Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(9): 30566–30580.
DOI: 10.3934/math.20261211
Received: 06 August 2026
Revised: 03 September 2026
Accepted: 09 September 2026
Published: 20 September 2026

Research article

Inclusion relations between Abel and Cesàro summability in the interval-valued setting

Bağdagül Kartal Erdoğan*

Erciyes University, Department of Mathematics, 38039 Kayseri, Türkiye

* **Correspondence:** Email: bagdagulkartal@erciyes.edu.tr.

Abstract: In this paper, the relationship between Abel and Cesàro convergence for interval-valued sequences and summability for interval-valued series is investigated. It is first shown that Cesàro convergence implies Abel convergence for interval-valued sequences. A Tauberian converse is then established by showing that Abel convergence implies Cesàro convergence when the endpoint sequences are bounded. These results are further extended to interval-valued series through their sequences of partial sums. In particular, interval-valued analogues of several classical Tauberian theorems are obtained by showing that Abel summability implies Cesàro summability under boundedness, nonnegativity, and the lower Landau condition on the sequence of partial sums. Consequently, classical Abel-Cesàro inclusion results and Tauberian theorems are extended to the interval-valued setting, providing a basis for further investigations of interval-valued summability methods.

Keywords: interval-valued sequences; Abel summability; Cesàro summability; Tauberian theorem; Hausdorff metric
Mathematics Subject Classification: 40A05, 40D25, 40G05, 40G10

1. Introduction

Summability methods play an important role in mathematical analysis, particularly in the study of divergent sequences and series. Among the classical summability methods, Abel and Cesàro summability have been extensively investigated because of their close connection with Fourier analysis, Tauberian theory, and other areas of analysis. The fundamental relations between these methods were established in Hardy's book [1], which remains one of the principal references in summability theory. Besides the Abel and Cesàro methods, various generalizations and extensions of summability theory have been developed. In [2, 3], absolute Cesàro summability methods have been studied. Absolute matrix summability methods for infinite series have also been extensively studied in [4, 5]. These studies have contributed significantly to the development of summability theory. Many practical problems involve uncertainty arising from measurement errors, incomplete information, or numerical approximations. Interval analysis, introduced by Moore [6] and further developed by Markov [7] and Moore et al. [8], provides an effective framework for dealing with such uncertainties by replacing real numbers with closed intervals. Since its introduction, interval analysis has developed into an important research area [9, 10], and various applications in quasilinear spaces and signal processing have also been investigated [11, 12]. Motivated by these developments, more recently, Kılınç and Yıldırım [13] introduced the space of interval-valued Cesàro convergent sequences, while Kartal Erdogan and Topçu [14] established the space of interval-valued Abel convergent sequences and studied its fundamental properties. Also, related results for interval-valued Riesz convergence were obtained in [15] by Tuncer and Akgün. In [13–15], the fundamental properties of interval-valued Cesàro, Abel, and Riesz convergent sequence spaces, respectively, were investigated. In the present paper, the relations between Abel and Cesàro convergence (for sequences) and summability (for series) in the interval-valued setting are studied. Accordingly, this study focuses on the interaction between Abel and Cesàro methods rather than on the structure of the corresponding sequence spaces.

In the interval-valued setting, both Abel and Cesàro methods provide useful ways to study the convergence of sequences and summability of series. Cesàro summability is based on the arithmetic means of the terms, while Abel summability is defined by means of the corresponding power series. The relation between these two methods is important since it shows under which conditions one type of summability implies the other. In particular, the Abel-Cesàro relation allows classical inclusion and Tauberian results to be considered for interval-valued sequences and series. Although several interval-valued summability spaces have been introduced and investigated, the corresponding Abel-Cesàro and Tauberian relations in the interval-valued setting appear to be less extensively studied.

The main results establish inclusion relations between interval-valued Abel and Cesàro methods for both sequences and series. It is shown that interval-valued Cesàro convergence implies interval-valued Abel convergence, while the converse implication is obtained under suitable Tauberian assumptions. These results are subsequently extended to interval-valued series by considering their sequences of partial sums, leading to interval-valued versions of several classical Tauberian theorems. The proofs make explicit use of the endpoint representation of intervals and the componentwise form of the Hausdorff metric. This enables the corresponding classical real-valued results to be applied to the endpoint sequences.

The remainder of the paper is organized as follows. Section 2 presents the necessary definitions and preliminary results. Section 3 establishes the inclusion relations between interval-valued Abel and Cesàro convergence for sequences and extends these results to interval-valued series. Finally, Section 4 concludes the paper and discusses possible directions for future research.

2. Preliminaries

In this section, the necessary definitions and relevant results are given, and a general proposition for the interval-valued setting is established.

Definition 1 ([8]). *An interval X is represented by*

$$X = \left[\underline{X}, \overline{X}\right] = \left\{x \in \mathbb{R} : \underline{X} \leq x \leq \overline{X}\right\},$$

where $\underline{X}$ and $\overline{X}$ denote the lower and upper endpoints of X, respectively.

