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Research article

Existence of the solution for boundary value problem of fractional differential equations with derivatives involving p-Laplacian operator

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Abstract: This paper was concerned with the existence of the solution for a kind of nonlinear fractional differential equation with derivatives and p-Laplacian operator on the infinite interval. By using Schauder's fixed point theorem, we gave the existence of the solution. In the end, some examples were given to illustrate our main result.

Keywords: boundary value problem; existence; Riemann-Liouville fractional derivative; infinite interval; p-Laplacian operator
Mathematics Subject Classification: 34B15, 34B18

1. Introduction

Fractional differential equations are powerful tools for modeling complex dynamics owing to the memory and hereditary properties of fractional operators, which offer superior descriptions of anomalous diffusion and viscoelastic behavior in comparison with integer-order models; see [1, 2] and the references therein. Serving as a natural extension of local boundary restrictions, nonlocal boundary conditions capture global cumulative and coupling effects of solutions, matching well the intrinsic nonlocality of fractional calculus and enhancing the modeling flexibility for practical engineering systems. Boundary value problems (BVPs for short) on the half-line are appropriate to characterize long-time unbounded evolutionary processes.

Moreover, the p-Laplacian operator extends linear fractional equations to nonlinear settings. BVPs with a p-Laplacian operator also have been discussed; see [3–5] and so on. Meanwhile research works on the existence of solutions for differential equations involving the p-Laplacian operator with nonlocal boundary value condition on the infinite interval are few; see [6, 7].

Recently, several works have addressed the existence of solutions for a class of fractional order BVPs on infinite intervals; see [8–11]. Fixed point theorems furnish feasible analytical tools for such problems. Different methods about fixed point approaches containing, but not limited to, generalized

Caputo fractional BVPs have been employed; readers can refer [12–15].

In [6], authors apply the Leray-Schauder continuation principle to establish the existence of at least one solution to the third order p-Laplacian BVP on an unbounded domain. Here, the order of the equation is an integer.

In [7], authors consider the following fractional BVP with p-Laplacian operator on an infinite interval

$$\begin{cases} D_{0^+}^{\beta}(p(t)\phi_p(D_{0^+}^{\alpha}u(t))) + f(t, u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha}u(t)) = 0,\ t \in [0, +\infty), \\ u(0) = 0,\ D_{0^+}^{\alpha-1}u(0) = \sum\limits_{i=1}^{m} a_i \int_0^{\xi_i} u(t)dt,\ D_{0^+}^{\alpha}u(0) = 0, \\ \lim\limits_{t\to+\infty} D_{0^+}^{\beta-1}(p(t)\phi_p(D_{0^+}^{\alpha}u(t))) = 0, \end{cases} \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1$. Clearly, $(\phi_p)^{-1} = \phi_q,\ \frac{1}{p} + \frac{1}{q} = 1$. $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are standard Riemann-Liouville fractional derivatives. $1 < \alpha,\beta \le 2$, $a_i \ge 0$; $0 < \xi_1 < \xi_2 < \cdots < \xi_m < +\infty$; $f(t, u, v, w) \in C([0, +\infty) \times R^3, R)$, $p(t) \in C[0, +\infty)$ and $\phi_q(\frac{t^{\beta-1}}{p(t)}) \in L^1(0, +\infty)$.

In [13], the existence of a solution of the following problem has been studied.

$$\begin{cases} {}^{C}D^{\tau}y(x) = Y(x, y(x)),\ \tau \in (1, 2],\ 0 \le x \le z,\ w \in (0, z), \\ \alpha y(0) + \beta y(w) = -\gamma y(z), \alpha, \beta, \gamma \ge 0, \alpha + \beta + \gamma \ne 0, \\ \delta y'(0) + \epsilon y'(w) = -\zeta y'(z), \delta, \epsilon, \zeta \ge 0,\ \delta + \epsilon + \zeta \ne 0, \end{cases}$$

where ${}^{C}D^{\tau}$ is the Caputo derivative of order τ, and $Y : [0, z] \times R \to R$ is a continuous function.

In this work, authors develop a generalized framework for establishing existence and uniqueness results for nonlinear fractional BVPs with mixed-boundary conditions, by employing a combination of Krasnoselskii's fixed-point theorem and the Banach contraction principle. This research work is of high importance.

Inspired by the above discussions, this paper studies a kind of fractional p-Laplacian BVP subject to generalized nonlocal boundary conditions at infinity and establishes criteria for the existence of the solution.

