5}	1.106005
	1/128	6.441168×10^{-6}	1.210317	6.439910×10^{-6}	1.209223

Example 4.3. *We choose $u_d(\mathbf{x},t)$ and $f(\mathbf{x},t)$ in (1.1) and (1.2), where $T=1, \Omega=[-1,1]^2$. The initial values and boundary conditions required to satisfy (1.2) are as follows:*

$$\begin{aligned} u_d(\mathbf{x},t) =& t^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right) - \frac{24}{\Gamma(5-\theta)}t^{4-\theta}\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right) \\ &- \frac{1}{2}t^4\pi^4\left[\cos(\pi x_1)\cos^2\left(\frac{\pi}{2}x_2\right) + \cos(\pi x_2)\cos^2\left(\frac{\pi}{2}x_1\right) + \cos(\pi x_1)\cos(\pi x_2)\right], \\ f(\mathbf{x},t) =& \frac{24}{\Gamma(5-\theta)}t^{4-\theta}\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right) + \frac{1}{2}t^4\pi^4\left[\cos(\pi x_1)\cos^2\left(\frac{\pi}{2}x_2\right)\right. \\ &\left. + \cos(\pi x_2)\cos^2\left(\frac{\pi}{2}x_1\right) + \cos(\pi x_1)\cos(\pi x_2)\right] + (1-t)^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right). \end{aligned}$$

Then, $u(\mathbf{x},t)$ and $v(\mathbf{x},t)$ are determined by

$$u(\mathbf{x},t) = t^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right), \quad v(\mathbf{x},t) = -(1-t)^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right).$$

First, we set the parameters for the temporal convergence order, with $\tau = 2^{-l}, l = 4,5,6,7,$ and $N = 120$. Observe the error e_u and e_v, when $\theta = 0.3, 0.5, 0.8$. We can observe that as the time-step size becomes smaller and smaller, the time convergence order obtained approaches $3-\theta$ more and more closely.

Second, under the discrete numerical value N of the same space, the temporal convergence order approaches $3-\theta$ as θ increases. We can further observe that for all the parameters of θ in the table, the convergence order of the error is in excellent agreement with the theoretical value of $3-\theta$ from Table 6. Our numerical calculation results are in agreement with the derived theory.

Table 6. The temporal errors of e_u and e_v vary according to the different values of the parameter θ in Example 4.3.

	τ	e_u	$Order_u$	e_v	$Order_v$
$\theta = 0.3$	$1/16$	8.112237×10^{-6}	–	7.540524×10^{-6}	–
	$1/32$	1.342384×10^{-6}	2.595301	1.296783×10^{-6}	2.539727
	$1/64$	2.172654×10^{-7}	2.627267	2.136569×10^{-7}	2.601569
	$1/128$	3.468671×10^{-8}	2.647003	3.440247×10^{-8}	2.634711
$\theta = 0.5$	$1/16$	2.564060×10^{-5}	–	2.390105×10^{-5}	–
	$1/32$	4.750199×10^{-6}	2.432370	4.594572×10^{-6}	2.379071
	$1/64$	8.638064×10^{-7}	2.459208	8.499601×10^{-7}	2.434463
	$1/128$	1.554312×10^{-7}	2.474431	1.542050×10^{-7}	2.462544
$\theta = 0.8$	$1/16$	1.109654×10^{-4}	–	1.036439×10^{-4}	–
	$1/32$	2.486511×10^{-5}	2.157915	2.407129×10^{-5}	2.106249
	$1/64$	5.493049×10^{-6}	2.178443	5.407387×10^{-6}	2.154310
	$1/128$	1.204813×10^{-6}	2.188797	1.195637×10^{-6}	2.177151

Finally, we will verify the spectral convergence accuracy of the space by using a semi-log plot. For $\theta = 0.3, 0.5$ *and* 0.7*, we set* $K = 2^{12}$ *and* $N = 11, 13, \cdots, 21$*. By observing Figure 3, it can be seen that the discretization error decreases as the polynomial order* N *in space increases, and eventually stabilizes.*

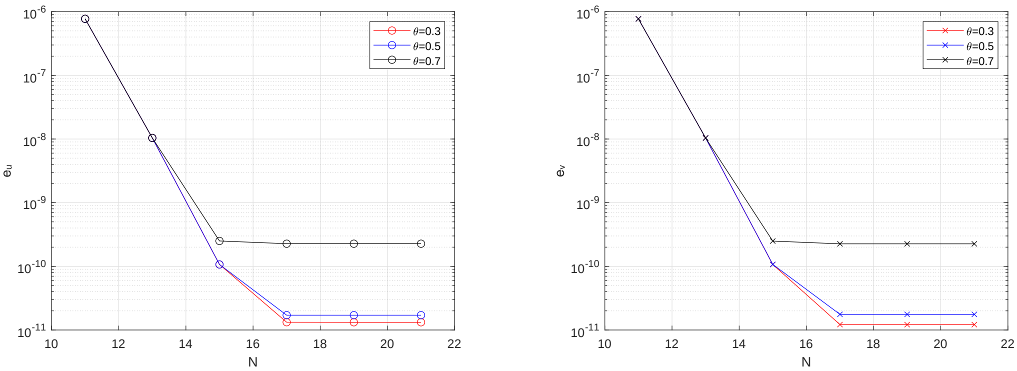

Figure 3. The deviation of the state variable (left) and the control variable (right) changes in response to the spatial polynomial's approximation order N corresponding to the parameter θ in Example 4.3.

In the following Examples 4.4 and 4.5, we have extended the algorithm to address nonlinear OCPs. Regarding the adjoint equation of the nonlinear example, we have elaborately derived them as follows.

Define the Lagrangian

$$\begin{aligned}\ell(u, v, p) &= \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega_T)} + \frac{\alpha}{2}\|v\|^2_{L^2(\Omega_T)} - \int_{\Omega_T} {}_0D_t^\theta u \cdot p + \Delta u \cdot \Delta p + (u^3 - u) \cdot p - (f + v) \cdot p \, d\mathbf{x}dt \\ &= \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega_T)} + \frac{\alpha}{2}\|v\|^2_{L^2(\Omega_T)} - \int_{\Omega_T} u \cdot {}_tD_T^\theta p + \Delta u \cdot \Delta p + (u^3 - u) \cdot p - (f + v) \cdot p \, d\mathbf{x}dt.\end{aligned}$$

Take the partial derivative of ℓ with respect to u and set it equal to zero:

$$\ell_u(u,v,p)\cdot\bar{h}=\int_{\Omega_T}(u-u_d)\cdot\bar{h}-{}_tD_T^{\theta}p\cdot\bar{h}-\Delta p\cdot\Delta\bar{h}-(3u^2-1)p\cdot\bar{h}\,d\mathbf{x}dt=0,\quad \forall p,\bar{h}\in V.$$

Therefore, the state of the nonlinear example and the structure of the adjoint equation are as follows:

$$\begin{aligned}&({}_0D_t^{\theta}u,\bar{v})+(\Delta u,\Delta\bar{v})+(u^3-u,\bar{v})=(v+f,\bar{v}),\quad \bar{v}\in Y.\\&({}_tD_T^{\theta}p,\bar{v})+(\Delta p,\Delta\bar{v})+((3u^2-1)p,\bar{v})=(u-u_d,\bar{v}),\quad \bar{v}\in Y.\end{aligned}\tag{4.1}$$

Based on the theory of nonlinear OCPs, we present one-dimensional and two-dimensional numerical examples satisfying the initial and boundary conditions in (1.2).

Example 4.4. *To distinguish it from the OCP numerical solution image, we have chosen a new 1D nonlinear example. We choose $u_d(\mathbf{x},t)$ and $f(\mathbf{x},t)$ in (4.1) as follows, where $T=1,\Omega=[-1,1]$:*

$$\begin{aligned}u_d(\mathbf{x},t)=&\,t^4\sin^2(\pi x)-\frac{24}{\Gamma(5-\theta)}(1-t)^{4-\theta}\sin^2(\pi x)-8\pi^4(1-t)^4\sin^2(\pi x)+8\pi^4(1-t)^4\cos^2(\pi x)\\&-3\left(t^4\sin^2(\pi x)\right)^2(1-t)^4\sin^2(\pi x)+(1-t)^4\sin^2(\pi x),\\f(\mathbf{x},t)=&\,\frac{24}{\Gamma(5-\theta)}t^{4-\theta}\sin^2(\pi x)+8\pi^4t^4\sin^2(\pi x)-8\pi^4t^4\cos^2(\pi x)+(1-t)^4\sin^2(\pi x)\\&+\left(t^4\sin^2(\pi x)\right)^3-t^4\sin^2(\pi x).\end{aligned}$$

Then, $u(\mathbf{x},t)$ and $v(\mathbf{x},t)$ are determined by

$$u(\mathbf{x},t)=t^4\sin^2(\pi x),\quad v(\mathbf{x},t)=-(1-t)^4\sin^2(\pi x).$$

First, set the parameters to be consistent with those of Example 4.1; we choose $\tau=2^{-l}, l=4,5,6,7$ and $N=120$ for $\theta=0.3,0.5,0.7$, respectively. Table 7 lists the maximum errors e_u,e_v, CPU times, and the corresponding temporal convergence rates obtained from the proposed scheme for varying values of θ. The results confirm that the observed temporal convergence order closely aligns with the theoretical order of $3-\theta$, demonstrating good agreement with the expected behavior.

Second, in Figure 4, we have plotted the errors obtained when $N=15,17,\cdots,27$, and $K=2^{12}$. Based on different θ values such as $\theta=0.3,0.5$, and 0.7, in the context of spectral methods for partial differential equations, the behavior of the numerical error demonstrates a linear dependence on the polynomial degree employed in the spatial approximation. In Figure 5, we plot the corresponding numerical solutions for the state variables and control variables under the conditions of $N=120$ and $\tau=1/32$.

Finally, as can be seen from Table 7 and Figure 4, the numerical methodology presented in this work is designed to effectively solve a class of nonlinear partial differential equations characterized by high-order spatial operators and fractional derivatives in time.

Table 7. The temporal errors of e_u and e_v vary according to the different values of the parameter θ in Example 4.4.

	τ	e_u	$Order_u$	e_v	$Order_v$	$CPUtime(s)$
$\theta = 0.3$	$1/16$	1.649421×10^{-5}	–	1.575419×10^{-5}	–	2.352165
	$1/32$	2.729980×10^{-6}	2.594997	2.712759×10^{-6}	2.537902	3.739323
	$1/64$	4.418781×10^{-7}	2.627170	4.472132×10^{-7}	2.600726	6.452709
	$1/128$	7.055415×10^{-8}	2.646845	7.203688×10^{-8}	2.634155	13.671961
$\theta = 0.5$	$1/16$	5.212809×10^{-5}	–	4.986935×10^{-5}	–	2.419523
	$1/32$	9.660334×10^{-6}	2.431916	9.603457×10^{-6}	2.376527	3.628740
	$1/64$	1.756833×10^{-6}	2.459095	1.778034×10^{-6}	2.433270	6.365028
	$1/128$	3.161244×10^{-7}	2.474412	3.227253×10^{-7}	2.461905	13.461021
$\theta = 0.7$	$1/16$	1.414298×10^{-4}	–	1.353004×10^{-4}	–	2.406026
	$1/32$	2.969768×10^{-5}	2.251664	2.950500×10^{-5}	2.197135	4.266826
	$1/64$	6.138663×10^{-6}	2.274353	6.208045×10^{-6}	2.248748	7.300583
	$1/128$	1.258636×10^{-6}	2.286063	1.283909×10^{-6}	2.273595	12.836731

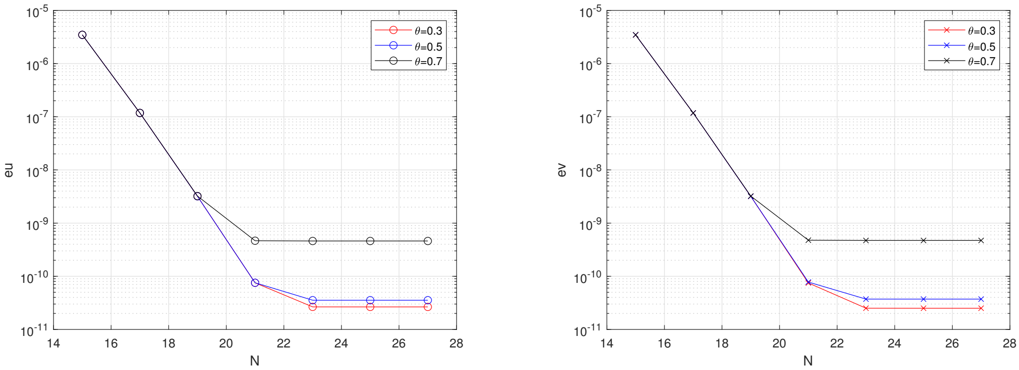

Figure 4. The deviation of the state variable (left) and the control variable (right) changes in response to the spatial polynomial's approximation order N corresponding to the parameter θ in Example 4.4.