Definition 2 ([8]). *An interval of the form $[x, x]$, where $x \in \mathbb{R}$, is called a degenerate (or derived) interval. Such an interval is identified with the real number x, that is, $x = [x, x]$.*

Considering $X = [\underline{X}, \overline{X}]$ and $Y = [\underline{Y}, \overline{Y}]$, the inequality $X < Y$ is equivalent to $\overline{X} < \underline{Y}$, while $X \subseteq Y$ is characterized by $\underline{Y} \leq \underline{X}, \overline{X} \leq \overline{Y}$. Arithmetic operations within the interval setting are formulated using endpoint calculations as follows:

$$X + Y = [\underline{X} + \underline{Y},\ \overline{X} + \overline{Y}], \qquad X - Y = [\underline{X} - \overline{Y},\ \overline{X} - \underline{Y}],$$

$$C = \{\underline{XY},\ \underline{X}\overline{Y},\ \overline{X}\underline{Y},\ \overline{XY}\}, \qquad \text{then} \qquad X \cdot Y = [\min C,\ \max C],$$

$$\frac{X}{Y} = X \cdot \frac{1}{Y} = [\underline{X},\ \overline{X}] \cdot \left[\frac{1}{\overline{Y}},\ \frac{1}{\underline{Y}}\right]; \quad 0 \notin Y,$$

$$\alpha X = \alpha[\underline{X},\ \overline{X}] = \begin{cases} [\alpha\underline{X},\ \alpha\overline{X}], & \alpha \geq 0, \\ [\alpha\overline{X},\ \alpha\underline{X}], & \alpha < 0. \end{cases}$$

Definition 3 ([8]). *Let*

$$I = \left\{ X = [\underline{X}, \overline{X}] :\ \underline{X}, \overline{X} \in \mathbb{R},\ \underline{X} \leq \overline{X} \right\}$$

denote the family of all closed intervals on $\mathbb{R}$. An interval-valued sequence is defined as a mapping $f : \mathbb{N} \to I$, where $\mathbb{N} = \{0, 1, 2, ...\}$ denotes the set of natural numbers.

The family of all sequences whose terms are closed intervals is denoted by $w(I)$ and specified by

$$w(I) = \{ (X_n) :\ X_n \in I,\ n \in \mathbb{N} \}.$$

Definition 4 ([8]). *The set I of all real intervals forms a metric space under the Hausdorff metric h, where*

$$h(X, Y) = \max\{|\underline{X} - \underline{Y}|,\ |\overline{X} - \overline{Y}|\}.$$

Definition 5 ([16]). *Let (X_n) be an interval-valued sequence, and let $X_0 = \left[\underline{X_0}, \overline{X_0}\right]$ be an interval. The sequence (X_n) is said to converge to X_0 if, for every $\varepsilon > 0$, there exists an integer N such that $h(X_n, X_0) < \varepsilon$ whenever $n > N$. In this case, the convergence is denoted by $\lim_{n\to\infty} X_n = X_0$, or simply by $X_n \to X_0$, as $n \to \infty$.*

Definition 6 ([13]). *An interval-valued sequence is said to be Cesàro convergent to the interval $V = \left[\underline{V}, \overline{V}\right]$ if and only if*

$$\lim_{n\to\infty} h\left(\frac{1}{n+1}\sum_{k=0}^{n} U_k,\ V\right) = 0.$$

$C_I(1)$ represents the space of interval-valued Cesàro convergent sequences. That is,

$$C_I(1) = \left\{(U_k) \in w(I) :\ \lim_{n\to\infty} h\left(\frac{1}{n+1}\sum_{k=0}^{n}[\underline{U}_k, \overline{U}_k],\ [\underline{V}, \overline{V}]\right) = 0,\ V = [\underline{V}, \overline{V}] \in I\right\}.$$

The notion of Cesàro convergence for interval-valued sequences extends to Cesàro summability for interval-valued series through their sequences of partial sums, in complete analogy with the classical theory.

Definition 7. *Let $\sum_{k=0}^{\infty} A_k$ be an interval-valued series, where $A_k \in I$ for each $k \geq 0$, and let*

$$S_n = \sum_{k=0}^{n} A_k$$

denote its sequence of partial sums.

Following the classical Cesàro summability method, the series $\sum_{k=0}^{\infty} A_k$ is Cesàro summable to the interval $S = [\underline{S}, \overline{S}] \in I$ if its sequence of partial sums (S_n) is Cesàro convergent to S according to Definition 6; that is,

$$\lim_{n\to\infty} h\left(\frac{1}{n+1}\sum_{k=0}^{n} S_k,\, S\right) = 0.$$

In this case, it is denoted by

$$\sum_{k=0}^{\infty} A_k = S \quad (C, 1).$$

Definition 8 ([14])**.** *Let $U = (U_k)$ be an interval-valued sequence. The sequence U is said to be Abel convergent to the interval $V = \left[\underline{V}, \overline{V}\right]$ if $\lim_{x\to 1^-} h\left(F_U(x), V\right) = 0$. Here, for $0 < x < 1$,*

$$F_U(x) = (1-x)\sum_{k=0}^{\infty} U_k x^k = \left[\underline{F}_U(x),\, \overline{F}_U(x)\right] = \left[(1-x)\sum_{k=0}^{\infty} \underline{U}_k x^k,\, (1-x)\sum_{k=0}^{\infty} \overline{U}_k x^k\right].$$

The set of interval-valued sequences converging in the Abel sense is denoted by A_I and explicitly defined as

$$A_I = \left\{(U_k) \in w(I) : \lim_{x\to 1^-} h\left(F_U(x),\, V\right) = 0 \quad \text{for some } V \in I\right\}.$$

Similarly, Abel summability for interval-valued series is defined by means of the Abel convergence of their sequences of partial sums.

