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Research article

On proportional Caputo-hybrid fractional maclaurin-type inequalities for convex stochastic processes

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Abstract: In this paper, we investigated Maclaurin-type inequalities for convex stochastic processes by means of proportional Caputo-hybrid fractional operators. To this end, we introduced the left-sided and right-sided stochastic mean-square proportional Caputo-hybrid operators and established a new integral identity involving these operators. This identity served as the main analytical tool for deriving our estimates. Based on this representation formula, we obtained new fractional Caputo-hybrid Maclaurin-type inequalities for convex stochastic processes under appropriate differentiability and integrability assumptions. The obtained results may be regarded as stochastic fractional extensions of the classical Maclaurin-type inequalities. In this way, the present work enriched the theory of stochastic fractional calculus and contributed to the ongoing development of generalized integral inequalities for stochastic processes.

Keywords: stochastic proportional Caputo-hybrid operators; Maclaurin's inequality; convex stochastic processes; Hölder's inequality; power mean inequality
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1. Introduction

Convexity is one of the central notions in mathematical analysis, optimization theory, approximation theory, and numerical analysis. Its importance stems from the rich geometric and analytic structure it provides, together with its decisive role in the derivation of sharp estimates for integral means, approximation errors, and stability bounds [24]. Among the most celebrated results related to convexity is the classical Hermite–Hadamard inequality, which gives a two-sided estimate for the integral average of a convex function on a compact interval in terms of its values at the midpoint and at the endpoints. This inequality has become a cornerstone in the theory of integral inequalities and has inspired a vast literature devoted to its refinements, extensions, and analogues under various generalized convexity assumptions and through different integral operators; see [25, 26].

In parallel with the development of generalized convexity, fractional calculus has emerged as a powerful extension of classical calculus, allowing differentiation and integration of non-integer order. Unlike classical operators, fractional operators naturally incorporate nonlocality, hereditary effects, and memory, making them particularly suitable for the modeling of complex systems arising in physics, engineering, biology, economics, and control theory. Because of these features, the interaction between convexity theory and fractional calculus has generated an active and fruitful research area, especially in the context of Hermite–Hadamard-, midpoint-, trapezium-, Simpson-, and Maclaurin-type inequalities.

Another major direction of modern analysis concerns stochastic processes, which provide an appropriate mathematical framework for describing systems evolving under uncertainty. Since many real-world phenomena are inherently random, extending deterministic inequalities to stochastic environments is both mathematically natural and practically relevant. The notion of convex stochastic processes, first formalized by Nikodem, opened the door to stochastic analogues of many classical inequalities.

Definition 1.1 ([27]). A stochastic process $\mathrm{F} : \mathcal{I} \times \Upsilon \to \mathbb{R}$ is a convex stochastic process, if for every $\mathfrak{e}_1, \mathfrak{e}_2 \in \mathcal{I}$ and $\jmath \in (0, 1)$, the following inequality is satisfied:

$$\mathrm{F}\left(\jmath\mathfrak{e}_1 + (1-\jmath)\,\mathfrak{e}_2, \cdot\right) \leq \jmath\mathrm{F}(\mathfrak{e}_1, \cdot) + (1-\jmath)\,\mathrm{F}(\mathfrak{e}_2, \cdot) \quad \text{(almost everywhere)}.$$

Subsequently, several authors established Hermite–Hadamard- and related inequalities for convex and generalized convex stochastic processes, thereby enriching the theory of stochastic analysis and uncertainty quantification. In particular, the Hermite-Hadamard inequality was established for convex stochastic processes by Kotrys in [19] and for s-convex stochastic processes by Set et al. in [28]. Materano et al. provided Simpson-type inequalities for s-convex and quasi-convex stochastic processes in [21], and Ostrowski-type inequalities for s-convex and quasi-convex stochastic processes in [22]. Afzal and Botmart investigated Jensen and Hermite-Hadamard inequalities for h-Godunova-Levin stochastic processes in [2], while Sharma et al studied the notion of strongly generalized convex stochastic processes and multidimensional general h-harmonic preinvex stochastic processes in [32] and [33], respectively.

In [14], Hafiz proposed the mean-square stochastic analogue of the Riemann–Liouville fractional integral, stated as follows.

Definition 1.2 ([14]). Let $\mathrm{F} : \mathcal{I} \times \Upsilon \to \mathbb{R}$ be a stochastic process. For any order $\gamma > 0$, the left-sided and right-sided stochastic mean-square fractional integrals of F are defined, respectively, by

$$\mathrm{R}^{\gamma}_{\mathrm{e}_1^+}[\mathrm{F}](\ell) = \frac{1}{\Gamma(\gamma)} \int_{\mathrm{e}_1}^{\ell} (\ell - \jmath)^{\gamma-1} \mathrm{F}(\jmath, \cdot)\,\mathrm{d}\jmath, \qquad \text{for a.e. } \ell > \mathrm{e}_1,$$

and

$$\mathrm{R}^{\gamma}_{\mathrm{e}_2^-}[\mathrm{F}](\ell) = \frac{1}{\Gamma(\gamma)} \int_{\ell}^{\mathrm{e}_2} (\jmath - \ell)^{\gamma-1} \mathrm{F}(\jmath, \cdot)\,\mathrm{d}\jmath, \qquad \text{for a.e. } \ell < \mathrm{e}_2.$$

Later, Agahi and Babakhani established in [3] a fractional counterpart of the Hermite–Hadamard inequality, namely

$$\mathrm{F}\left(\frac{\mathrm{e}_1 + \mathrm{e}_2}{2}, \cdot\right) \leq \frac{\Gamma(\gamma + 1)}{2(\mathrm{e}_2 - \mathrm{e}_1)^{\gamma}} \left(\mathrm{R}^{\gamma}_{\mathrm{e}_2^-}[\mathrm{F}](\mathrm{e}_1) + \mathrm{R}^{\gamma}_{\mathrm{e}_1^+}[\mathrm{F}](\mathrm{e}_2)\right) \leq \frac{\mathrm{F}(\mathrm{e}_1, \cdot) + \mathrm{F}(\mathrm{e}_2, \cdot)}{2}.$$

During the last few years, an increasing amount of attention has been devoted to the study of stochastic fractional inequalities under generalized convexity assumptions. In particular, Hermite–Hadamard-type inequalities for s-convex stochastic processes have been investigated via stochastic tempered fractional integrals [20], yielding midpoint- and trapezium-type estimates through suitable auxiliary identities. Likewise, parametrized inequalities for s-convex stochastic processes have been derived by means of stochastic k-Caputo fractional derivatives in [7], where a unified parametric identity serves as a generating mechanism for several well-known quadrature-type inequalities. Moreover, the notion of harmonically s-convex stochastic processes has also been successfully combined with Riemann–Liouville fractional integrals to establish new Hermite–Hadamard- and midpoint-type inequalities in [8]. Furthermore, the authors in [23] introduced the notion of k-Riemann-Liouville fractional mean square and established Maclaurin-type inequalities for k-Riemann–Liouville Maclaurin-type inequalities for s-convex stochastic processes. Hernández and coauthors investigated Hermite–Hadamard inequality for convex and (m, h_1, h_2)-convex stochastic processes via Katugampola fractional integrals in [15, 16], respectively. In [17], Jarad et al. introduced a new stochastic fractional integral and related inequalities of Jensen-Mercer and Hermite-Hadamard-Mercer type for convex stochastic processes. Set et al [31] established Hermite-Hadamard type inequalities for coordinates convex stochastic processes. For more studies, we refer the reader to [1, 9, 10] These developments clearly show that the synthesis of stochastic analysis, generalized convexity, and fractional operators continues to produce meaningful extensions of classical deterministic results.

Maclaurin's quadrature rule constitutes an important tool in numerical integration and approximation theory. In contrast to many classical quadrature formulas that involve endpoint evaluations, Maclaurin's rule is based on suitably chosen interior points of the interval, which gives it a distinctive practical and theoretical advantage. Indeed, the use of interior nodes makes this rule especially appropriate for functions that are not well defined, not easily computable, or exhibit singular behavior near the boundary of the interval. For this reason, Maclaurin-type inequalities are particularly valuable in situations where endpoint information is unavailable or unreliable. Their fractional extensions are even more appealing, since they combine the interior-point nature of Maclaurin's rule with the nonlocal and memory-sensitive structure of fractional operators, thereby providing a flexible framework for the analysis of stochastic systems with boundary irregularities and hereditary effects.

Recently, fractional calculus has witnessed substantial advancements through the introduction of generalized and hybrid operators that merge local and nonlocal dynamics. Among these extensions, the proportional Caputo-hybrid fractional operator has emerged as a particularly versatile tool. By unifying the proportional derivative (which captures local proportional control and exponential scaling) with the classic Caputo fractional operator (which models nonlocal memory and hereditary properties), this hybrid operator offers enhanced mathematical flexibility.

From both theoretical and applied modeling perspectives, utilizing the proportional Caputo-hybrid fractional operator within a stochastic framework provides several key advantages:

- **Dual Memory and Scaling Dynamics in Random Environments:** Stochastic processes in real-world systems (e.g., financial markets, anomalous diffusion, and biological networks) frequently display memory-dependent behaviors subject to instantaneous random scaling. The proportional parameter allows tuning local rate variations, while the fractional order controls continuous memory retention, making it exceptionally well-suited for capturing complex uncertainty and mean-square dynamics.
- **Physically Meaningful Initial Conditions:** In contrast to Riemann-Liouville type operators, the Caputo-hybrid operator accommodates initial conditions expressed in terms of standard (integer-order) mean-square derivatives. This property significantly simplifies the physical interpretation and formulation of stochastic fractional differential equations and integral inequalities.

Concurrently, integral inequalities involving proportional fractional operators have attracted significant attention. For instance, Sarikaya provided Hermite–Hadamard and Simpson-type inequalities for differentiable convex functions utilizing the proportional Caputo-hybrid operator in [29, 30], respectively. Demir established two versions of Milne-type integral inequalities for twice-differentiable convex functions in [11, 12], while another version of the same inequality was provided by Al-Hazmy et al. in [5]. Other versions of Newton-type inequalities were presented by Demir and Tunç [13] and Alqahtani et al. [6]. A more comprehensive study of proportional Caputo-hybrid fractional inequalities was provided by Alanzi et al. in [4].

Motivated by these considerations, the main purpose of this paper is to investigate Maclaurin-type inequalities for convex stochastic processes by means of proportional Caputo-hybrid fractional operators. More precisely, we introduce the left- and right-sided stochastic mean-square proportional Caputo-hybrid operators and establish a new integral identity involving these operators. This identity plays a fundamental role in deriving our main estimates. On the basis of this representation formula, we obtain new fractional Caputo-hybrid Maclaurin-type inequalities for convex stochastic processes under suitable differentiability and integrability assumptions. Our results may be viewed as stochastic fractional extensions of classical Maclaurin-type inequalities, and they contribute to the growing body of work connecting generalized integral inequalities with stochastic fractional calculus.

The remainder of this paper is organized as follows. In Section 2, we recall the basic concepts from stochastic analysis that will be needed throughout the paper. Section 3 introduces the stochastic mean-square proportional Caputo-hybrid operators and establishes an auxiliary integral identity that constitutes the backbone of our approach. In Section 4, we derive the main fractional Caputo-hybrid Maclaurin-type inequalities for convex stochastic processes. Numerical examples along with graphical representations confirming the validity of our obtained result are provided in Section 5. Finally, we conclude with remarks on the significance of the obtained results and possible directions for future investigations.

2. Preliminaries

In this section, we recall several basic notions from the theory of stochastic processes that will be used throughout the paper.

Definition 2.1 ([34]). Let $(\Upsilon, \mathbb{A}, \mathbb{P})$ be a probability space. A mapping

$$\mathrm{F} : \Upsilon \to \mathbb{R}$$

is called a random variable whenever it is $\mathbb{A}$-measurable. Furthermore, if $\mathcal{I} \subseteq \mathbb{R}$ is an interval, then a mapping

$$\mathrm{F} : \mathcal{I} \times \Upsilon \to \mathbb{R}$$

is said to be a stochastic process provided that, for each fixed $\jmath \in \mathcal{I}$, the function $\mathrm{F}(\jmath, \cdot)$ is a random variable.