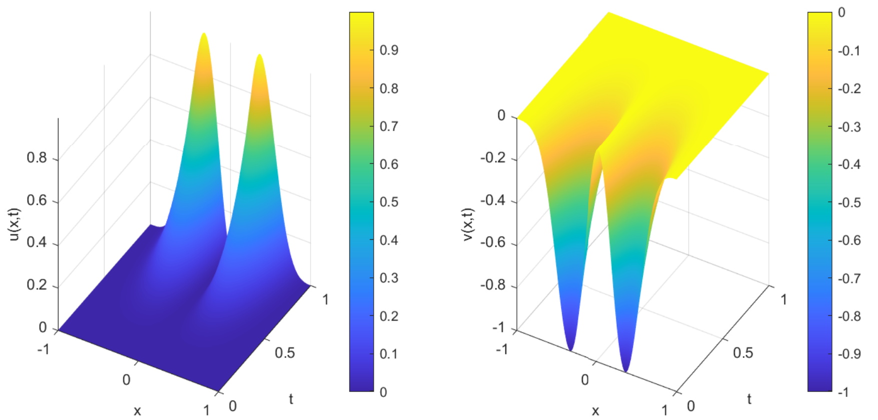

Figure 5. The numerical solution of the state variable (left) and the control variable (right) when $\theta = 0.5$ in Example 4.4.

Example 4.5. *We choose a 2D nonlinear test case with $u_d(\mathbf{x}, t)$ and $f(\mathbf{x}, t)$ in (4.1), where $T = 1$, $\Omega = [-1, 1]^2$:*

$$\begin{aligned}
u_d(\mathbf{x}, t) =& t^4 \cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right) - \frac{24}{\Gamma(5-\theta)} t^{4-\theta} \cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right) \\
& - \frac{1}{2}t^4\pi^4 \left[\cos(\pi x_1)\cos^2\left(\frac{\pi}{2}x_2\right) + \cos(\pi x_2)\cos^2\left(\frac{\pi}{2}x_1\right) + \cos(\pi x_1)\cos(\pi x_2)\right] \\
& - 3t^8(1-t)^4 \cos^6\left(\frac{\pi}{2}x_1\right)\cos^6\left(\frac{\pi}{2}x_2\right) + (1-t)^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right). \\
f(\mathbf{x}, t) =& \frac{24}{\Gamma(5-\theta)} t^{4-\theta} \cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right) + \frac{1}{2}t^4\pi^4 \left[\cos(\pi x_1)\cos^2\left(\frac{\pi}{2}x_2\right)\right. \\
& \left. + \cos(\pi x_2)\cos^2\left(\frac{\pi}{2}x_1\right) + \cos(\pi x_1)\cos(\pi x_2)\right] + t^{12}\cos^6\left(\frac{\pi}{2}x_1\right)\cos^6\left(\frac{\pi}{2}x_2\right) \\
& - t^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right) + (1-t)^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right).
\end{aligned}$$

Then, $u(\mathbf{x}, t)$ and $v(\mathbf{x}, t)$ are determined by

$$u(\mathbf{x}, t) = t^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right), \quad v(\mathbf{x}, t) = -(1-t)^4\cos^2\left(\frac{\pi}{2}x_1\right)\cos^2\left(\frac{\pi}{2}x_2\right).$$

First, we choose the same parameters as those used in Example 4.3, where $\tau = 2^{-l}$ with $l = 4, 5, 6, 7$ and $N = 120$ for $\theta = 0.3, 0.5, 0.8$. After careful observation of Table 8, we find that the calculated error estimates are in excellent agreement with the time convergence order and the results of Example 4.3.

Table 8. The temporal errors of e_u and e_v vary according to the different values of the parameter θ in Example 4.5.

	τ	e_u	$Order_u$	e_v	$Order_v$
$\theta = 0.3$	1/16	8.066245×10^{-6}	–	7.632457×10^{-6}	–
	1/32	1.334745×10^{-6}	2.595332	1.312747×10^{-6}	2.539558
	1/64	2.160263×10^{-7}	2.627286	2.162912×10^{-7}	2.601542
	1/128	3.448879×10^{-8}	2.647007	3.482671×10^{-8}	2.634709
$\theta = 0.5$	1/16	2.550133×10^{-5}	–	2.419367×10^{-5}	–
	1/32	4.724275×10^{-6}	2.432407	4.651164×10^{-6}	2.378966
	1/64	8.590767×10^{-7}	2.459234	8.604389×10^{-7}	2.434447
	1/128	1.545787×10^{-7}	2.474445	1.561033×10^{-7}	2.462571
$\theta = 0.8$	1/16	1.104239×10^{-4}	–	1.049200×10^{-4}	–
	1/32	2.474358×10^{-5}	2.157926	2.436819×10^{-5}	2.106219
	1/64	5.466125×10^{-6}	2.178464	5.474096×10^{-6}	2.154306
	1/128	1.198896×10^{-6}	2.188811	1.210387×10^{-6}	2.177151

Second, we verify the spectral convergence accuracy of the space through Figure 6. For the three possible values of parameter $\theta = 0.3, 0.5$, and 0.7, we set $K = 2^{12}$ and $N = 11, 13, \cdots, 21$. Figure 6 shows that the behavior of discretization errors reveals a remarkably linear relationship with the spatial polynomial order N. Table 9 depicts the variation of the CPU time with respect to the spatial polynomial approximation order N for a fixed number of time steps $K = 2^{12}$. It can be observed that the

CPU time increases monotonically with increasing N for all considered fractional orders θ = 0.3, 0.5, and 0.8, which is consistent with the increasing computational complexity of the Legendre-Galerkin spectral discretization. Moreover, the influence of the fractional order on the CPU time is relatively limited, and the spatial approximation order plays a dominant role in determining the computational cost.

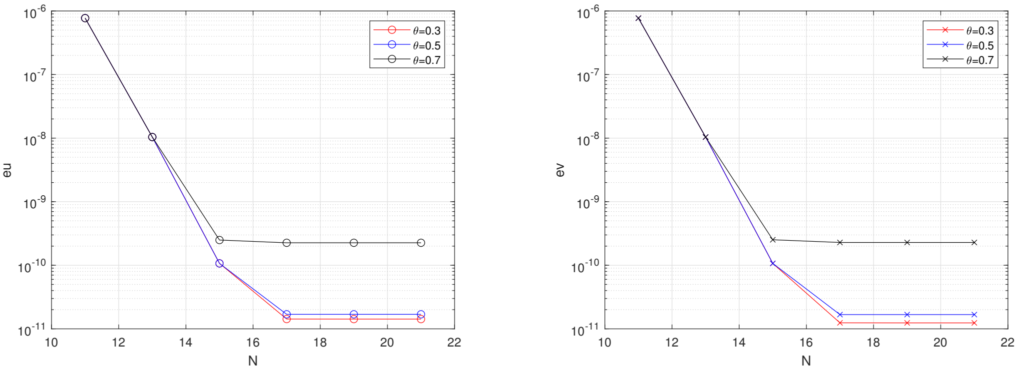

Figure 6. The deviation of the state variable (left) and the control variable (right) changes in response to the spatial polynomial's approximation order N corresponding to the parameter θ in Example 4.5.

Table 9. The CPU time (s) for $K = 2^{12}$ with different spatial polynomial approximation orders N under various values of the parameter θ in Example 4.5.

	$N = 11$	$N = 13$	$N = 15$	$N = 17$	$N = 19$	$N = 21$
$\theta = 0.3$	281.6302	523.8291	701.6348	1157.9560	1443.1330	2870.2160
$\theta = 0.5$	315.7960	554.1653	736.7113	1272.1050	1431.5850	2082.3110
$\theta = 0.8$	411.6289	550.7774	617.9113	1338.7970	1737.7100	1908.8530

Finally, all of the aforementioned results are consistent with those obtained for the linear optimal control problem. The nonlinear examples further demonstrate that the proposed discretization, combined with a standard fixed-point iteration, is capable of handling nonlinear optimal control problems in practice. Nevertheless, these examples are intended only as preliminary numerical illustrations, highlighting the potential applicability of the proposed approach beyond the linear setting. A rigorous theoretical treatment of the nonlinear case remains a challenging direction for future research.

In summary, all numerical examples verify the error estimates derived from the theoretical analysis. In the two-dimensional case, the computational cost is estimated to be $O(K(N_1N_2)^3+K^2(N_1N_2)^2)$, where N_1 and N_2 denote the polynomial degrees in the x_1 and x_2 directions, respectively. Tables 2 and 3 reveal that achieving the same absolute error precision requires less CPU time with our method compared to the L1 scheme, confirming its superior efficiency. Therefore, for the purpose of acquiring results quickly under predetermined accuracy requirements, opting for a high-order scheme is recommended. The proposed fully discrete scheme based on the L2 approximation achieves the expected temporal convergence order of $3-\theta$, and the Legendre spectral Galerkin approximation exhibits spectral accuracy in space. These results demonstrate that the proposed method is an effective approach for solving the linear time-fractional optimal control problem.

5. Concluding remarks

In this work, a uniformly accurate high-order temporal finite difference scheme and spatial Legendre-Galerkin method is developed for the efficient and accurate solution of a type of OCP governed by time-fractional spatial fourth-order partial differential equations. The proposed schemes are based on the spatial discretization on the Legendre-Galerkin method and temporal discretization on uniform-temporal-accuracy high-order finite difference by using an L2 scheme approximate to the time-fractional derivative. Our approaches resolve the numerical challenge due to the problem of theoretical convergence order reduction using low-order numerical schemes in the first time layer by using block-by-block techniques to achieve high-precision time convergence order only for the first two time layers. The first-order optimality condition for the time-fractional spatial fourth-order OCP is rigorously theoretically derived. Based on the first-order optimality condition of the OCP combined with time finite difference discretization and spatial spectral method discretization, a fully discrete numerical scheme is established to solve the problem. The convergence of the OCP is strictly proved by using a first-order optimal condition and three proposed auxiliary equations.

In future work, we will introduce fast methods to improve the computational efficiency of the presented scheme. Furthermore, we will use the numerical scheme proposed in this article to solve the OCP of time-fractional Cahn-Hilliard equation.