Definition 9. *Let $\sum_{k=0}^{\infty} A_k$ be an interval-valued series, where $A_k \in I$ for each $k \geq 0$, and let*

$$S_n = \sum_{k=0}^{n} A_k$$

denote its sequence of partial sums.

Following the classical Abel summability method, the series $\sum_{k=0}^{\infty} A_k$ is Abel summable to the interval $S = [\underline{S}, \overline{S}] \in I$ if its sequence of partial sums (S_n) is Abel convergent to S according to Definition 8; that is,

$$\lim_{x\to 1^-} h\left((1-x)\sum_{n=0}^{\infty} S_n x^n,\, S\right) = 0.$$

In this case, we write

$$\sum_{k=0}^{\infty} A_k = S \ (A).$$

Definition 10 ([1]). *A real sequence (a_n) is said to satisfy the lower Landau condition, denoted by*

$$a_n = O_L(1),$$

if there exists a constant $M > 0$ such that

$$a_n \geq -M$$

for all sufficiently large n.

Remark 1. *For an interval-valued sequence $U = (U_n)$, where $U_n = [\underline{U}_n, \overline{U}_n]$, the notation $U_n = O_L(1)$ is understood componentwise. Since $\underline{U}_n \leq \overline{U}_n$ for all $n \geq 0$, if the lower endpoint satisfies $\underline{U}_n = O_L(1)$, then the upper endpoint automatically satisfies $\overline{U}_n = O_L(1)$ as well.*

Definition 11 ([9]). *The class of bounded interval-valued sequences is denoted by I_{ℓ_∞} and introduced by*

$$I_{\ell_\infty} = \left\{(U_k) \in w(I) : \sup_k \{|\underline{U}_k|, |\overline{U}_k|\} < \infty\right\}.$$

Remark 2. *Let $U = (U_k)$ be an interval-valued sequence with $U_k = [\underline{U}_k, \overline{U}_k]$ for each $k \geq 0$, and let $0 = [0, 0]$. Recall that the Hausdorff metric distance from U_k to the zero interval is given by*

$$h(U_k,\ [0, 0]) = \max\{|\underline{U}_k|,\ |\overline{U}_k|\}.$$

Therefore, $U = (U_k)$ is bounded in the metric space (I, h) if and only if $\sup_k h(U_k, [0, 0]) < \infty$, which is precisely equivalent to the simultaneous boundedness of both endpoint sequences $(\underline{U}_k)$ and $(\overline{U}_k)$ in $\mathbb{R}$ (i.e., $U \in I_{\ell_\infty}$).

Lemma 1 ([1]). *Let (a_n) be a sequence of real numbers. If (a_n) is Cesàro convergent to L, that is,*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_k = L,$$

then (a_n) is Abel convergent to L, i.e.,

$$\lim_{x\to 1^-} (1 - x) \sum_{n=0}^{\infty} a_n x^n = L.$$

Lemma 2 ([1]). *Let (a_n) be a bounded sequence of real numbers. If (a_n) is Abel convergent to L, that is,*

$$\lim_{x\to 1^-} (1 - x) \sum_{n=0}^{\infty} a_n x^n = L,$$

then (a_n) is Cesàro convergent to L, i.e.,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_k = L.$$

Lemma 3 ([1]). *Let $\sum a_n$ be a series of real numbers with sequence of partial sums (s_n). If $\sum a_n = s\ (A)$ and $s_n = O(1)$, then $\sum a_n = s\ (C, 1)$.*

Lemma 4 ([1]). *Let $\sum a_n$ be a series of real numbers with sequence of partial sums (s_n). If $\sum a_n = s\,(A)$ and $s_n \geq 0$ for all $n \geq 0$, then $\sum a_n = s\,(C, 1)$.*

Lemma 5 ([1]). *Let $\sum a_n$ be a series of real numbers with sequence of partial sums (s_n). If $\sum a_n = s\,(A)$ and $s_n = O_L(1)$, then $\sum a_n = s\,(C, 1)$.*

Proposition 1. *Let T be a regular summability method defined componentwise by linear positive operators. Let*

$$U = (U_k) = ([\underline{U}_k, \overline{U}_k])$$

be an interval-valued sequence, and let $V = [\underline{V}, \overline{V}] \in I$. Then U is T convergent (or T summable) to V if and only if the real-valued endpoint sequences $(\underline{U}_k)$ and $(\overline{U}_k)$ are T convergent (or T summable) to $\underline{V}$ and $\overline{V}$, respectively.