Definition 2.2 ([34]). A stochastic process $\mathrm{F} : \mathcal{I} \times \Upsilon \to \mathbb{R}$ is said to be *continuous in probability* on $\mathcal{I}$ if, for every $\jmath_0 \in \mathcal{I}$,

$$\mathbb{P}\text{-} \lim_{\jmath \to \jmath_0} \mathrm{F}(\jmath, \cdot) = \mathrm{F}(\jmath_0, \cdot),$$

where $\mathbb{P}$-lim denotes convergence in probability.

Definition 2.3 ([34]). A stochastic process $\mathrm{F} : \mathcal{I} \times \Upsilon \to \mathbb{R}$ is called *mean-square continuous* on $\mathcal{I}$ if, for every $\jmath_0 \in \mathcal{I}$,

$$\lim_{\jmath \to \jmath_0} \mathbb{E}\Big[(\mathrm{F}(\jmath, \cdot) - \mathrm{F}(\jmath_0, \cdot))^2\Big] = 0,$$

where $\mathbb{E}[\cdot]$ stands for the expectation operator.

Definition 2.4 ([34]). We say that the stochastic process $\mathrm{F} : \mathcal{I} \times \Upsilon \to \mathbb{R}$ is *differentiable in probability* at a point $\jmath_0 \in \mathcal{I}$ if there exists a random variable

$$\mathrm{F}'(\jmath_0, \cdot)$$

such that

$$\mathbb{P}\text{-} \lim_{\jmath \to \jmath_0} \frac{\mathrm{F}(\jmath, \cdot) - \mathrm{F}(\jmath_0, \cdot)}{\jmath - \jmath_0} = \mathrm{F}'(\jmath_0, \cdot).$$

Definition 2.5 ([34]). A stochastic process $\mathrm{F} : \mathcal{I} \times \Upsilon \to \mathbb{R}$ is said to be *mean-square differentiable* at $\jmath_0 \in \mathcal{I}$ if there exists a random variable

$$\mathrm{F}'(\jmath_0, \cdot)$$

such that

$$\lim_{\jmath \to \jmath_0} \mathbb{E}\left[\left(\frac{\mathrm{F}(\jmath, \cdot) - \mathrm{F}(\jmath_0, \cdot)}{\jmath - \jmath_0} - \mathrm{F}'(\jmath_0, \cdot)\right)^2\right] = 0.$$

Definition 2.6 ([34]). Let $\mathrm{F} : \mathcal{I} \times \Upsilon \to \mathbb{R}$ be a stochastic process such that

$$\mathbb{E}[\mathrm{F}(\jmath, \cdot)^2] < \infty, \qquad \jmath \in \mathcal{I}.$$

A random variable $\mathrm{G} : \Upsilon \to \mathbb{R}$ is called the *mean-square integral* of F over the interval $[\mathfrak{e}_1, \mathfrak{e}_2] \subseteq \mathcal{I}$ if, for every partition

$$\Delta : \quad \mathfrak{e}_1 = \jmath_0 < \jmath_1 < \cdots < \jmath_m = \mathfrak{e}_2,$$

and every choice of sample points

$$\xi_k \in [\jmath_{k-1}, \jmath_k], \qquad k = 1, 2, \ldots, m,$$

the corresponding Riemann sums converge to G in the mean-square sense as the mesh of the partition tends to zero, that is,

$$\lim_{\|\Delta\|\to 0} \mathbb{E}\left[\left(\sum_{k=1}^{m} \mathrm{F}(\xi_k, \cdot)\,(\jmath_k - \jmath_{k-1}) - \mathrm{G}(\cdot)\right)^2\right] = 0,$$

where

$$\|\Delta\| = \max_{1\le k\le m} (\jmath_k - \jmath_{k-1}).$$

In this case, we write

$$\mathrm{G}(\cdot) = \int_{\mathrm{e}_1}^{\mathrm{e}_2} \mathrm{F}(\ell, \cdot)\,\mathrm{d}\ell \quad \text{(a.e.)}.$$

3. An auxiliary identity

In this section, we first introduce the notion of stochastic proportional Caputo-hybrid operators. We then establish a new integral identity that serves as the foundation for deriving Maclaurin-type inequalities for convex stochastic processes.

Definition 3.1. The left-sided and right-sided stochastic mean-square proportional Caputo-hybrid operators of order γ are expressed as :

$$\mathbb{D}^{\gamma}_{\mathrm{e}_1^+}[\mathrm{F}](\mathrm{e}_2) = \frac{1}{\Gamma(1-\gamma)} \int_{\mathrm{e}_1}^{\mathrm{e}_2} \left(\mathscr{B}_1(\gamma, \mathrm{e}_2 - t)\,\mathrm{F}(\jmath, \cdot) + \mathscr{B}_0(\gamma, \mathrm{e}_2 - t)\,\mathrm{F}'(\jmath, \cdot)\right)(\mathrm{e}_2 - t)^{-\gamma}\,\mathrm{d}\jmath,$$

and

$$\mathbb{D}^{\gamma}_{\mathrm{e}_2^-}[\mathrm{F}](\mathrm{e}_1) = \frac{1}{\Gamma(1-\gamma)} \int_{\mathrm{e}_1}^{\mathrm{e}_2} \left(\mathscr{B}_1(\gamma, \jmath - \mathrm{e}_1)\,\mathrm{F}(\jmath, \cdot) + \mathscr{B}_0(\gamma, \jmath - \mathrm{e}_1)\,\mathrm{F}'(\jmath, \cdot)\right)(\jmath - \mathrm{e}_1)^{-\gamma}\,\mathrm{d}\jmath,$$

respectively, where $\gamma \in [0, 1)$ and $\mathscr{B}_0(\gamma, \jmath) = (1-\gamma)^2 \jmath^{1-\gamma}$ and $\mathscr{B}_1(\gamma, \jmath) = \gamma^2 \jmath^{\gamma}$.

Lemma 3.2. *Let* $\mathrm{F} : [\mathrm{e}_1, \mathrm{e}_2] \times \Upsilon \to \mathbb{R}$ *be a mean-square differentiable stochastic. If* $\mathrm{F}, \mathrm{F}', \mathrm{F}''$ *are mean-square integrable on* $[\mathrm{e}_1, \mathrm{e}_2] \times \Upsilon$*, then the following equality holds almost everywhere:*

$$\gamma^2(\mathrm{e}_2 - \mathrm{e}_1)^{\gamma}\,\mathscr{M}(\mathrm{e}_1, \mathrm{e}_2, \mathrm{F}) + \frac{1-\gamma}{2(\mathrm{e}_2 - \mathrm{e}_1)^{\gamma-1}}\mathscr{M}(\mathrm{e}_1, \mathrm{e}_2, \mathrm{F}') - \frac{\Gamma(1-\gamma)}{2(\mathrm{e}_2 - \mathrm{e}_1)^{1-\gamma}}\left(\mathbb{D}^{\gamma}_{\mathrm{e}_2^-}[\mathrm{F}](\mathrm{e}_1) + \mathbb{D}^{\gamma}_{\mathrm{e}_1^+}[\mathrm{F}](\mathrm{e}_2)\right)$$

$$= \frac{\gamma^2(\mathrm{e}_2 - \mathrm{e}_1)^{1+\gamma}}{2}\left(\int_0^{\frac{1}{6}} \jmath\left(\mathrm{F}'(((1-\jmath)\,\mathrm{e}_1 + \jmath\mathrm{e}_2, \cdot), \cdot) - \mathrm{F}'(\jmath\mathrm{e}_1 + (1-\jmath)\,\mathrm{e}_2, \cdot)\right)\mathrm{d}\jmath\right.$$

$$+ \int_{\frac{1}{6}}^{\frac{1}{2}} \left(\jmath - \frac{3}{8}\right)\left(\mathrm{F}'((1-\jmath)\,\mathrm{e}_1 + \jmath\mathrm{e}_2, \cdot) - \mathrm{F}'(\jmath\mathrm{e}_1 + (1-\jmath)\,\mathrm{e}_2, \cdot)\right)\mathrm{d}\jmath$$

$$+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\jmath-\frac{5}{8}\right)\left(\mathrm{F}'\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}'\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath$$

$$\left.+\int_{\frac{5}{6}}^{1}(\jmath-1)\left(\mathrm{F}'\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}'\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath\right)$$

$$+\frac{(1-\gamma)\,(\mathrm{e}_2-\mathrm{e}_1)^{2-\gamma}}{4}\left(\int_0^{\frac{1}{6}}\jmath^{2-2\gamma}\left(\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath\right.$$

$$+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\jmath^{2-2\gamma}-\frac{3}{8}\right)\left(\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath$$

$$+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\jmath^{2-2\gamma}-\frac{5}{8}\right)\left(\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath$$

$$\left.+\int_{\frac{5}{6}}^{1}\left(\jmath^{2-2\gamma}-1\right)\left(\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath\right),$$

where $\gamma\in[0,1)$*, and*

$$\mathscr{M}\left(\mathrm{e}_1,\mathrm{e}_2,\mathrm{F}\right)=\frac{1}{8}\left[3\mathrm{F}\left(\frac{5\mathrm{e}_1+\mathrm{e}_2}{6},\cdot\right)+2\mathrm{F}\left(\frac{\mathrm{e}_1+\mathrm{e}_2}{2},\cdot\right)+3\mathrm{F}\left(\frac{\mathrm{e}_1+5\mathrm{e}_2}{6},\cdot\right)\right].$$

Proof. Let

$$\mathrm{N}_1=\int_0^{\frac{1}{6}}\jmath\left(\mathrm{F}'\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}'\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath,$$

$$\mathrm{N}_2=\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\jmath-\frac{3}{8}\right)\left(\mathrm{F}'\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}'\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath,$$

$$\mathrm{N}_3=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\jmath-\frac{5}{8}\right)\left(\mathrm{F}'\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)-\mathrm{F}'\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right)\mathrm{d}\jmath,$$

$$N_4 = \int_{\frac{5}{6}}^{1} (\jmath - 1)\left(F'\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) - F'\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right) d\jmath,$$

$$M_1 = \int_{0}^{\frac{1}{6}} \jmath^{2-2\gamma} \left(F''\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) - F''\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right) d\jmath,$$

$$M_2 = \int_{\frac{1}{6}}^{\frac{1}{2}} \left(\jmath^{2-2\gamma} - \frac{3}{8}\right)\left(F''\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) - F''\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right) d\jmath,$$

$$M_3 = \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\jmath^{2-2\gamma} - \frac{5}{8}\right)\left(F''\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) - F''\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right) d\jmath$$

and

$$M_4 = \int_{\frac{5}{6}}^{1} \left(\jmath^{2-2\gamma} - 1\right)\left(F''\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) - F''\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right) d\jmath.$$

Using the integration by parts, N_1 gives

$$\begin{aligned} N_1 =& \frac{1}{e_2 - e_1} \jmath\left(F\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) + F\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right)\Big|_{0}^{\frac{1}{6}} \\ &- \frac{1}{e_2 - e_1} \int_{0}^{\frac{1}{6}} \left(F\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) + F\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right) d\jmath \\ =& \frac{1}{6\,(e_2 - e_1)} \left(F\left(\frac{5e_1 + e_2}{6}, \cdot\right) + F\left(\frac{e_1 + 5e_2}{6}, \cdot\right)\right) \\ &- \frac{1}{e_2 - e_1} \int_{0}^{\frac{1}{6}} \left(F\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) + F\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right) d\jmath. \end{aligned} \tag{3.1}$$