Author contributions

Guiying Jiang: Investigation, formal analysis, writing—original draft, resources, data curation, and validation; Ziqiang Wang: Investigation, formal analysis, editing and writing—review; Junying Cao: Methodology, project administration, validation, data curation, and formal funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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Appendix A: Proof of Lemma 2.4

Proof. First, analyze the situation of $k = K-1, K-2$:

$$\bar{E}_0(P_N^{K-2}, \bar{v}_N) + \bar{E}_1(P_N^{K-1}, \bar{v}_N) + \bar{E}_2(P_N^{K}, \bar{v}_N) + \alpha_0(\Delta P_N^{K-1}, \Delta\bar{v}_N) = \alpha_0(U_N^{K-1} - u_d^{K-1}, \bar{v}_N), \tag{A.1}$$

$$E_0(P_N^{K-2}, \bar{v}_N) + E_1(P_N^{K-1}, \bar{v}_N) + E_2(P_N^{K}, \bar{v}_N) + \alpha_0(\Delta P_N^{K-2}, \Delta\bar{v}_N) = \alpha_0(U_N^{K-2} - u_d^{K-2}, \bar{v}_N). \tag{A.2}$$

Let $\bar{v}_N = -E_1 P_N^{K-1}$ in (A.1) and $\bar{v}_N = \bar{E}_0 P_N^{K-2}$ in (A.2). Adding them, we have

$$\begin{aligned}
&-\bar{E}_1E_1\left\|P_N^{K-1}\right\|_0^2 + \bar{E}_0E_0\left\|P_N^{K-2}\right\|_0^2 - \alpha_0E_1\left\|\Delta P_N^{K-1}\right\|_0^2 + \alpha_0\bar{E}_0\left\|\Delta P_N^{K-2}\right\|_0^2\\
=&(-\alpha_0E_1(U_N^{K-1} - u_d^{K-1}) + E_1\bar{E}_2P_N^K, P_N^{K-1}) + (\alpha_0\bar{E}_0(U_N^{K-2} - u_d^{K-2}) - \bar{E}_0E_2P_N^K, P_N^{K-2}).
\end{aligned}$$

By Young's inequality, we obtain

$$\begin{aligned}
&-\bar{E}_1E_1\|P_N^{K-1}\|_0^2 + \bar{E}_0E_0\|P_N^{K-2}\|_0^2 - \alpha_0E_1\|\Delta P_N^{K-1}\|_0^2 + \alpha_0\bar{E}_0\|\Delta P_N^{K-2}\|_0^2\\
\le&\frac{(-\alpha_0E_1)^2}{2\kappa_1}\|U_N^{K-1} - u_d^{K-1}\|_0^2 + \frac{\kappa_1}{2}\|P_N^{K-1}\|_0^2 + \frac{(\alpha_0\bar{E}_0)^2}{2\kappa_2}\|U_N^{K-2} - u_d^{K-2}\|_0^2 + \frac{\kappa_2}{2}\|P_N^{K-2}\|_0^2\\
&+\frac{(E_1\bar{E}_2)^2}{2\kappa_1}\|P_N^K\|_0^2 + \frac{\kappa_1}{2}\|P_N^{K-1}\|_0^2 + \frac{(-\bar{E}_0E_2)^2}{2\kappa_2}\|P_N^K\|_0^2 + \frac{\kappa_2}{2}\|P_N^{K-2}\|_0^2.
\end{aligned}$$

Letting $\kappa_1 = -E_1\bar{E}_1/2$, $\kappa_2 = \bar{E}_0E_0/2$, we can obtain

$$\begin{aligned}
&-\bar{E}_1E_1\|P_N^{K-1}\|_0^2 + \bar{E}_0E_0\|P_N^{K-2}\|_0^2 - 2\alpha_0E_1\|\Delta P_N^{K-1}\|_0^2 + 2\alpha_0\bar{E}_0\|\Delta P_N^{K-2}\|_0^2\\
\le&2\Big[\frac{(\bar{E}_2E_1)^2}{-\bar{E}_1E_1} + \frac{(\bar{E}_0E_2)^2}{\bar{E}_0E_0}\Big]\|P_N^K\|_0^2 + 2\frac{(-\alpha_0E_1)^2}{-\bar{E}_1E_1}\|U_N^{K-1} - u_d^{K-1}\|_0^2 + 2\frac{(\alpha_0\bar{E}_0)^2}{\bar{E}_0E_0}\|U_N^{K-2} - u_d^{K-2}\|_0^2.
\end{aligned}\tag{A.3}$$

Because $-\bar{E}_1E_1 > 0$, $\bar{E}_0E_0 > 0$, $-2\alpha_0E_1 > 0$, $2\alpha_0\bar{E}_0 > 0$, we have

$$\begin{aligned} &-\bar{E}_1E_1\|P_N^{K-1}\|_0^2 - 2\alpha_0E_1\|\Delta P_N^{K-1}\|_0^2 \\ \leq& 2\Big[\frac{(\bar{E}_2E_1)^2}{-\bar{E}_1E_1} + \frac{(\bar{E}_0E_2)^2}{\bar{E}_0E_0}\Big]\|P_N^K\|_0^2 + 2\frac{(-\alpha_0E_1)^2}{-\bar{E}_1E_1}\|U_N^{K-1} - u_d^{K-1}\|_0^2 + 2\frac{(\alpha_0\bar{E}_0)^2}{\bar{E}_0E_0}\|U_N^{K-2} - u_d^{K-2}\|_0^2. \end{aligned} \tag{A.4}$$

By dividing both sides of the above inequality (A.4) by $-\bar{E}_1E_1$, we obtain

$$\|P_N^{K-1}\|_0^2 + \frac{2\alpha_0}{\bar{E}_1}\|\Delta P_N^{K-1}\|_0^2 \leq A_1\|P_N^K\|_0^2 + B_1[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\|U_N^i - u_d^i\|_0^2, \tag{A.5}$$

where

$$\begin{aligned} A_1 &= 2\Big(\frac{\bar{E}_2^2}{\bar{E}_1^2} - \frac{\bar{E}_0E_2^2}{\bar{E}_1E_0E_1}\Big) = \frac{15\theta^2 - 48\theta + 36}{8(\theta^3 - 3\theta + 2)}, \\ B_1 &= 2\Big(\frac{1}{\bar{E}_1^2} - \frac{\bar{E}_0}{\bar{E}_1E_0E_1}\Big) = \Big\{(2-\theta)^2\Big[0.5 + \frac{2^{2\theta-3}(1-\theta)}{\theta+2}\Big]\Big\}. \end{aligned} \tag{A.6}$$

Because $\beta_0 \in (3/2, 2)$, $\beta_0^{-1} \in (1/2, 2/3)$, $2/\bar{E}_1 = 1/(1-\theta) \in (1, +\infty)$, we have $\beta_0^{-1} < 2/\bar{E}_1$. Arranging the inequality (A.5), we can get

$$\begin{aligned} &\|P_N^{K-1}\|_0^2 + \alpha_0\beta^{-1}\|\Delta P_N^{K-1}\|_0^2 < \|P_N^{K-1}\|_0^2 + \frac{2\alpha_0}{\bar{E}_1}\|\Delta P_N^{K-1}\|_0^2 \\ \leq& A_1\|P_N^K\|_0^2 + B_1[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\|U_N^i - u_d^i\|_0^2. \end{aligned} \tag{A.7}$$

Therefore, $\bar{P}_N^{K-1} = P_N^{K-1} - gP_N^K$, and $0 < g < 2/3$, and we obtain

$$\begin{aligned} &\left\|\bar{P}_N^{K-1}\right\|_0^2 + \alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2 = \left\|P_N^{K-1} - gP_N^K\right\|_0^2 + \alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2 \\ &= (P_N^{K-1}, P_N^{K-1}) - g(P_N^{K-1}, P_N^K) - g(P_N^K, P_N^{K-1}) + g^2\|P_N^K\|_0^2 + \alpha_0\beta_0^{-1}\|\Delta P_N^{K-1}\|_0^2 \\ &\leq \|P_N^{K-1}\|_0^2 + 2g\|P_N^{K-1}\|_0\|P_N^K\|_0 + g^2\|P_N^K\|_0^2 + \alpha_0\beta_0^{-1}\|\Delta P_N^{K-1}\|_0^2 \\ &\leq 2\Big(\left\|P_N^{K-1}\right\|_0^2 + g^2\left\|P_N^K\right\|_0^2\Big) + \alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2 \\ &\leq 2\Big(\left\|P_N^{K-1}\right\|_0^2 + \alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2\Big) + 2g^2\left\|P_N^K\right\|_0^2 \\ &\leq 2(A_1 + g^2)\left\|P_N^K\right\|_0^2 + 2B_1[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\left\|U_N^i - u_d^i\right\|_0^2. \end{aligned} \tag{A.8}$$

Similarly, it can be seen from (A.3) that

$$\begin{aligned} &\bar{E}_0E_0\|P_N^{K-2}\|_0^2 + 2\alpha_0\bar{E}_0\|\Delta P_N^{K-2}\|_0^2 \\ \leq& 2\Big[\frac{(\bar{E}_2E_1)^2}{-\bar{E}_1E_1} + \frac{(\bar{E}_0E_2)^2}{\bar{E}_0E_0}\Big]\|P_N^K\|_0^2 + 2\frac{(-\alpha_0E_1)^2}{-\bar{E}_1E_1}\|U_N^{K-1} - u_d^{K-1}\|_0^2 + 2\frac{(\alpha_0\bar{E}_0)^2}{\bar{E}_0E_0}\|U_N^{K-2} - u_d^{K-2}\|_0^2. \end{aligned}$$

By dividing both sides of the above inequality by $\bar{E}_0E_0$, we get

$$\|P_N^{K-2}\|_0^2 + \frac{2\alpha_0}{E_0}\|\Delta P_N^{K-2}\|_0^2 \leq A_2\|P_N^K\|_0^2 + B_2[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\|U_N^i - u_d^i\|_0^2, \tag{A.9}$$

where

$$A_2=2\Big(\frac{E_1\bar{E}_2^2}{-E_0\bar{E}_1\bar{E}_0}+\frac{E_2^2}{E_0^2}\Big)=\frac{(15\theta-18)(2\theta-4)}{4-\theta^3-3\theta^2},$$
$$B_2=2\Big(\frac{E_1}{-E_0\bar{E}_1\bar{E}_0}+\frac{1}{E_0^2}\Big)=\Big\{\frac{8(1-\theta)(2-\theta)^2}{2+\theta}+\frac{2^{2\theta+1}(1-\theta)^2(2-\theta)^2}{(2+\theta)^2}\Big\}. \tag{A.10}$$

According to $2/E_0=2^{\theta+1}/(2+\theta)\in(1,4/3)$, we get $\beta_0^{-1}<2/E_0$. Arranging the inequality (A.9), we can get

$$\|P_N^{K-2}\|_0^2+\alpha\beta^{-1}\|\Delta P_N^{K-2}\|_0^2\le A_2\|P_N^K\|_0^2+B_2[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\|U_N^i-u_d^i\|_0^2. \tag{A.11}$$

Therefore, $\bar{P}_N^{K-2}=P_N^{K-2}-gP_N^{K-1}$, and $0<g<2/3$, and utilizing (A.7) and (A.11), we obtain

$$\begin{aligned}
&\left\|\bar{P}_N^{K-2}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-2}\right\|_0^2=\left\|P_N^{K-2}-gP_N^{K-1}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-2}\right\|_0^2\\
&=(P_N^{K-2},P_N^{K-2})-g(P_N^{K-2},P_N^{K-1})-g(P_N^{K-1},P_N^{K-2})+g^2\|P_N^{K-1}\|_0^2+\alpha_0\beta_0^{-1}\|\Delta P_N^{K-2}\|_0^2\\
&\le\|P_N^{K-2}\|_0^2+2g\|P_N^{K-2}\|_0\|P_N^{K-1}\|_0+g^2\|P_N^{K-1}\|_0^2+\alpha_0\beta_0^{-1}\|\Delta P_N^{K-2}\|_0^2\\
&\le 2\Big(\left\|P_N^{K-2}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-2}\right\|_0^2\Big)+2g^2\Big(\left\|P_N^{K-1}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2\Big)\\
&\le 2(A_2+g^2A_1)\left\|P_N^K\right\|_0^2+2(B_2+g^2B_1)[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\left\|U_N^i-u_d^i\right\|_0^2.
\end{aligned} \tag{A.12}$$

According to (A.8) and (A.12), we have

$$\left\|\bar{P}_N^{K-1}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2\le S_1\left\|P_N^K\right\|_0^2+S_2[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\left\|U_N^i-u_d^i\right\|_0^2, \tag{A.13}$$
$$\left\|\bar{P}_N^{K-2}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-2}\right\|_0^2\le S_1\left\|P_N^K\right\|_0^2+S_2[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2}\left\|U_N^i-u_d^i\right\|_0^2, \tag{A.14}$$

where $S_1=\max\{2(A_1+g^2),2(A_2+g^2A_1)\}$, and S_2 is defined as in (2.12).