Proof. Let $T(U)$ denote the transformed interval sequence of U under the summability method T. Since T is defined componentwise by linear positive operators, the interval arithmetic properties ensure that

$$T(U) = [\, T((\underline{U}_k)),\, T((\overline{U}_k))\,].$$

By Definition 4, the Hausdorff metric distance between $T(U)$ and V is expressed as

$$h(T(U),\, V) = \max\{|T((\underline{U}_k)) - \underline{V}|,\, |T((\overline{U}_k)) - \overline{V}|\}.$$

Consequently, $h\,(T(U), V) \longrightarrow 0$ holds if and only if both

$$\left|T((\underline{U}_k)) - \underline{V}\right| \longrightarrow 0 \quad \text{and} \quad \left|T((\overline{U}_k)) - \overline{V}\right| \longrightarrow 0$$

simultaneously. Thus, U is T convergent (or T summable) to V if and only if the real-valued endpoint sequences $(\underline{U}_k)$ and $(\overline{U}_k)$ are T convergent (or T summable) to $\underline{V}$ and $\overline{V}$, respectively. □

3. Inclusion relations between A_I and $C_I(1)$

In this section, the inclusion relations between interval-valued Abel and Cesàro convergence for sequences and the corresponding summability methods for series are investigated.

Theorem 1. *Let $U = (U_k)$ be an interval-valued Cesàro convergent sequence. If*

$$\frac{1}{n+1}\sum_{k=0}^{n} U_k \to V = [\underline{V},\, \overline{V}] \quad (n \to \infty),$$

then U is Abel convergent to V.

Proof. Let

$$U_k = [\underline{U}_k, \overline{U}_k]$$

for $k \geq 0$. Since U is Cesàro convergent to $V = [\underline{V}, \overline{V}]$, Proposition 1 implies that $(\underline{U}_k)$ and $(\overline{U}_k)$ are Cesàro convergent to $\underline{V}$ and $\overline{V}$, respectively. By Lemma 1,

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty} \underline{U}_k\, x^k = \underline{V},$$

and

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}\overline{U}_k\, x^k = \overline{V}.$$

Therefore,

$$\lim_{x\to 1^-} h(F_U(x),\, V) = 0,$$

which shows that U is Abel convergent to V. □

The following examples illustrate Theorem 1 and demonstrate that the converse of Theorem 1 does not hold in general without additional Tauberian conditions.

Example 1. *Consider the interval-valued sequence $U = (U_k)_{k=0}^{\infty}$ defined by*

$$U_k = \left[(-1)^k,\ 1 + \frac{1}{k+1}\right] \quad (k \geq 0).$$

Let us examine both the Cesàro and Abel convergence of (U_k):
(1) ***Cesàro Convergence:*** *For the lower endpoint sequence, $\underline{U}_k = (-1)^k$ for $k \geq 0$. The arithmetic mean yields*

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k = 0 = \underline{V}.$$

For the upper endpoint sequence,

$$\overline{U}_k = 1 + \frac{1}{k+1} \qquad \text{for } k \geq 0.$$

Using the arithmetic mean of the upper endpoints, we explicitly have:

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(1+\frac{1}{k+1}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+1}\right) = 1+0 = 1 = \overline{V},$$

since

$$\sum_{k=0}^{n}\frac{1}{k+1} = O(\ln n) \quad \text{and} \quad \lim_{n\to\infty}\frac{\ln n}{n+1} = 0.$$

Thus, by Proposition 1, U is Cesàro convergent to $V = [0, 1]$.
(2) ***Abel Convergence:*** *For $0 < x < 1$, the power series representation yields*

$$(1-x)\sum_{k=0}^{\infty}\underline{U}_k\, x^k = (1-x)\sum_{k=0}^{\infty}(-1)^k x^k = (1-x)\frac{1}{1+x} \xrightarrow{x\to 1^-} 0 = \underline{V},$$

and for the upper endpoint,

$$(1-x)\sum_{k=0}^{\infty}\overline{U}_k x^k = (1-x)\sum_{k=0}^{\infty}\left(1+\frac{1}{k+1}\right)x^k = (1-x)\left(\frac{1}{1-x}+\frac{1}{x}\ln\left(\frac{1}{1-x}\right)\right) = 1+\frac{1-x}{x}\ln\left(\frac{1}{1-x}\right).$$

Since

$$\lim_{x\to 1^-}(1-x)\ln\left(\frac{1}{1-x}\right) = 0,$$

it follows that

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}\overline{U}_k\, x^k = 1 = \overline{V}.$$

Therefore,

$$\lim_{x\to 1^-} h(F_U(x),\ V) = 0,$$

confirming Theorem 1.

The next counterexample shows that Abel convergence of an interval-valued sequence does not necessarily imply its Cesàro convergence.