Similarly, we obtain

$$\begin{aligned} N_2 =& \frac{1}{e_2 - e_1} \left(\jmath - \frac{3}{8}\right)\left(F\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) + F\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right)\Big|_{\frac{1}{6}}^{\frac{1}{2}} \\ &- \frac{1}{e_2 - e_1} \int_{\frac{1}{6}}^{\frac{1}{2}} \left(F\left((1-\jmath)\,e_1 + \jmath e_2, \cdot\right) + F\left(\jmath e_1 + (1-\jmath)\,e_2, \cdot\right)\right) d\jmath \end{aligned}$$

$$=\frac{1}{4(e_2-e_1)}F\left(\frac{e_1+e_2}{2},\cdot\right)+\frac{5}{24(e_2-e_1)}\left(F\left(\frac{5e_1+e_2}{6},\cdot\right)+F\left(\frac{e_1+5e_2}{6},\cdot\right)\right)$$
$$-\frac{1}{e_2-e_1}\int_{\frac{1}{6}}^{\frac{1}{2}}\left(F\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)d\jmath, \tag{3.2}$$

$$N_3=\frac{1}{e_2-e_1}\left(\jmath-\frac{5}{8}\right)\left(F\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)\Big|_{\frac{1}{2}}^{\frac{5}{6}}$$
$$-\frac{1}{e_2-e_1}\int_{\frac{1}{2}}^{\frac{5}{6}}\left(F\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)d\jmath$$
$$=\frac{5}{24(e_2-e_1)}\left(F\left(\frac{e_1+5e_2}{6},\cdot\right)+F\left(\frac{5e_1+e_2}{6},\cdot\right)\right)+\frac{1}{4(e_2-e_1)}F\left(\frac{e_1+e_2}{2},\cdot\right)$$
$$-\frac{1}{e_2-e_1}\int_{\frac{1}{2}}^{\frac{5}{6}}\left(F\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)d\jmath, \tag{3.3}$$

$$N_4=\frac{1}{e_2-e_1}\left(\jmath-1\right)\left(F\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)\Big|_{\frac{5}{6}}^{1}$$
$$-\frac{1}{e_2-e_1}\int_{\frac{5}{6}}^{1}\left(F\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)d\jmath$$
$$=\frac{1}{6(e_2-e_1)}\left(F\left(\frac{e_1+5e_2}{6},\cdot\right)+F\left(\frac{5e_1+e_2}{6},\cdot\right)\right)$$
$$-\frac{1}{e_2-e_1}\int_{\frac{5}{6}}^{1}\left(F\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)d\jmath, \tag{3.4}$$

$$M_1=\frac{1}{e_2-e_1}\jmath^{2-2\gamma}\left(F'\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F'\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)\Big|_{0}^{\frac{1}{6}}$$
$$-\frac{2-2\gamma}{e_2-e_1}\int_{0}^{\frac{1}{6}}\jmath^{1-2\gamma}\left(F'\left(\left(1-\jmath\right)e_1+\jmath e_2,\cdot\right)+F'\left(\jmath e_1+\left(1-\jmath\right)e_2,\cdot\right)\right)d\jmath$$

$$
\begin{aligned}
&= \frac{1}{e_2 - e_1}\left(\frac{1}{6}\right)^{2-2\gamma}\left(F'\left(\frac{5e_1 + e_2}{6}, \cdot\right) + F'\left(\frac{e_1 + 5e_2}{6}, \cdot\right)\right) \\
&\quad - \frac{2-2\gamma}{e_2 - e_1}\int_0^{\frac{1}{6}} \jmath^{1-2\gamma}\left(F'\left((1-\jmath)e_1 + \jmath e_2, \cdot\right) + F'\left(\jmath e_1 + (1-\jmath)e_2, \cdot\right)\right) d\jmath,
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
M_2 &= \frac{1}{e_2 - e_1}\left(\jmath^{2-2\gamma} - \frac{3}{8}\right)\left(F'\left((1-\jmath)e_1 + \jmath e_2, \cdot\right) + F'\left(\jmath e_1 + (1-\jmath)e_2, \cdot\right)\right)\Big|_{\frac{1}{6}}^{\frac{1}{2}} \\
&\quad - \frac{2-2\gamma}{e_2 - e_1}\int_{\frac{1}{6}}^{\frac{1}{2}} \jmath^{1-2\gamma}\left(F'\left((1-\jmath)e_1 + \jmath e_2, \cdot\right) + F'\left(\jmath e_1 + (1-\jmath)e_2, \cdot\right)\right) d\jmath \\
&= \frac{2}{e_2 - e_1}\left(\left(\frac{1}{2}\right)^{2-2\gamma} - \frac{3}{8}\right)F\left(\frac{e_1 + e_2}{2}, \cdot\right) \\
&\quad - \frac{1}{e_2 - e_1}\left(\left(\frac{1}{6}\right)^{2-2\gamma} - \frac{3}{8}\right)\left(F\left(\frac{5e_1 + e_2}{6}, \cdot\right) + F\left(\frac{e_1 + 5e_2}{6}, \cdot\right)\right) \\
&\quad - \frac{2-2\gamma}{e_2 - e_1}\int_{\frac{1}{6}}^{\frac{1}{2}} \jmath^{1-2\gamma}\left(F'\left((1-\jmath)e_1 + \jmath e_2, \cdot\right) + F'\left(\jmath e_1 + (1-\jmath)e_2, \cdot\right)\right) d\jmath,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
M_3 &= \frac{1}{e_2 - e_1}\left(\jmath^{2-2\gamma} - \frac{5}{8}\right)\left(F'\left((1-\jmath)e_1 + \jmath e_2, \cdot\right) + F'\left(\jmath e_1 + (1-\jmath)e_2, \cdot\right)\right)\Big|_{\frac{1}{2}}^{\frac{5}{6}} \\
&\quad - \frac{2-2\gamma}{e_2 - e_1}\int_{\frac{1}{2}}^{\frac{5}{6}} \jmath^{1-2\gamma}\left(F'\left((1-\jmath)e_1 + \jmath e_2, \cdot\right) + F'\left(\jmath e_1 + (1-\jmath)e_2, \cdot\right)\right) d\jmath \\
&= \frac{1}{e_2 - e_1}\left(\left(\frac{5}{6}\right)^{2-2\gamma} - \frac{5}{8}\right)\left(F\left(\frac{e_1 + 5e_2}{6}, \cdot\right) + F\left(\frac{5e_1 + e_2}{6}, \cdot\right)\right) \\
&\quad - \frac{2}{e_2 - e_1}\left(\left(\frac{1}{2}\right)^{2-2\gamma} - \frac{5}{8}\right)F\left(\frac{e_1 + e_2}{2}, \cdot\right) \\
&\quad - \frac{2-2\gamma}{e_2 - e_1}\int_{\frac{1}{2}}^{\frac{5}{6}} \jmath^{1-2\gamma}\left(F'\left((1-\jmath)e_1 + \jmath e_2, \cdot\right) + F'\left(\jmath e_1 + (1-\jmath)e_2, \cdot\right)\right) d\jmath
\end{aligned} \tag{3.7}
$$

and

$$
M_4 = \frac{1}{e_2 - e_1}\left(\jmath^{2-2\gamma} - 1\right)\left(F'\left((1-\jmath)e_1 + \jmath e_2, \cdot\right) + F'\left(\jmath e_1 + (1-\jmath)e_2, \cdot\right)\right)\Big|_{\frac{5}{6}}^{1}
$$

$$-\frac{1}{e_2-e_1}\int_{\frac{5}{6}}^{1} \jmath^{1-2\gamma}\left(F'\left((1-\jmath)e_1+\jmath e_2,\cdot\right)+F'\left(\jmath e_1+(1-\jmath)e_2,\cdot\right)\right)d\jmath$$
$$=\frac{1}{e_2-e_1}\left(1-\left(\frac{5}{6}\right)^{2-2\gamma}\right)\left(F\left(\frac{e_1+5e_2}{6},\cdot\right)+F\left(\frac{5e_1+e_2}{6},\cdot\right)\right)$$
$$-\frac{2-2\gamma}{e_2-e_1}\int_{\frac{5}{6}}^{1} \jmath^{1-2\gamma}\left(F'\left((1-\jmath)e_1+\jmath e_2,\cdot\right)+F'\left(\jmath e_1+(1-\jmath)e_2,\cdot\right)\right)d\jmath. \tag{3.8}$$

Summing (3.1)–(3.4), then using the change of variable, we get

$$\sum_{k=1}^{4}N_k=\frac{1}{4(e_2-e_1)}\left(3F\left(\frac{5e_1+e_2}{6},\cdot\right)+2F\left(\frac{e_1+e_2}{2},\cdot\right)+3F\left(\frac{e_1+5e_2}{6},\cdot\right)\right) \tag{3.9}$$
$$-\frac{1}{e_2-e_1}\int_{0}^{1}\left(F\left((1-\jmath)e_1+\jmath e_2,\cdot\right)+F\left(\jmath e_1+(1-\jmath)e_2,\cdot\right)\right)d\jmath$$
$$=\frac{1}{4(e_2-e_1)}\left(3F\left(\frac{5e_1+e_2}{6},\cdot\right)+2F\left(\frac{e_1+e_2}{2},\cdot\right)+3F\left(\frac{e_1+5e_2}{6},\cdot\right)\right)-\frac{2}{(e_2-e_1)^2}\int_{e_1}^{e_2}F(u,\cdot)\,du.$$

Now, adding (3.5)–(3.8), then using the change of variable, we get

$$\sum_{k=1}^{4}M_k=\frac{1}{4(e_2-e_1)}\left(3F'\left(\frac{5e_1+e_2}{6},\cdot\right)+2F'\left(\frac{e_1+e_2}{2},\cdot\right)+3F'\left(\frac{e_1+5e_2}{6},\cdot\right)\right) \tag{3.10}$$
$$-\frac{2-2\gamma}{e_2-e_1}\int_{0}^{1} \jmath^{1-2\gamma}\left(F'\left((1-\jmath)e_1+\jmath e_2,\cdot\right)+F'\left(\jmath e_1+(1-\jmath)e_2,\cdot\right)\right)d\jmath$$
$$=\frac{1}{4(e_2-e_1)}\left(3F'\left(\frac{5e_1+e_2}{6},\cdot\right)+2F'\left(\frac{e_1+e_2}{2},\cdot\right)+3F'\left(\frac{e_1+5e_2}{6},\cdot\right)\right)$$
$$-\frac{2-2\gamma}{(e_2-e_1)^{3-2\gamma}}\left(\int_{e_1}^{e_2}(u-e_1)^{1-2\gamma}F'(u,\cdot)\,du+\int_{e_1}^{e_2}(e_2-u)^{1-2\gamma}F'(u,\cdot)\,du\right).$$