When $k=K-3$, letting $\bar{v}_N=2\bar{P}_N^{k-3}$ in the third row of (2.11), we get

$$2\left\|\bar{P}_N^{K-3}\right\|_0^2+2\alpha_0\beta_0^{-1}(\Delta P_N^{K-3},\Delta\bar{P}_N^{K-3})=2\sum_{s=K-2}^{K}\bar{d}_s^{K-3}(\bar{P}_N^s,\bar{P}_N^{K-3})+2\alpha_0\beta_0^{-1}(U_N^{K-3}-u_d^{K-3},\bar{P}_N^{K-3}). \tag{A.15}$$

Using (2.10), we derive $2(\Delta P_N^k,\Delta\bar{P}_N^k)$ as follows:

$$\begin{aligned}
&2(\Delta P_N^k,\Delta\bar{P}_N^k)=(\Delta P_N^k+\Delta P_N^k,\Delta\bar{P}_N^k)=(\Delta P_N^k+g\Delta P_N^{k+1},\Delta P_N^k-g\Delta P_N^{k+1})+(\Delta\bar{P}_N^k,\Delta\bar{P}_N^k)\\
&=(\Delta P_N^k,\Delta\bar{P}_N^k)-g^2(\Delta P_N^{k+1},\Delta P_N^{k+1})+(\Delta\bar{P}_N^k,\Delta\bar{P}_N^k)=\left\|\Delta P_N^k\right\|_0^2+\left\|\Delta\bar{P}_N^k\right\|_0^2-g^2\left\|\Delta P_N^{k+1}\right\|_0^2.
\end{aligned} \tag{A.16}$$

Therefore, $2(\Delta P_N^{K-3},\Delta\bar{P}_N^{K-3})=\left\|\Delta P_N^{K-3}\right\|_0^2+\left\|\Delta\bar{P}_N^{K-3}\right\|_0^2-g^2\left\|\Delta P_N^{K-2}\right\|_0^2$, and by (A.15), the integration by parts, we have

$$\begin{aligned}
&2\left\|\bar{P}_N^{K-3}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-3}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta\bar{P}_N^{K-3}\right\|_0^2-\alpha_0\beta_0^{-1}g^2\left\|\Delta P_N^{K-2}\right\|_0^2\\
&=2\sum_{s=K-2}^{K}\bar{d}_s^{K-3}(\bar{P}_N^s,\bar{P}_N^{K-3})+2\alpha_0\beta_0^{-1}((\Delta)^{-1}(U_N^{K-3}-u_d^{K-3}),\Delta\bar{P}_N^{K-3}),
\end{aligned}$$

where $(\Delta)^{-1}$ is the inverse operator corresponding to Δ. Applying the basic inequality, we can derive

$$\begin{aligned}&2\left\|\bar{P}_N^{K-3}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-3}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta\bar{P}_N^{K-3}\right\|_0^2-\alpha_0\beta_0^{-1}g^2\left\|\Delta P_N^{K-2}\right\|_0^2\\&\leq\sum_{s=K-2}^{K}\bar{d}_s^{K-3}\Big(\left\|\bar{P}_N^{s}\right\|_0^2+\left\|\bar{P}_N^{K-3}\right\|_0^2\Big)+\alpha_0\beta_0^{-1}\left\|(\Delta)^{-1}(U_N^{K-3}-u_d^{K-3})\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta\bar{P}_N^{K-3}\right\|_0^2.\end{aligned}\tag{A.17}$$

The term $\alpha_0\beta_0^{-1}\left\|\Delta\bar{P}_N^{K-3}\right\|_0^2$ at both ends are cancelled out. Using (1), (6), and (7) in Lemma 2.3, (A.17) can be organized as

$$\begin{aligned}&\left\|\bar{P}_N^{K-3}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-3}\right\|_0^2\\&\leq\alpha_0\beta_0^{-1}g^2\left\|\Delta P_N^{K-2}\right\|_0^2+\bar{d}_{K-2}^{K-3}\left\|P_N^{K-2}\right\|_0^2+\bar{d}_{K-1}^{K-3}\left\|P_N^{K-1}\right\|_0^2+\bar{d}_K^{K-3}\left\|P_N^{K}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|(\Delta)^{-1}(U_N^{K-3}-u_d^{K-3})\right\|_0^2\\&\leq g(\left\|P_N^{K-2}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-2}\right\|_0^2)+\bar{d}_{K-1}^{K-3}(\left\|P_N^{K-1}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2)\\&\quad+\bar{d}_K^{K-3}(\left\|P_N^{K}\right\|_0^2+\frac{\alpha_0}{\bar{d}_K^{K-3}\beta_0}\left\|(\Delta)^{-1}(U_N^{K-3}-u_d^{K-3})\right\|_0^2).\end{aligned}$$

According to (3), (4) in Lemma 2.3 and (1) in Lemma 2.2, we get

$$\frac{\alpha_0}{\bar{d}_K^{K-3}\beta_0}\leq\frac{3\alpha_0(K-k)^\theta}{2(1-\theta)(2-\theta)\beta_0}\leq\frac{3\Gamma(1-\theta)T^\theta}{2\beta_0},\quad g+\bar{d}_{K-1}^{K-3}+\bar{d}_K^{K-3}\leq1,\tag{A.18}$$

so

$$\begin{aligned}&\left\|\bar{P}_N^{K-3}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-3}\right\|_0^2\leq g(\left\|P_N^{K-2}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-2}\right\|_0^2)+\bar{d}_{K-1}^{K-3}(\left\|P_N^{K-1}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2)\\&\quad+\bar{d}_K^{K-3}\Big(\left\|P_N^{K}\right\|_0^2+\frac{3\Gamma(1-\theta)T^\theta}{2\beta_0}\left\|(\Delta)^{-1}(U_N^{K-3}-u_d^{K-3})\right\|_0^2\Big)\\&\leq g(\left\|P_N^{K-2}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-2}\right\|_0^2)+\bar{d}_{K-1}^{K-3}(\left\|P_N^{K-1}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2)\\&\quad+\bar{d}_K^{K-3}\Big(\left\|P_N^{K}\right\|_0^2+\frac{3\Gamma(1-\theta)T^\theta\bar{C}_u}{2\beta_0}\left\|(U_N^{K-3}-u_d^{K-3})\right\|_0^2\Big)\\&\leq g(\left\|P_N^{K-2}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-2}\right\|_0^2)+\bar{d}_{K-1}^{K-3}(\left\|P_N^{K-1}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{K-1}\right\|_0^2)\\&\quad+\bar{d}_K^{K-3}\Big(\left\|P_N^{K}\right\|_0^2+\bar{C}_u\Gamma(1-\theta)T^{2\theta}\left\|(U_N^{K-3}-u_d^{K-3})\right\|_0^2\Big),\\&\leq S_1\left\|P_N^{K}\right\|_0^2+S_2\bar{C}_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=K-1,K-2,K-3}\left\|U_N^i-u_d^i\right\|_0^2,\end{aligned}$$

where $\bar{C}_u>0$ and its value depend on the function u.

When $k\leq K-4$, letting $\bar{v}_N=2\bar{P}_N^k$ into the fourth line of (2.11), we get

$$2\left\|\bar{P}_N^{k}\right\|_0^2+2\alpha_0\beta_0^{-1}(\Delta P_N^k,\Delta\bar{P}_N^k)=2g(\bar{P}_N^{k+1},\bar{P}_N^k)+2\sum_{s=k+2}^{K-1}\bar{d}_s^k(\bar{P}_N^s,\bar{P}_N^k)+2\bar{d}_K^k(P_N^K,\bar{P}_N^k)+2\alpha_0\beta_0^{-1}(U_N^k-u_d^k,\bar{P}_N^k).$$

According to (A.16) and integration by parts, we have

$$\begin{aligned}&2\left\|\bar{P}_N^{k}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{k}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta\bar{P}_N^{k}\right\|_0^2-\alpha_0\beta_0^{-1}g^2\left\|\Delta P_N^{k+1}\right\|_0^2\\&=2g(\bar{P}_N^{k+1},\bar{P}_N^k)+2\sum_{s=k+2}^{K-1}\bar{d}_s^k(\bar{P}_N^s,\bar{P}_N^k)+2\bar{d}_K^k(P_N^K,\bar{P}_N^k)+2\alpha_0\beta_0^{-1}((\Delta)^{-1}(U_N^k-u_d^k),\Delta\bar{P}_N^k).\end{aligned}$$

Applying the basic inequality, we have

$$\begin{aligned}
&2\left\|\bar{P}_N^k\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^k\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta\bar{P}_N^k\right\|_0^2-\alpha_0\beta_0^{-1}g^2\left\|\Delta P_N^{k+1}\right\|_0^2\\
&\le g\Big(\left\|\bar{P}_N^{k+1}\right\|_0^2+\left\|\bar{P}_N^k\right\|_0^2\Big)+\sum_{s=k+2}^{K-1}\bar{d}_s^k\Big(\left\|\bar{P}_N^s\right\|_0^2+\left\|\bar{P}_N^k\right\|_0^2\Big)+\bar{d}_K^k\Big(\left\|P_N^K\right\|_0^2+\left\|\bar{P}_N^k\right\|_0^2\Big)\\
&\quad+\alpha_0\beta_0^{-1}\left\|(\Delta)^{-1}(U_N^k-u_d^k)\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta\bar{P}_N^k\right\|_0^2.
\end{aligned}\tag{A.19}$$

By rearranging (A.19), we immediately obtain

$$\begin{aligned}
&2\left\|\bar{P}_N^k\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^k\right\|_0^2-\alpha_0\beta_0^{-1}g^2\left\|\Delta P_N^{k+1}\right\|_0^2\\
\le& g\left\|\bar{P}_N^{k+1}\right\|_0^2+\Big(g+\bar{d}_K^k+\sum_{s=k+2}^{K-1}\bar{d}_s^k\Big)\left\|\bar{P}_N^k\right\|_0^2+\sum_{s=k+2}^{K-1}\bar{d}_s^k\left\|\bar{P}_N^s\right\|_0^2+\bar{d}_K^k\left\|P_N^K\right\|_0^2+\alpha_0\beta_0^{-1}\left\|(\Delta)^{-1}(U_N^k-u_d^k)\right\|_0^2.
\end{aligned}$$

From (3) in Lemma 2.3, we have $g+\sum_{s=k+2}^{K-1}\bar{d}_s^k+\bar{d}_K^k\le 1$, and then,

$$\begin{aligned}
&\left\|\bar{P}_N^k\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^k\right\|_0^2-\alpha_0\beta_0^{-1}g^2\left\|\Delta P_N^{k+1}\right\|_0^2\\
\le& g\left\|\bar{P}_N^{k+1}\right\|_0^2+\sum_{s=k+2}^{K-1}\bar{d}_s^k\left\|\bar{P}_N^s\right\|_0^2+\bar{d}_K^k\left\|P_N^K\right\|_0^2+\alpha_0\beta_0^{-1}\left\|(\Delta)^{-1}(U_N^k-u_d^k)\right\|_0^2.
\end{aligned}$$