Example 2. *Define the interval-valued sequence* $U = (U_k)_{k=0}^{\infty}$ *by*

$$U_k = [(-1)^k\, k,\ (-1)^k\, k+1] \quad (k \geq 0).$$

Note that the endpoint sequences are unbounded, i.e., $U \notin I_{l_\infty}$.
(1) ***Abel Convergence:*** *Recall the formal power series identity for* $0 < x < 1$*:*

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

Replacing x *by* $-x$*, we obtain*

$$\sum_{k=0}^{\infty}(-1)^k kx^k = \frac{-x}{(1+x)^2}.$$

Now, let us calculate the Abel mean for the lower endpoint sequence $\underline{U}_k = (-1)^k k$*:*

$$\lim_{x\to 1^-}\underline{F}_U(x) = \lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty}(-1)^k kx^k = \lim_{x\to 1^-}(1-x)\frac{-x}{(1+x)^2} = 0\cdot\left(-\frac{1}{4}\right) = 0.$$

For the upper endpoint sequence $\overline{U}_k = (-1)^k k + 1$*, we split the sum:*

$$\overline{F}_U(x) = (1-x)\sum_{k=0}^{\infty}\left((-1)^k\, k+1\right)x^k = (1-x)\frac{-x}{(1+x)^2} + (1-x)\frac{1}{1-x} = \frac{-x(1-x)}{(1+x)^2} + 1.$$

Taking the limit as $x \to 1^-$*, we obtain*

$$\lim_{x\to 1^-}\overline{F}_U(x) = 0 + 1 = 1.$$

Consequently, by Proposition 1, we have

$$\lim_{x\to 1^-} h(F_U(x),\ [0,1]) = 0.$$

Hence, U *is Abel convergent to* $V = [0,1]$*, i.e.,* $U \in A_I$.
(2) ***Cesàro Divergence:*** *Now, let us evaluate the Cesàro mean of the lower endpoint sequence* $\underline{U}_k = (-1)^k k$*. Let*

$$C_n = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\, k.$$

- *If n is even, say, $n = 2m$ ($m \geq 0$):*

$$\sum_{k=0}^{2m} (-1)^k k = 0 + (-1+2) + (-3+4) + \cdots + (-(2m-1) + 2m) = m.$$

Thus,

$$C_{2m} = \frac{m}{2m+1},$$

which implies, $\lim_{m\to\infty} C_{2m} = \frac{1}{2}$.

- *If n is odd, say, $n = 2m+1$ ($m \geq 0$):*

$$\sum_{k=0}^{2m+1} (-1)^k k = m - (2m+1) = -m - 1.$$

Thus,

$$C_{2m+1} = \frac{-m-1}{2m+2} = -\frac{1}{2},$$

which implies, $\lim_{m\to\infty} C_{2m+1} = -\frac{1}{2}$.

Since the subsequence limits $\lim_{m\to\infty} C_{2m} = \frac{1}{2}$ *and* $\lim_{m\to\infty} C_{2m+1} = -\frac{1}{2}$ *do not coincide, the limit*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \underline{U}_k$$

does not exist. Therefore, (U_k) *is not Cesàro convergent, i.e.,* $U \notin C_I(1)$.

To overcome this limitation and establish a converse result, suitable Tauberian conditions must be imposed.

Theorem 2. *Let $U = (U_k)$ be an interval-valued Abel convergent sequence, and assume that $U \in I_{\ell_\infty}$. Then $U \in C_I(1)$. Equivalently,*

$$A_I \cap I_{\ell_\infty} \subset C_I(1).$$

Proof. Let $U_k = [\underline{U}_k, \overline{U}_k]$ for $k \geq 0$. Since U is Abel convergent, there exists an interval $V = [\underline{V}, \overline{V}]$ such that

$$\lim_{x\to 1^-} h(F_U(x),\ V) = 0.$$

By Proposition 1, the endpoint sequences $(\underline{U}_k)$ and $(\overline{U}_k)$ are Abel convergent to $\underline{V}$ and $\overline{V}$, respectively. Since $U \in I_{\ell_\infty}$, both endpoint sequences are bounded by Remark 2. Hence, by Lemma 2,

$$\frac{1}{n+1} \sum_{k=0}^{n} \underline{U}_k \longrightarrow \underline{V} \qquad \text{and} \qquad \frac{1}{n+1} \sum_{k=0}^{n} \overline{U}_k \longrightarrow \overline{V} \qquad (n \to \infty).$$

Therefore,

$$\frac{1}{n+1} \sum_{k=0}^{n} U_k \longrightarrow [\underline{V},\ \overline{V}] = V \quad (n \to \infty).$$

Thus, U is Cesàro convergent to V, i.e., $U \in C_I(1)$. □

Having established the relationship for interval-valued sequences, the corresponding convergence implications are now extended to interval-valued series by means of their sequences of partial sums.

Theorem 3. *Let $\sum_{n=0}^{\infty} A_n$ be an interval-valued series with sequence of partial sums (S_n). If*

$$\sum_{n=0}^{\infty} A_n = S \ (A) \qquad \textit{and} \qquad (S_n) \in I_{\ell_\infty},$$

then

$$\sum_{n=0}^{\infty} A_n = S \ (C, 1).$$

Proof. Let $A_n = [\underline{A}_n, \overline{A}_n]$ and $S_n = [\underline{S}_n, \overline{S}_n]$, where $S_n = \sum_{k=0}^{n} A_k$. Suppose that

$$\sum_{n=0}^{\infty} A_n = S = [\underline{S}, \overline{S}] \ (A).$$

Then, by Definition 9,

$$\lim_{x \to 1^-} h\left((1 - x) \sum_{n=0}^{\infty} S_n x^n, S\right) = 0.$$

By applying Proposition 1 to the sequence of partial sums (S_n), the endpoint sequences $(\underline{S}_n)$ and $(\overline{S}_n)$ are Abel convergent to $\underline{S}$ and $\overline{S}$, respectively. Hence,

$$\lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} \underline{S}_n x^n = \underline{S} \qquad \text{and} \qquad \lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} \overline{S}_n x^n = \overline{S}.$$