Multiplying (3.9) by $\frac{\gamma^2(e_2-e_1)^{1+\gamma}}{2}$ and (3.10) by $\frac{(1-\gamma)(e_2-e_1)^{2-\gamma}}{4}$, summing the resulting equalities, we obtain

$$\frac{\gamma^2(e_2-e_1)^{1+\gamma}}{2}\sum_{k=1}^{4}N_k+\frac{(1-\gamma)(e_2-e_1)^{2-\gamma}}{4}\sum_{k=1}^{4}M_k$$
$$=\gamma^2(e_2-e_1)^{\gamma}\left(\frac{3F\left(\frac{5e_1+e_2}{6},\cdot\right)+2F\left(\frac{e_1+e_2}{2},\cdot\right)+3F\left(\frac{e_1+5e_2}{6},\cdot\right)}{8}\right)$$

$$
\begin{aligned}
&+\frac{1-\gamma}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\gamma-1}}\left(\frac{3\mathrm{F}'\left(\frac{5\mathrm{e}_1+\mathrm{e}_2}{6},\cdot\right)+2\mathrm{F}'\left(\frac{\mathrm{e}_1+\mathrm{e}_2}{2},\cdot\right)+3\mathrm{F}'\left(\frac{\mathrm{e}_1+5\mathrm{e}_2}{6},\cdot\right)}{8}\right)\\
&-\frac{\gamma^2}{\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1-\gamma}}\int_{\mathrm{e}_1}^{\mathrm{e}_2}\mathrm{F}(\ell,\cdot)\,\mathrm{d}\ell-\frac{(1-\gamma)^2}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1-\gamma}}\left(\int_{\mathrm{e}_1}^{\mathrm{e}_2}\left(\ell-\mathrm{e}_1\right)^{1-2\gamma}\mathrm{F}'(\ell,\cdot)\,\mathrm{d}\ell+\int_{\mathrm{e}_1}^{\mathrm{e}_2}\left(\mathrm{e}_2-\ell\right)^{1-2\gamma}\mathrm{F}'(\ell,\cdot)\,\mathrm{d}\ell\right)\\
=&\gamma^2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\gamma}\left(\frac{3\mathrm{F}\left(\frac{5\mathrm{e}_1+\mathrm{e}_2}{6},\cdot\right)+2\mathrm{F}\left(\frac{\mathrm{e}_1+\mathrm{e}_2}{2},\cdot\right)+3\mathrm{F}\left(\frac{\mathrm{e}_1+5\mathrm{e}_2}{6},\cdot\right)}{8}\right)\\
&+\frac{1-\gamma}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\gamma-1}}\left(\frac{3\mathrm{F}'\left(\frac{5\mathrm{e}_1+\mathrm{e}_2}{6},\cdot\right)+2\mathrm{F}'\left(\frac{\mathrm{e}_1+\mathrm{e}_2}{2},\cdot\right)+3\mathrm{F}'\left(\frac{\mathrm{e}_1+5\mathrm{e}_2}{6},\cdot\right)}{8}\right)\\
&-\frac{1}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1-\gamma}}\left(\int_{\mathrm{e}_1}^{\mathrm{e}_2}\left(\gamma^2\left(\ell-\mathrm{e}_1\right)^{\gamma}\mathrm{F}(\ell,\cdot)+(1-\gamma)^2\left(\ell-\mathrm{e}_1\right)^{1-\gamma}\mathrm{F}'(\ell,\cdot)\right)\left(\ell-\mathrm{e}_1\right)^{-\gamma}\mathrm{d}\ell\right.\\
&\left.+\int_{\mathrm{e}_1}^{\mathrm{e}_2}\left(\gamma^2\left(\mathrm{e}_2-\ell\right)^{\gamma}\mathrm{F}(\ell,\cdot)+(1-\gamma)^2\left(\mathrm{e}_2-\ell\right)^{1-\gamma}\mathrm{F}'(\ell,\cdot)\right)\left(\mathrm{e}_2-\ell\right)^{-\gamma}\mathrm{d}\ell\right)\\
=&\gamma^2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\gamma}\left(\frac{3\mathrm{F}\left(\frac{5\mathrm{e}_1+\mathrm{e}_2}{6},\cdot\right)+2\mathrm{F}\left(\frac{\mathrm{e}_1+\mathrm{e}_2}{2},\cdot\right)+3\mathrm{F}\left(\frac{\mathrm{e}_1+5\mathrm{e}_2}{6},\cdot\right)}{8}\right)\\
&+\frac{1-\gamma}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\gamma-1}}\left(\frac{3\mathrm{F}'\left(\frac{5\mathrm{e}_1+\mathrm{e}_2}{6},\cdot\right)+2\mathrm{F}'\left(\frac{\mathrm{e}_1+\mathrm{e}_2}{2},\cdot\right)+3\mathrm{F}'\left(\frac{\mathrm{e}_1+5\mathrm{e}_2}{6},\cdot\right)}{8}\right)\\
&-\frac{\Gamma(1-\gamma)}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1-\gamma}}\left(\mathbb{D}^{\gamma}_{\mathrm{e}_2^-}[\mathrm{F}]\left(\mathrm{e}_1\right)+\mathbb{D}^{\gamma}_{\mathrm{e}_1^+}[\mathrm{F}]\left(\mathrm{e}_2\right)\right),
\end{aligned}
$$

which is the desired result. □

4. Fractional Caputo-hybrid maclaurin-type inequalities

This section is devoted to the derivation of new proportional Caputo-hybrid Maclaurin-type inequalities for convex stochastic processes by means of the integral identity obtained in the preceding section.

Theorem 4.1. *Let* $\mathrm{F}:[\mathrm{e}_1,\mathrm{e}_2]\times\Upsilon\to\mathbb{R}$ *be a stochastic process in the interval* $[\mathrm{e}_1,\mathrm{e}_2]$ *with* $\mathrm{e}_1,\mathrm{e}_2$*, such that* $\mathrm{F},\mathrm{F}',\mathrm{F}''$ *and mean-square differentiable stochastic processes on* $[\mathrm{e}_1,\mathrm{e}_2]\times\Upsilon$*. If* $|\mathrm{F}'|$ *and* $|\mathrm{F}''|$ *are convex stochastic processes on* $[\mathrm{e}_1,\mathrm{e}_2]\times\Upsilon$*, then the following inequality holds almost everywhere*

$$
\begin{aligned}
&\left|\gamma^2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\gamma}\mathscr{M}\left(\mathrm{e}_1,\mathrm{e}_2,\mathrm{F}\right)+\frac{1-\gamma}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\gamma-1}}\mathscr{M}\left(\mathrm{e}_1,\mathrm{e}_2,\mathrm{F}'\right)-\frac{\Gamma(1-\gamma)}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1-\gamma}}\left(\mathbb{D}^{\gamma}_{\mathrm{e}_2^-}[\mathrm{F}]\left(\mathrm{e}_1\right)+\mathbb{D}^{\gamma}_{\mathrm{e}_1^+}[\mathrm{F}]\left(\mathrm{e}_2\right)\right)\right|\\
\le&\frac{25\gamma^2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1+\gamma}}{576}\left(\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|+\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|\right)\\
&+\frac{(1-\gamma)\left(\mathrm{e}_2-\mathrm{e}_1\right)^{2-\gamma}}{4}\left(\frac{1+5^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\frac{3+2\gamma}{6(3-2\gamma)}+\Omega_1(\gamma)+\Omega_2(\gamma)\right)\left(\left|\mathrm{F}''\left(\mathrm{e}_1,\cdot\right)\right|+\left|\mathrm{F}''\left(\mathrm{e}_2,\cdot\right)\right|\right),
\end{aligned}
$$

where $\Omega(\gamma)$ *and* $\Omega_2(\gamma)$ *are defined as in* (4.1) *and* (4.2)*, respectively.*

Proof. From Lemma 3.2, absolute value, and stochastic convexity of $|F'|$ and $|F''|$, we have

$$\begin{aligned}
&\left|\gamma^2(e_2-e_1)^\gamma \mathscr{M}(e_1,e_2,F)+\frac{1-\gamma}{2(e_2-e_1)^{\gamma-1}}\mathscr{M}(e_1,e_2,F')-\frac{\Gamma(1-\gamma)}{2(e_2-e_1)^{1-\gamma}}\left(\mathbb{D}^\gamma_{e_2^-}[F](e_1)+\mathbb{D}^\gamma_{e_1^+}[F](e_2)\right)\right|\\
\leq&\frac{\gamma^2(e_2-e_1)^{1+\gamma}}{2}\left(\int_0^{\frac{1}{6}} \jmath\left(|F'((1-\jmath)e_1+\jmath e_2,\cdot)|+|F'(\jmath e_1+(1-\jmath)e_2,\cdot)|\right)d\jmath\right.\\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\left(|F'((1-\jmath)e_1+\jmath e_2,\cdot)|+|F'(\jmath e_1+(1-\jmath)e_2,\cdot)|\right)d\jmath\\
&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\left(|F'((1-\jmath)e_1+\jmath e_2,\cdot)|+|F'(\jmath e_1+(1-\jmath)e_2,\cdot)|\right)d\jmath\\
&\left.+\int_{\frac{5}{6}}^{1}(1-\jmath)\left(|F'((1-\jmath)e_1+\jmath e_2,\cdot)|+|F'(\jmath e_1+(1-\jmath)e_2,\cdot)|\right)d\jmath\right)\\
&+\frac{(1-\gamma)(e_2-e_1)^{2-\gamma}}{4}\left(\int_0^{\frac{1}{6}} \jmath^{2-2\gamma}\left(|F''((1-\jmath)e_1+\jmath e_2,\cdot)|+|F''(\jmath e_1+(1-\jmath)e_2,\cdot)|\right)d\jmath\right.\\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|\left(|F''((1-\jmath)e_1+\jmath e_2,\cdot)|+|F''(\jmath e_1+(1-\jmath)e_2,\cdot)|\right)d\jmath\\
&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|\left(|F''((1-\jmath)e_1+\jmath e_2,\cdot)|+|F''(\jmath e_1+(1-\jmath)e_2,\cdot)|\right)d\jmath\\
&\left.+\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)\left(|F''((1-\jmath)e_1+\jmath e_2,\cdot)|+|F''(\jmath e_1+(1-\jmath)e_2,\cdot)|\right)d\jmath\right)
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{\gamma^2(\mathrm{e}_2-\mathrm{e}_1)^{1+\gamma}}{2}\left(\int_0^{\frac{1}{6}} \jmath\left(|\mathrm{F}'(\mathrm{e}_1,\cdot)|+|\mathrm{F}'(\mathrm{e}_2,\cdot)|\right)\mathrm{d}\jmath+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\left(|\mathrm{F}'(\mathrm{e}_1,\cdot)|+|\mathrm{F}'(\mathrm{e}_2,\cdot)|\right)\mathrm{d}\jmath\right.\\
& \left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\left(|\mathrm{F}'(\mathrm{e}_1,\cdot)|+|\mathrm{F}'(\mathrm{e}_2,\cdot)|\right)\mathrm{d}\jmath+\int_{\frac{5}{6}}^{1}(1-\jmath)\left(|\mathrm{F}'(\mathrm{e}_1,\cdot)|+|\mathrm{F}'(\mathrm{e}_2,\cdot)|\right)\mathrm{d}\jmath\right)\\
& +\frac{(1-\gamma)(\mathrm{e}_2-\mathrm{e}_1)^{2-\gamma}}{4}\left(\int_0^{\frac{1}{6}}\jmath^{2-2\gamma}\left(|\mathrm{F}''(\mathrm{e}_1,\cdot)|+|\mathrm{F}''(\mathrm{e}_2,\cdot)|\right)\mathrm{d}\jmath+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|\left(|\mathrm{F}''(\mathrm{e}_1,\cdot)|+|\mathrm{F}''(\mathrm{e}_2,\cdot)|\right)\mathrm{d}\jmath\right.\\
& \left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|\left(|\mathrm{F}''(\mathrm{e}_1,\cdot)|+|\mathrm{F}''(\mathrm{e}_2,\cdot)|\right)\mathrm{d}\jmath+\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)\left(|\mathrm{F}''(\mathrm{e}_1,\cdot)|+|\mathrm{F}''(\mathrm{e}_2,\cdot)|\right)\mathrm{d}\jmath\right)\\
= & \frac{25\gamma^2(\mathrm{e}_2-\mathrm{e}_1)^{1+\gamma}}{576}\left(|\mathrm{F}'(\mathrm{e}_1,\cdot)|+|\mathrm{F}'(\mathrm{e}_2,\cdot)|\right)\\
& +\frac{(1-\gamma)(\mathrm{e}_2-\mathrm{e}_1)^{2-\gamma}}{4}\left(\frac{1+5^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\frac{3+2\gamma}{6(3-2\gamma)}+\Omega_1(\gamma)+\Omega_2(\gamma)\right)\left(|\mathrm{F}''(\mathrm{e}_1,\cdot)|+|\mathrm{F}''(\mathrm{e}_2,\cdot)|\right),
\end{aligned}
$$

where we have used

$$\int_0^{\frac{1}{6}}\jmath\,\mathrm{d}\jmath=\int_{\frac{5}{6}}^{1}(1-\jmath)\,\mathrm{d}\jmath=\frac{1}{72},$$

$$\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\mathrm{d}\jmath=\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\mathrm{d}\jmath=\frac{17}{576},$$

$$\int_0^{\frac{1}{6}}\jmath^{2-2\gamma}\mathrm{d}\jmath=\frac{1}{6^{3-2\gamma}(3-2\gamma)},$$