Similarly, by (A.18), we get

$$\begin{aligned}
&\left\|\bar{P}_N^k\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^k\right\|_0^2\\
&\le g\Big(\left\|\bar{P}_N^{k+1}\right\|_0^2+\alpha_0\beta_0^{-1}g\left\|\Delta P_N^{k+1}\right\|_0^2\Big)+\sum_{s=k+2}^{K-1}\bar{d}_s^k\left\|\bar{P}_N^s\right\|_0^2+\bar{d}_K^k\left\|P_N^K\right\|_0^2+\alpha_0\beta_0^{-1}\left\|(\Delta)^{-1}(U_N^k-u_d^k)\right\|_0^2\\
&\le g\Big(\left\|\bar{P}_N^{k+1}\right\|_0^2+\alpha_0\beta_0^{-1}g\left\|\Delta P_N^{k+1}\right\|_0^2\Big)+\sum_{s=k+2}^{K-1}\bar{d}_s^k\Big(\left\|\bar{P}_N^s\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^s\right\|_0^2\Big)\\
&\quad+\bar{d}_K^k\Big(\left\|P_N^K\right\|_0^2+\frac{\alpha_0}{\bar{d}_K^k\beta_0}\left\|(\Delta)^{-1}(U_N^k-u_d^k)\right\|_0^2\Big)\\
&\le g\Big(\left\|\bar{P}_N^{k+1}\right\|_0^2+\alpha_0\beta_0^{-1}g\left\|\Delta P_N^{k+1}\right\|_0^2\Big)+\sum_{s=k+2}^{K-1}\bar{d}_s^k\Big(\left\|\bar{P}_N^s\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^s\right\|_0^2\Big)\\
&\quad+\bar{d}_K^k\Big(\left\|P_N^K\right\|_0^2+\frac{3\alpha_0(K-k)^\theta}{2(1-\theta)(2-\theta)\beta_0}\left\|(\Delta)^{-1}(U_N^k-u_d^k)\right\|_0^2\Big)\\
&\le g\Big(\left\|\bar{P}_N^{k+1}\right\|_0^2+\alpha_0\beta_0^{-1}g\left\|\Delta P_N^{k+1}\right\|_0^2\Big)+\sum_{s=k+2}^{K-1}\bar{d}_s^k\Big(\left\|\bar{P}_N^s\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^s\right\|_0^2\Big)\\
&\quad+\bar{d}_K^k\Big(\left\|P_N^K\right\|_0^2+\frac{3\Gamma(1-\theta)T^\theta}{2\beta_0}\left\|(\Delta)^{-1}(U_N^k-u_d^k)\right\|_0^2\Big)\\
&\le g\Big(\left\|\bar{P}_N^{k+1}\right\|_0^2+\alpha_0\beta_0^{-1}g\left\|\Delta P_N^{k+1}\right\|_0^2\Big)+\sum_{s=k+2}^{K-1}\bar{d}_s^k\Big(\left\|\bar{P}_N^s\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^s\right\|_0^2\Big)
\end{aligned}\tag{A.20}$$

$$
\begin{aligned}
&+\bar{d}_K^k\Big(\left\|P_N^K\right\|_0^2+\frac{3\Gamma(1-\theta)T^{\theta}\bar{C}_u}{2\beta_0}\left\|U_N^k-u_d^k\right\|_0^2\Big)\\
&\leq g\Big(\left\|\bar{P}_N^{k+1}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^{k+1}\right\|_0^2\Big)+\sum_{s=k+2}^{K-1}\bar{d}_s^k\Big(\left\|\bar{P}_N^s\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^s\right\|_0^2\Big)\\
&+\bar{d}_K^k\Big(\left\|P_N^K\right\|_0^2+\bar{C}_u\Gamma(1-\theta)T^{2\theta}\left\|U_N^k-u_d^k\right\|_0^2\Big).
\end{aligned}
$$

Next, we prove the estimated value by employing mathematical induction:

$$
\left\|\bar{P}_N^k\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^k\right\|_0^2\leq S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|(U_N^i-u_d^i)\right\|_0^2, \tag{A.21}
$$

where $C_u=\max\{1,\bar{C}_u\}$.

According to (A.13), (A.14), and (A.19)–(A.21), the solution is obviously valid for $k=K-1,K-2,K-3$, and $K-4$. Suppose that (A.21) is also true when $j=K-5,K-6,\cdots,1$:

$$
\|\bar{P}_N^j\|_0^2+\alpha_0\beta_0^{-1}\|\Delta P_N^j\|_0^2\leq S_1\left\|P_N^K\right\|_0^2+S_2C_u\left[\Gamma(1-\theta)\right]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2,\quad 1\leq j\leq K-1. \tag{A.22}
$$

From Eq (A.22), we can draw the conclusion that

$$
\begin{aligned}
&\left\|\bar{P}_N^0\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^0\right\|_0^2\\
&\leq g\Big(\left\|\bar{P}_N^1\right\|_0^2+\alpha_0\beta_0^{-1}g\left\|\Delta P_N^1\right\|_0^2\Big)+\sum_{s=2}^{K-1}\bar{d}_s^0\Big(\left\|\bar{P}_N^s\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta P_N^s\right\|_0^2\Big)+\bar{d}_K^0\Big(\left\|P_N^K\right\|_0^2+\frac{\alpha_0}{\bar{d}_K^0\beta_0}\left\|(\Delta)^{-1}(U_N^0-u_d^0)\right\|_0^2\Big)\\
&\leq\Big(g+\sum_{s=2}^{K-1}\bar{d}_s^0+\bar{d}_K^0\Big)\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big).
\end{aligned}
$$

Therefore, based on the principle of mathematical induction, it can be concluded that when $0\leq k\leq K-1$, both Eq (A.21) hold strictly.

Finally, we estimate the value of $\|P_N^k\|_0$. According to Eq (A.21), we can conclude that

$$
\begin{aligned}
&\sqrt{\alpha_0\beta_0^{-1}}\left\|\Delta P_N^k\right\|_0\leq\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}},\\
&\left\|\bar{P}_N^k\right\|_0\leq\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}.
\end{aligned} \tag{A.23}
$$

Using (A.23) and the triangle inequality,

$$
\begin{aligned}
&\left\|P_N^k\right\|_0=\left\|\bar{P}_N^k+gP_N^{k+1}\right\|_0\\
&\leq g\left\|P_N^{k+1}\right\|_0+\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}\\
&\leq g\left[g\left\|P_N^{k+2}\right\|_0+\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}\right]\\
&\quad+\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}\leq\cdots\\
&\leq g^{K-k}\left\|P_N^K\right\|_0+(1+g+\cdots+g^{K-k-2}+g^{K-k-1})\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq g^{K-k}\left\|P_N^K\right\|_0+\frac{1-g^{K-k}}{1-g}\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}$$
$$\leq g^{K-k}\left\|P_N^K\right\|_0+3\Big(S_1\left\|P_N^K\right\|_0^2+S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}.$$

Because $P_N^K=0$, we conclude that

$$\left\|P_N^k\right\|_0+\sqrt{\alpha_0\beta_0^{-1}}\|\Delta P_N^k\|_0\leq 4\Big(S_2C_u[\Gamma(1-\theta)]^2T^{2\theta}\max_{0\leq i\leq K-1}\left\|U_N^i-u_d^i\right\|_0^2\Big)^{\frac{1}{2}}, 0\leq k\leq K-1.$$

The proof of Lemma 2.4 has been completed. □

Appendix B: Proof of Lemma 3.1

Proof. Denote

$$P_N^k(u)-p(\mathbf{x},t_k)=p_N^k(u)-\mathbb{Q}_{2,N}^0p(\mathbf{x},t_k)+\mathbb{Q}_{2,N}^0p(\mathbf{x},t_k)-p(\mathbf{x},t_k)\doteq\lambda_{P_N(u)}^k+\varepsilon_p^k,$$

where $\lambda_{P_N(u)}^k=P_N^k(u)-\mathbb{Q}_{2,N}^0p(\mathbf{x},t_k),\varepsilon_p^k=\mathbb{Q}_{2,N}^0p(\mathbf{x},t_k)-p(\mathbf{x},t_k)$, for $k=0,1,\cdots,K-1$.

When $k=K-1,K-2$, from the definition of R_τ^k and Lemma 2.1, $\forall\bar{v}_N\in Y_N$, we have

$$\sum_{i=0}^{2}\bar{E}_i(\lambda_{P_N(u)}^{K-2+i}+\varepsilon_p^{K-2+i},\bar{v}_N)+\alpha_0(\Delta\lambda_{P_N(u)}^{K-1},\Delta\bar{v}_N)=\alpha_0(R_\tau^{K-1},\bar{v}_N),\tag{B.1}$$
$$\sum_{i=0}^{2}E_i(\lambda_{P_N(u)}^{K-2+i}+\varepsilon_p^{K-2+i},\bar{v}_N)+\alpha_0(\Delta\lambda_{P_N(u)}^{K-2},\Delta\bar{v}_N)=\alpha_0(R_\tau^{K-2},\bar{v}_N).\tag{B.2}$$

Let $\bar{v}_N=-E_1\lambda_{P_N(u)}^{K-1}$ in (B.1) and $\bar{v}_N=\bar{E}_0\lambda_{P_N(u)}^{K-2}$ in (B.2). Add the two together and apply Young's inequality:

$$\begin{aligned}
&-\bar{E}_1E_1\left\|\lambda_{P_N(u)}^{K-1}\right\|_0^2+\bar{E}_0E_0\left\|\lambda_{P_N(u)}^{K-2}\right\|_0^2-\alpha_0E_1\left\|\Delta\lambda_{P_N(u)}^{K-1}\right\|_0^2+\alpha_0\bar{E}_0\left\|\Delta\lambda_{P_N(u)}^{K-2}\right\|_0^2\\
=\;&-\alpha_0E_1(R_\tau^{K-1},\lambda_{P_N(u)}^{K-1})+\alpha_0\bar{E}_0(R_\tau^{K-2},\lambda_{P_N(u)}^{K-2})+\bar{E}_0E_1(\varepsilon_p^{K-2},\lambda_{P_N(u)}^{K-1})\\
&-E_1\bar{E}_0(\varepsilon_p^{K-1},\lambda_{P_N(u)}^{K-2})+\bar{E}_1E_1(\varepsilon_p^{K-1},\lambda_{P_N(u)}^{K-1})-\bar{E}_0E_0(\varepsilon_p^{K-2},\lambda_{P_N(u)}^{K-2})+\bar{E}_2E_1(\lambda_{P_N(u)}^{K},\lambda_{P_N(u)}^{K-1})\\
&-E_2\bar{E}_0(\lambda_{P_N(u)}^{K},\lambda_{P_N(u)}^{K-2})+E_1\bar{E}_2(\varepsilon_p^{K},\lambda_{P_N(u)}^{K-1})-E_2\bar{E}_0(\varepsilon_p^{K},\lambda_{P_N(u)}^{K-2})\\
\leq\;&\frac{(-\alpha_0E_1)^2}{2\epsilon_1}\left\|R_\tau^{K-1}\right\|_0^2+\frac{(\alpha_0\bar{E}_0)^2}{2\epsilon_2}\left\|R_\tau^{K-2}\right\|_0^2+\frac{5\epsilon_1}{2}\left\|\lambda_{P_N(u)}^{K-1}\right\|_0^2+\frac{5\epsilon_2}{2}\left\|\lambda_{P_N(u)}^{K-2}\right\|_0^2\\
&+\Big[\frac{(\bar{E}_1E_1)^2}{2\epsilon_1}+\frac{(-E_1\bar{E}_0)^2}{2\epsilon_2}\Big]\left\|\varepsilon_p^{K-1}\right\|_0^2+\Big[\frac{(\bar{E}_0E_1)^2}{2\epsilon_1}+\frac{(-E_0\bar{E}_0)^2}{2\epsilon_2}\Big]\left\|\varepsilon_p^{K-2}\right\|_0^2\\
&+\Big[\frac{(\bar{E}_2E_1)^2}{2\epsilon_1}+\frac{(-E_2\bar{E}_0)^2}{2\epsilon_2}\Big](\left\|\lambda_{P_N(u)}^{K}\right\|_0^2+\left\|\varepsilon_p^{K}\right\|_0^2).
\end{aligned}\tag{B.3}$$

Let $\epsilon_1=-\bar{E}_1E_1/5$ and $\epsilon_2=\bar{E}_0E_0/5$ in (B.3), and we get

$$-\bar{E}_1E_1\left\|\lambda_{P_N(u)}^{K-1}\right\|_0^2+\bar{E}_0E_0\left\|\lambda_{P_N(u)}^{K-2}\right\|_0^2-2\alpha_0E_1\left\|\Delta\lambda_{P_N(u)}^{K-1}\right\|_0^2+2\alpha_0\bar{E}_0\left\|\Delta\lambda_{P_N(u)}^{K-2}\right\|_0^2$$

$$\leq \frac{5(-\alpha_0 E_1)^2}{-\bar{E}_1 E_1}\left\|R_\tau^{K-1}\right\|_0^2 + \frac{5(\alpha_0 \bar{E}_0)^2}{E_0 \bar{E}_0}\left\|R_\tau^{K-2}\right\|_0^2 + \frac{5(E_2^2 \bar{E}_0 \bar{E}_1 - \bar{E}_2^2 E_1 E_0)}{E_0 \bar{E}_1}(\left\|\lambda_{P_N(u)}^K\right\|_0^2 + \left\|\varepsilon_p^K\right\|_0^2) \quad (B.4)$$
$$+\frac{5(E_1^2 \bar{E}_0 - \bar{E}_1 E_0 E_1)}{E_0}\left\|\varepsilon_p^{K-1}\right\|_0^2 + \frac{5(E_0 \bar{E}_0 \bar{E}_1 - \bar{E}_0^2 E_1)}{\bar{E}_1}\left\|\varepsilon_p^{K-2}\right\|_0^2.$$

