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Research article

Approximate quasi-periodic solutions for nonlinear fractional equations with finite temporal translations

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Abstract: We investigate a class of nonlinear fractional lattice equations with finite temporal translation interactions under quasi-periodic external forcing. The model combines a fractional time operator, nonlinear interactions generated by a potential function, and finite translations of the unknown function in the temporal variable. In particular, the translation terms are represented by the finite-difference operator $(\Delta_\tau u)(t) = u(t + \tau) - u(t)$, and are not interpreted as a conventional time-delay differential equation. The present formulation should not be confused with a conventional time-delay differential equation. Working in a quasi-periodic functional framework associated with finitely generated frequency modules, we study approximate quasi-periodic response states. Using a variational approach based on averaged energy functionals and Ekeland's variational principle, we construct finite-frequency quasi-periodic solutions corresponding to approximations of the external forcing. We also analyze the lack of compactness inherent in the Besicovitch framework. The results provide a variational description of approximate quasi-periodic responses in nonlinear fractional systems with finite temporal translations and multi-frequency excitation.

Keywords: nonlinear fractional equations; quasi-periodic dynamics; finite temporal translations; nonlocal temporal operators; memory effects; variational methods; Besicovitch spaces
Mathematics Subject Classification: 34K37, 37K10, 37K60, 26A33, 70K75

Notation summary

Symbol	Meaning
κ	Fractional coefficient
β	Fractional order
τ	Translation parameter
Λ	Frequency module
$\mathcal{W}$	Nonlinear potential
f	Quasi-periodic forcing

1. Introduction

Nonlinear lattice systems arise naturally in mathematical physics as discrete models for wave propagation, crystal dynamics, and coupled oscillator chains. They provide simplified but structurally rich descriptions of spatially extended systems where energy is transferred through nearest-neighbor interactions. Such models have been extensively studied in connection with nonlinear wave phenomena and localized excitations, including discrete breathers; see, for instance, the seminal work of Aubry [1] and the monograph by Kevrekidis [2] for a comprehensive account.

A central feature of lattice dynamics is the interplay between nonlinearity and discrete spatial coupling, typically represented through difference operators. When combined with nonlinear interaction potentials, these systems exhibit a wide range of complex behaviors, including coherent structures, localized oscillations, and long-time recurrent dynamics.

In recent years, fractional differential equations have emerged as an effective tool for modeling systems with memory and hereditary effects. Fractional time derivatives naturally encode nonlocal temporal dependence, making them particularly suitable for describing viscoelastic materials, anomalous diffusion, and dissipative processes with long-range temporal correlations. Classical foundations and modern developments in fractional calculus can be found in [3–5].

Fractional nonlinear Schrödinger-type models have also attracted considerable attention in mathematical physics and nonlinear optics due to their ability to describe anomalous diffraction, Lévy-type transport, and nonlocal wave propagation phenomena. In this context, the interplay between fractional dispersion, nonlinear interactions, and complex external lattices has generated a rich variety of localization and stability effects. For instance, Huang et al. [6] investigated gap solitons in fractional dimensions under quasi-periodic lattices and established the existence of stable in-phase and out-of-phase localized modes within the Bloch spectral gaps. Their analysis demonstrated that both the lattice parameters and the Lévy index strongly influence the band-gap structure and the stability of nonlinear localized states.

Related studies have explored the role of parity-time (PT)-symmetric structures in fractional nonlinear media. Su et al. [7] analyzed a variable-coefficient fractional nonlinear Schrödinger equation with Wadati and PT-symmetric potentials, showing how fractional diffraction parameters govern transitions between breather waves and solitons as well as the stability of localized states under perturbations and collisions. More recently, Wang et al. [8] studied cubic–quintic fractional nonlinear Schrödinger models with PT-symmetric lattices and identified stability regions for both fundamental and dipole gap solitons inside finite Bloch gaps. Their results highlighted the delicate interaction between fractional diffraction, nonlinear competing effects, and PT-symmetry breaking mechanisms.

Fractional quasi-periodic media also exhibit remarkable localization phenomena analogous to Anderson localization. In this direction, Huang et al. [9] investigated localization-delocalization transitions in fractional-dimensional quasi-periodic lattices and showed that the transition threshold depends inversely on the Lévy index. Their work further revealed the occurrence of anti-Anderson localization in weakly disordered regimes and demonstrated that focusing nonlinearities may enhance localization properties. These studies collectively underline the importance of combining fractional effects with quasi-periodic structures in order to capture complex wave phenomena in nonlocal media.

Another important class of external influences in dynamical systems is given by quasi-periodic and almost periodic forcing terms. These arise from the superposition of multiple incommensurate frequencies and appear naturally in systems driven by several independent oscillatory sources. The mathematical theory of almost periodic functions goes back to the pioneering work of Bohr [10] and has been systematically developed in later contributions such as [11]. More recent perspectives on multi-periodic structures and frequency modules can be found in [12].

From a variational viewpoint, problems involving almost periodic or quasi-periodic structures have been studied using methods in the mean and generalized compactness frameworks. Foundational contributions in this direction include the works of Blot [13–15] and subsequent developments in the analysis of oscillatory phenomena [16]. The variational approach adopted in this paper also relies on the Ekeland variational principle, for which we refer to [17], as well as on classical ideas from optimal control and functional analysis [18, 19].

From a mathematical viewpoint, the model studied in this work combines three ingredients: a fractional temporal operator, nonlinear finite translations of the unknown function, and quasi-periodic external forcing. For a fixed $\tau \in \mathbb{R}$, we introduce the translation operator

$$(\mathcal{T}_\tau u)(t) = u(t + \tau)$$

and the associated finite-difference operator

$$\Delta_\tau = \mathcal{T}_\tau - I.$$

Thus,

$$(\Delta_\tau u)(t) = u(t + \tau) - u(t).$$

The parameter τ is a prescribed translation length in the independent variable. It does not represent a delay variable, and the present formulation is not a delay differential equation in the usual sense.

We study the following continuous-time fractional functional equation:

$$\kappa \mathcal{D}_t^\beta x(t) = \mathcal{W}'\big(x(t + \tau) - x(t)\big) - \mathcal{W}'\big(x(t) - x(t - \tau)\big) + f(t), \qquad t \in \mathbb{R}. \tag{1.1}$$

Here $x : \mathbb{R} \to \mathbb{R}$ is the unknown function, $\kappa > 0$, and $\mathcal{D}_t^\beta$ is the fractional operator defined spectrally through the Fourier multiplier

$$\widehat{\mathcal{D}_t^\beta u}(\lambda) = |\lambda|^\beta \widehat{u}(\lambda), \qquad \lambda \in \Lambda.$$

The two nonlinear terms in (1.1) are generated by finite translations of the same continuous-time function. They should therefore be distinguished from nearest-neighbor interactions in a spatial lattice, where a discrete site index $n \in \mathbb{Z}$ is present and the coupling has the form $x_{n+1} - x_n$ or $x_n - x_{n-1}$. They should also be distinguished from conventional delay differential equations, whose formulation explicitly involves a prescribed past state together with a delay-equation phase space and an initial-history condition.

The present paper does not introduce a discrete spatial index or a delay phase space. Instead, the translation operators are treated directly as bounded operators on the quasi-periodic Besicovitch spaces introduced below.

For a fixed $\tau \in \mathbb{R}$, the finite temporal translation operator is therefore defined by

$$(\Delta_\tau u)(t) := u(t + \tau) - u(t).$$

The interaction potential $\mathcal{W} : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy $\mathcal{W} \in C^1(\mathbb{R})$, and $\mathcal{W}'$ denotes its derivative.