$$\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)\mathrm{d}\jmath=\frac{5^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\frac{3+2\gamma}{6(3-2\gamma)},$$

$$\Omega_1(\gamma)=\int_{\frac{1}{6}}^{\frac{1}{2}}\left|J^{2-2\gamma}-\frac{3}{8}\right|\mathrm{d}J=\begin{cases}\dfrac{3^{3-2\gamma}-1}{6^{3-2\gamma}(3-2\gamma)}-\dfrac{1}{8}, & \text{if}\quad 0\le\left(\dfrac{3}{8}\right)^{\frac{1}{2-2\gamma}}<\dfrac{1}{6},\\ \dfrac{1+3^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\dfrac{1}{4}+\dfrac{3(1-\gamma)}{2(3-2\gamma)}\left(\dfrac{3}{8}\right)^{\frac{1}{2-2\gamma}}, & \text{if}\quad \dfrac{1}{6}\le\left(\dfrac{3}{8}\right)^{\frac{1}{2-2\gamma}}\le\dfrac{1}{2},\\ \dfrac{1}{8}-\dfrac{3^{3-2\gamma}-1}{6^{3-2\gamma}(3-2\gamma)}, & \text{if}\quad \dfrac{1}{2}<\left(\dfrac{3}{8}\right)^{\frac{1}{2-2\gamma}}\le 1,\end{cases}\tag{4.1}$$

and

$$\Omega_2(\gamma)=\int_{\frac{1}{2}}^{\frac{5}{6}}\left|J^{2-2\gamma}-\frac{5}{8}\right|\mathrm{d}J=\begin{cases}\dfrac{5^{3-2\gamma}-3^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\dfrac{5}{24}, & \text{if}\quad 0<\left(\dfrac{5}{8}\right)^{\frac{1}{2-2\gamma}}\le\dfrac{1}{2},\\ \dfrac{3^{3-2\gamma}+5^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\dfrac{5}{6}+\dfrac{5(1-\gamma)}{2(3-2\gamma)}\left(\dfrac{5}{8}\right)^{\frac{1}{2-2\gamma}}, & \text{if}\quad \dfrac{1}{2}<\left(\dfrac{5}{8}\right)^{\frac{1}{2-2\gamma}}<\dfrac{5}{6},\\ \dfrac{5}{24}-\dfrac{5^{3-2\gamma}-3^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)} & \text{if}\quad \dfrac{5}{6}\le\left(\dfrac{5}{8}\right)^{\frac{1}{2-2\gamma}}\le 1.\end{cases}\tag{4.2}$$

The proof is completed. □

Corollary 4.2. *If we attempt to tend γ to 1, Theorem 4.1 yields*

$$\left|\frac{1}{8}\left(3\mathrm{F}\left(\frac{5\mathrm{e}_1+\mathrm{e}_2}{6},\cdot\right)+2\mathrm{F}\left(\frac{\mathrm{e}_1+\mathrm{e}_2}{2},\cdot\right)+3\mathrm{F}\left(\frac{\mathrm{e}_1+5\mathrm{e}_2}{6},\cdot\right)\right)-\frac{1}{\mathrm{e}_2-\mathrm{e}_1}\int_{\mathrm{e}_1}^{\mathrm{e}_2}\mathrm{F}(u,\cdot)\,\mathrm{d}u\right|$$
$$\le\frac{25(\mathrm{e}_2-\mathrm{e}_1)}{576}\left(|\mathrm{F}'(\mathrm{e}_1,\cdot)|+|\mathrm{F}'(\mathrm{e}_2,\cdot)|\right).$$

Corollary 4.3. *In Theorem 4.1, if we take $\gamma=0$, we obtain*

$$\left|\frac{1}{8}\left(3\mathrm{F}'\left(\frac{5\mathrm{e}_1+\mathrm{e}_2}{6},\cdot\right)+2\mathrm{F}'\left(\frac{\mathrm{e}_1+\mathrm{e}_2}{2},\cdot\right)+3\mathrm{F}'\left(\frac{\mathrm{e}_1+5\mathrm{e}_2}{6},\cdot\right)\right)-\frac{\mathrm{F}(\mathrm{e}_2,\cdot)-\mathrm{F}(\mathrm{e}_1,\cdot)}{\mathrm{e}_2-\mathrm{e}_1}\right|$$
$$\le\frac{29(\mathrm{e}_2-\mathrm{e}_1)}{1296}\left(|\mathrm{F}''(\mathrm{e}_1,\cdot)|+|\mathrm{F}''(\mathrm{e}_2,\cdot)|\right).$$

Theorem 4.4. *Let $\mathrm{F}:[\mathrm{e}_1,\mathrm{e}_2]\times\Upsilon\to\mathbb{R}$ be a stochastic process in the interval $[\mathrm{e}_1,\mathrm{e}_2]$ with $\mathrm{e}_1,\mathrm{e}_2$, such that $\mathrm{F},\mathrm{F}',\mathrm{F}''$ are mean-square differentiable stochastic processes on $[\mathrm{e}_1,\mathrm{e}_2]\times\Upsilon$. If $|\mathrm{F}'|^q$ and $|\mathrm{F}''|^q$, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ are convex stochastic process on $[\mathrm{e}_1,\mathrm{e}_2]\times\Upsilon$, then the following inequality holds almost everywhere*

$$\left|\gamma^2(\mathrm{e}_2-\mathrm{e}_1)^{\gamma}\mathscr{M}(\mathrm{e}_1,\mathrm{e}_2,\mathrm{F})+\frac{1-\gamma}{2(\mathrm{e}_2-\mathrm{e}_1)^{\gamma-1}}\mathscr{M}(\mathrm{e}_1,\mathrm{e}_2,\mathrm{F}')-\frac{\Gamma(1-\gamma)}{2(\mathrm{e}_2-\mathrm{e}_1)^{1-\gamma}}\left(\mathbb{D}^{\gamma}_{\mathrm{e}_2^-}[\mathrm{F}](\mathrm{e}_1)+\mathbb{D}^{\gamma}_{\mathrm{e}_1^+}[\mathrm{F}](\mathrm{e}_2)\right)\right|$$

$$
\begin{aligned}
\leq & \gamma^2 (\mathrm{e}_2 - \mathrm{e}_1)^{1+\gamma} \left(\left(\frac{1}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\left(\frac{11\left|\mathrm{F}'(\mathrm{e}_1,\cdot)\right|^q + \left|\mathrm{F}'(\mathrm{e}_2,\cdot)\right|^q}{72} \right)^{\frac{1}{q}} + \left(\frac{\left|\mathrm{F}'(\mathrm{e}_1,\cdot)\right|^q + 11\left|\mathrm{F}'(\mathrm{e}_2,\cdot)\right|^q}{72} \right)^{\frac{1}{q}} \right) \right. \\
& + \left(\frac{5^{p+1} + 3^{p+1}}{24^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\left(\frac{2\left|\mathrm{F}'(\mathrm{e}_1,\cdot)\right|^q + \left|\mathrm{F}'(\mathrm{e}_2,\cdot)\right|^q}{9} \right)^{\frac{1}{q}} + \left(\frac{\left|\mathrm{F}'(\mathrm{e}_1,\cdot)\right|^q + 2\left|\mathrm{F}'(\mathrm{e}_2,\cdot)\right|^q}{9} \right)^{\frac{1}{q}} \right) \Bigg) \\
& + \frac{(1-\gamma)(\mathrm{e}_2 - \mathrm{e}_1)^{2-\gamma}}{4} \left(\left(\frac{6^{(2\gamma-2)p+1}}{((2-2\gamma)p+1)} \right)^{\frac{1}{p}} \right. \\
& \times \left(\left(\frac{11\left|\mathrm{F}''(\mathrm{e}_1,\cdot)\right|^q + \left|\mathrm{F}''(\mathrm{e}_2,\cdot)\right|^q}{72} \right)^{\frac{1}{q}} + \left(\frac{\left|\mathrm{F}''(\mathrm{e}_1,\cdot)\right|^q + 11\left|\mathrm{F}''(\mathrm{e}_2,\cdot)\right|^q}{72} \right)^{\frac{1}{q}} \right) \\
& + \left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left| \jmath^{2-2\gamma} - \frac{3}{8} \right|^p d\jmath \right)^{\frac{1}{p}} + \left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left| \jmath^{2-2\gamma} - \frac{5}{8} \right|^p d\jmath \right)^{\frac{1}{p}} \right) \\
& \times \left(\left(\frac{2\left|\mathrm{F}''(\mathrm{e}_1,\cdot)\right|^q + \left|\mathrm{F}''(\mathrm{e}_2,\cdot)\right|^q}{9} \right)^{\frac{1}{q}} + \left(\frac{\left|\mathrm{F}''(\mathrm{e}_1,\cdot)\right|^q + 2\left|\mathrm{F}''(\mathrm{e}_2,\cdot)\right|^q}{9} \right)^{\frac{1}{q}} \right) \\
& + \left(\frac{\mathrm{B}\left(\frac{1}{2-2\gamma}, p+1\right) - \mathrm{B}_{\left(\frac{5}{6}\right)^{2-2\gamma}}\left(\frac{1}{2-2\gamma}, p+1\right)}{2-2\gamma} \right)^{\frac{1}{p}} \\
& \times \left. \left(\left(\frac{\left|\mathrm{F}''(\mathrm{e}_1,\cdot)\right|^q + 11\left|\mathrm{F}''(\mathrm{e}_2,\cdot)\right|^q}{72} \right)^{\frac{1}{q}} + \left(\frac{11\left|\mathrm{F}''(\mathrm{e}_1,\cdot)\right|^q + \left|\mathrm{F}''(\mathrm{e}_2,\cdot)\right|^q}{72} \right)^{\frac{1}{q}} \right) \right),
\end{aligned}
$$

where $\mathrm{B}(\cdot,\cdot)$ *and* $\mathrm{B}_\sigma(\cdot,\cdot)$ *denote the beta and the incomplete beta functions, respectively.*

Proof. From Lemma 3.2, absolute value, Hölder's inequality, and stochastic convexity of $|\mathrm{F}'|^q$ and $|\mathrm{F}''|^q$, we have

$$
\begin{aligned}
& \left| \gamma^2 (\mathrm{e}_2 - \mathrm{e}_1)^{\gamma} \mathscr{M}(\mathrm{e}_1, \mathrm{e}_2, \mathrm{F}) + \frac{1-\gamma}{2(\mathrm{e}_2 - \mathrm{e}_1)^{\gamma-1}} \mathscr{M}(\mathrm{e}_1, \mathrm{e}_2, \mathrm{F}') - \frac{\Gamma(1-\gamma)}{2(\mathrm{e}_2 - \mathrm{e}_1)^{1-\gamma}} \left(\mathbb{D}^{\gamma}_{\mathrm{e}_2^-}[\mathrm{F}](\mathrm{e}_1) + \mathbb{D}^{\gamma}_{\mathrm{e}_1^+}[\mathrm{F}](\mathrm{e}_2) \right) \right| \\
\leq & \frac{\gamma^2 (\mathrm{e}_2 - \mathrm{e}_1)^{1+\gamma}}{2} \left(\left(\int_0^{\frac{1}{6}} \jmath^p d\jmath \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{6}} \left|\mathrm{F}'((1-\jmath)\mathrm{e}_1 + \jmath\mathrm{e}_2, \cdot)\right|^q d\jmath \right)^{\frac{1}{q}} + \left(\int_0^{\frac{1}{6}} \jmath^p d\jmath \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{6}} \left|\mathrm{F}'(\jmath\mathrm{e}_1 + (1-\jmath)\mathrm{e}_2, \cdot)\right|^q d\jmath \right)^{\frac{1}{q}} \right. \\
& + \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left| \jmath - \frac{3}{8} \right|^p d\jmath \right)^{\frac{1}{p}} \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left|\mathrm{F}'((1-\jmath)\mathrm{e}_1 + \jmath\mathrm{e}_2, \cdot)\right|^q d\jmath \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left| \jmath - \frac{3}{8} \right|^p d\jmath \right)^{\frac{1}{p}} \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left|\mathrm{F}'(\jmath\mathrm{e}_1 + (1-\jmath)\mathrm{e}_2, \cdot)\right|^q d\jmath \right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|^{p}\mathrm{d}\jmath\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{F}'\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|^{p}\mathrm{d}\jmath\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{F}'\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{5}{6}}^{1}(1-\jmath)^{p}\mathrm{d}\jmath\right)^{\frac{1}{p}}\left(\int_{\frac{5}{6}}^{1}\left|\mathrm{F}'\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{\frac{5}{6}}^{1}(1-\jmath)^{p}\mathrm{d}\jmath\right)^{\frac{1}{p}}\left(\int_{\frac{5}{6}}^{1}\left|\mathrm{F}'\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}\Bigg)$$

