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Research article

Well-posedness and cold start criteria for differential equations with a modified Riemann–Liouville fractional operator

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Abstract: This paper investigates the start-up dynamics of a fractional system subjected to a weakly singular power-law input. For $1 < \alpha < 2$, a modified Riemann–Liouville operator is introduced to prescribe the initial displacement while avoiding the singular contribution generated by a constant state. The associated fractional trace is explicitly identified, and the initial-value problem is reformulated as an equivalent Volterra integral equation. We establish the existence, uniqueness, and continuous dependence of solutions under locally integrable forcing for the fractional system. For the power-law excitation $At^{-\mu}$ for $0 < \mu < 1$, an explicit response representation in terms of Mittag–Leffler functions is obtained. The short-time behavior reveals that a nonzero fractional trace produces a singular ordinary velocity, thereby selecting the zero-trace branch when finite velocity is required. On this branch, a sharp cold-start criterion is established: $f'(0) = 0$ if and only if $\alpha - \mu > 1$. The critical and subcritical regimes are further classified, showing that zero initial velocity may coexist with an unbounded but locally integrable acceleration. The analysis is extended to more general excitations of the form $t^{-\mu}h(t)$, demonstrating that the leading local behavior of the modulation function determines the effective start-up exponent. Numerical experiments with a piecewise-linear product-integration scheme reproduce the zero-trace cold start and the singular ordinary velocity associated with a nonzero trace. Mesh grading near $t = 0$ yields essentially second-order convergence in the latter case. These results identify the compatibility between weakly singular forcing and zero ordinary initial velocity and show how the associated start-up singularity affects both the solution behavior and numerical convergence.

Keywords: modified Riemann–Liouville derivative; cold start; Volterra equation; weakly singular forcing; Mittag–Leffler function
Mathematics Subject Classification: 34A12, 44A10, 45D05

1. Introduction

Fractional differential equations provide effective models for mechanical systems with hereditary memory, nonlocal damping, and viscoelastic behavior [1–3]. For fractional-order $\alpha \in (1, 2)$, the fractional dynamics interpolate between relaxation and wave- or oscillator-type evolution [4]. Linear responses are commonly represented by Mittag–Leffler functions [5, 6].

A substantial body of literature has investigated free and forced fractional oscillators, including damping, resonance, impulse responses, and Mittag–Leffler or Green-function representations [7–10]. These studies provide the analytical foundation for fractional oscillatory responses under regular, harmonic, and impulsive loading. Their main focus, however, is not the compatibility between zero ordinary initial velocity and a locally integrable load that is singular at the initial time. The compatibility issue is closely related to the treatment of initial data. Equations with Caputo fractional derivatives are naturally associated with integer-order initial values, whereas Riemann–Liouville formulations generally involve fractional traces [11]. Previous studies have clarified the mathematical and physical interpretation of such fractional initial data and memory states [12–14].

Physically, a cold start refers to the situation in which, at the onset of the external excitation, the system is at the prescribed displacement f_0 and at rest in the ordinary kinematic sense, namely, $f(0) = f_0$ and $f'(0) = 0$. Under singular start-up forcing, formally imposing this condition is not equivalent to proving that the resulting motion possesses a zero velocity trace. Cold start is therefore treated as a compatibility property of the response rather than as an automatically admissible initial datum. The basic excitation is $g(t) = At^{-\mu}, 0 < \mu < 1$, which is unbounded at the origin but belongs to $L^1(0, T)$ for every $T > 0$ (see [15]). The studies reviewed above do not provide a unified criterion determining when a weakly singular start-up load is compatible with zero ordinary initial velocity, how the critical and subcritical regimes differ, or whether the criterion persists for more general inputs $t^{-\mu}h(t)$.

In this paper, we shall investigate the following linear model to address some of the questions raised above:

$$\begin{cases} \mathfrak{D}_0^{\alpha} f(t) + \lambda f(t) = At^{-\mu}, \quad t > 0, \quad 1 < \alpha < 2, \ 0 < \mu < 1, \\ f(0) = f_0. \end{cases} \tag{1.1}$$

We assume $\lambda \geq 0$ throughout. For $\lambda > 0$, λ represents the linear restoring coefficient, and Eq (1.1) belongs to the class of forced fractional oscillator equations [7,8] while $\lambda = 0$ is included as the limiting case without the restoring term. We use the modified Riemann–Liouville (mRL) fractional derivatives $\mathfrak{D}_0^{\alpha}$ in order to remove the singular contribution generated solely by the constant displacement while preserving the algebraic start-up behavior generated by the dynamics. We use the Laplace transform and an equivalent Volterra integral formulation of the problem to establish the analytical results. The short-time asymptotic behavior of $f'(t)$ is then analyzed to determine the cold-start criterion. We further apply a piecewise-linear product-integration scheme to the Volterra integral equation to examine how the start-up regularity affects numerical convergence on uniform and graded meshes.