The nonlinear terms

$$\mathcal{W}'(x(t + \tau) - x(t)) \quad \text{and} \quad \mathcal{W}'(x(t) - x(t - \tau))$$

represent nonlinear interactions between translated values of the unknown function separated by the fixed increment τ. Thus, the difference structure is generated by finite translations in the independent variable rather than by a discrete spatial lattice.

The external forcing term $f : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and quasi-periodic in time. More precisely, there exist rationally independent frequencies $\omega_1, \ldots, \omega_m \in \mathbb{R}$ and a continuous function $F : \mathbb{T}^m \to \mathbb{R}$ such that

$$f(t) = F(\omega_1 t, \ldots, \omega_m t).$$

Equivalently, f admits the Fourier expansion

$$f(t) = \sum_{\lambda \in \Lambda} \hat{f}_\lambda e^{i\lambda t},$$

where $\Lambda \subset \mathbb{R}$ denotes the finitely generated frequency module associated with $\{\omega_1, \ldots, \omega_m\}$.

Remark 1.1. **Finite translations versus spatial lattice coupling and delay.** The operators $x(t+\tau)$ and $x(t - \tau)$ in (1.1) are finite translations of the independent variable. The parameter τ is fixed throughout the analysis. It is important to distinguish this formulation from two standard settings. First, a spatial lattice equation contains a discrete site index, for example $x_n(t)$, and nearest-neighbor coupling involves $x_{n+1}(t) - x_n(t)$. No such discrete spatial index is present here. Second, a delay differential equation is formulated as an evolution problem with a prescribed history on an interval preceding the initial time and requires the corresponding delay-equation phase space. No such history-space formulation is introduced here.

Instead, we work with the translation operator

$$(\mathcal{T}_\tau u)(t) = u(t + \tau)$$

and the bounded finite-difference operator

$$\Delta_\tau = \mathcal{T}_\tau - I.$$

The fractional operator $\mathcal{D}_t^\beta$ and the finite translation operator $\mathcal{T}_\tau$ therefore play distinct mathematical roles.

In this framework, Eq (1.1) describes a continuous-time fractional functional equation with finite translations and quasi-periodic forcing.

Classical nonlinear lattice equations provide an important background for difference-type interaction structures in nonlinear dynamics; see, for example, [1, 2]. The present problem is not a spatial lattice equation of that type, however. In particular, there is no discrete site variable n in the formulation below. Instead, the difference structure is generated by finite translations of a continuous-time function.

The analysis of (1.1) presents several mathematical difficulties. First, the nonlinear interaction involves finite translations of the unknown function and is therefore nonlocal with respect to the independent variable. Second, the presence of a fractional operator introduces nonlocal temporal effects and requires a spectral formulation rather than classical differential techniques. Third, the quasi-periodic setting naturally leads to a lack of compactness in the underlying functional spaces, since Besicovitch-type frameworks are invariant under translations and do not exhibit decay at infinity. As a consequence, standard compactness methods used in variational analysis are not directly applicable.

To overcome these challenges, we develop a variational framework in fractional Besicovitch spaces associated with finitely generated frequency modules. This setting allows for a Fourier-based representation of quasi-periodic functions and provides a natural Hilbert space structure in which energy methods can be implemented. The associated energy functional is analyzed using harmonic decomposition techniques combined with variational principles in the sense of Ekeland.

A key structural feature of the problem is the intrinsic failure of compactness, which manifests through translation invariance in the frequency domain. This leads to the possibility that minimizing sequences do not converge strongly, even when bounded in the natural energy space. As a result, one is naturally led to an approximate notion of solutions rather than exact quasi-periodic states.

Main contribution.

This work provides a variational analysis of nonlinear fractional functional equations with finite translations and quasi-periodic forcing. The combination of fractional temporal dynamics, nonlinear translation interactions, and quasi-periodic forcing has, to the best of our knowledge, not been previously addressed in this form.

We establish the existence of *approximate quasi-periodic solutions* to (1.1) whose frequency spectrum is contained in the prescribed module Λ. More precisely, we prove that, for any given level of accuracy, one can construct finite-frequency quasi-periodic solutions associated with suitably approximated forcing terms while preserving the underlying frequency structure.

A central and genuinely new aspect of our analysis is the identification of an *intrinsic lack of compactness* in the fractional Besicovitch framework. We show that this phenomenon is not merely technical but structural, stemming from the translation invariance of the frequency representation. As a consequence, bounded minimizing sequences of the associated variational functional do not, in general, admit strongly convergent subsequences.

This explains why the present variational argument does not, by itself, provide convergence to an exact quasi-periodic solution of the original forced equation. Instead, the natural conclusion is the existence of finite-frequency quasi-periodic solutions corresponding to forcing approximations that converge to the prescribed forcing in the Besicovitch norm.

Our results therefore introduce a framework in which, in the presence of fractional effects and quasi-periodic forcing, approximate quasi-periodic solutions arise naturally from finite-frequency variational approximations. This perspective clarifies the role of the functional framework and provides a rigorous explanation for the limitations of classical compactness-based variational methods in this setting.

The remainder of the paper is organized as follows. Section 2 introduces the functional framework based on Besicovitch and fractional Sobolev-type spaces. Section 3 presents the structural assumptions on the nonlinear potential. Section 4 collects the auxiliary analytical results used throughout the paper. Section 5 contains the main existence theorem together with its variational proof. Section 6 is devoted to illustrative examples and physical interpretations of the model. The paper concludes with a final section summarizing the main findings and perspectives.

2. Functional spaces

We introduce the functional setting used to analyze quasi-periodic solutions of (1.1). The natural framework is provided by Besicovitch-type spaces associated with finitely generated frequency modules, which allow for a Fourier representation adapted to quasi-periodic functions while avoiding compactness assumptions.

Let $\Lambda \subset \mathbb{R}$ be a finitely generated $\mathbb{Z}$-module of the form

$$\Lambda = \left\{ \sum_{j=1}^{m} k_j \omega_j : k_j \in \mathbb{Z} \right\}, \quad \omega_1, \ldots, \omega_m \in \mathbb{R}.$$

We denote by $\mathcal{QP}(\mathbb{R}, \Lambda)$ the space of trigonometric polynomials

$$u(t) = \sum_{\lambda \in F} c_\lambda e^{i\lambda t}, \qquad F \subset \Lambda \text{ finite},$$

which is stable under translations and algebraic operations.

For $p \geq 1$, the Besicovitch mean of a locally integrable function u (when it exists) is defined by

$$\mathcal{M}(u) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(t)\, dt.$$

The associated Besicovitch space is given by

$$\mathcal{B}^p(\mathbb{R}, \Lambda) = \overline{\mathcal{QP}(\mathbb{R}, \Lambda)}^{\|\cdot\|_{B^p}}, \qquad \|u\|_{B^p} = \left(\mathcal{M}(|u|^p)\right)^{1/p}.$$

By construction, every element of $\mathcal{B}^p(\mathbb{R}, \Lambda)$ is obtained as a mean-limit of trigonometric polynomials with frequencies in Λ.

In the Hilbert case $p = 2$, $\mathcal{B}^2(\mathbb{R}, \Lambda)$ is endowed with the inner product

$$\langle u, v \rangle = \mathcal{M}(u\bar{v}),$$

and admits a Fourier representation indexed by Λ. For $u \in \mathcal{QP}(\mathbb{R}, \Lambda)$, the Fourier coefficients are given by

$$\widehat{u}(\lambda) = \mathcal{M}(u(t)e^{-i\lambda t}), \quad \lambda \in \Lambda,$$

and satisfy Parseval's identity

$$\|u\|_{B^2}^2 = \sum_{\lambda \in \Lambda} |\widehat{u}(\lambda)|^2,$$

which extends by density to all $u \in \mathcal{B}^2(\mathbb{R}, \Lambda)$.