$$+\frac{(1-\gamma)\,(\mathrm{e}_2-\mathrm{e}_1)^{2-\gamma}}{4}$$

$$\times\left(\left(\int_{0}^{\frac{1}{6}}\jmath^{(2-2\gamma)p}\mathrm{d}\jmath\right)^{\frac{1}{p}}\left[\left(\int_{0}^{\frac{1}{6}}\left|\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{0}^{\frac{1}{6}}\left|\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]\right.$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|^{p}\mathrm{d}\jmath\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|^{p}\mathrm{d}\jmath\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]$$

$$\left.+\left(\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)^{p}\mathrm{d}\jmath\right)^{\frac{1}{p}}\left[\left(\int_{\frac{5}{6}}^{1}\left|\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{\frac{5}{6}}^{1}\left|\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right|^{q}\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]\right)\Bigg]$$

$$\leq\frac{\gamma^{2}\,(\mathrm{e}_2-\mathrm{e}_1)^{1+\gamma}}{2}\left(\left(\frac{1}{6^{p+1}\,(p+1)}\right)^{\frac{1}{p}}\left(\left(\int_{0}^{\frac{1}{6}}\left((1-\jmath)\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^{q}+\jmath\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^{q}\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.\right.$$

$$\left.\left.+\left(\int_{0}^{\frac{1}{6}}\left(\jmath\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^{q}+(1-\jmath)\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^{q}\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right)\right)$$

$$+\left(\frac{5^{p+1}+3^{p+1}}{24^{p+1}\left(p+1\right)}\right)^{\frac{1}{p}}\left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\left(1-\jmath\right)\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\jmath\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\jmath\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\left(1-\jmath\right)\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{5^{p+1}+3^{p+1}}{24^{p+1}\left(p+1\right)}\right)^{\frac{1}{p}}\left(\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\left(1-\jmath\right)\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\jmath\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\jmath\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\left(1-\jmath\right)\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{1}{6^{p+1}\left(p+1\right)}\right)^{\frac{1}{p}}\left(\left(\int_{\frac{5}{6}}^{1}\left(\left(1-\jmath\right)\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\jmath\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{5}{6}}^{1}\left(\jmath\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\left(1-\jmath\right)\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right)$$

$$+\frac{\left(1-\gamma\right)\left(\mathrm{e}_2-\mathrm{e}_1\right)^{2-\gamma}}{4}\left(\left(\frac{1}{6^{(2-2\gamma)p+1}\left(\left(2-2\gamma\right)p+1\right)}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(\int_{0}^{\frac{1}{6}}\left(\left(1-\jmath\right)\left|\mathrm{F}''\left(\mathrm{e}_1,\cdot\right)\right|^q+\jmath\left|\mathrm{F}''\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{0}^{\frac{1}{6}}\left(\jmath\left|\mathrm{F}''\left(\mathrm{e}_1,\cdot\right)\right|^q+\left(1-\jmath\right)\left|\mathrm{F}''\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|^p\mathrm{d}\jmath\right)^{\frac{1}{p}}\left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\left(1-\jmath\right)\left|\mathrm{F}''\left(\mathrm{e}_1,\cdot\right)\right|^q+\jmath\left|\mathrm{F}''\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\jmath\left|F''\left(e_1,\cdot\right)\right|^q+\left(1-\jmath\right)\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\Bigg)$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|^p d\jmath\right)^{\frac{1}{p}}\Bigg(\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\left(1-\jmath\right)\left|F''\left(e_1,\cdot\right)\right|^q+\jmath\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\jmath\left|F''\left(e_1,\cdot\right)\right|^q+\left(1-\jmath\right)\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\Bigg)$$

$$+\left(\frac{B\left(\frac{1}{2-2\gamma},p+1\right)-B_{\left(\frac{5}{6}\right)^{2-2\gamma}}\left(\frac{1}{2-2\gamma},p+1\right)}{2-2\gamma}\right)^{\frac{1}{p}}\Bigg(\left(\int_{\frac{5}{6}}^{1}\left(\left(1-\jmath\right)\left|F''\left(e_1,\cdot\right)\right|^q+\jmath\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{5}{6}}^{1}\left(\jmath\left|F''\left(e_1,\cdot\right)\right|^q+\left(1-\jmath\right)\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\Bigg)\Bigg)\Bigg)$$

$$=\gamma^2\left(e_2-e_1\right)^{1+\gamma}\Bigg(\left(\frac{1}{6^{p+1}\left(p+1\right)}\right)^{\frac{1}{p}}\left(\left(\frac{11\left|F'\left(e_1,\cdot\right)\right|^q+\left|F'\left(e_2,\cdot\right)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{\left|F'\left(e_1,\cdot\right)\right|^q+11\left|F'\left(e_2,\cdot\right)\right|^q}{72}\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{5^{p+1}+3^{p+1}}{24^{p+1}\left(p+1\right)}\right)^{\frac{1}{p}}\left(\left(\frac{2\left|F'\left(e_1,\cdot\right)\right|^q+\left|F'\left(e_2,\cdot\right)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{\left|F'\left(e_1,\cdot\right)\right|^q+2\left|F'\left(e_2,\cdot\right)\right|^q}{9}\right)^{\frac{1}{q}}\right)\Bigg)$$

$$+\frac{\left(1-\gamma\right)\left(e_2-e_1\right)^{2-\gamma}}{4}\Bigg(\left(\frac{6^{(2\gamma-2)p+1}}{\left(\left(2-2\gamma\right)p+1\right)}\right)^{\frac{1}{p}}$$

$$\times\left(\left(\frac{11\left|F''\left(e_1,\cdot\right)\right|^q+\left|F''\left(e_2,\cdot\right)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{\left|F''\left(e_1,\cdot\right)\right|^q+11\left|F''\left(e_2,\cdot\right)\right|^q}{72}\right)^{\frac{1}{q}}\right)$$

$$+\left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|^p d\jmath\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|^p d\jmath\right)^{\frac{1}{p}}\right)$$

$$\times\left(\left(\frac{2\left|F''\left(e_1,\cdot\right)\right|^q+\left|F''\left(e_2,\cdot\right)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{\left|F''\left(e_1,\cdot\right)\right|^q+2\left|F''\left(e_2,\cdot\right)\right|^q}{9}\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{B\left(\frac{1}{2-2\gamma},p+1\right)-B_{\left(\frac{5}{6}\right)^{2-2\gamma}}\left(\frac{1}{2-2\gamma},p+1\right)}{2-2\gamma}\right)^{\frac{1}{p}}$$

$$\times\left(\left(\frac{|F''(e_1,\cdot)|^q+11|F''(e_2,\cdot)|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{11|F''(e_1,\cdot)|^q+|F''(e_2,\cdot)|^q}{72}\right)^{\frac{1}{q}}\right)\Bigg).$$

The proof is completed. □

Theorem 4.5. *Let* $F:[e_1,e_2]\times\Upsilon\to\mathbb{R}$ *be a stochastic process in the interval* $[e_1,e_2]$ *with* e_1,e_2, *such that* F,F',F'' *are mean-square differentiable stochastic processes on* $[e_1,e_2]\times\Upsilon$. *If* $|F'|^q$ *and* $|F''|^q$ *where* $q>1$ *are convex stochastic processes on* $[e_1,e_2]\times\Upsilon$, *then the following inequality holds almost everywhere*

$$\left|\gamma^2(e_2-e_1)^\gamma\mathscr{M}(e_1,e_2,F)+\frac{1-\gamma}{2(e_2-e_1)^{\gamma-1}}\mathscr{M}(e_1,e_2,F')-\frac{\Gamma(1-\gamma)}{2(e_2-e_1)^{1-\gamma}}\left(\mathbb{D}^{\gamma}_{e_2^-}[F](e_1)+\mathbb{D}^{\gamma}_{e_1^+}[F](e_2)\right)\right|$$

$$\leq\gamma^2(e_2-e_1)^{1+\gamma}\left(\left(\frac{1}{72}\right)^{1-\frac{1}{q}}\left(\left(\frac{8|F'(e_1,\cdot)|^q+|F'(e_2,\cdot)|^q}{648}\right)^{\frac{1}{q}}+\left(\frac{|F'(e_1)|^q+8|F'(e_2,\cdot)|^q}{648}\right)^{\frac{1}{q}}\right)\right.$$

$$\left.+\left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left(\left(\frac{863|F'(e_1,\cdot)|^q+361|F'(e_2,\cdot)|^q}{41472}\right)^{\frac{1}{q}}+\left(\frac{361|F'(e_1,\cdot)|^q+863|F'(e_2,\cdot)|^q}{41472}\right)^{\frac{1}{q}}\right)\right)$$

$$+\frac{(1-\gamma)(e_2-e_1)^{2-\gamma}}{4}\left(\left(\frac{1}{6^{3-2\gamma}(3-2\gamma)}\right)^{\frac{1}{p}}\left(\left(\frac{(21-10\gamma)|F''(e_1,\cdot)|^q+(24-12\gamma)|F''(e_2,\cdot)|^q}{6^{4-2\gamma}(3-2\gamma)(4-2\gamma)}\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\frac{(24-12\gamma)|F''(e_1,\cdot)|^q+(21-10\gamma)|F''(e_2,\cdot)|^q}{6^{4-2\gamma}(3-2\gamma)(4-2\gamma)}\right)^{\frac{1}{q}}\right)$$

$$+(\Omega_1(\gamma))^{1-\frac{1}{q}}\left(((\Omega_1(\gamma)-\Omega_3(\gamma))|F''(e_1,\cdot)|^q+\Omega_3(\gamma)|F''(e_2,\cdot)|^q)^{\frac{1}{q}}\right.$$

$$\left.+(\Omega_3(\gamma)|F''(e_1,\cdot)|^q+(\Omega_1(\gamma)-\Omega_3(\gamma))|F''(e_2,\cdot)|^q)^{\frac{1}{q}}\right)$$

$$+(\Omega_2(\gamma))^{1-\frac{1}{q}}\left(((\Omega_2(\gamma)-\Omega_4(\gamma))|F''(e_1,\cdot)|^q+\Omega_4(\gamma)|F''(e_2,\cdot)|^q)^{\frac{1}{q}}\right.$$

$$\left.+(\Omega_4(\gamma)|F''(e_1,\cdot)|^q+(\Omega_2(\gamma)-\Omega_4(\gamma))|F''(e_2,\cdot)|^q)^{\frac{1}{q}}\right)$$

$$+\left(\frac{5^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\frac{3+2\gamma}{6(3-2\gamma)}\right)^{1-\frac{1}{q}}(\mathscr{C}_1(\gamma)|F''(e_1,\cdot)|^q+\mathscr{C}_2(\gamma)|F''(e_2,\cdot)|^q)^{\frac{1}{q}}$$

$$\left.+(\mathscr{C}_2(\gamma)|F''(e_1,\cdot)|^q+\mathscr{C}_1(\gamma)|F''(e_2,\cdot)|^q)^{\frac{1}{q}}\right),$$

where $\Omega_1(\gamma),\Omega_2(\gamma),\mathscr{C}_1(\gamma),\mathscr{C}_2(\gamma),\Omega_3(\gamma)$, *and* $\Omega_4(\gamma)$ *are defined by* (4.1)–(4.6)*, respectively.*