The paper is organized as follows. Section 2 introduces the fractional operators, Laplace transform formula, and Mittag–Leffler function. Section 3 establishes the Volterra formulation, well-posedness, and the resolvent representation. Section 4 treats the pure power-law input and gives the full cold-start classification, and extends the analysis to regularly modulated weak singularities. Section 5 applies a piecewise-linear product-integration scheme to the equivalent Volterra integral equation and examines how the start-up regularity influences numerical convergence on uniform and graded meshes. Section 6 summarizes the main findings and their significance.

2. Preliminaries

For $\beta > 0$, the left-sided Riemann–Liouville fractional integral of a locally integrable function h is

$$I_0^{\beta} h(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} h(\tau)\ \mathrm{d}\tau,\ t > 0.$$

For $1 < \alpha < 2$, the Riemann–Liouville fractional derivative is defined by

$$\begin{aligned} D_0^{\alpha,\mathrm{RL}} h(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} I_0^{2-\alpha} h(t) \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-\tau)^{1-\alpha} h(\tau) d\tau. \end{aligned}$$

These definitions are standard [16–18]. In particular,

$$D_0^{\alpha,\mathrm{RL}} c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}$$

for a constant c. While mathematically valid, this creates a physical discrepancy: A simple non-zero initial displacement in a mechanical cold start will unrealistically introduce an infinite response at $t = 0$.

In what follows, we denote by $W^{2,1}(0, T)$ the Sobolev space, that is

$$W^{2,1}(0, T) = \left\{ f \in L^1(0, T) :\ D^2 f \in L^1(0, T) \right\}.$$

For $1 < \alpha < 2$ and $T > 0$, set

$$H_f(t) := I_0^{2-\alpha}(f(t) - f(0)),$$

and define

$$\mathcal{D}_{\alpha,T} := \left\{ f \in L^1(0, T) : H_f \in W^{2,1}(0, T) \right\}.$$

To circumvent the singular contribution generated solely by a constant initial displacement, we use the following modified Riemann–Liouville fractional derivative, motivated by Jumarie's formulation [19].

Definition 2.1. *Let $f \in \mathcal{D}_{\alpha,T}$. The modified Riemann–Liouville fractional derivative $\mathfrak{D}_0^{\alpha} f(t)$ of $f(t)$ is defined by*

$$\begin{aligned} \mathfrak{D}_0^{\alpha} f(t) &= D_0^{\alpha,\mathrm{RL}}(f(t) - f(0)) \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_0^t (t-\tau)^{1-\alpha} (f(\tau) - f(0))\, \mathrm{d}\tau, \end{aligned}$$

for almost every $t \in (0, T)$.

This definition of mRL-fractional derivative removes only the constant displacement contribution. It does not suppress other start-up modes. For example, if $f(t) = f_0 + v_0 t$, then

$$\mathfrak{D}_0^\alpha f(t) = \frac{v_0}{\Gamma(2-\alpha)} t^{1-\alpha}.$$

Thus, an ordinary linear term is detected as a weak singularity.

Lemma 2.1. *Let* $1 < \alpha < 2$ *and suppose that* $f|_{[0,T]} \in \mathcal{D}_{\alpha,T}$ *for every* $T > 0$*. Assume that there exist constants* $M > 0$*,* $\omega \geq 0$*, and* $t_0 > 0$ *such that*

$$|f(t)| + |H_f''(t)| \leq M e^{\omega t}$$

for almost every $t \geq t_0$*. Then, for* $\mathrm{Re}(s) > \omega$*,*

$$\mathcal{L}\{\mathfrak{D}_0^\alpha f\}(s) = s^\alpha F(s) - f_0 s^{\alpha-1} - \vartheta,$$

where $F(s) = \mathcal{L}\{f\}(s)$*. In particular, if* $\vartheta = 0$*, then*

$$\mathcal{L}\{\mathfrak{D}_0^\alpha f\}(s) = s^\alpha F(s) - f_0 s^{\alpha-1}.$$

Proof. Since f is continuous at $t = 0$ and $1 < \alpha < 2$, we have $H_f(0) = 0$. By the Laplace transform of the fractional integral,

$$\mathcal{L}\{H_f\}(s) = s^{\alpha-2}\left(F(s) - \frac{f_0}{s}\right).$$

The assumptions of the lemma ensure that the classical Laplace transform formula for H_f'' is applicable. Hence, using $H_f'(0) = \vartheta$ and $\mathfrak{D}_0^\alpha f = H_f''$, we obtain

$$\begin{aligned}\mathcal{L}\{\mathfrak{D}_0^\alpha f\}(s) &= s^2 \mathcal{L}\{H_f\}(s) - sH_f(0) - H_f'(0)\\ &= s^\alpha F(s) - f_0 s^{\alpha-1} - \vartheta.\end{aligned}$$