For further details on Besicovitch spaces and their Sobolev-type extensions, we can refer to [13–16] and the references therein.

For $\beta \in (1, 2]$, we introduce the fractional Sobolev–Besicovitch space

$$\mathcal{H}^\beta(\mathbb{R}, \Lambda) = \left\{ u \in \mathcal{B}^2(\mathbb{R}, \Lambda) : \sum_{\lambda \in \Lambda} (1 + |\lambda|^\beta) |\widehat{u}(\lambda)|^2 < \infty \right\},$$

which is a Hilbert space with norm

$$\|u\|_{\mathcal{H}^\beta}^2 = \sum_{\lambda \in \Lambda} (1 + |\lambda|^\beta) |\widehat{u}(\lambda)|^2.$$

This definition is purely spectral and avoids pointwise formulations of fractional derivatives, which are not well adapted to non-decaying functions.

The fractional derivative is defined as a Fourier multiplier:

$$\widehat{\mathcal{D}_t^\beta u}(\lambda) = |\lambda|^\beta \widehat{u}(\lambda), \qquad \lambda \in \Lambda,$$

which yields a continuous linear operator

$$\mathcal{D}_t^\beta : \mathcal{H}^\beta(\mathbb{R}, \Lambda) \to \mathcal{B}^2(\mathbb{R}, \Lambda).$$

Translations act isometrically on $\mathcal{B}^2(\mathbb{R}, \Lambda)$ via

$$(\mathcal{T}_\tau u)(t) = u(t + \tau),$$

and therefore continuously on $\mathcal{H}^\beta(\mathbb{R}, \Lambda)$. As a consequence, the discrete difference operator $u(t) \mapsto u(t + \tau) - u(t)$ is bounded in both spaces.

Given $\mathcal{W} \in C^1(\mathbb{R})$, the associated Nemytskii operator

$$\mathcal{N}_{\mathcal{W}}(u)(t) = \mathcal{W}(u(t))$$

is well-defined on $\mathcal{B}^2(\mathbb{R}, \Lambda)$ under standard growth assumptions, and is Fréchet differentiable with derivative $\mathcal{W}'(u)v$.

A key feature of this framework is the absence of compactness.

Theorem 2.1. *The embedding*

$$\mathcal{H}^\beta(\mathbb{R}, \Lambda) \hookrightarrow \mathcal{B}^2(\mathbb{R}, \Lambda)$$

is continuous but not compact.

Proof. Continuity follows from the inequality

$$\|u\|_{\mathcal{B}^2} \le \|u\|_{\mathcal{H}^\beta}.$$

To prove non-compactness, consider a nonconstant function $u \in \mathcal{QP}(\mathbb{R}, \Lambda)$ and define $u_n(t) = u(t + n)$. Then (u_n) is bounded in $\mathcal{H}^\beta(\mathbb{R}, \Lambda)$, since translation preserves Fourier amplitudes. However, for any $\lambda_0 \in \Lambda \setminus \{0\}$ such that $\widehat{u}(\lambda_0) \neq 0$, one has

$$\widehat{u_n}(\lambda_0) = e^{i\lambda_0 n} \widehat{u}(\lambda_0),$$

which does not converge. Hence (u_n) admits no strongly convergent subsequence in $\mathcal{B}^2(\mathbb{R}, \Lambda)$. $\square$

This lack of compactness is intrinsic: it reflects the translation invariance and absence of decay in Besicovitch spaces, and cannot be removed without additional structural assumptions.

3. Structural assumptions

Throughout the paper, we consider the nonlinear interaction potential $\mathcal{W} : \mathbb{R} \to \mathbb{R}$ and impose the following assumptions:

(A1) $\mathcal{W} \in C^1(\mathbb{R}, \mathbb{R})$.
(A2) $\mathcal{W}'$ is globally Lipschitz; that is, there exists $L > 0$ such that

$$|\mathcal{W}'(x) - \mathcal{W}'(y)| \leq L|x - y|, \qquad x, y \in \mathbb{R}.$$

(A3) $\mathcal{W}$ is bounded from above:

$$\sup_{x\in\mathbb{R}} \mathcal{W}(x) < +\infty.$$

The assumptions above concern only the nonlinear interaction law $\mathcal{W}$ and are independent of the interpretation of the translation operator. In particular, the parameter τ below is a fixed temporal translation parameter and does not represent a time-delay state variable.

Lemma 3.1. *Under* (A2)*, there exists* $C > 0$ *such that*

$$|\mathcal{W}(x)| \leq C(1 + |x|^2), \qquad x \in \mathbb{R}.$$

Proof. By the mean value theorem, for every $x \in \mathbb{R}$,

$$\mathcal{W}(x) - \mathcal{W}(0) = \mathcal{W}'(\xi)x$$

for some ξ between 0 and x. Hence

$$|\mathcal{W}(x)| \leq |\mathcal{W}(0)| + |\mathcal{W}'(\xi)|\,|x|.$$

By the global Lipschitz continuity of $\mathcal{W}'$,

$$|\mathcal{W}'(\xi)| \leq |\mathcal{W}'(0)| + L|\xi| \leq |\mathcal{W}'(0)| + L|x|.$$

Consequently,

$$|\mathcal{W}(x)| \leq |\mathcal{W}(0)| + |\mathcal{W}'(0)||x| + L|x|^2,$$

and therefore

$$|\mathcal{W}(x)| \leq C(1 + |x|^2)$$

for a suitable constant $C > 0$. □

Proposition 3.2. *Let*

$$\mathcal{N}_{\mathcal{W}'}(u)(t) = \mathcal{W}'(u(t)).$$

Under (A2)*, the Nemytskii operator*

$$\mathcal{N}_{\mathcal{W}'} : \mathcal{B}^2(\mathbb{R}, \Lambda) \longrightarrow \mathcal{B}^2(\mathbb{R}, \Lambda)$$

is well-defined and globally Lipschitz:

$$\|\mathcal{N}_{\mathcal{W}'}(u) - \mathcal{N}_{\mathcal{W}'}(v)\|_{\mathcal{B}^2} \leq L\|u - v\|_{\mathcal{B}^2}.$$

Proof. Since $\mathcal{W}'$ is globally Lipschitz, it has at most linear growth. In particular,

$$|\mathcal{W}'(x)| \leq |\mathcal{W}'(0)| + L|x|, \qquad x \in \mathbb{R}.$$

Therefore, for $u \in \mathcal{B}^2(\mathbb{R}, \Lambda)$,

$$|\mathcal{W}'(u(t))|^2 \leq 2|\mathcal{W}'(0)|^2 + 2L^2|u(t)|^2.$$

Taking the Besicovitch mean gives

$$\|\mathcal{W}'(u)\|_{\mathcal{B}^2} \leq C(1 + \|u\|_{\mathcal{B}^2}),$$

so that $\mathcal{W}'(u) \in \mathcal{B}^2(\mathbb{R}, \Lambda)$.

Moreover, the global Lipschitz assumption directly yields

$$|\mathcal{W}'(u(t)) - \mathcal{W}'(v(t))| \leq L|u(t) - v(t)|.$$

Consequently,

$$\|\mathcal{W}'(u) - \mathcal{W}'(v)\|_{\mathcal{B}^2} \leq L\|u - v\|_{\mathcal{B}^2}.$$

□

For the variational arguments below, the nonlinear functional will be considered on the fractional Hilbert space $\mathcal{H}^\beta(\mathbb{R}, \Lambda)$, where the spectral regularity provides the appropriate framework for the corresponding quadratic energy. We therefore do not require a C^1 Nemytskii mapping on the whole Besicovitch space $\mathcal{B}^2$.