Proof. From Lemma 3.2, absolute value, power mean inequality, and stochastic convexity of $|F'|^q$ and $|F''|^q$, we have

$$\left|\gamma^2(\mathrm{e}_2-\mathrm{e}_1)^\gamma \mathscr{M}(\mathrm{e}_1,\mathrm{e}_2,\mathrm{F})+\frac{1-\gamma}{2(\mathrm{e}_2-\mathrm{e}_1)^{\gamma-1}}\mathscr{M}(\mathrm{e}_1,\mathrm{e}_2,\mathrm{F}')-\frac{\Gamma(1-\gamma)}{2(\mathrm{e}_2-\mathrm{e}_1)^{1-\gamma}}\left(\mathbb{D}^\gamma_{\mathrm{e}_2^-}[\mathrm{F}](\mathrm{e}_1)+\mathbb{D}^\gamma_{\mathrm{e}_1^+}[\mathrm{F}](\mathrm{e}_2)\right)\right|$$

$$\leq\frac{\gamma^2(\mathrm{e}_2-\mathrm{e}_1)^{1+\gamma}}{2}\left(\left(\int_0^{\frac{1}{6}}\jmath\mathrm{d}\jmath\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{1}{6}}\jmath\left|\mathrm{F}'((1-\jmath)\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{6}}\jmath\left|\mathrm{F}'(\jmath\mathrm{e}_1+(1-\jmath)\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]\right.$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\mathrm{d}\jmath\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\left|\mathrm{F}'((1-\jmath)\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\left|\mathrm{F}'(\jmath\mathrm{e}_1+(1-\jmath)\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\mathrm{d}\jmath\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\left|\mathrm{F}'((1-\jmath)\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\left|\mathrm{F}'(\jmath\mathrm{e}_1+(1-\jmath)\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]$$

$$\left.+\left(\int_{\frac{5}{6}}^{1}(1-\jmath)\mathrm{d}\jmath\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{5}{6}}^{1}(1-\jmath)\left|\mathrm{F}'((1-\jmath)\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}+\left(\int_{\frac{5}{6}}^{1}(1-\jmath)\left|\mathrm{F}'(\jmath\mathrm{e}_1+(1-\jmath)\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]\right]$$

$$+\frac{(1-\gamma)(\mathrm{e}_2-\mathrm{e}_1)^{2-\gamma}}{4}\left(\left(\int_0^{\frac{1}{6}}\jmath^{2-2\gamma}\mathrm{d}\jmath\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{1}{6}}\jmath^{2-2\gamma}\left|\mathrm{F}''((1-\jmath)\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_0^{\frac{1}{6}}\jmath^{2-2\gamma}\left|\mathrm{F}''(\jmath\mathrm{e}_1+(1-\jmath)\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|\mathrm{d}\jmath\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|\left|\mathrm{F}''((1-\jmath)\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|\left|\mathrm{F}''(\jmath\mathrm{e}_1+(1-\jmath)\mathrm{e}_2,\cdot)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|\mathrm{d}\jmath\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|\left|\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|\left|\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)\mathrm{d}\jmath\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)\left|\mathrm{F}''\left((1-\jmath)\,\mathrm{e}_1+\jmath\mathrm{e}_2,\cdot\right)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)\left|\mathrm{F}''\left(\jmath\mathrm{e}_1+(1-\jmath)\,\mathrm{e}_2,\cdot\right)\right|^q\mathrm{d}\jmath\right)^{\frac{1}{q}}\right)\right]\\
\leq&\frac{\gamma^2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1+\gamma}}{2}\left(\left(\frac{1}{72}\right)^{1-\frac{1}{q}}\left(\left(\int_0^{\frac{1}{6}}\jmath\left((1-\jmath)\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\jmath\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\int_0^{\frac{1}{6}}\jmath\left(\jmath\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+(1-\jmath)\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right)\\
&+\left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\left((1-\jmath)\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\jmath\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\left(\jmath\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+(1-\jmath)\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right)\\
&+\left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\left((1-\jmath)\left|\mathrm{F}'\left(\mathrm{e}_1,\cdot\right)\right|^q+\jmath\left|\mathrm{F}'\left(\mathrm{e}_2,\cdot\right)\right|^q\right)\mathrm{d}\jmath\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\left(\jmath\left|F'\left(e_1,\cdot\right)\right|^q+(1-\jmath)\left|F'\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\Bigg)$$

$$+\left(\frac{1}{72}\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{5}{6}}^{1}(1-\jmath)\left((1-\jmath)\left|F'\left(e_1,\cdot\right)\right|^q+\jmath\left|F'\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{5}{6}}^{1}(1-\jmath)\left(\jmath\left|F'\left(e_1,\cdot\right)\right|^q+(1-\jmath)\left|F'\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right)$$

$$+\frac{(1-\gamma)\left(e_2-e_1\right)^{2-\gamma}}{4}\left(\left(\frac{1}{6^{3-2\gamma}(3-2\gamma)}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(\int_{0}^{\frac{1}{6}}\jmath^{2-2\gamma}\left((1-\jmath)\left|F''\left(e_1,\cdot\right)\right|^q+\jmath\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}+\left(\int_{0}^{\frac{1}{6}}\jmath^{2-2\gamma}\left(\jmath\left|F''\left(e_1,\cdot\right)\right|^q+(1-\jmath)\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|d\jmath\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|\left((1-\jmath)\left|F''\left(e_1,\cdot\right)\right|^q+\jmath\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath^{2-2\gamma}-\frac{3}{8}\right|\left(\jmath\left|F''\left(e_1,\cdot\right)\right|^q+(1-\jmath)\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|d\jmath\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|\left((1-\jmath)\left|F''\left(e_1,\cdot\right)\right|^q+\jmath\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath^{2-2\gamma}-\frac{5}{8}\right|\left(\jmath\left|F''\left(e_1,\cdot\right)\right|^q+(1-\jmath)\left|F''\left(e_2,\cdot\right)\right|^q\right)d\jmath\right)^{\frac{1}{q}}\right)$$

$$
\begin{aligned}
&+\left(\frac{5^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\frac{3+2\gamma}{6(3-2\gamma)}\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)\left((1-\jmath)\left|F''(e_1,\cdot)\right|^{q}+\jmath\left|F''(e_2,\cdot)\right|^{q}\right)d\jmath\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{5}{6}}^{1}\left(1-\jmath^{2-2\gamma}\right)\left(\jmath\left|F''(e_1,\cdot)\right|^{q}+(1-\jmath)\left|F''(e_2,\cdot)\right|^{q}\right)d\jmath\right)^{\frac{1}{q}}\right)\\
\leq&\gamma^{2}(e_2-e_1)^{1+\gamma}\left(\left(\frac{1}{72}\right)^{1-\frac{1}{q}}\left(\left(\frac{8\left|F'(e_1,\cdot)\right|^{q}+\left|F'(e_2,\cdot)\right|^{q}}{648}\right)^{\frac{1}{q}}+\left(\frac{\left|F'(e_1,\cdot)\right|^{q}+8\left|F'(e_2,\cdot)\right|^{q}}{648}\right)^{\frac{1}{q}}\right)\right.\\
&\left.+\left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left(\left(\frac{863\left|F'(e_1,\cdot)\right|^{q}+361\left|F'(e_2,\cdot)\right|^{q}}{41472}\right)^{\frac{1}{q}}+\left(\frac{361\left|F'(e_1,\cdot)\right|^{q}+863\left|F'(e_2,\cdot)\right|^{q}}{41472}\right)^{\frac{1}{q}}\right)\right)\\
&+\frac{(1-\gamma)(e_2-e_1)^{2-\gamma}}{4}\left(\left(\frac{1}{6^{3-2\gamma}(3-2\gamma)}\right)^{\frac{1}{p}}\left(\left(\frac{(21-10\gamma)\left|F''(e_1,\cdot)\right|^{q}+(24-12\gamma)\left|F''(e_2,\cdot)\right|^{q}}{6^{4-2\gamma}(3-2\gamma)(4-2\gamma)}\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\frac{(24-12\gamma)\left|F''(e_1,\cdot)\right|^{q}+(21-10\gamma)\left|F''(e_2,\cdot)\right|^{q}}{6^{4-2\gamma}(3-2\gamma)(4-2\gamma)}\right)^{\frac{1}{q}}\right)\\
&+(\Omega_1(\gamma))^{1-\frac{1}{q}}\left(\left((\Omega_1(\gamma)-\Omega_3(\gamma))\left|F''(e_1,\cdot)\right|^{q}+\Omega_3(\gamma)\left|F''(e_2,\cdot)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\Omega_3(\gamma)\left|F''(e_1,\cdot)\right|^{q}+(\Omega_1(\gamma)-\Omega_3(\gamma))\left|F''(e_2,\cdot)\right|^{q}\right)^{\frac{1}{q}}\right)\\
&+(\Omega_2(\gamma))^{1-\frac{1}{q}}\left(\left((\Omega_2(\gamma)-\Omega_4(\gamma))\left|F''(e_1,\cdot)\right|^{q}+\Omega_4(\gamma)\left|F''(e_2,\cdot)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\Omega_4(\gamma)\left|F''(e_1,\cdot)\right|^{q}+(\Omega_2(\gamma)-\Omega_4(\gamma))\left|F''(e_2,\cdot)\right|^{q}\right)^{\frac{1}{q}}\right)\\
&+\left(\frac{5^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\frac{3+2\gamma}{6(3-2\gamma)}\right)^{1-\frac{1}{q}}\left(\left(\mathscr{C}_1(\gamma)\left|F''(e_1,\cdot)\right|^{q}+\mathscr{C}_2(\gamma)\left|F''(e_2,\cdot)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\mathscr{C}_2(\gamma)\left|F''(e_1,\cdot)\right|^{q}+\mathscr{C}_1(\gamma)\left|F''(e_2,\cdot)\right|^{q}\right)^{\frac{1}{q}}\right)\right),
\end{aligned}
$$

where we have used (4.1), (4.2), and

$$
\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|\jmath d\jmath=\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|(1-\jmath)\,d\jmath=\frac{361}{41472},
$$

$$
\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\jmath-\frac{3}{8}\right|(1-\jmath)\,d\jmath=\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\jmath-\frac{5}{8}\right|\jmath d\jmath=\frac{863}{41472},
$$

$$\mathscr{C}_1(\gamma) = \int_{\frac{5}{6}}^{1} \left(1 - \jmath^{2-2\gamma}\right)(1-\jmath)\,\mathrm{d}\jmath = \frac{9-2\gamma}{6(3-2\gamma)(4-2\gamma)}\left(\frac{5}{6}\right)^{3-2\gamma} + \frac{1}{72} - \frac{1}{(3-2\gamma)(4-2\gamma)}, \tag{4.3}$$

$$\mathscr{C}_2(\gamma) = \int_{\frac{5}{6}}^{1} \left(1 - \jmath^{2-2\gamma}\right)\jmath\,\mathrm{d}\jmath = \frac{1}{4-2\gamma}\left(\frac{5}{6}\right)^{4-2\gamma} - \frac{14+11\gamma}{36(4-2\gamma)}, \tag{4.4}$$

$$\Omega_3(\gamma) = \int_{\frac{1}{6}}^{\frac{1}{2}} \left|\jmath^{2-2\gamma} - \frac{3}{8}\right|(1-\jmath)\,\mathrm{d}\jmath = \begin{cases} \dfrac{3^{4-2\gamma}-1}{6^{4-2\gamma}(4-2\gamma)} - \dfrac{1}{24} & \text{if} \quad 0 \le \left(\dfrac{3}{8}\right)^{\frac{1}{2-2\gamma}} < \dfrac{1}{6}, \\ \dfrac{1}{1536} + \dfrac{4^{4-2\gamma}+12^{4-2\gamma}-2\times 9^{4-2\gamma}}{24^{4-2\gamma}(4-2\gamma)} & \text{if} \quad \dfrac{1}{6} \le \left(\dfrac{3}{8}\right)^{\frac{1}{2-2\gamma}} \le \dfrac{1}{2}, \\ \dfrac{1}{24} - \dfrac{3^{4-2\gamma}-1}{6^{4-2\gamma}(4-2\gamma)} & \text{if} \quad \dfrac{1}{2} < \left(\dfrac{3}{8}\right)^{\frac{1}{2-2\gamma}} < 1, \end{cases} \tag{4.5}$$

and

$$\Omega_4(\gamma) = \int_{\frac{1}{2}}^{\frac{5}{6}} \left|\jmath^{2-2\gamma} - \frac{5}{8}\right|\jmath\,\mathrm{d}\jmath = \begin{cases} \dfrac{5^{4-2\gamma}-3^{4-2\gamma}}{6^{4-2\gamma}(4-2\gamma)} - \dfrac{5}{36} & \text{if} \quad 0 < \left(\dfrac{5}{8}\right)^{\frac{1}{2-2\gamma}} \le \dfrac{1}{2}, \\ \dfrac{12^{4-2\gamma}+20^{4-2\gamma}-2\times 15^{4-2\gamma}}{24^{4-2\gamma}(4-2\gamma)} - \dfrac{235}{4608} & \text{if} \quad \dfrac{1}{2} < \left(\dfrac{5}{8}\right)^{\frac{1}{2-2\gamma}} < \dfrac{5}{6}, \\ \dfrac{5}{36} - \dfrac{5^{4-2\gamma}-3^{4-2\gamma}}{6^{4-2\gamma}(4-2\gamma)} & \text{if} \quad \dfrac{5}{6} \le \left(\dfrac{5}{8}\right)^{\frac{1}{2-2\gamma}} \le 1. \end{cases} \tag{4.6}$$

The proof is completed. □

5. Numerical validation

In this section, to demonstrate the validity and applicability of our main theoretical results, we construct genuine stochastic processes involving nontrivial random variables.