This proves the result. □

In addition, denote by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ \alpha > 0,$$

the two-parameter Mittag–Leffler function, where β is a complex parameter for which the denominator is finite. A standard Laplace transform pair is

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \tag{2.1}$$

for suitable s and for parameters satisfying the usual convergence assumptions.

3. Initial value problem

Let $T > 0$, $g \in L^1(0,T)$, and $\lambda \geq 0$. We consider

$$\begin{cases} \mathfrak{D}_0^\alpha f(t) + \lambda f(t) = g(t), \quad 0 < t < T, \\ f(0) = f_0, \\ \left(I_0^{2-\alpha}(f - f_0)\right)'(0) = \vartheta. \end{cases} \tag{3.1}$$

Definition 3.1. *A function $f(t) \in C[0,T]$ is a strong solution of* (3.1) *if $H(t) = I_0^{2-\alpha}(f(t) - f_0) \in W^{2,1}(0,T)$, with $H(0) = 0, H'(0) = \vartheta$, and $H''(t) + \lambda f(t) = g(t)$ for almost every $t \in (0,T)$.*

Theorem 3.1. *A function $f(t) \in C[0,T]$ is a strong solution of* (3.1) *if and only if*

$$f(t) = f_0 + \frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [g(\tau) - \lambda f(\tau)] \, \mathrm{d}\tau, \qquad 0 \leq t \leq T. \tag{3.2}$$

Proof. Assume first that (3.2) holds. Using the semigroup property of fractional integrals,

$$\begin{aligned} H(t) = I_0^{2-\alpha}(f - f_0) &= I_0^{2-\alpha}\left(\frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1}\right) + I_0^2(g - \lambda f) \\ &= \vartheta t + I_0^2(g - \lambda f) \\ &= \vartheta t + \int_0^t (t-\tau)(g(\tau) - \lambda f(\tau)) \, \mathrm{d}\tau. \end{aligned} \tag{3.3}$$

Since $g - \lambda f \in L^1(0,T)$, it follows that $H(t) \in W^{2,1}(0,T)$ and

$$H'(t) = \vartheta + \int_0^t (g - \lambda f) \, \mathrm{d}\tau, \; H''(t) = g(t) - \lambda f(t), \text{ for } a.e.\, t \in (0,T).$$

Thus, $H(0) = 0, H'(0) = \vartheta$, and $H''(t) + \lambda f(t) = g(t)$ for almost every $t \in (0,T)$. Hence f is a strong solution.

Conversely, suppose that f is a strong solution. Integrating the identity $H'' = g - \lambda f$ twice and using $H(0) = 0$ and $H'(0) = \vartheta$ gives

$$H(t) = \vartheta t + I_0^2(g - \lambda f)(t). \tag{3.4}$$

Notice that the right-hand side of (3.4) is also

$$I_0^{2-\alpha}\left(\frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1} + I_0^\alpha(g - \lambda f)(t)\right).$$

The operator $I_0^{2-\alpha}$ is injective on $L^1(0,T)$, and (3.2) follows.

The proof of Theorem 3.1 is completed. □

Remark 3.1. *Strong solution refers to the Sobolev regularity of the fractional primitive $H(t)$, rather than to classical C^2 regularity of f.*

Theorem 3.2 (Well-posedness)**.** *Let $T > 0$, $g \in L^1(0,T)$, $f_0, \vartheta \in \mathbb{R}$, and $\lambda \geq 0$. Then problem* (3.1) *has a unique strong solution in $C[0,T]$. The solution depends continuously on f_0, ϑ, and g.*

Proof. Define

$$(\Phi f)(t) = f_0 + \frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1} + I_0^\alpha (g(t) - \lambda f(t)).$$

Since $\alpha > 1$ and $g(t) - \lambda f(t) \in L^1(0, T)$, the function $I_0^\alpha (g - \lambda f)$ is continuous. Thus, Φ maps $C[0, T]$ into itself. For $q > 0$, define the equivalent Bielecki norm

$$\|f\|_q = \sup_{0 \le t \le T} e^{-qt} |f(t)|, \ \forall f \in C[0, T].$$

For any $f, h \in C[0, T]$,

$$\begin{aligned} e^{-qt} |(\Phi f)(t) - (\Phi h)(t)| &\le \frac{\lambda \|f - h\|_q}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} e^{-q(t-\tau)} \, \mathrm{d}\tau \\ &\le \frac{\lambda}{q^\alpha} \|f - h\|_q. \end{aligned}$$

Choosing $q > \lambda^{1/\alpha}$, it makes Φ a contraction. Thus, the Banach fixed-point theorem gives existence and uniqueness.