4. Auxiliary results

In this section, we establish the basic properties of the translation operator and the nonlinear interaction operator associated with (4.1).

We consider

$$\kappa \mathcal{D}_t^\beta x(t) = \mathcal{W}'(x(t+\tau) - x(t)) - \mathcal{W}'(x(t) - x(t-\tau)) + f(t), \tag{4.1}$$

where $\kappa > 0, \beta \in (1, 2]$, and $f \in \mathcal{B}^2(\mathbb{R}, \Lambda)$.

Throughout this section, $\tau \in \mathbb{R}$ is regarded as a fixed translation parameter. The operators appearing below act on the independent temporal variable and are ordinary translation and finite-difference operators. No time-delay differential equation is introduced.

We begin with the translation operator.

Lemma 4.1. *For $\tau \in \mathbb{R}$, define*

$$(\mathcal{T}_\tau u)(t) := u(t + \tau).$$

Then $\mathcal{T}_\tau$ is an isometry on $\mathcal{B}^2(\mathbb{R}, \Lambda)$ and on $\mathcal{H}^\beta(\mathbb{R}, \Lambda)$.

Proof. By translation invariance of the Besicovitch mean,

$$\|\mathcal{T}_\tau u\|_{\mathcal{B}^2} = \|u\|_{\mathcal{B}^2}.$$

Moreover, if

$$\widehat{\mathcal{T}_\tau u}(\lambda) = e^{i\lambda\tau}\widehat{u}(\lambda),$$

then

$$|\widehat{\mathcal{T}_\tau u}(\lambda)| = |\widehat{u}(\lambda)|.$$

Consequently,

$$\|\mathcal{T}_\tau u\|_{\mathcal{H}^\beta} = \|u\|_{\mathcal{H}^\beta}.$$

□

Lemma 4.2. *For $\tau \in \mathbb{R}$, define the finite-difference operator*

$$\Delta_\tau := \mathcal{T}_\tau - I, \qquad (\Delta_\tau u)(t) = u(t+\tau) - u(t).$$

Then Δ_τ is a bounded linear operator on both $\mathcal{B}^2(\mathbb{R}, \Lambda)$ and $\mathcal{H}^\beta(\mathbb{R}, \Lambda)$, with

$$\|\Delta_\tau u\|_{\mathcal{B}^2} \leq 2\|u\|_{\mathcal{B}^2}, \qquad \|\Delta_\tau u\|_{\mathcal{H}^\beta} \leq 2\|u\|_{\mathcal{H}^\beta}.$$

Proof. By Lemma 4.1, it comes that:

$$\|\Delta_\tau u\|_{\mathcal{B}^2} \leq \|\mathcal{T}_\tau u\|_{\mathcal{B}^2} + \|u\|_{\mathcal{B}^2} = 2\|u\|_{\mathcal{B}^2}.$$

The estimate in $\mathcal{H}^\beta$ follows in exactly the same way. □

Proposition 4.3. *Define*

$$\mathcal{L}(u) = \mathcal{N}_{\mathcal{W}'}(\Delta_\tau u) - \mathcal{T}_{-\tau}\mathcal{N}_{\mathcal{W}'}(\Delta_\tau u),$$

that is,

$$\mathcal{L}(u)(t) = \mathcal{W}'\big(u(t+\tau) - u(t)\big) - \mathcal{W}'\big(u(t) - u(t-\tau)\big).$$

Then $\mathcal{L}$ is a well-defined locally Lipschitz mapping from $\mathcal{B}^2(\mathbb{R}, \Lambda)$ into itself. More precisely,

$$\|\mathcal{L}(u) - \mathcal{L}(v)\|_{\mathcal{B}^2} \leq 4L\|u - v\|_{\mathcal{B}^2}.$$

Proof. Using the global Lipschitz continuity of $\mathcal{W}''$, the boundedness of Δ_τ, and the isometry of $\mathcal{T}_{-\tau}$, we obtain

$$\begin{aligned}\|\mathcal{L}(u) - \mathcal{L}(v)\|_{\mathcal{B}^2} &\leq L\|\Delta_\tau(u-v)\|_{\mathcal{B}^2} + L\|\Delta_\tau(u-v)\|_{\mathcal{B}^2} \\ &\leq 4L\|u - v\|_{\mathcal{B}^2}.\end{aligned}$$

Hence $\mathcal{L}$ is well-defined and Lipschitz continuous on $\mathcal{B}^2(\mathbb{R}, \Lambda)$. □

The preceding proposition is the only regularity property of the nonlinear interaction operator needed for the finite-dimensional Galerkin construction below. In particular, no interpretation of $\mathcal{L}$ as a time-delay operator is required.

5. Main results

We consider the nonlinear fractional lattice equation

$$\kappa \mathcal{D}_t^{\beta} x(t) = \mathcal{W}'(x(t+\tau) - x(t)) - \mathcal{W}'(x(t) - x(t-\tau)) + f(t), \tag{5.1}$$

where $\kappa > 0, \beta \in (1, 2]$, $\tau \in \mathbb{R}$, $\Lambda \subset \mathbb{R}$ is a finitely generated $\mathbb{Z}$-module, and $f \in \mathcal{B}^2(\mathbb{R}, \Lambda)$ satisfies

$$\mathcal{M}(f) = 0.$$

The nonlinear interaction operator is defined by

$$\mathcal{L}(x)(t) = \mathcal{W}'(x(t+\tau) - x(t)) - \mathcal{W}'(x(t) - x(t-\tau)).$$

Throughout this section, assumptions (A1)–(A3) are in force.

5.1. Finite-frequency approximation

We first recall the finite-frequency approximation of a Besicovitch almost periodic function.

Proposition 5.1. *Let $f \in \mathcal{B}^2(\mathbb{R}, \Lambda)$ satisfy $\mathcal{M}(f) = 0$. Then there exists a sequence of finite Fourier polynomials*

$$f_N(t) = \sum_{\lambda \in \Gamma_N \setminus \{0\}} \widehat{f}(\lambda) e^{i\lambda t},$$

where $\Gamma_N \subset \Lambda$ is finite and

$$\Gamma_N = -\Gamma_N, \qquad \Gamma_N \subset \Gamma_{N+1}, \qquad \bigcup_{N \geq 1} \Gamma_N = \Lambda$$

in the sense that every frequency occurring in a finite Fourier polynomial belongs to some Γ_N, such that

$$\|f_N - f\|_{\mathcal{B}^2(\mathbb{R},\Lambda)} \longrightarrow 0 \qquad \text{as } N \to \infty.$$

Moreover,

$$\mathcal{M}(f_N) = 0 \qquad \text{for every } N.$$

Proof. Since $f \in \mathcal{B}^2(\mathbb{R}, \Lambda)$, finite Fourier polynomials with frequencies in Λ are dense in $\mathcal{B}^2(\mathbb{R}, \Lambda)$. Hence there exists a sequence of finite Fourier polynomials converging to f in the Besicovitch norm.

Because $\mathcal{M}(f) = 0$, the zero Fourier coefficient of f vanishes. We may therefore remove the zero-frequency component from each approximating polynomial. Thus, after relabeling, we obtain

$$f_N(t) = \sum_{\lambda \in \Gamma_N \setminus \{0\}} \widehat{f}(\lambda) e^{i\lambda t},$$

with

$$\mathcal{M}(f_N) = 0.$$

The convergence

$$\|f_N - f\|_{\mathcal{B}^2} \to 0$$

follows from the density of finite Fourier polynomials in $\mathcal{B}^2(\mathbb{R}, \Lambda)$. □

5.2. Fractional primitives of finite-frequency forcings

The zero-mean condition is precisely what allows us to invert the fractional operator on each finite-dimensional Fourier space.