According to Lemmas 2.1 and 2.6, the inequality (B.4) becomes

$$\begin{aligned}
&\left\|\lambda_{P_N(u)}^{K-1}\right\|_0^2 + \alpha_0\beta_0^{-1}\left\|\Delta\lambda_{P_N(u)}^{K-1}\right\|_0^2 < \left\|\lambda_{P_N(u)}^{K-1}\right\|_0^2 + \frac{2\alpha_0}{\bar{E}_1}\left\|\Delta\lambda_{P_N(u)}^{K-1}\right\|_0^2 \\
&\leq \left(\frac{5\alpha_0^2}{\bar{E}_1^2} - \frac{5\alpha_0^2 \bar{E}_0}{E_0 \bar{E}_1 E_1}\right)\max_{i=K-1,K-2}\left\|R_\tau^i\right\|_0^2 + \frac{5(E_2^2 \bar{E}_0 \bar{E}_1 - \bar{E}_2^2 E_1 E_0)}{-E_1 E_0 \bar{E}_1^2}(\left\|\lambda_{P_N(u)}^K\right\|_0^2 + \left\|\varepsilon_p^K\right\|_0^2) \\
&\quad +\frac{5(E_1 \bar{E}_0 - \bar{E}_1 E_0)}{-\bar{E}_1 E_0}\left\|\varepsilon_p^{K-1}\right\|_0^2 + \frac{5(E_0 \bar{E}_0 \bar{E}_1 - \bar{E}_0^2 E_1)}{-\bar{E}_1^2 E_1}\left\|\varepsilon_p^{K-2}\right\|_0^2 \qquad (B.5) \\
&= \frac{5}{2}B_1[\Gamma(1-\theta)]^2 T^{2\theta}\max_{i=K-1,K-2}\left\|R_\tau^i\right\|_0^2 + \frac{5}{2}A_1(\left\|\varepsilon_p^K\right\|_0^2 + \left\|\lambda_{P_N(u)}^K\right\|_0^2) + D_1\left\|\varepsilon_p^{K-1}\right\|_0^2 + D_2\left\|\varepsilon_p^{K-2}\right\|_0^2 \\
&\leq C(\tau^{3-\theta} + N^{-l})^2,
\end{aligned}$$

where A_1, B_1, D_1, and D_2 are given in Theorem 2.2.

Based on the norm $|||\cdot|||_2$ in (2.8), (B.5), and Lemma 2.6, we have

$$\begin{aligned}
|||P_N^{K-1}(u) - p(\mathbf{x}, t_{K-1})|||_2 &\leq |||\lambda_{P_N(u)}^{K-1}|||_2 + |||\varepsilon_p^{K-1}|||_2 \leq \left\|\lambda_{P_N(u)}^{K-1}\right\|_2 + \left\|\varepsilon_p^{K-1}\right\|_2 \\
&\leq C(\tau^{3-\theta} + N^{-l}) + CN^{2-l} \leq C(\tau^{3-\theta} + N^{2-l}).
\end{aligned}$$

Similarly, according to (B.4), we get

$$\begin{aligned}
&\left\|\lambda_{P_N(u)}^{K-2}\right\|_0^2 + \alpha_0\beta_0^{-1}\left\|\Delta\lambda_{P_N(u)}^{K-2}\right\|_0^2 < \left\|\lambda_{P_N(u)}^{K-2}\right\|_0^2 + \frac{2\alpha_0}{E_0}\left\|\Delta\lambda_{P_N(u)}^{K-2}\right\|_0^2 \\
&\leq \left(\frac{5\alpha_0^2}{E_0^2} - \frac{5\alpha_0^2 E_1}{E_0 \bar{E}_1 \bar{E}_0}\right)\max_{i=K-1,K-2}\left\|R_\tau^i\right\|_0^2 + \frac{5(E_2^2 \bar{E}_0 \bar{E}_1 - \bar{E}_2^2 E_1 E_0)}{E_0^2 \bar{E}_1 \bar{E}_0}(\left\|\lambda_{P_N(u)}^K\right\|_0^2 + \left\|\varepsilon_p^K\right\|_0^2) \\
&\quad + \frac{5(E_1^2 \bar{E}_0 - \bar{E}_1 E_0 E_1)}{\bar{E}_0 E_0^2}\left\|\varepsilon_p^{K-1}\right\|_0^2 + \frac{5(E_0 \bar{E}_1 - \bar{E}_0 E_1)}{\bar{E}_1 E_0}\left\|\varepsilon_p^{K-2}\right\|_0^2 \\
&= \frac{5}{2}B_2[\Gamma(1-\theta)]^2 T^{2\theta}\max_{i=K-1,K-2}\left\|R_\tau^i\right\|_0^2 + \frac{5}{2}A_2(\left\|\varepsilon_p^K\right\|_0^2 + \left\|\lambda_{P_N(u)}^K\right\|_0^2) + D_3\left\|\varepsilon_p^{K-1}\right\|_0^2 + D_1\left\|\varepsilon_p^{K-2}\right\|_0^2 \\
&\leq C(\tau^{3-\theta} + N^{-l})^2,
\end{aligned}$$

where A_2, B_2, D_1, and D_3 are given in Theorem 2.2, so we get

$$\begin{aligned}
|||P_N^{K-2}(u) - p(\mathbf{x}, t_{K-2})|||_2 &\leq |||\lambda_{P_N(u)}^{K-2}|||_2 + |||\varepsilon_p^{K-2}|||_2 \leq \left\|\lambda_{P_N(u)}^{K-2}\right\|_2 + \left\|\varepsilon_p^{K-2}\right\|_2 \\
&\leq C(\tau^{3-\theta} + N^{-l}) + CN^{2-l} \leq C(\tau^{3-\theta} + N^{2-l}).
\end{aligned}$$

Similarly, with (B.1) and (B.2) for $k \leq K-3$ we obtain

$$(\lambda_{P_N(u)}^k + \varepsilon_p^k, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta\lambda_{P_N(u)}^k, \Delta\bar{v}_N) = \sum_{s=k+1}^{K} d_s^k(\lambda_{P_N(u)}^s + \varepsilon_p^s, \bar{v}_N) + \alpha_0\beta_0^{-1}(R_\tau^k, \bar{v}_N), \forall \bar{v}_N \in Y_N. \quad (B.6)$$

Based on Lemma 2.1 and (B.6), the following conclusion can be drawn:

$$\begin{aligned}
&(\lambda^k_{P_N(u)}, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta\lambda^k_{P_N(u)}, \Delta\bar{v}_N) \\
&= \sum_{s=k+1}^{K} d^k_s(\lambda^s_{P_N(u)}, \bar{v}_N) - \Big[(\varepsilon^k_p, \bar{v}_N) - \sum_{s=k+1}^{K} d^k_s(\varepsilon^k_p, \bar{v}_N) - \alpha_0\beta_0^{-1}(R^k_\tau, \bar{v}_N)\Big] \\
&= \sum_{s=k+1}^{K} d^k_s(\lambda^s_{P_N(u)}, \bar{v}_N) - (\mathbb{Q}^0_{2,N}p(\mathbf{x}, t_k) - p(\mathbf{x}, t_k), \bar{v}_N) + \sum_{s=k+1}^{K} d^k_s(\mathbb{Q}^0_{2,N}p(\mathbf{x}, t_s) - p(\mathbf{x}, t_s), \bar{v}_N) + \alpha_0\beta_0^{-1}(R^k_\tau, \bar{v}_N) \\
&= \sum_{s=k+1}^{K} d^k_s(\lambda^s_{P_N(u)}, \bar{v}_N) + (I - \mathbb{Q}^0_{2,N})(p(\mathbf{x}, t_k), \bar{v}_N) - (I - \mathbb{Q}^0_{2,N}) \sum_{s=k+1}^{K} d^k_s(p(\mathbf{x}, t_s), \bar{v}_N) + \alpha_0\beta_0^{-1}(R^k_\tau, \bar{v}_N) \qquad \text{(B.7)} \\
&= \sum_{s=k+1}^{K} d^k_s(\lambda^s_{P_N(u)}, \bar{v}_N) + (I - \mathbb{Q}^0_{2,N})(p(\mathbf{x}, t_k) - \sum_{s=k+1}^{K} d^k_s p(\mathbf{x}, t_s), \bar{v}_N) + \alpha_0\beta_0^{-1}(R^k_\tau, \bar{v}_N) \\
&= \sum_{s=k+1}^{K} d^k_s(\lambda^s_{P_N(u)}, \bar{v}_N) + (I - \mathbb{Q}^0_{2,N})(\alpha_0\beta_0^{-1}({}_tD^\theta_T p(\mathbf{x}, t_k) - R^k_\tau), \bar{v}_N) + \alpha_0\beta_0^{-1}(R^k_\tau, \bar{v}_N) \\
&= \sum_{s=k+1}^{K} d^k_s(\lambda^s_{P_N(u)}, \bar{v}_N) + \alpha_0\beta_0^{-1}((I - \mathbb{Q}^0_{2,N})({}_tD^\theta_T p(\mathbf{x}, t_k) - R^k_\tau) + R^k_\tau, \bar{v}_N) \\
&= \sum_{s=k+1}^{K} d^k_s(\lambda^s_{P_N(u)}, \bar{v}_N) + \alpha_0\beta_0^{-1}(\zeta^k_N p, \bar{v}_N), \quad \forall \bar{v}_N \in Y_N,
\end{aligned}$$

where $\zeta^k_N p = (I - \mathbb{Q}^0_{2,N})({}_tD^\theta_T p(\mathbf{x}, t_k) - R^k_\tau) + R^k_\tau$, I is the identity operator, and R^k_τ is given in Lemma 2.1.

Using the triangle inequality leads to

$$\left\|\zeta^k_N p\right\|_0 \le \left\|(I - \mathbb{Q}^0_{2,N})_t D^\theta_T p(\mathbf{x}, t_k)\right\|_0 + \left\|(I - \mathbb{Q}^0_{2,N})R^k_\tau\right\|_0 + \left\|R^k_\tau\right\|_0.$$

According to Lemma 2.1, we can draw a conclusion: $\left\|R^k_\tau\right\|_0 \le C\tau^{3-\theta}$, and according to Lemma 2.6,

$$\left\|(I - \mathbb{Q}^0_{2,N})\ {}_tD^\theta_T p(\mathbf{x}, t_k)\right\|_0 \le CN^{-l}.$$

Combining Lemmas 2.1 and 2.6, we have $\left\|(I - \mathbb{Q}^0_{2,N})R^k_\tau\right\|_0 \le CN^{-l}\tau^{3-\theta}$. Thus, we have

$$\left\|\zeta^k_N p\right\|_0 \le C(\tau^{3-\theta} + N^{-l} + \tau^{3-\theta}N^{-l}).$$

Let $\hat{\lambda}^i_{P_N(u)} = \lambda^i_{P_N(u)} - g\lambda^{i+1}_{P_N(u)}$, $i = 0, 1, \cdots, K-1$, and the expression (B.7) can be rewritten as

$$\begin{aligned}
&(\hat{\lambda}^k_{P_N(u)}, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta\lambda^k_{P_N(u)}, \Delta\bar{v}_N) \\
=\ & g(\hat{\lambda}^{k+1}_{P_N(u)}, \bar{v}_N) + \textstyle\sum_{s=k+2}^{K-1} \bar{d}^k_s(\hat{\lambda}^s_{P_N(u)}, \bar{v}_N) + \bar{d}^k_K(\hat{\lambda}^K_{P_N(u)}, \bar{v}_N) + \alpha_0\beta_0^{-1}(\zeta^k_N p, \bar{v}_N), k \le K-4. \qquad \text{(B.8)}
\end{aligned}$$