Let f and $\widetilde{f}$ be two solutions corresponding to the data (f_0, ϑ, g) and $(\widetilde{f_0}, \widetilde{\vartheta}, \widetilde{g})$, respectively. Then, in light of (3.2), one obtains that

$$\left(1 - \frac{\lambda}{q^\alpha}\right) \|f - \widetilde{f}\|_q \le |f_0 - \widetilde{f_0}| + \frac{T^{\alpha-1}}{\Gamma(\alpha)} |\vartheta - \widetilde{\vartheta}| + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \|g - \widetilde{g}\|_{L^1(0,T)}.$$

Thus, we have

$$\|f - \widetilde{f}\|_q \le C \left(|f_0 - \widetilde{f_0}| + |\vartheta - \widetilde{\vartheta}| + \|g - \widetilde{g}\|_{L^1} \right),$$

where $C > 0$ is a constant. The theorem is derived. □

Theorem 3.3. *Under the assumptions of Theorem 3.2, the unique solution of problem* (3.1) *is*

$$f(t) = f_0 E_{\alpha,1}(-\lambda t^\alpha) + \vartheta t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t - \tau)^\alpha) g(\tau) \, \mathrm{d}\tau. \tag{3.5}$$

Proof. Writing (3.2) as $f = b - \lambda I_0^\alpha f$, where $b(t) = f_0 + \frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1} + I_0^\alpha g(t)$, or equivalently $f = \left(I + \lambda I_0^\alpha\right)^{-1} b$. Hence, it yields the Neumann series

$$f = \sum_{n=0}^{\infty} (-\lambda)^n I_0^{n\alpha} b = \sum_{n=0}^{\infty} (-\lambda)^n I_0^{n\alpha} \left(f_0 + \frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1} + I_0^\alpha g(t) \right), \tag{3.6}$$

which converges uniformly on $[0, T]$. For the first term in (3.6), the corresponding Neumann series converges as

$$\begin{aligned} \sum_{n=0}^{\infty} (-\lambda)^n I_0^{n\alpha} f_0 &= \sum_{n=0}^{\infty} (-\lambda)^n f_0 I_0^{n\alpha} \cdot 1 \\ &= \sum_{n=0}^{\infty} (-\lambda)^n f_0 \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \end{aligned}$$

$$= f_0 \sum_{n=0}^{\infty} \frac{(-\lambda)^n t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Similarly, the remaining two terms can be expressed as

$$\vartheta \sum_{n=0}^{\infty} \frac{(-\lambda)^n t^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}, \text{ and } \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\lambda)^n (t-\tau)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} \right) g(\tau)\, \mathrm{d}\tau.$$

Summing the Neumann series of the three terms and applying the rwo-parameter Mittag–Leffler function yields (3.5). □

4. Weakly singular power-law forcing

We now take $g(t)$ to be a specific weakly singular power-law factor.

4.1. Pure power-law forcing

Theorem 4.1. *Let* $1 < \alpha < 2$, $0 < \mu < 1$, $\lambda \geq 0$, *and* $A, f_0, \vartheta \in \mathbb{R}$. *Taking* $g(t) = At^{-\mu}$, *the unique solution of problem* (3.1) *is*

$$f(t) = f_0 E_{\alpha,1}(-\lambda t^{\alpha}) + \vartheta t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) + A\Gamma(1-\mu) t^{\alpha-\mu} E_{\alpha,\alpha-\mu+1}(-\lambda t^{\alpha}), \quad 0 < t < T. \tag{4.1}$$

Theorem 4.2. *Under the assumptions of Theorem 4.1 and* $A \neq 0$, *for the solution* $f(t)$ *of* (3.1),

$$f'(0) = 0, \text{ if and only if } \vartheta = 0, \alpha - \mu > 1.$$

Proof. Let $q = \alpha - \mu$. Then $0 < q < 2$ and $q > \alpha - 1$. In light of (4.1), expanding the Mittag–Leffler functions yields

$$\begin{aligned} f(t) = {} & f_0 \left[1 - \frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + O(t^{2\alpha}) \right] + \vartheta \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{\lambda t^{2\alpha-1}}{\Gamma(2\alpha)} + O(t^{3\alpha-1}) \right] \\ & + A\Gamma(1-\mu) \left[\frac{t^q}{\Gamma(q+1)} - \frac{\lambda t^{q+\alpha}}{\Gamma(q+\alpha+1)} + O(t^{q+2\alpha}) \right]. \end{aligned} \tag{4.2}$$