Proposition 5.2. *Let*

$$g(t) = \sum_{\lambda \in \Gamma_N \setminus \{0\}} \widehat{g}(\lambda) e^{i\lambda t}$$

be a finite Fourier polynomial satisfying $\mathcal{M}(g) = 0$. Then there exists a unique zero-mean finite Fourier polynomial

$$Q(t) = \sum_{\lambda \in \Gamma_N \setminus \{0\}} \widehat{Q}(\lambda) e^{i\lambda t}$$

such that

$$\kappa \mathcal{D}_t^\beta Q = g.$$

More precisely,

$$\widehat{Q}(\lambda) = \frac{\widehat{g}(\lambda)}{\kappa |\lambda|^\beta}, \qquad \lambda \in \Gamma_N \setminus \{0\}.$$

Proof. Since $\mathcal{M}(g) = 0$, the zero Fourier coefficient of g vanishes. By the definition of the fractional operator through its Fourier multiplier,

$$\widehat{\mathcal{D}_t^\beta Q}(\lambda) = |\lambda|^\beta \widehat{Q}(\lambda), \qquad \lambda \neq 0.$$

We therefore define

$$\widehat{Q}(\lambda) = \frac{\widehat{g}(\lambda)}{\kappa |\lambda|^\beta}, \qquad \lambda \in \Gamma_N \setminus \{0\}.$$

Since Γ_N is finite, Q is a finite Fourier polynomial and

$$\widehat{\kappa \mathcal{D}_t^\beta Q}(\lambda) = \kappa |\lambda|^\beta \frac{\widehat{g}(\lambda)}{\kappa |\lambda|^\beta} = \widehat{g}(\lambda).$$

Hence

$$\kappa \mathcal{D}_t^\beta Q = g.$$

Finally, the equation does not determine the zero Fourier coefficient of Q, since

$$|\lambda|^\beta = 0 \qquad \text{for } \lambda = 0.$$

The normalization

$$\mathcal{M}(Q) = 0$$

therefore fixes the additive constant and gives uniqueness. □

5.3. Finite-dimensional variational formulation

For each N, define

$$X_N = \operatorname{span}\left\{e^{i\lambda t} : \lambda \in \Gamma_N\right\}.$$

We denote by P_N the corresponding Fourier projection,

$$P_N u = \sum_{\lambda \in \Gamma_N} \widehat{u}(\lambda) e^{i\lambda t}.$$

For each N, let $Q_N \in X_N$ be the unique zero-mean fractional primitive associated with f_N, namely

$$\kappa \mathcal{D}_t^{\beta} Q_N = f_N, \qquad \mathcal{M}(Q_N) = 0. \tag{5.2}$$

The main idea is to remove the forcing term by translating the unknown by Q_N.

Lemma 5.3. *Let Q_N be defined by* (5.2). *For $u \in X_N$, define $x = u + Q_N$ and introduce the functional*

$$\mathscr{E}_N(u) = \mathcal{M}\left(\frac{\kappa}{2}\left|\mathcal{D}_t^{\beta/2} u(t)\right|^2 - \mathcal{W}(u(t+\tau) - u(t) + Q_N(t+\tau) - Q_N(t))\right).$$

Then $\mathscr{E}_N$ is of class C^1 on X_N. Moreover, for every $v \in X_N$,

$$\begin{aligned} D\mathscr{E}_N(u) \cdot v = \mathcal{M}\Big(&\kappa \mathcal{D}_t^{\beta/2} u \, \mathcal{D}_t^{\beta/2} v \\ &- \mathcal{W}'(\mathcal{T}(u + Q_N))\mathcal{T} v\Big), \end{aligned} \tag{5.3}$$

where $\mathcal{T} u(t) = u(t+\tau) - u(t)$.

Proof. The map $u \mapsto \mathcal{D}_t^{\beta/2} u$ is linear and continuous on the finite-dimensional space X_N. Moreover, by Proposition 4.3, the nonlinear superposition operator induced by $\mathcal{W}$ is of class C^1. The translation operator $\mathcal{T}$ is bounded by Lemma 4.2. Consequently, the composition

$$u \mapsto \mathcal{W}(\mathcal{T}(u + Q_N))$$

is of class C^1.

Differentiating the quadratic term gives

$$D\left[\frac{\kappa}{2}\mathcal{M}\left(|\mathcal{D}_t^{\beta/2} u|^2\right)\right] \cdot v = \kappa \mathcal{M}\left(\mathcal{D}_t^{\beta/2} u \, \mathcal{D}_t^{\beta/2} v\right).$$

For the potential term, the chain rule yields

$$D\mathcal{M}\left[\mathcal{W}(\mathcal{T}(u + Q_N))\right] \cdot v = \mathcal{M}\left[\mathcal{W}'(\mathcal{T}(u + Q_N))\mathcal{T} v\right].$$

Combining the two identities proves the stated formula. □

Proposition 5.4. *For every N, the finite-dimensional functional $\mathscr{E}_N$ admits a critical point $u_N \in X_N$. Consequently,*

$$x_N = u_N + Q_N$$

is a finite-frequency quasi-periodic solution of

$$\kappa \mathcal{D}_t^{\beta} x_N = \mathcal{L}(x_N) + f_N.$$

Proof. Since X_N is finite-dimensional, all norms on X_N are equivalent. The global Lipschitz assumption (A2) implies that $\mathcal{W}'$ has at most linear growth. Hence, by the lemma in Section 3 (or the corresponding quadratic growth estimate),

$$|\mathcal{W}(z)| \leq C(1 + |z|^2).$$

Therefore, $\mathcal{E}_N$ is well-defined and continuous on X_N.

By assumption (A3), $\mathcal{W}$ is bounded from above. Thus there exists $C_N > 0$ such that

$$-\mathcal{M}\left[\mathcal{W}(\mathcal{T}(u + Q_N))\right] \geq -C_N.$$

Consequently,

$$\mathcal{E}_N(u) \geq \frac{\kappa}{2}\|\mathcal{D}_t^{\beta/2}u\|_2^2 - C_N.$$

On the zero-mean subspace of X_N, the fractional seminorm $\|\mathcal{D}_t^{\beta/2}u\|_2$ is a norm, because all nonzero frequencies satisfy $|\lambda| > 0$. Hence $\mathcal{E}_N$ is coercive on that subspace.

It follows that $\mathcal{E}_N$ attains its minimum at some $u_N \in X_N$. Since $\mathcal{E}_N$ is C^1, this minimizer satisfies

$$D\mathcal{E}_N(u_N) \cdot v = 0, \qquad \forall v \in X_N.$$

□

5.4. *Euler–Lagrange equation*

We now identify the equation satisfied by the critical point.