Since $(S_n) \in I_{\ell_\infty}$, Remark 2 implies that both endpoint sequences $(\underline{S}_n)$ and $(\overline{S}_n)$ are bounded. Therefore, by Lemma 3,

$$\sum_{n=0}^{\infty} \underline{A}_n = \underline{S} \quad (C, 1) \qquad \text{and} \qquad \sum_{n=0}^{\infty} \overline{A}_n = \overline{S} \quad (C, 1).$$

Applying Proposition 1 again to the sequence of partial sums, we obtain

$$\sum_{n=0}^{\infty} A_n = S \ (C, 1).$$

Thus, the interval-valued series is $(C, 1)$ summable to S. □

Example 3. *Consider the interval-valued series $\sum_{n=0}^{\infty} A_n$ whose sequence of partial sums (S_n) is defined by*

$$S_n = [(-1)^n (n + 1),\ (-1)^n (n + 1) + 1], \qquad n \geq 0.$$

Note that the sequence (S_n) is unbounded, i.e., $(S_n) \notin I_{\ell_\infty}$.

For $0 < x < 1$, the Abel means of the lower and upper endpoint sequences satisfy

$$\lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} (-1)^n (n + 1) x^n = \lim_{x \to 1^-} \frac{1 - x}{(1 + x)^2} = 0,$$

and

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}((-1)^n(n+1)+1)\,x^n = \lim_{x\to 1^-}\left(\frac{1-x}{(1+x)^2}+1\right) = 1.$$

Hence, by Proposition 1, the sequence (S_n) is Abel convergent to $S = [0,1]$. Thus, the series $\sum_{n=0}^{\infty} A_n$ is Abel summable to $S = [0,1]$.

On the other hand, the Cesàro means of the lower endpoint sequence

$$\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k(k+1)$$

have subsequential limits $1/2$ and $-1/2$, according as n is even or odd. Hence, the lower endpoint sequence is not Cesàro convergent. Consequently, the series $\sum_{n=0}^{\infty} A_n$ is not Cesàro summable, showing that boundedness is essential in Theorem 3.

Theorem 4. *Let $\sum_{n=0}^{\infty} A_n$ be an interval-valued series with sequence of partial sums (S_n). If*

$$\sum_{n=0}^{\infty} A_n = S\ (A) \qquad \textit{and} \qquad [0,0] \le S_n \quad \textit{for every } n \ge 0,$$

then

$$\sum_{n=0}^{\infty} A_n = S\ (C,1).$$

Proof. Let $A_n = [\underline{A}_n, \overline{A}_n]$ and $S_n = [\underline{S}_n, \overline{S}_n]$, where $S_n = \sum_{k=0}^{n} A_k$. Suppose that

$$\sum_{n=0}^{\infty} A_n = S = [\underline{S}, \overline{S}]\ (A).$$

By Definition 9,

$$\lim_{x\to 1^-} h\left((1-x)\sum_{n=0}^{\infty} S_n x^n, S\right) = 0.$$

By the componentwise characterization established in Proposition 1, the Abel convergence of (S_n) to S yields

$$\sum_{n=0}^{\infty} \underline{A}_n = \underline{S}\ (A) \qquad \text{and} \qquad \sum_{n=0}^{\infty} \overline{A}_n = \overline{S}\ (A).$$

Since $[0,0] \le S_n$ for every $n \ge 0$, we have $0 \le \underline{S}_n \le \overline{S}_n$ for every $n \ge 0$. Thus, both endpoint sequences of partial sums are nonnegative.

Therefore, applying Lemma 4 to the lower and upper endpoint series, we obtain

$$\sum_{n=0}^{\infty} \underline{A}_n = \underline{S}\ (C,1) \qquad \text{and} \qquad \sum_{n=0}^{\infty} \overline{A}_n = \overline{S}\ (C,1).$$

By applying Proposition 1 again to the sequence of partial sums (S_n), we conclude that (S_n) is Cesàro convergent to $S = [\underline{S}, \overline{S}]$. Hence, by Definition 7,

$$\sum_{n=0}^{\infty} A_n = S\ (C,1),$$

which completes the proof. □

Example 4. *Consider the interval-valued series $\sum_{n=0}^{\infty} A_n$ whose sequence of partial sums (S_n) is defined by*

$$S_n = [(-1)^n (n+1) - 1,\ (-1)^n (n+1)], \qquad n \geq 0.$$

Note that (S_n) does not satisfy the nonnegativity condition $[0,0] \leq S_n$ for every $n \geq 0$. For example, $S_1 = [-3,-2]$ and $[0,0] \nleq S_1$.