Consequently,

$$f'(t) = \frac{\vartheta}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{A\Gamma(1-\mu)}{\Gamma(q)} t^{q-1} - \frac{\lambda f_0}{\Gamma(\alpha)} t^{\alpha-1} + o(t^{\alpha-1}), \quad (t \to 0). \tag{4.3}$$

- If $\vartheta \neq 0$, then

$$f'(t) \sim \frac{\vartheta}{\Gamma(\alpha-1)} t^{\alpha-2}.$$

  Thus, $|f'(t)| \to \infty$ as $t \to 0$.
- If $\vartheta = 0$, then

$$\begin{cases} |f'(t)| \to \infty, & 0 < q < 1, \\ f'(0) = A\Gamma(1-\mu), & q = 1, \\ f'(0) = 0, & 1 < q < 2. \end{cases}$$

This finishes the proof of Theorem 4.2. $\square$

Remark 4.1. *If $\vartheta = 0$, then as $\alpha \to 1^+$, the solution* (4.1) *converges to*

$$f(t) = f_0 e^{-\lambda t} + A \int_0^t e^{-\lambda(t-\tau)} \tau^{-\mu} \, \mathrm{d}\tau,$$

which solves

$$f'(t) + \lambda f(t) = A t^{-\mu}, \qquad f(0) = f_0.$$

In particular, for $\lambda = 0$,

$$f(t) = f_0 + \frac{A}{1-\mu} t^{1-\mu}.$$

Consequently, the cold-start condition consists of two independent requirements. Finite ordinary velocity forces $\vartheta = 0$, while $f'(0) = 0$ holds precisely when $\alpha - \mu > 1$.

Corollary 4.1. *Assume $A \neq 0$, $\vartheta = 0$, and $1 < q = \alpha - \mu < 2$. Then*

$$f \in C^1[0,T], \; f' \in AC[0,T].$$

Furthermore,

$$f''(t) = \frac{A\Gamma(1-\mu)}{\Gamma(q-1)} t^{q-2} - \frac{\lambda f_0}{\Gamma(\alpha-1)} t^{\alpha-2} + o(t^{\alpha-2}), \quad t \to 0.$$

Thus, a cold start does not require finite initial acceleration. Although $f''(t)$ is unbounded near $t = 0$, its singularity is integrable. Therefore, the velocity remains continuous and can satisfy $f'(0) = 0$.

Remark 4.2. *Assume $\vartheta = 0$ and set $q = \alpha - \mu$.*

(1) If $q > 1$, then $f'(0) = 0$ and the response coincides with the solution of the Caputo problem

$${}^C D_0^\alpha f(t) + \lambda f(t) = A t^{-\mu}, \; f(0) = f_0, \; f'(0) = 0.$$

(2) If $q = 1$, then

$$f'(0) = A\Gamma(2-\alpha),$$

and the same response solves the homogeneous Caputo problem

$${}^C D_0^\alpha f(t) + \lambda f(t) = 0, \; f(0) = f_0, \; f'(0) = A\Gamma(2-\alpha).$$

(3) If $q < 1$, then no finite ordinary initial velocity exists. Hence the response cannot be represented as a standard Caputo initial-value problem with finite classical velocity.

4.2. Generalized start-up forcing

We now consider

$$g(t) = t^{-\mu} h(t), \; 0 < \mu < 1, \tag{4.4}$$

where $h \in C[0,T]$.

Theorem 4.3. *Let f be the solution corresponding to* (4.4)*, and set $q = \alpha - \mu$. Then, as $t \to 0$,*

$$f(t) - f_0 = \frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{h(0)\Gamma(1-\mu)}{\Gamma(q+1)} t^q + o(t^q), \tag{4.5}$$

and

$$f'(t) = \frac{\vartheta}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{h(0)\Gamma(1-\mu)}{\Gamma(q)} t^{q-1} + o(t^{q-1}). \tag{4.6}$$

Proof. From (3.2), we have

$$f(t) - f_0 = \frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1} + I_0^\alpha(t^{-\mu} h(t)) - \lambda I_0^\alpha f(t).$$

After the substitution $\tau = tr$,

$$\begin{aligned} I_0^\alpha(t^{-\mu} h(t)) &= \frac{t^q}{\Gamma(\alpha)} \int_0^1 (1-r)^{\alpha-1} r^{-\mu} h(tr)\, \mathrm{d}r \\ &= \frac{h(0)\Gamma(1-\mu)}{\Gamma(q+1)} t^q + o(t^q). \end{aligned}$$

Since f is continuous, $I_0^\alpha f = O(t^\alpha) = o(t^q)$ yields (4.5).