Lemma 5.5. *Let u_N be a critical point of $\mathcal{E}_N$ and put $x_N = u_N + Q_N$. Then,*

$$\kappa\mathcal{D}_t^{\beta}x_N = \mathcal{L}(x_N) + f_N, \quad in \quad X_N.$$

Proof. Since u_N is a critical point,

$$D\mathcal{E}_N(u_N) \cdot v = 0 \qquad \forall v \in X_N.$$

Therefore,

$$\mathcal{M}\left(\kappa\mathcal{D}_t^{\beta/2}u_N\mathcal{D}_t^{\beta/2}v - \mathcal{W}'(\mathcal{T}x_N)\mathcal{T}v\right) = 0.$$

Using the self-adjointness of the fractional Fourier multiplier,

$$\mathcal{M}\left(\mathcal{D}_t^{\beta/2}u_N\mathcal{D}_t^{\beta/2}v\right) = \mathcal{M}\left(\mathcal{D}_t^{\beta}u_N\, v\right).$$

Moreover, translation invariance of the Besicovitch mean gives

$$\mathcal{M}\left[\mathcal{W}'(\mathcal{T}x_N)\mathcal{T}v\right] = \mathcal{M}\left[\left(\mathcal{W}'(\mathcal{T}x_N) - \mathcal{W}'(\mathcal{T}x_N(\cdot - \tau))\right)v\right].$$

Since

$$\mathcal{T}x_N(t) = x_N(t + \tau) - x_N(t),$$

we have

$$\mathcal{T}x_N(t - \tau) = x_N(t) - x_N(t - \tau).$$

Hence

$$\mathcal{W}'(\mathcal{T} x_N(t)) - \mathcal{W}'(\mathcal{T} x_N(t-\tau)) = \mathcal{L}(x_N)(t).$$

Thus

$$\mathcal{M}\left[\left(\kappa\mathcal{D}_t^\beta u_N - \mathcal{L}(x_N)\right)v\right] = 0 \qquad \forall v \in X_N.$$

Since

$$x_N = u_N + Q_N$$

and, by (5.2),

$$\kappa\mathcal{D}_t^\beta Q_N = f_N,$$

we obtain

$$\kappa\mathcal{D}_t^\beta x_N = \kappa\mathcal{D}_t^\beta u_N + f_N = \mathcal{L}(x_N) + f_N.$$

This proves the result. □

5.5. Main existence theorem

We can now formulate the main result.

Theorem 5.6. *Let $\kappa > 0$, $\beta \in (1, 2]$, $\tau \in \mathbb{R}$, and assume* (A1)–(A3). *Let*

$$f \in \mathcal{B}^2(\mathbb{R}, \Lambda)$$

satisfy

$$\mathcal{M}(f) = 0.$$

Then there exists a sequence of finite-frequency quasi-periodic functions

$$x_N \in \mathcal{QP}(\mathbb{R}, \Lambda_N)$$

and a sequence of finite Fourier approximations

$$f_N \in \mathcal{QP}(\mathbb{R}, \Lambda_N)$$

such that

$$\|f_N - f\|_{\mathcal{B}^2} \longrightarrow 0 \qquad \text{as } N \to \infty,$$

and

$$\kappa\mathcal{D}_t^\beta x_N = \mathcal{L}(x_N) + f_N$$

for every N.

Proof. By Proposition 5.1, there exists a sequence of finite Fourier polynomials

$$f_N \in \mathcal{QP}(\mathbb{R}, \Lambda_N)$$

such that

$$\mathcal{M}(f_N) = 0$$

and

$$\|f_N - f\|_{\mathcal{B}^2} \longrightarrow 0.$$

For every N, Proposition 5.2 provides a unique zero-mean finite Fourier polynomial

$$Q_N \in \mathcal{QP}(\mathbb{R}, \Lambda_N)$$

satisfying

$$\kappa \mathcal{D}_t^{\beta} Q_N = f_N.$$

We then consider the finite-dimensional functional

$$\mathscr{E}_N(u) = \mathcal{M}\left(\frac{\kappa}{2}|\mathcal{D}_t^{\beta/2} u|^2 - \mathcal{W}(\mathcal{T}(u + Q_N))\right)$$

on X_N.

By Proposition 5.4, this functional admits a critical point $u_N \in X_N$. Define

$$x_N = u_N + Q_N.$$

Since both u_N and Q_N have frequencies contained in Γ_N, we have

$$x_N \in \mathcal{QP}(\mathbb{R}, \Lambda_N).$$

By Lemma 5.5, x_N satisfies

$$\kappa \mathcal{D}_t^{\beta} x_N = \mathcal{L}(x_N) + f_N.$$

Finally,

$$\|f_N - f\|_{\mathcal{B}^2} \to 0,$$

which proves the theorem. □

Remark 5.7. Theorem 5.6 is an approximation result. It does not assert the existence of an exact quasi-periodic solution corresponding to the original forcing f. Rather, it shows that the forcing can be approximated in the Besicovitch norm by finite-frequency forcings f_N, and that each approximating problem admits a finite-frequency quasi-periodic solution x_N.

In particular, the result gives

$$\kappa \mathcal{D}_t^{\beta} x_N - \mathcal{L}(x_N) = f_N \longrightarrow f \quad \text{in } \mathcal{B}^2.$$

Thus the sequence $\{x_N\}$ solves a sequence of finite-dimensional quasi-periodic problems whose right-hand sides converge to the prescribed forcing.

Remark 5.8. No convergence of the sequence $\{x_N\}$ is claimed in the present theorem. Indeed, the Besicovitch space is not compactly embedded into a stronger topology in a way that would allow one to extract a strongly convergent subsequence by a standard Rellich-type argument. Therefore, additional assumptions would be required to pass to the limit in the nonlinear term

$$\mathcal{L}(x_N) = \mathcal{W}'(x_N(\cdot + \tau) - x_N) - \mathcal{W}'(x_N - x_N(\cdot - \tau)).$$

The present result consequently establishes existence at every finite frequency level, together with convergence of the corresponding forcing terms, but not existence of a limiting exact solution.

Remark 5.9. **Variational structure.**

For the homogeneous problem $f \equiv 0$, the equation

$$\kappa \mathcal{D}_t^{\beta} x = \mathcal{L}(x)$$

is the Euler–Lagrange equation associated with the functional

$$\mathscr{E}_0(u) = \mathcal{M}\left(\frac{\kappa}{2}|\mathcal{D}_t^{\beta/2} u|^2 - \mathcal{W}(\mathcal{T}u)\right).$$

Indeed,

$$D\mathscr{E}_0(u) \cdot v = \mathcal{M}\left(\kappa \mathcal{D}_t^{\beta/2} u\, \mathcal{D}_t^{\beta/2} v - \mathcal{W}'(\mathcal{T}u)\mathcal{T}v\right),$$

and the corresponding Euler–Lagrange equation is

$$\kappa \mathcal{D}_t^{\beta} u = \mathcal{W}'(u(t+\tau) - u(t)) - \mathcal{W}'(u(t) - u(t-\tau)).$$

For the forced finite-frequency problem, the fractional primitive Q_N satisfying

$$\kappa \mathcal{D}_t^{\beta} Q_N = f_N$$

is used to translate the variational functional. The critical point of the translated functional then produces the solution

$$x_N = u_N + Q_N$$

of the forced equation.

Remark 5.10. Theorem 5.6 combines the finite-frequency approximation of quasi-periodic forcing with the variational structure of the nonlinear fractional lattice equation. The argument separates the problem into two levels. At the finite-dimensional level, the fractional primitive removes the forcing and allows the construction of a critical point of a C^1 functional. At the infinite-dimensional level, the sequence of finite-frequency forcings converges to the prescribed forcing in the Besicovitch norm.

The lack of compactness prevents us from concluding, under the present assumptions alone, that the corresponding solutions converge to an exact quasi-periodic solution. The theorem should therefore be understood as an existence theorem for finite-frequency quasi-periodic approximations.

6. Illustrative examples and physical interpretation

The examples presented in this section are intended to illustrate the analytical framework and the qualitative behavior predicted by the theory. Unless explicitly stated otherwise, the figures are schematic representations designed to visualize the spectral organization and quasi-periodic structure of the solutions, rather than numerical simulations of the nonlinear fractional lattice equation.

In this section, we present several examples illustrating the analytical framework developed in the previous sections and clarifying the physical interpretation of Theorem 5.6. The purpose of these examples is threefold. First, they provide explicit realizations of nonlinear fractional lattice systems subjected to quasi-periodic forcing. Second, they show how the variational approximation procedure naturally preserves the spectral structure of the forcing. Third, they illustrate the combined influence of nonlinearity, memory effects, and multi-frequency excitation on the resulting lattice dynamics.