For $0 < x < 1$, the Abel means of the lower and upper endpoint sequences satisfy

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} ((-1)^n (n+1) - 1) x^n = \lim_{x \to 1^-} \left(\frac{1-x}{(1+x)^2} - 1 \right) = -1,$$

while

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \lim_{x \to 1^-} \frac{1-x}{(1+x)^2} = 0.$$

Hence, by Proposition 1, the sequence (S_n) is Abel convergent to $S = [-1,0]$. Thus, the series $\sum_{n=0}^{\infty} A_n$ is Abel summable to $S = [-1,0]$.

On the other hand, the Cesàro means of the upper endpoint sequence

$$\frac{1}{n+1} \sum_{k=0}^{n} (-1)^k (k+1)$$

have subsequential limits $1/2$ and $-1/2$, according as n is even or odd. Hence, the upper endpoint sequence is not Cesàro convergent. Consequently, the series $\sum_{n=0}^{\infty} A_n$ is not Cesàro summable, showing that the nonnegativity condition in Theorem 4 cannot be omitted.

Theorem 5. *Let $\sum_{n=0}^{\infty} A_n$ be an interval-valued series with sequence of partial sums (S_n). If*

$$\sum_{n=0}^{\infty} A_n = S \ (A) \quad \text{and} \quad \underline{S}_n = O_L(1),$$

then

$$\sum_{n=0}^{\infty} A_n = S \ (C,1).$$

Proof. Let $A_n = [\underline{A}_n, \overline{A}_n]$ and $S_n = [\underline{S}_n, \overline{S}_n]$, where $S_n = \sum_{k=0}^{n} A_k$. Suppose that

$$\sum_{n=0}^{\infty} A_n = S = [\underline{S}, \overline{S}] \ (A).$$

By Definition 9, the sequence of partial sums (S_n) is Abel convergent to S, and by Proposition 1, the endpoint sequences $(\underline{S}_n)$ and $(\overline{S}_n)$ are Abel convergent to $\underline{S}$ and $\overline{S}$, respectively. Hence,

$$\sum_{n=0}^{\infty} \underline{A}_n = \underline{S} \ (A) \quad \text{and} \quad \sum_{n=0}^{\infty} \overline{A}_n = \overline{S} \ (A).$$

Since $\underline{S}_n = O_L(1)$, Remark 1 implies that the endpoint sequences $(\underline{S}_n)$ and $(\overline{S}_n)$ satisfy the lower Landau condition. Therefore, applying Lemma 5 to the lower and upper endpoint series, we obtain

$$\sum_{n=0}^{\infty} \underline{A}_n = \underline{S}\ (C,1) \quad \text{and} \quad \sum_{n=0}^{\infty} \overline{A}_n = \overline{S}\ (C,1).$$

By Proposition 1, the Cesàro summability of the endpoint series implies that the sequence of partial sums (S_n) is Cesàro convergent to $S = [\underline{S}, \overline{S}]$. Hence, by Definition 7,

$$\sum_{n=0}^{\infty} A_n = S\ (C,1),$$

which completes the proof. □

Example 5. *Consider the interval-valued series $\sum_{n=0}^{\infty} A_n$ whose sequence of partial sums (S_n) is defined by*

$$S_n = [(-1)^n (n+1),\ (-1)^n (n+1) + 2], \qquad n \geq 0.$$

The lower endpoint sequence

$$\underline{S}_n = (-1)^n (n+1)$$

is not bounded below. Hence, $(\underline{S}_n)$ does not satisfy the lower Landau condition $\underline{S}_n = O_L(1)$. For $0 < x < 1$, the Abel means of the lower and upper endpoint sequences satisfy

$$\lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} (-1)^n (n+1)\, x^n = \lim_{x\to 1^-} \frac{1-x}{(1+x)^2} = 0,$$

and

$$\lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} ((-1)^n (n+1) + 2) x^n = \lim_{x\to 1^-} \left(\frac{1-x}{(1+x)^2} + 2 \right) = 2.$$

Hence, the sequence (S_n) is Abel convergent to $S = [0,2]$. Thus, the series $\sum_{n=0}^{\infty} A_n$ is Abel summable to $S = [0,2]$. On the other hand, as shown in Example 3, the Cesàro means of the sequence $((-1)^n(n+1))$ do not converge. Therefore, the series $\sum_{n=0}^{\infty} A_n$ is not Cesàro summable. This shows that the lower Landau condition in Theorem 5 cannot be omitted.

Remark 3. *Theorems 3–5 show that Abel summability of an interval-valued series implies Cesàro summability when the sequence of partial sums satisfies one of the following Tauberian conditions: boundedness, nonnegativity, or the lower Landau condition. These results provide interval-valued counterparts of the corresponding classical Tauberian theorems through the componentwise characterization established in Proposition 1. A summary of the established inclusion relations and their associated Tauberian conditions is given in Table 1.*

Table 1. Summary of inclusion relations and Tauberian conditions for interval-valued sequences and series.