According to (7) in Lemma 2.3, it can be deduced from (B.8) that the situation of $k = K-3$ also holds:

$$(\hat{\lambda}^{K-3}_{P_N(u)}, \bar{v}_N) + \alpha_0\beta_0^{-1}(\Delta\lambda^{K-3}_{P_N(u)}, \Delta\bar{v}_N) = (\sum_{s=k-2}^{K} \bar{d}^{K-3}_s \hat{\lambda}^s_{P_N(u)}, \bar{v}_N) + \alpha_0\beta_0^{-1}(\zeta^{K-3}_N p, \bar{v}_N).$$

By following a similar approach to Theorem 2.4, one can obtain

$$
\begin{aligned}
&|||\lambda^k_{P_N(u)}|||_2^2 = \left\|\lambda^k_{P_N(u)}\right\|_0^2 + \alpha_0\beta_0^{-1}\left\|\Delta\lambda^k_{P_N(u)}\right\|_0^2 \\
\leq & 16(S_1\left\|\lambda^K_{P_N(u)}\right\|_0^2 + S_2 C_u[\Gamma(1-\theta)]^2 T^{2\theta}\max_{0\leq i\leq K-1}\left\|\zeta^i_N p\right\|_0^2) \leq C(\tau^{3-\theta} + N^{-l} + \tau^{3-\theta}N^{-l})^2.
\end{aligned}
$$

Based on the above estimation and the norm $|||\cdot|||_2$, we arrive at

$$
\begin{aligned}
|||P^k_N(u) - p(\mathbf{x}, t_k)|||_2 \leq |||\lambda^k_{P_N(u)}|||_2 + |||\varepsilon^k_p|||_2 &\leq \left\|\lambda^k_{P_N(u)}\right\|_2 + \left\|\varepsilon^k_p\right\|_2 \\
&\leq C(\tau^{3-\theta} + N^{-l} + \tau^{3-\theta}N^{-l}) + CN^{2-l} \leq C(\tau^{3-\theta} + N^{2-l}),
\end{aligned}
$$

where the constant C has no relation to N and τ. Therefore, the proof of Lemma 3.1 has been completed. □

Appendix C: Proof of Lemma 3.2

Proof. Denote $U^k_N(v) - u(\mathbf{x}, t_k) = U^k_N(v) - \mathbb{Q}^0_{2,N}u(\mathbf{x}, t_k) + \mathbb{Q}^0_{2,N}u(\mathbf{x}, t_k) - u(\mathbf{x}, t_k) \doteq \lambda^k_{U_N(v)} + \varepsilon^k_u$, where $\lambda^k_{U_N(v)} = U^k_N(v) - \mathbb{Q}^0_{2,N}u(\mathbf{x}, t_k), \varepsilon^k_u = \mathbb{Q}^0_{2,N}u(\mathbf{x}, t_k) - u(\mathbf{x}, t_k)$, for $k = 1, 2, \cdots, K$. Similarly, with (2.9), the fully discrete format of the state equation is as follows:

$$
\begin{aligned}
&(\bar{A}_1 U^0_N(v), \bar{v}_N) + (\bar{B}_1 U^1_N(v), \bar{v}_N) + (\bar{C}_1 U^2_N(v), \bar{v}_N) + \alpha_0(\Delta U^1_N(v), \Delta\bar{v}_N) = \alpha_0(f^1 + v^1, \bar{v}_N),\ k = 1, \\
&(\bar{A}_2 U^0_N(v), \bar{v}_N) + (\bar{B}_2 U^1_N(v), \bar{v}_N) + (\bar{C}_2 U^2_N(v), \bar{v}_N) + \alpha_0(\Delta U^2_N(v), \Delta\bar{v}_N) = \alpha_0(f^2 + v^2, \bar{v}_N),\ k = 2, \\
&(U^k_N(v), \bar{v}_N(v)) + \alpha_0\beta_0^{-1}(\Delta U^k_N(v), \Delta\bar{v}_N) = \Big(\sum_{i=1}^{k} D^k_{k-i}U^{k-i}_N(v), \bar{v}_N\Big) + \alpha_0\beta_0^{-1}(f^k + v^k, \bar{v}_N),\ k \geq 3.
\end{aligned}
\tag{C.1}
$$

When $k = 1, 2$, from the definition of $\bar{R}^k_\tau$ and Lemma 2.1, $\forall\bar{v}_N \in Y_N$, we have

$$
\bar{A}_k(\lambda^0_{U_N(v)} + \varepsilon^0_u, \bar{v}_N) + \bar{B}_k(\lambda^1_{U_N(v)} + \varepsilon^1_u, \bar{v}_N) + \bar{C}_k(\lambda^2_{U_N(v)} + \varepsilon^2_u, \bar{v}_N) + \alpha_0(\Delta\lambda^k_{U_N(v)}, \Delta\bar{v}_N) = \alpha_0(\bar{R}^k_\tau, \bar{v}_N). \tag{C.2}
$$

Let $\bar{v}_N = -\bar{B}_2\lambda^1_{U_N(v)}$ in (C.2) for $k = 1$ and $\bar{v}_N = \bar{C}_1\lambda^2_{U_N(v)}$ in (C.2) for $k = 2$. Add the two together and apply Young's inequality:

$$
\begin{aligned}
&-\bar{B}_1\bar{B}_2\left\|\lambda^1_{U_N(v)}\right\|_0^2 + \bar{C}_1\bar{C}_2\left\|\lambda^2_{U_N(v)}\right\|_0^2 - \alpha_0\bar{B}_2\left\|\Delta\lambda^1_{U_N(v)}\right\|_0^2 + \alpha_0\bar{C}_1\left\|\Delta\lambda^2_{U_N(v)}\right\|_0^2 \\
=&-\alpha_0\bar{B}_2(\bar{R}^1_\tau, \lambda^1_{U_N(v)}) + \alpha_0\bar{C}_1(\bar{R}^2_\tau, \lambda^2_{U_N(v)}) + \bar{A}_1\bar{B}_2(\varepsilon^0_u, \lambda^1_{U_N(v)}) + \bar{B}_1\bar{B}_2(\varepsilon^1_u, \lambda^1_{U_N(v)}) + \bar{C}_1\bar{B}_2(\varepsilon^2_u, \lambda^1_{U_N(v)}) \\
&-\bar{C}_1\bar{A}_2(\varepsilon^0_u, \lambda^2_{U_N(v)}) - \bar{C}_1\bar{B}_2(\varepsilon^1_u, \lambda^2_{U_N(v)}) - \bar{C}_1\bar{C}_2(\varepsilon^2_u, \lambda^2_{U_N(v)}) + \bar{A}_1\bar{B}_2(\lambda^0_{U_N(v)}, \lambda^1_{U_N(v)}) - \bar{C}_1\bar{A}_2(\lambda^0_{U_N(v)}, \lambda^2_{U_N(v)}) \\
\leq&\frac{(-\alpha_0\bar{B}_2)^2}{2\epsilon_1}\left\|\bar{R}^1_\tau\right\|_0^2 + \frac{(\alpha_0\bar{C}_1)^2}{2\epsilon_2}\left\|\bar{R}^2_\tau\right\|_0^2 + \Big[\frac{(\bar{B}_2\bar{A}_1)^2}{2\epsilon_1} + \frac{(-\bar{A}_2\bar{C}_1)^2}{2\epsilon_2}\Big]\left\|\lambda^0_{U_N(v)}\right\|_0^2 \\
&+\frac{5\epsilon_1}{2}\left\|\lambda^1_{U_N(v)}\right\|_0^2 + \frac{5\epsilon_2}{2}\left\|\lambda^2_{U_N(v)}\right\|_0^2 + \Big[\frac{(\bar{B}_2\bar{A}_1)^2}{2\epsilon_1} + \frac{(-\bar{A}_2\bar{C}_1)^2}{2\epsilon_2}\Big]\left\|\varepsilon^0_u\right\|_0^2 \\
&+\Big[\frac{(\bar{B}_1\bar{B}_2)^2}{2\epsilon_1} + \frac{(-\bar{B}_2\bar{C}_1)^2}{2\epsilon_2}\Big]\left\|\varepsilon^1_u\right\|_0^2 + \Big[\frac{(\bar{B}_2\bar{C}_1)^2}{2\epsilon_1} + \frac{(-\bar{C}_2\bar{C}_1)^2}{2\epsilon_2}\Big]\left\|\varepsilon^2_u\right\|_0^2.
\end{aligned}
\tag{C.3}
$$

Let $\epsilon_1 = -\bar{B}_1\bar{B}_2/5$ and $\epsilon_2 = \bar{C}_1\bar{C}_2/5$ in (C.3), and we get

$$\begin{aligned}
&-\bar{B}_1\bar{B}_2\left\|\lambda^1_{U_N(v)}\right\|_0^2+\bar{C}_2\bar{C}_1\left\|\lambda^2_{U_N(v)}\right\|_0^2-2\alpha_0\bar{B}_2\left\|\Delta\lambda^1_{U_N(v)}\right\|_0^2+2\alpha_0\bar{C}_1\left\|\Delta\lambda^2_{U_N(v)}\right\|_0^2\\
\le&\frac{5(-\alpha_0\bar{B}_2)^2}{-\bar{B}_2\bar{B}_1}\left\|\bar{R}^1_\tau\right\|_0^2+\frac{5(\alpha_0\bar{C}_1)^2}{\bar{C}_1\bar{C}_2}\left\|\bar{R}^2_\tau\right\|_0^2+5\frac{\bar{A}_2^2\bar{B}_1\bar{C}_1-\bar{A}_1^2\bar{B}_2\bar{C}_2}{\bar{B}_1\bar{C}_2}\left\|\lambda^0_{U_N(v)}\right\|_0^2\\
&+5\frac{\bar{A}_2^2\bar{B}_1\bar{C}_1-\bar{A}_1^2\bar{B}_2\bar{C}_2}{\bar{B}_1\bar{C}_2}\left\|\varepsilon^0_u\right\|_0^2+5\frac{\bar{B}_2^2\bar{C}_1-\bar{B}_2\bar{B}_1\bar{C}_2}{\bar{C}_2}\left\|\varepsilon^1_u\right\|_0^2+5\frac{\bar{C}_2\bar{C}_1\bar{B}_1-\bar{C}_1^2\bar{B}_2}{\bar{B}_1}\left\|\varepsilon^2_u\right\|_0^2.
\end{aligned}\tag{C.4}$$

According to Lemmas 2.1 and 2.6, the inequality (C.4) becomes

$$\begin{aligned}
&\left\|\lambda^1_{U_N(v)}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta\lambda^1_{U_N(v)}\right\|_0^2<\left\|\lambda^1_{U_N(v)}\right\|_0^2+\frac{2\alpha_0}{\bar{B}_1}\left\|\Delta\lambda^1_{U_N(v)}\right\|_0^2\\
\le\ &5\frac{\bar{A}_2^2\bar{B}_1\bar{C}_1-\bar{A}_1^2\bar{B}_2\bar{C}_2}{\bar{B}_1\bar{C}_2}\left\|\lambda^0_{U_N(v)}\right\|_0^2+\Big(\frac{5\alpha_0^2}{\bar{B}_1^2}-\frac{5\alpha_0^2\bar{C}_1}{\bar{C}_2\bar{B}_1\bar{B}_2}\Big)\max_{i=1,2}\left\|\bar{R}^i_\tau\right\|_0^2\\
&+\left[5\frac{\bar{A}_2^2\bar{B}_1\bar{C}_1-\bar{A}_1^2\bar{B}_2\bar{C}_2}{\bar{B}_1\bar{C}_2}\left\|\varepsilon^0_u\right\|_0^2+5\frac{\bar{B}_2^2\bar{C}_1-\bar{B}_2\bar{B}_1\bar{C}_2}{\bar{C}_2}\left\|\varepsilon^1_u\right\|_0^2+5\frac{\bar{C}_2\bar{C}_1\bar{B}_1-\bar{C}_1^2\bar{B}_2}{\bar{B}_1}\left\|\varepsilon^2_u\right\|_0^2\right]/(-\bar{B}_1\bar{B}_2)\\
=\ &\frac{5}{2}M_1\left\|\lambda^0_{U_N(v)}\right\|_0^2+\frac{5}{2}M_2[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=1,2}\left\|\bar{R}^i_\tau\right\|_0^2+\frac{5}{2}M_1\left\|\varepsilon^0_u\right\|_0^2+N_1\left\|\varepsilon^1_u\right\|_0^2+N_2\left\|\varepsilon^2_u\right\|_0^2\\
\le\ &C(\tau^{3-\theta}+N^{-l})^2,
\end{aligned}\tag{C.5}$$

where M_1, M_2, N_1, and N_2 are defined as follows:

$$M_1=\frac{(15\theta-18)(\theta-2)}{8(1-\theta)^2(\theta+2)},\quad M_2=(2-\theta)^2\left[\frac{1}{2}+\frac{4^\theta(1-\theta)}{8(\theta+2)}\right],N_1=\frac{5(2-\theta)}{(1-\theta)(\theta+2)},\quad N_2=\frac{5(2-\theta)}{16(1-\theta)^2}.$$