Differentiating (3.2) for $t > 0$ gives

$$f'(t) = \frac{\vartheta}{\Gamma(\alpha-1)} t^{\alpha-2} + I_0^{\alpha-1}(t^{-\mu} h(t) - \lambda f(t)).$$

By the same argument as above, we obtain (4.6). □

Corollary 4.2. *Assume $h(0) \neq 0$. Then*

$$f'(0) = 0 \Longleftrightarrow \vartheta = 0 \ \text{ and } \ \alpha - \mu > 1. \tag{4.7}$$

If $\vartheta = 0$ and $\alpha - \mu = 1$, then

$$f'(0) = h(0)\Gamma(1-\mu).$$

Proposition 4.1. *Assume $\vartheta = 0$ and*

$$h(t) = h_0 + h_1 t^\rho + o(t^\rho),\ h_0 \neq 0,\ \rho > 0, \tag{4.8}$$

and $f(t) = f_0 + O(t^q)$ with $q = \alpha - \mu$, then

$$f(t) = f_0 + \frac{h_0\Gamma(1-\mu)}{\Gamma(q+1)} t^q + \begin{cases} \dfrac{h_1\Gamma(\rho-\mu+1)}{\Gamma(q+\rho+1)} t^{q+\rho} + o(t^{q+\rho}), & 0 < \rho < \mu, \\ \dfrac{h_1 - \lambda f_0}{\Gamma(\alpha+1)} t^\alpha + o(t^\alpha), & \rho = \mu, \\ -\dfrac{\lambda f_0}{\Gamma(\alpha+1)} t^\alpha + o(t^\alpha), & \rho > \mu \text{ and } f_0 \neq 0. \end{cases} \tag{4.9}$$

If $\rho > \mu$ and $f_0 = 0$, the next nonzero term is determined by the smaller of $q + \rho$ and $q + \alpha$.

Remark 4.3. *Suppose that $h(t) = ct^{\rho} + o(t^{\rho}), c \neq 0, \rho > 0$, Then, on the zero-trace branch $\vartheta = 0$, $f'(0) = 0$, if and only if $\alpha + \rho - \mu > 1$. Thus, when $h(0) = 0$, the cold-start threshold is determined by the first nonvanishing local term of h.*

Remark 4.4. *Under the assumptions above, a cold start with $f'(0) = 0$ may coexist with an unbounded but integrable start-up acceleration.*

The results above show that the behavior of $f'(t)$ near $t = 0$ is determined by its leading nonzero term. This is similar to Voronovskaja-type analysis in approximation theory, where the leading error term is used to describe the asymptotic behavior of an approximation; see, for example, [20, 21].

5. A discrete Volterra solver and numerical experiments

We compute the numerical solution from the equivalent Volterra equation, using the closed-form Mittag–Leffler solution only to evaluate the numerical error. Since the forcing $g(t) = At^{-\mu}$ is unbounded at $t = 0$, a quadrature rule that samples $g(0)$ is not applicable. We therefore evaluate $I_0^{\alpha}(At^{-\mu})$ analytically and discretize the remaining history integral by a piecewise-linear product-integration method on uniform and graded meshes; see, for example, [22, 23] for related numerical approaches.

Set $q = \alpha - \mu$. From the Volterra formulation (3.2) and

$$I_0^{\alpha}(At^{-\mu}) = \frac{A\Gamma(1-\mu)}{\Gamma(q+1)} t^{q},$$

we obtain

$$f(t) = b(t) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \qquad b(t) = f_0 + \frac{\vartheta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{A\Gamma(1-\mu)}{\Gamma(q+1)} t^{q}. \tag{5.1}$$

Thus, the singular value $g(0)$ does not enter the numerical scheme.

5.1. Piecewise-linear product-integration scheme

Let

$$0 = t_0 < t_1 < \cdots < t_N = T, \qquad t_n = T\left(\frac{n}{N}\right)^{r}, \qquad r \geq 1,$$

where $r = 1$ gives a uniform mesh, while $r > 1$ places more points near $t = 0$. Denote $h_j = t_{j+1} - t_j$ and let F_n approximate $f(t_n)$. On each interval $[t_j, t_{j+1}]$, we set

$$(\Pi_j F)(\tau) = \frac{t_{j+1} - \tau}{h_j} F_j + \frac{\tau - t_j}{h_j} F_{j+1}.$$