Result	Type	Hypothesis / Tauberian condition	Implication
Theorem 1	Sequence	Cesàro convergent: $U \in C_I(1)$	$U \in A_I$
Theorem 2	Sequence	$U \in A_I$ and Boundedness: $U \in I_{l_\infty}$	$U \in C_I(1)$
Theorem 3	Series	$\sum A_n = S\ (A)$ and Boundedness: $(S_n) \in I_{l_\infty}$	$\sum A_n = S\ (C, 1)$
Theorem 4	Series	$\sum A_n = S\ (A)$ and Nonnegativity: $[0, 0] \leq S_n$	$\sum A_n = S\ (C, 1)$
Theorem 5	Series	$\sum A_n = S\ (A)$ and Lower Landau Condition: $\underline{S}_n = O_L(1)$	$\sum A_n = S\ (C, 1)$

4. Conclusions

In this paper, the inclusion relations between Abel and Cesàro convergence for interval-valued sequences and the corresponding summability relations for interval-valued series have been investigated. For interval-valued sequences, it has first been shown that Cesàro convergence implies Abel convergence. A Tauberian converse has also been established by showing that Abel convergence implies Cesàro convergence when the sequence is bounded. For interval-valued series, the corresponding Abel-Cesàro summability relations have been investigated through the sequences of partial sums. In particular, it has been shown that Abel summability implies Cesàro summability when the sequence of partial sums satisfies one of the following Tauberian conditions: boundedness, nonnegativity, or the lower Landau condition. All these results are obtained through the componentwise characterization of interval-valued convergence and summability established in Proposition 1, together with the Hausdorff metric. Thus, the classical Abel-Cesàro convergence and summability results are extended to the interval-valued setting. The results obtained in this paper provide a basis for further studies of other summability methods for interval-valued sequences and series.

Acknowledgments

The author would like to sincerely thank the reviewers for their careful and constructive comments and valuable suggestions. Their feedback has contributed significantly to improving the presentation and quality of this manuscript.

Use of Generative-AI tools declaration

Artificial Intelligence (AI) tools were used solely for language editing and proofreading. They were not used to develop the mathematical results or proofs. All mathematical content, results, and conclusions were developed and verified by the author, who takes full responsibility for the final content of the manuscript.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

References

1. G. H. Hardy, *Divergent series*, London: Oxford University Press, 1949.
2. H. S. Özarslan, A note on generalized absolute Cesàro summability, *Adv. Pure Appl. Math.*, **5** (2014), 1–3. https://doi.org/10.1515/apam-2013-0037
3. H. S. Özarslan, On the generalized absolute Cesàro summability methods, *Russ. Math.*, **65** (2021), 29–33. https://doi.org/10.3103/S1066369X21110049
4. H. S. Özarslan, B. K. Erdoğan, A note on summability of infinite series, *Earthline J. Math. Sci.*, **14** (2024), 589–594. https://doi.org/10.34198/ejms.14424.589594
5. H. S. Özarslan, B. K. Erdoğan, Generalized absolute matrix summability factors, *Publ. Inst. Math. (Beograd) (N.S.)*, **117** (2025), 77–82. https://doi.org/10.2298/PIM2531077O
6. R. E. Moore, *Interval analysis*, Englewood Cliffs: Prentice-Hall, 1966.
7. S. M. Markov, On directed interval arithmetic and its applications, *J. Univers. Comput. Sci.*, **1** (1996), 514–526. https://doi.org/10.1007/978-3-642-80350-5_43
8. R. E. Moore, R. B. Kearfott, M. J. Cloud, *Introduction to interval analysis*, Philadelphia: SIAM, 2009. https://doi.org/10.1137/1.9780898717716
9. M. Şengönül, A. Eryılmaz, On the sequence spaces of interval numbers, *Thai J. Math.*, **8** (2010), 503–510.
10. M. Şengönül, An application of Markov's extended interval arithmetic to interval-valued sequence spaces: a special example, *J. Appl. Pure Math.*, **6** (2024), 105–117. https://doi.org/10.23091/japm.2024.105
11. H. Levent, Y. Yılmaz, Hahn-Banach extension theorem for interval-valued functions and existence of quasilinear functionals, *New Trends Math. Sci.*, **6** (2018), 19–28, https://doi.org/10.20852/ntmsci.2018.266
12. H. Levent, Y. Yılmaz, Analysis of signals with inexact data by using interval-valued functions, *J. Anal.*, **30** (2022), 1635–1651. https://doi.org/10.1007/s41478-022-00422-0
13. G. Kılınç, M. S. Yıldırım, On the interval valued Cesàro convergent sequences space, *Adyu. J. Sci.*, **13** (2023), 1–17. https://doi.org/10.37094/adyujsci.1296930
14. B. Kartal Erdoğan, O. Topçu, The space of interval-valued Abel-convergent sequences and their fundamental properties, *Mathematics*, **14** (2026), 1048. https://doi.org/10.3390/math14061048
15. A. N. Tuncer, A. C. Akgün, On the space of interval-valued Riesz convergent sequences and its applications to fundamental properties, *Mathematics*, **14** (2026), 2094. https://doi.org/10.3390/math14122094
16. K. Chiao, Fundamental properties of interval vector max-norm, *Tamsui Oxf. J. Math. Sci.*, **18** (2002), 219–233.

AIMS Press