Based on the norm $|||\cdot|||_2$ in (2.8), (C.5), and Lemma 2.6, we have

$$||U_N^1(v)-u(\mathbf{x},t_1)|||_2\le|||\lambda^1_{U_N(v)}|||_2+|||\varepsilon^1_u|||_2\le\|\lambda^1_{U_N(v)}\|_2+\|\varepsilon^1_u\|_2\le C(\tau^{3-\theta}+N^{-l})+CN^{2-l}\le C(\tau^{3-\theta}+N^{2-l}).$$

Similarly, according to (C.4), we get

$$\begin{aligned}
&\left\|\lambda^2_{U_N(v)}\right\|_0^2+\alpha_0\beta_0^{-1}\left\|\Delta\lambda^2_{U_N(v)}\right\|_0^2<\left\|\lambda^2_{U_N(v)}\right\|_0^2+\frac{2\alpha_0}{\bar{C}_2}\left\|\Delta\lambda^2_{U_N(v)}\right\|_0^2\\
\le&5\frac{\bar{A}_2^2\bar{B}_1\bar{C}_1-\bar{A}_1^2\bar{B}_2\bar{C}_2}{\bar{B}_1\bar{C}_2^2\bar{C}_1}\left\|\lambda^0_{U_N(v)}\right\|_0^2+\Big(\frac{5\alpha_0^2}{\bar{C}_2^2}-\frac{5\alpha_0^2\bar{B}_2}{\bar{C}_2\bar{B}_1\bar{C}_1}\Big)\max_{i=1,2}\left\|\bar{R}^i_\tau\right\|_0^2\\
&+\left[5\frac{\bar{A}_2^2\bar{B}_1\bar{C}_1-\bar{A}_1^2\bar{B}_2\bar{C}_2}{\bar{B}_1\bar{C}_2}\left\|\varepsilon^0_u\right\|_0^2+5\frac{\bar{B}_2^2\bar{C}_1-\bar{B}_2\bar{B}_1\bar{C}_2}{\bar{C}_2}\left\|\varepsilon^1_u\right\|_0^2+5\frac{\bar{C}_2\bar{C}_1\bar{B}_1-\bar{C}_1^2\bar{B}_2}{\bar{B}_1}\left\|\varepsilon^2_u\right\|_0^2\right]/(\bar{C}_1\bar{C}_2)\\
=&\frac{5}{2}M_3\left\|\lambda^0_{U_N(v)}\right\|_0^2+\frac{5}{2}M_4[\Gamma(1-\theta)]^2T^{2\theta}\max_{i=1,2}\left\|\bar{R}^i_\tau\right\|_0^2+\frac{5}{2}M_3\left\|\varepsilon^0_u\right\|_0^2+N_3\left\|\varepsilon^1_u\right\|_0^2+N_1\left\|\varepsilon^2_u\right\|_0^2\\
\le&C(\tau^{3-\theta}+N^{-l})^2,
\end{aligned}$$

where M_3, M_4, and N_3 are defined as follows:

$$M_3=\frac{2(15\theta-18)(\theta-2)}{(1-\theta)(\theta+2)^2},\ M_4=(1-\theta)(2-\theta)^2\left[\frac{2\cdot4^\theta(1-\theta)}{(\theta+2)^2}+\frac{8}{(\theta+2)}\right],\ N_3=\frac{5(32-16\theta)}{(\theta+2)^2}.$$

We then get

$$\|U_N^2(v)-u(\mathbf{x},t_2)\||_2 \le \||\lambda^2_{U_N(v)}\||_2+\||\varepsilon^2_u\||_2 \le \|\lambda^2_{U_N(v)}\|_2+\|\varepsilon^2_u\|_2 \le C(\tau^{3-\theta}+N^{-l})+CN^{2-l} \le C(\tau^{3-\theta}+N^{2-l}).$$

Similarly (C.2), for $k \le K-3$ we obtain

$$(\lambda^k_{U_N(v)}+\varepsilon^k_u,\bar{v}_N)+\alpha_0\beta_0^{-1}(\Delta\lambda^k_{U_N(v)},\Delta\bar{v}_N)=\sum_{i=1}^{k} D^k_{k-i}(\lambda^{k-i}_{U_N(v)}+\varepsilon^{k-i}_u,\bar{v}_N)+\alpha_0\beta_0^{-1}(\bar{R}^k_\tau,\bar{v}_N), \forall \bar{v}_N \in Y_N. \quad \text{(C.6)}$$

Based on Lemma 2.1 and (C.6), the following conclusion can be drawn:

$$\begin{aligned}
&(\lambda^k_{U_N(v)},\bar{v}_N)+\alpha_0\beta_0^{-1}(\Delta\lambda^k_{U_N(v)},\Delta\bar{v}_N)\\
&=\sum_{i=1}^{k} D^k_{k-i}(\lambda^{k-i}_{U_N(v)},\bar{v}_N)-\Big[(\varepsilon^k_u,\bar{v}_N)-\sum_{i=1}^{k} D^k_{k-i}(\varepsilon^{k-i}_u,\bar{v}_N)-\alpha_0\beta_0^{-1}(\bar{R}^k_\tau,\bar{v}_N)\Big]\\
&=\sum_{i=1}^{k} D^k_{k-i}(\lambda^{k-i}_{U_N(v)},\bar{v}_N)-(\mathbb{Q}^0_{2,N}u(\mathbf{x},t_k)-u(\mathbf{x},t_k),\bar{v}_N)+\sum_{i=1}^{k} D^k_{k-i}(\mathbb{Q}^0_{2,N}u(\mathbf{x},t_{k-i})-u(\mathbf{x},t_{k-i}),\bar{v}_N)+\alpha_0\beta_0^{-1}(\bar{R}^k_\tau,\bar{v}_N)\\
&=\sum_{i=1}^{k} D^k_{k-i}(\lambda^{k-i}_{U_N(v)},\bar{v}_N)+(I-\mathbb{Q}^0_{2,N})(u(\mathbf{x},t_k),\bar{v}_N)-\ (I-\mathbb{Q}^0_{2,N})\sum_{i=1}^{k} D^k_{k-i}(u(\mathbf{x},t_{k-i}),\bar{v}_N)+\alpha_0\beta_0^{-1}(\bar{R}^k_\tau,\bar{v}_N)\\
&=\sum_{i=1}^{k} D^k_{k-i}(\lambda^{k-i}_{U_N(v)},\bar{v}_N)+(I-\mathbb{Q}^0_{2,N})\big(u(\mathbf{x},t_k)-\sum_{i=1}^{k} D^k_{k-i}u(\mathbf{x},t_{k-i}),\bar{v}_N\big)+\alpha_0\beta_0^{-1}(\bar{R}^k_\tau,\bar{v}_N)\\
&=\sum_{i=1}^{k} D^k_{k-i}(\lambda^{k-i}_{U_N(v)},\bar{v}_N)+(I-\mathbb{Q}^0_{2,N})\big(\alpha_0\beta_0^{-1}({}_0D^\theta_t u(\mathbf{x},t_k)-\bar{R}^k_\tau,\bar{v}_N)\big)+\alpha_0\beta_0^{-1}(\bar{R}^k_\tau,\bar{v}_N)\\
&=\sum_{i=1}^{k} D^k_{k-i}(\lambda^{k-i}_{U_N(v)},\bar{v}_N)+\alpha_0\beta_0^{-1}\big((I-\mathbb{Q}^0_{2,N})({}_0D^\theta_t u(\mathbf{x},t_k)-\bar{R}^k_\tau)+\bar{R}^k_\tau,\bar{v}_N\big)\\
&=\sum_{i=1}^{k} D^k_{k-i}(\lambda^{k-i}_{U_N(v)},\bar{v}_N)+\alpha_0\beta_0^{-1}(\zeta^k_N u,\bar{v}_N), \quad \forall \bar{v}_N \in Y_N, \qquad \text{(C.7)}
\end{aligned}$$

where $\zeta^k_N u=(I-\mathbb{Q}^0_{2,N})({}_0D^\theta_t u(\mathbf{x},t_k)-\bar{R}^k_\tau)+\bar{R}^k_\tau$.

Using the triangle inequality leads to

$$\left\|\zeta^k_N u\right\|_0 \le \left\|(I-\mathbb{Q}^0_{2,N})_0D^\theta_t u(\mathbf{x},t_k)\right\|_0+\left\|(I-\mathbb{Q}^0_{2,N})\bar{R}^k_\tau\right\|_0+\left\|\bar{R}^k_\tau\right\|_0.$$

According to Lemma 2.1, we can draw a conclusion: $\left\|\bar{R}^k_\tau\right\|_0 \le C\tau^{3-\theta}$, and according to Lemma 2.6,

$$\left\|(I-\mathbb{Q}^0_{2,N})_0D^\theta_t u(\mathbf{x},t_k)\right\|_0 \le CN^{-l}.$$

Combining Lemmas 2.1 and 2.6, we have

$$\left\|(I-\mathbb{Q}^0_{2,N})\bar{R}^k_\tau\right\|_0 \le CN^{-l}\tau^{3-\theta}.$$

Thus, we have

$$\left\|\zeta^k_N u\right\|_0 \le C(\tau^{3-\theta}+N^{-l}+\tau^{3-\theta}N^{-l}).$$

Let $\hat{\lambda}^i_{U_N(v)} = \lambda^i_{U_N(v)} - \rho\lambda^{i-1}_{U_N(v)}$, $i = 2, \cdots, K$, where ρ is given in Theorem 3.1 of [32], and the expression (C.7) can be rewritten as:

$$(\hat{\lambda}^k_{U_N(v)}, \bar{v}_N)+\alpha_0\beta_0^{-1}(\Delta\lambda^k_{U_N(v)}, \Delta\bar{v}_N)=\rho(\hat{\lambda}^{k-1}_{U_N(v)}, \bar{v}_N)+\sum_{i=2}^{k-1}\bar{D}^k_{k-i}(\hat{\lambda}^{k-i}_{U_N(v)}, \bar{v}_N)+\bar{D}^k_0(\hat{\lambda}^0_{U_N(v)}, \bar{v}_N)+\alpha_0\beta_0^{-1}(\zeta^k_N u, \bar{v}_N).$$

By following a similar approach to Lemma 2.5, one can obtain

$$\begin{aligned}&|||\lambda^k_{U_N(v)}|||_2^2 = \left\|\lambda^k_{U_N(v)}\right\|_0^2 + \alpha_0\beta_0^{-1}\left\|\Delta\lambda^k_{U_N(v)}\right\|_0^2\\ \le& 16\left(S_2 C_{f,v}[\Gamma(1-\theta)]^2 T^{2\theta}\max_{1\le i\le K}\left\|\zeta^i_N u\right\|_0^2\right) \le C(\tau^{3-\theta} + N^{-l} + \tau^{3-\theta}N^{-l})^2.\end{aligned}$$

Based on the above estimation and the norm $|||\cdot|||_2$, we arrive at

$$\begin{aligned}&|||U_N(v)^k - u(\mathbf{x}, t_k)|||_2 \le |||\lambda^k_{U_N(v)}|||_2 + |||\varepsilon^k_u|||_2 \le \left\|\lambda^k_{U_N(v)}\right\|_2 + \left\|\varepsilon^k_u\right\|_2\\ \le& C(\tau^{3-\theta} + N^{-l} + \tau^{3-\theta}N^{-l}) + CN^{2-l} \le C(\tau^{3-\theta} + N^{2-l}),\end{aligned}$$

where the constant C has no relation to N and τ.

Therefore, the proof of Lemma 3.2 has been completed. □

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