For $x_{n,j} = t_n - t_j$, define

$$A_{n,j} = \frac{1}{h_j}\left[\frac{x_{n,j}^{\alpha+1} - x_{n,j+1}^{\alpha+1}}{\alpha+1} - \frac{x_{n,j+1}(x_{n,j}^{\alpha} - x_{n,j+1}^{\alpha})}{\alpha}\right], \tag{5.2}$$

$$B_{n,j} = \frac{1}{h_j}\left[\frac{x_{n,j}(x_{n,j}^{\alpha} - x_{n,j+1}^{\alpha})}{\alpha} - \frac{x_{n,j}^{\alpha+1} - x_{n,j+1}^{\alpha+1}}{\alpha+1}\right]. \tag{5.3}$$

Then

$$I_0^\alpha f(t_n) \approx \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} (A_{n,j} F_j + B_{n,j} F_{j+1}).$$

The last interval contains the unknown value F_n. With $F_0 = f_0$, we obtain, for $n \geq 1$,

$$F_n = \frac{b(t_n) - \dfrac{\lambda}{\Gamma(\alpha)} \left[\displaystyle\sum_{j=0}^{n-2} (A_{n,j} F_j + B_{n,j} F_{j+1}) + A_{n,n-1} F_{n-1} \right]}{1 + \dfrac{\lambda}{\Gamma(\alpha)} B_{n,n-1}}. \tag{5.4}$$

The sum in (5.4) is taken as zero when $n = 1$. Moreover,

$$B_{n,n-1} = \frac{h_{n-1}^\alpha}{\alpha(\alpha+1)} > 0,$$

so the denominator in (5.4) is positive for$\lambda \geq 0$.

5.2. Mesh refinement and start-up behavior

We take

$$T = 1, \qquad \alpha = 1.8, \qquad \mu = 0.5, \qquad \lambda = A = 1, \qquad f_0 = 0,$$

so that $q = \alpha - \mu = 1.3$. We consider $\vartheta = 0$ and $\vartheta = 1$. For $N = 20, 40, 80, 160, 320$, let

$$E_N = \max_{0 \leq n \leq N} |F_n - f(t_n)|,$$

where the exact solution (4.1) is used only to evaluate the error. Assuming that $E_N \approx CN^{-p}$ as N increases, we estimate the observed convergence order by

$$\text{rate}_N = \log_2 \left(\frac{E_{N/2}}{E_N} \right), \qquad N \geq 40.$$

For $\vartheta = 0$, the uniform-mesh results are already close to second-order convergence. When $\vartheta = 1$, the $t^{\alpha-1}$ term causes a stronger variation near $t = 0$, and the convergence rate on a uniform mesh is lower. Using the graded mesh with $r = 2$ places more grid points near $t = 0$ and yields essentially second-order convergence in this test. These results can be clearly seen in Table 1. To better visualize the convergence of numerical solution, we plot these results in Figure 1.

Table 1. Maximum nodal errors for the product-integration Volterra solver.

	$\vartheta = 0, r = 1$		$\vartheta = 1, r = 1$		$\vartheta = 1, r = 2$	
N	E_N	rate	E_N	rate	E_N	rate
20	1.7226×10^{-4}	–	2.0822×10^{-4}	–	1.5866×10^{-4}	–
40	4.5708×10^{-5}	1.914	6.7780×10^{-5}	1.619	3.9590×10^{-5}	2.003
80	1.1959×10^{-5}	1.934	2.1678×10^{-5}	1.645	9.8929×10^{-6}	2.001
160	3.0976×10^{-6}	1.949	6.8227×10^{-6}	1.668	2.4730×10^{-6}	2.000
320	7.9623×10^{-7}	1.960	2.1176×10^{-6}	1.688	6.1823×10^{-7}	2.000

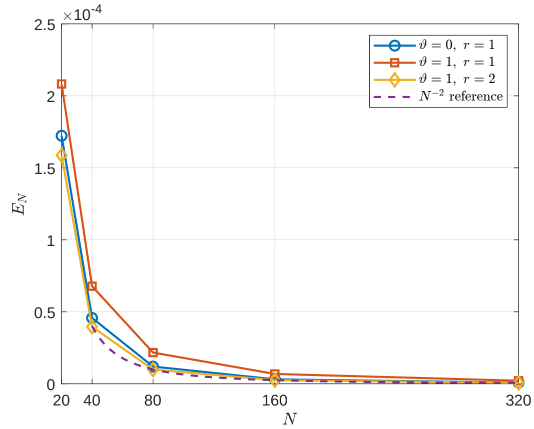

Figure 1. Log–log plot of the maximum nodal errors for the three test cases. The dashed line indicates the N^{-2} reference slope.

We next examine the numerical behavior near $t = 0$.