The examples considered below are inspired by nonlinear wave propagation models arising in discrete optics, nonlinear metamaterials, fractional transport phenomena, and quasi-periodic media. In particular, the interplay between fractional dispersion and quasi-periodic structures has recently attracted significant attention in the study of localization phenomena, gap solitons, and anomalous diffraction in fractional nonlinear Schrödinger systems; see, for example, [6–9].

6.1. *A Duffing-type fractional quasi-periodic lattice*

We consider the bounded nonlinear potential

$$\mathcal{W}(x) = 1 - \cos x, \qquad \mathcal{W}'(x) = \sin x.$$

Near the origin,

$$\mathcal{W}'(x) = x - \frac{x^3}{6} + O(x^5),$$

so the corresponding nonlinear interaction contains a linear contribution together with a cubic correction. The quadratic contribution corresponds to the linearized interaction between translated values of the continuous-time function, while the higher-order terms describe the associated nonlinear correction. Such interaction laws are classical in nonlinear lattice dynamics and are known to generate rich oscillatory phenomena, including localized nonlinear modes and energy transfer between frequencies.

We consider the finitely generated frequency module

$$\Lambda = \mathrm{span}_{\mathbb{Z}}\{1, \sqrt{2}\},$$

where the irrationality of $\sqrt{2}$ guarantees the absence of periodic resonance relations. Consequently, every nontrivial combination of the basic frequencies produces genuinely quasi-periodic oscillations.

A representative forcing term is given by

$$f(t) = \cos(t) + \cos(\sqrt{2}\,t),$$

which belongs to the quasi-periodic class

$$f \in \mathcal{QP}(\mathbb{R}, \Lambda),$$

and satisfies

$$\mathcal{M}(f) = 0.$$

The corresponding nonlinear fractional lattice equation takes the form

$$\kappa \mathcal{D}_t^{\beta} x(t) = \mathcal{W}'\big(x(t+\tau) - x(t)\big) - \mathcal{W}'\big(x(t) - x(t-\tau)\big) + f(t), \tag{6.1}$$

where $\beta \in (1, 2]$ denotes the order of the fractional derivative and $\tau > 0$ is a fixed temporal translation parameter.

Equation (6.1) combines three distinct mechanisms:

- The nonlinear finite-translation interactions generated by the potential $\mathcal{W}$,

- the hereditary memory effects induced by the fractional derivative,
- the quasi-periodic external excitation associated with multiple incommensurate frequencies.

The mathematical structure considered here is relevant to models in which fractional temporal dynamics, nonlinear finite translations, and multi-frequency excitation occur simultaneously. Possible applications include phenomenological descriptions of hereditary media and externally forced nonlocal dynamical systems.

6.2. *Quasi-periodic temporal structure*

The quasi-periodic forcing is generated by the superposition of oscillations with incommensurate frequencies. Unlike periodic forcing, the resulting signal never exactly repeats itself, although it remains highly ordered and spectrally rigid.

This behavior is illustrated in Figure 1.

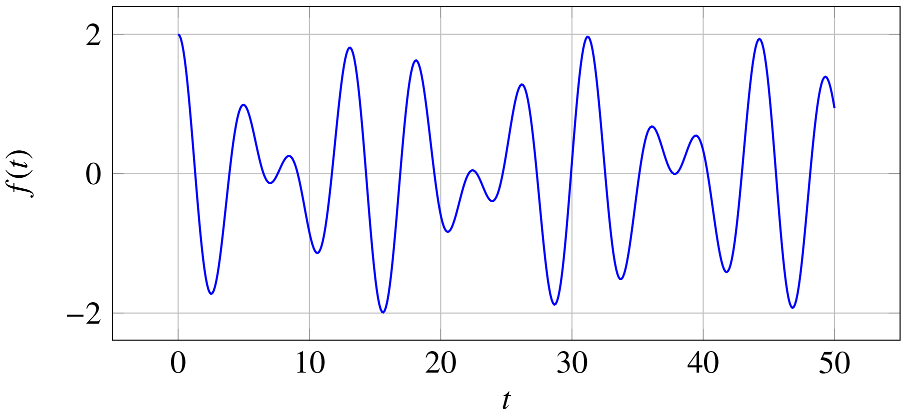

Figure 1. Quasi-periodic forcing generated by two incommensurate frequencies.

The forcing admits the Fourier representation

$$f(t) = \sum_{\lambda \in \Lambda} \widehat{f}_\lambda e^{i\lambda t},$$

with spectral support contained in the module

$$\Lambda = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}.$$

In the present example, the dominant frequencies are

$$\{\pm 1, \pm\sqrt{2}\}.$$

The discrete spectral nature of quasi-periodic functions plays a fundamental role in the variational framework developed in this work. Indeed, the Besicovitch setting allows one to characterize quasi-periodic states entirely through their Fourier coefficients and frequency modules.

The corresponding spectral structure is illustrated in Figure 2.

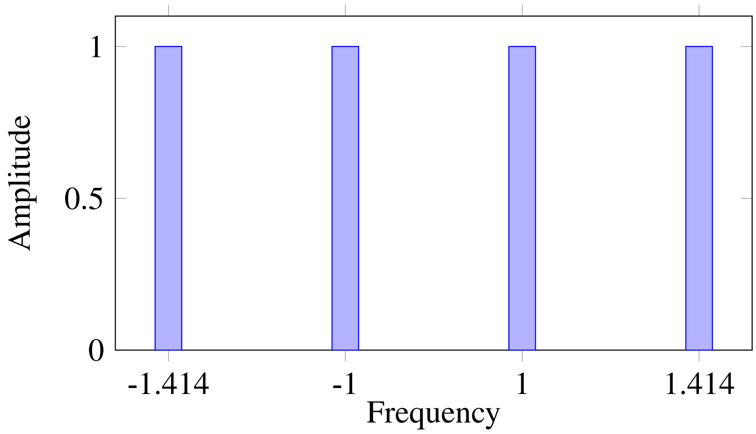

Figure 2. Discrete frequency spectrum associated with the quasi-periodic forcing.

6.3. Finite-mode approximation and variational construction

A central ingredient in the proof of Theorem 5.6 is the approximation of quasi-periodic functions by finite trigonometric sums. More precisely, for every $\varepsilon > 0$, one can construct

$$f_\varepsilon(t) = \sum_{j=1}^{N} a_j e^{i\lambda_j t}, \qquad \lambda_j \in \Lambda,$$

such that

$$\|f - f_\varepsilon\|_{\mathcal{B}^2} \leq \varepsilon.$$

This approximation preserves the spectral organization of the forcing while reducing the infinite-dimensional problem to a finite-frequency variational system. The resulting approximate solutions inherit the same quasi-periodic frequency module.

Figure 3 illustrates a typical spectral truncation.

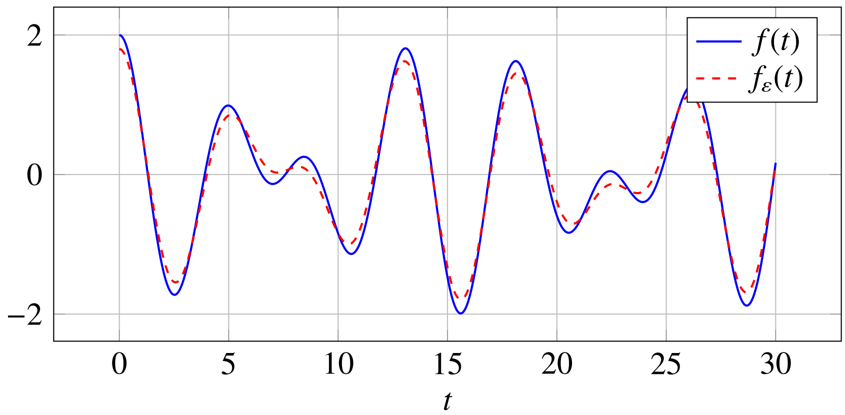

Figure 3. Schematic finite-frequency approximation of the quasi-periodic forcing.