Example 5.1. Let $(Y, \mathcal{A}, \mathbb{P})$ be a probability space, and let $X : Y \to \mathbb{R}^+$ be a positive random variable belonging to $L^2(Y, \mathcal{A}, \mathbb{P})$ such that $\mathbb{E}[X] = 1$ (for instance, X follows a uniform distribution $U(0, 2)$). Consider the interval $[\mathfrak{e}_1, \mathfrak{e}_2] = [0, 1]$ and define the stochastic process $\mathrm{F} : [0, 1] \times Y \to \mathbb{R}$ by

$$\mathrm{F}(l, \cdot) = \frac{l^3}{3} X(\cdot).$$

Since $l \mapsto \frac{l^3}{3}$ is convex on $[0, 1]$ and $X(\cdot) > 0$ almost surely, $\mathrm{F}(l, \cdot)$ is a convex stochastic process. Furthermore, F is mean-square differentiable with respect to l, with mean-square derivatives given by

$$\mathrm{F}'(l, \cdot) = l^2 X(\cdot) \quad \text{and} \quad \mathrm{F}''(l, \cdot) = 2lX(\cdot).$$

Observe that $|\mathrm{F}'(l,\cdot)|$ and $|\mathrm{F}''(l,\cdot)|$ are also convex stochastic processes on $[0,1]\times Y$. Direct calculations yield:

$$|\mathrm{F}'(0,\cdot)|+|\mathrm{F}'(1,\cdot)|=X(\cdot),\quad |\mathrm{F}''(0,\cdot)|+|\mathrm{F}''(1,\cdot)|=2X(\cdot).$$

Moreover, evaluating the mean-square functional $\mathscr{M}$ for F and F′, we get:

$$\mathscr{M}(0,1,\mathrm{F})=\frac{1}{12}X(\cdot)\quad\text{and}\quad \mathscr{M}(0,1,\mathrm{F}')=\frac{1}{3}X(\cdot).$$

Calculating the stochastic mean-square proportional Caputo-hybrid operators yields:

$$\mathbb{D}^{\gamma}_{1^-}[\mathrm{F}](0)=\frac{X(\cdot)}{\Gamma(1-\gamma)}\left(\frac{\gamma^2}{12}+\frac{(1-\gamma)^2}{4-2\gamma}\right),$$

and

$$\mathbb{D}^{\gamma}_{0^+}[\mathrm{F}](1)=\frac{X(\cdot)}{\Gamma(1-\gamma)}\left(\frac{\gamma^2}{12}+\frac{1-\gamma}{(3-2\gamma)(4-2\gamma)}\right).$$

Consequently, the left-hand side of Theorem 4.1 reduces to:

$$\mathbf{L}_1(\gamma,\cdot)=\left|\frac{\gamma(1-\gamma)(1-2\gamma)}{12(2-\gamma)(3-2\gamma)}\right|X(\cdot). \tag{5.1}$$

On the other hand, the right hand side of Theorem 4.1 becomes:

$$\mathbf{R}_1(\gamma,\cdot)=\left[\frac{25\gamma^2}{576}+\frac{1-\gamma}{2}\left(\frac{1+5^{3-2\gamma}}{6^{3-2\gamma}(3-2\gamma)}-\frac{3+2\gamma}{6(3-2\gamma)}+\Omega_1(\gamma)+\Omega_2(\gamma)\right)\right]X(\cdot).$$

Taking the expectation $\mathbb{E}[\cdot]$ on both sides (or evaluating pointwise a.e. since $X(\cdot)>0$), we obtain deterministic functions for comparison. A graphical depiction of both sides $\mathbb{E}[\mathbf{L}_1(\gamma,\cdot)]$ and $\mathbb{E}[\mathbf{R}_1(\gamma,\cdot)]$ is presented in Figure 1 using MATLAB.

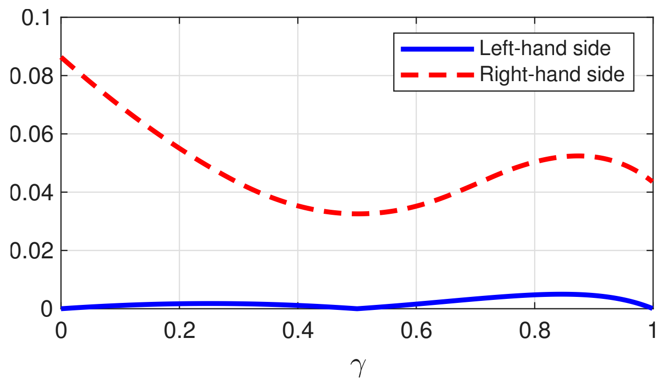

Figure 1. Graphical depiction of Theorem 4.1.

The graphical representation shows that the left-hand side remains below the right hand side for all $\gamma\in[0,1)$, which numerically confirms the validity of Theorem 4.1 for the chosen example.

Example 5.2. To provide a comprehensive numerical validation of Theorem 4.4 (Hölder's inequality variant), we consider the same stochastic process $\mathrm{F}:[0,1]\times Y\to\mathbb{R}$ considered in Example 5.1 defined on $[\mathfrak{e}_1,\mathfrak{e}_2]=[0,1]$ by $\mathrm{F}(l,\cdot)=\frac{l^3}{3}X(\cdot)$ with $X\sim U(0,2)$.

Using the conjugate exponents relation $\frac{1}{p}+\frac{1}{q}=1$, we express q as a function of $p>1$ via $q=\frac{p}{p-1}$. The left term remains the same as (5.1).

The Hölder upper bound function $\mathbf{R}_{2.2}(\gamma, p)$ as a function of the fractional order $\gamma \in [0,1)$ and exponent $p>1$ takes the continuous form:

$$\mathbf{R}_{2.2}(\gamma)=\Bigg\{\gamma^2\left(\frac{1+\sqrt{11}+\sqrt{19}+\sqrt{38}}{216}\right)+\frac{1-\gamma}{4}\Bigg[\frac{1+\sqrt{11}}{3\sqrt{2}}\left(\sqrt{\frac{6^{4\gamma-3}}{5-4\gamma}}+\sqrt{\frac{\mathrm{B}\left(\frac{1}{2-2\gamma},3\right)-\mathrm{B}_{\left(\frac{5}{6}\right)^{2-2\gamma}}\left(\frac{1}{2-2\gamma},3\right)}{2-2\gamma}}\right)$$
$$+\frac{2(1+\sqrt{2})}{3}\left(\sqrt{\frac{\left(\frac{1}{2}\right)^{5-4\gamma}-\left(\frac{1}{6}\right)^{5-4\gamma}}{5-4\gamma}-\frac{3\left[\left(\frac{1}{2}\right)^{3-2\gamma}-\left(\frac{1}{6}\right)^{3-2\gamma}\right]}{4(3-2\gamma)}+\frac{3}{64}}\right.$$
$$\left.+\sqrt{\frac{\left(\frac{5}{6}\right)^{5-4\gamma}-\left(\frac{1}{2}\right)^{5-4\gamma}}{5-4\gamma}-\frac{5\left[\left(\frac{5}{6}\right)^{3-2\gamma}-\left(\frac{1}{2}\right)^{3-2\gamma}\right]}{4(3-2\gamma)}+\frac{25}{192}}\right)\Bigg]\Bigg\}X(\cdot),$$

where $\mathrm{B}(\cdot,\cdot)$ and $\mathrm{B}_{\sigma}(\cdot,\cdot)$ denote the beta and the incomplete beta functions, respectively.

A representation comparing the deterministic expectation of the left-hand side $\mathbb{E}[\mathbf{L}_1(\gamma,\cdot)]$ with the expectation of the upper bound $\mathbb{E}[\mathbf{R}_{2.2}(\gamma)]$ over the domain $\gamma \in [0;1)$ is depicted in Figure 2.

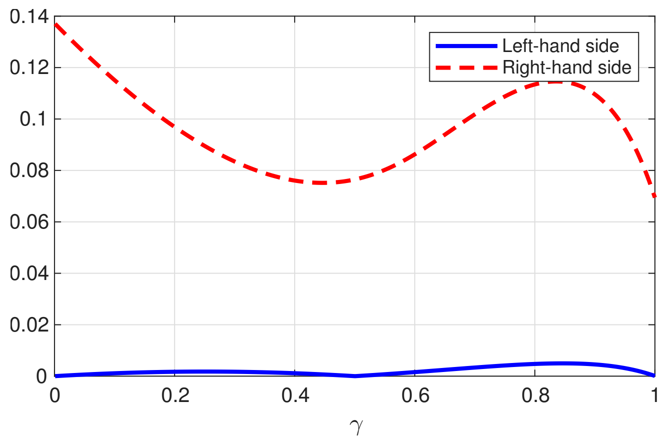

Figure 2. Graphical depiction of Theorem 4.4 for $p=q=2$.

Figure 2 confirms that $\mathbf{R}_{2.2}(\gamma,p) \geq \mathbb{E}[\mathbf{L}_1(\gamma,\cdot)]$ across the entire parametric interval, establishing the validity of Theorem 4.4.

6. Conclusions

In this paper, we have studied Maclaurin-type inequalities for convex stochastic processes within the framework of proportional Caputo-hybrid fractional operators. By introducing the left-sided and right-sided stochastic mean-square proportional Caputo-hybrid operators, we established a new integral identity that forms the backbone of our approach. This identity enabled us to derive new fractional Caputo-hybrid Maclaurin-type inequalities under suitable differentiability and integrability conditions.

The results obtained in this work extend the classical Maclaurin-type inequalities to a stochastic fractional setting and provide new estimates for convex stochastic processes in the mean-square sense. Hence, our findings contribute to the growing literature on the interplay between convexity, stochastic analysis, and fractional calculus. In particular, they show that proportional Caputo-hybrid operators offer an effective framework for deriving refined integral inequalities in random environments.

It is expected that the methods developed here may be useful for further investigations on other classes of generalized convex stochastic processes, as well as for related inequalities involving different fractional operators. Possible future directions include the study of analogous results for s-convex, harmonically convex, or preinvex stochastic processes, and the exploration of potential applications in stochastic modeling, numerical analysis, and fractional dynamical systems.

Author contributions

The authors collaborated equally on the conceptualization, analysis, and writing of this paper. All authors have reviewed the final version and consented to its submission.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors report no competing interests to disclose.

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