For the uniform meshes considered below, let $h = \frac{T}{N}$, and define

$$p_N = \frac{\log\left|\dfrac{F_2 - f_0}{F_1 - f_0}\right|}{\log(t_2/t_1)}, \qquad V_1 = \frac{F_1 - F_0}{t_1 - t_0}.$$

If $f(t) - f_0 \sim Ct^{\rho}$ near $t = 0$, then p_N approaches ρ under mesh refinement. Thus, p_N provides a numerical measure of the leading start-up exponent.

Figure 2 shows that the values of p_N approach $q = 1.3$ for $\vartheta = 0$, while $V_1 \sim C_0 h^{q-1} \to 0$ with $C_0 = \frac{A\Gamma(1-\mu)}{\Gamma(q+1)}$. On the other hand, for $\vartheta = 1$, the values of p_N approach $\alpha - 1 = 0.8$, and $V_1 \sim \frac{h^{\alpha-2}}{\Gamma(\alpha)}$ which increases as $h \to 0$. The discrete solution therefore reproduces the zero-velocity cold start in the zero-trace case and the singular ordinary velocity in the nonzero-trace case.

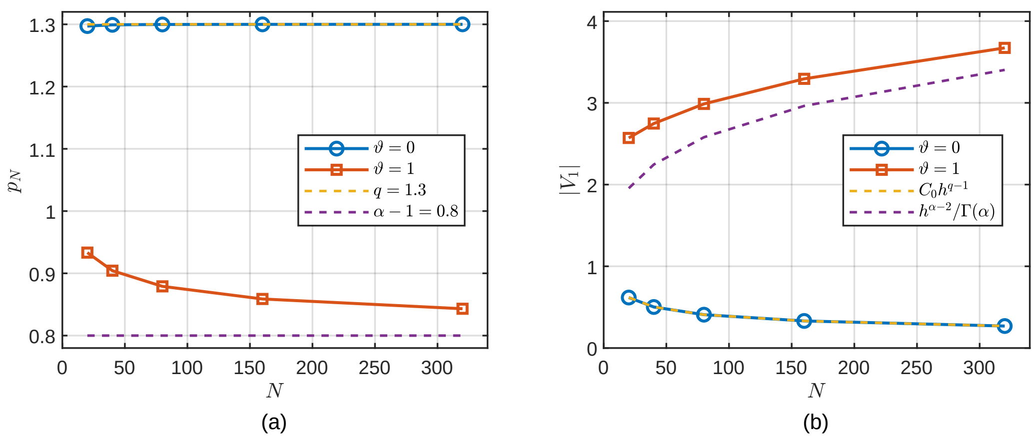

Figure 2. Numerical behavior near $t = 0$ on uniform meshes. (a) The numerical exponent p_N approaches $q = 1.3$ for $\vartheta = 0$ and $\alpha - 1 = 0.8$ for $\vartheta = 1$. (b) $|V_1|$ tends to zero for $\vartheta = 0$ and increases as the mesh is refined for $\vartheta = 1$.

6. Conclusions

The problem is well posed, but well-posedness alone does not imply a cold start. For the power-law forcing $At^{-\mu}$ with $A \neq 0$, the solution satisfies $f'(0) = 0$ exactly when $\vartheta = 0$ and $\alpha - \mu > 1$. These conditions have different roles: $\vartheta = 0$ eliminates the singular contribution of the fractional trace to the ordinary velocity, while $\alpha - \mu > 1$ ensures that the velocity generated by the weakly singular forcing vanishes at $t = 0$. On the zero-trace branch, $\alpha - \mu = 1$ gives a finite nonzero initial velocity, whereas $\alpha - \mu < 1$ gives an unbounded velocity. A cold start may still have an unbounded acceleration, because local integrability of the acceleration is sufficient for the velocity to remain continuous at the initial time. In the cold-start regime, the acceleration may still be unbounded near $t = 0$, but its singularity is locally integrable, so the velocity remains continuous at the initial time.

The numerical results show that the start-up behavior also affects the convergence of the numerical scheme.The product-integration method captures both the zero-trace cold start and the velocity singularity caused by a nonzero trace. In the reported tests, a nonzero trace reduces the convergence rate on a uniform mesh, while grading the mesh near $t = 0$ restores essentially second-order convergence. Compared with the response studies reviewed in the introduction, the present work identifies when a weakly singular start-up forcing is compatible with zero ordinary initial velocity and shows how the corresponding start-up singularity affects numerical convergence.

Author contributions

Junshan Li: Formal analysis, investigation, validation, visualization, and writing–original draft; Minling Zheng: Conceptualization, methodology, supervision, and writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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