The variational procedure developed in this paper then produces approximate quasi-periodic lattice states

$$x_\varepsilon \in \mathcal{QP}(\mathbb{R}, \Lambda),$$

whose oscillatory structure is governed by the same frequency module.

A typical approximate solution profile is represented in Figure 4.

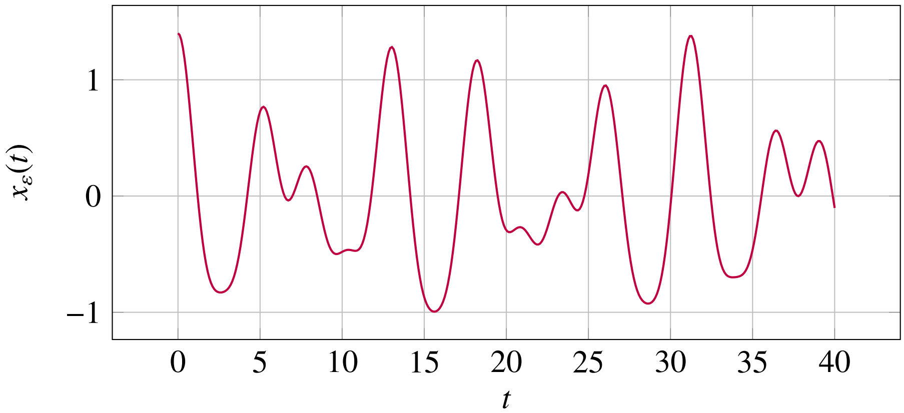

Figure 4. Schematic illustration of an approximate quasi-periodic response generated by finite spectral modes in Λ.

The appearance of the additional frequency

$$2.4142 \approx 1 + \sqrt{2}$$

reflects nonlinear frequency mixing induced by the sine-type interaction potential, which generates harmonic and combination frequencies through spectral coupling.

6.4. Spectral illustration

The theoretical analysis developed in this paper concerns fractional orders

$$\beta \in (1, 2].$$

The numerical implementation of such methods lies beyond the scope of the present work, whose primary objective is the variational analysis of quasi-periodic fractional lattice systems.

Nevertheless, these schemes provide a natural computational framework for approximating the dynamics predicted by the theoretical results established herein.

6.5. Example with three incommensurate frequencies

We now consider a more complex forcing involving three independent frequencies,

$$f_3(t) = \cos(t) + 0.7\cos(\sqrt{2}\,t) + 0.5\cos(\sqrt{3}\,t),$$

associated with the module

$$\Lambda_3 = \mathrm{span}_{\mathbb{Z}}\{1, \sqrt{2}, \sqrt{3}\}.$$

The resulting dynamics exhibit a substantially richer spectral structure due to the interaction of several irrational frequencies. Nonlinear coupling generates secondary combination modes such as

$$1 + \sqrt{2}, \qquad \sqrt{2} + \sqrt{3}, \qquad 1 + \sqrt{3},$$

which may appear in approximate lattice responses.

Such multi-frequency forcing models are particularly relevant in nonlinear optics and quasi-crystalline media, where several external oscillatory sources interact simultaneously.

6.6. Variational interpretation and lack of compactness

The variational formulation associated with the lattice equation is based on the energy functional

$$\mathscr{E}(u) = \mathcal{M}\left(\frac{\kappa}{2}|\mathcal{D}_t^{\beta/2}u|^2 - \mathcal{W}(u(t+\tau) - u(t))\right).$$

Critical points of $\mathscr{E}$ correspond formally to quasi-periodic lattice states. However, because Besicovitch spaces are translation invariant and lack compact embeddings, minimizing sequences generally fail to converge strongly.

Consequently, the variational procedure naturally leads to approximate critical points rather than exact minimizers.

This phenomenon is schematically represented in Figure 5.

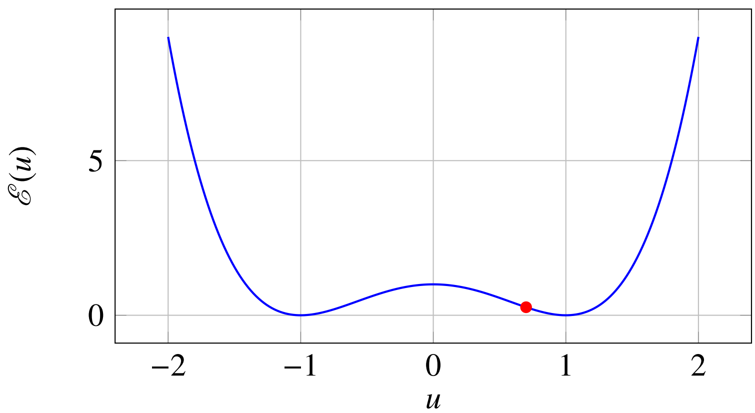

Figure 5. Schematic energy landscape illustrating the variational interpretation.

This lack of compactness is not merely technical. It reflects a genuine structural property of quasi-periodic fractional systems: Translations in the frequency domain preserve boundedness while preventing strong convergence. As a consequence, the natural solution concept in this framework becomes intrinsically approximate.

6.7. Physical interpretation

The previous examples illustrate how the mathematical framework developed in this work connects with several physically relevant mechanisms.

- The nonlinear potential models the interaction between finite translated values of the continuous-time state function, as encountered in discrete nonlinear media and lattice models with bounded interaction energies.
- The fractional derivative accounts for hereditary memory effects and long-range temporal correlations.

- The quasi-periodic forcing describes external excitations generated by several independent oscillatory sources.
- The frequency-module formulation provides a natural spectral description of quasi-periodic media.
- The variational approximation procedure reflects the physical fact that quasi-periodic states in complex media are often observed through finite spectral truncations and approximate resonant structures.

Altogether, these examples clarify how finite temporal translations, fractional memory effects, and quasi-periodic excitation combine to produce structured oscillatory dynamics.

7. Conclusions

In this work, we have investigated a class of nonlinear fractional functional equations involving finite translations of the independent variable under quasi-periodic external forcing. The model combines nonlinear finite-translation interactions with fractional temporal dynamics, leading to a class of nonlocal evolution systems that exhibit memory effects and complex oscillatory responses.

The interaction between fractional memory and quasi-periodic excitation generates structured oscillatory regimes governed by the spectral content of the forcing term.

Within a variational framework based on averaged energy functionals, we constructed finite-frequency quasi-periodic response states associated with finite-frequency approximations of the external excitation. The resulting sequence provides an approximate realization of the forced equation in the Besicovitch norm.

A key feature of the analysis is the presence of an intrinsic non-compactness phenomenon in the underlying functional setting, which reflects the persistence of oscillations and the absence of asymptotic localization in time. This structural property explains why standard compactness-based methods fail and highlights the role of memory and nonlocal interactions in shaping the long-term behavior of the system.

The results provide a variational description of quasi-periodic dynamics in nonlinear fractional equations with finite temporal translations and clarify how nonlocal temporal effects influence the emergence of sustained oscillatory regimes. In particular, the framework reveals how frequency structure is preserved in the nonlinear response despite the presence of memory-driven evolution.

Several directions remain open for further investigation. Of particular interest is the study of stability and bifurcation of quasi-periodic response states, as well as the possible transition to more complex or chaotic regimes under parameter variation. Another promising direction is the extension of the present framework to higher-dimensional lattices and more general classes of nonlinear nonlocal operators, where richer dynamical behaviors are expected to emerge.

Author contributions

Both authors contributed equally to this work.

Data availability

Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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