Here, we consider the following fractional BVP with p-Laplacian operator on an infinite interval

$$\begin{cases} D_{0^+}^{\beta}(p(t)\phi_p(D_{0^+}^{\alpha}u(t))) + f(t, u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha}u(t)) = 0, t \in (0, +\infty), \\ u(0) = 0,\ D_{0^+}^{\alpha-1}u(0) = \sum\limits_{i=1}^{m} a_i \int_0^{\xi_i} u(t)dt,\ D_{0^+}^{\alpha}u(0) = 0, \\ \lim\limits_{t\to+\infty} D_{0^+}^{\beta-1}(p(t)\phi_p(D_{0^+}^{\alpha}u(t))) = \lambda I^{\nu}(p(\eta)\phi_p(D_{0^+}^{\alpha}u(\eta))), \end{cases} \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s,\ p > 1$. Clearly, $(\phi_p)^{-1} = \phi_q, \frac{1}{p} + \frac{1}{q} = 1$. $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are standard Riemann-Liouville fractional derivatives, and I^{ν} is Riemann-Liouville fractional integral. $1 < \alpha,\beta \le 2$, $\nu > 0$, $a_i \ge 0$; $0 < \xi_1 < \xi_2 < \cdots < \xi_m < +\infty$, $\lambda > 0$, $\eta > 0$; $f(t, u, v, w) \in C([0, +\infty) \times R^3, R)$, $p(t) \in C[0, +\infty)$, and $\phi_q(\frac{t^{\beta-1}}{p(t)}) \in L^1(0, +\infty)$.

Remark 1.1. In this paper, we extend the equation in [6] to the fractional order equation.

Remark 1.2. If $\lambda = 0$, BVP (1.2) will be BVP (1.1) in [7]. It is worth pointing out that the norm space in this paper is different.

Compared with the boundary value conditions in BVP (1.1), this work presents a generalized eigenvalue extension to conventional fractional p-Laplacian models. The incorporation of a spectral parameter significantly broadens the modeling applicability. It unifies integer -order and fractional-order dynamical behaviors while affording flexible characterization of nonlinear diffusion, memory-dependent dynamics, as well as non-Newtonian rheological phenomena. Meanwhile, the newly coupled nonlocal asymptotic boundary condition substantially increases structural complexity, leading to greater difficulties in operator construction, compactness analysis, and a priori estimation. Therefore, this generalization is meaningful.

Throughout this paper, we give the following assumptions.

(H_1) $\Gamma(\alpha+1) > \sum\limits_{i=1}^{m} a_i\xi_i^{\alpha}$, $\Gamma(\beta+\nu) > \lambda\eta^{\beta+\nu-1}$.

(H_2) $p(t) > 0$ for $t \in [0,+\infty)$ and $\phi_q(\frac{t^{\beta-1}}{p(t)}) \in L^1[0,+\infty)$, $\lim\limits_{t\to+\infty} \frac{t^{\beta-1}}{p(t)} = 0$.

From (H_2), we conclude that $\sup\limits_{t\in[0,+\infty)} \phi_q(\frac{t^{\beta-1}}{p(t)}) < +\infty$.

(H_3) The following two conditions hold:

(i) There exist nonnegative functions $a(t),\ b(t),\ c(t),\ d(t) \in L^1[0,+\infty)$, such that

$$| f(t,x,y,z) | \leq a(t) + b(t)e^{-t} \mid x \mid^{p-1} + c(t) \mid y \mid^{p-1} + d(t) \mid z \mid^{p-1},$$

(ii) $\lim\limits_{t\to+\infty} a(t) = 0$; $\lim\limits_{t\to+\infty} (b(t)+c(t)+d(t)) = 0$.

The rest of the paper is organized as follows: In Section 2, the basic concepts, theorems, and lemmas are presented. The main result regarding the existence of the solution is proved through analytical skills in Section 3. In Section 4, we present some examples to support the main result.

2. Preliminaries

For convenience of readers, we first present some basic notations and results that will be used in the proof of our theorem. We suggest that one refers to [1,2] for details.

Definition 2.1. Let $x : (0,+\infty) \to R$ be a function and $p > 0$. The Riemann-Liouville fractional integral of order p of x is defined by

$$I_{0^+}^{p} x(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} x(s)ds,$$

provided the integral exists. The Riemann-Liouville fractional derivative of order p for a continuous function x is defined by

$$D_{0^+}^{p} x(t) = \frac{1}{\Gamma(n-p)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-p-1} x(s)ds,$$

provided the right side is pointwise defined on $(0,+\infty)$, where n is the smallest integer greater than or equal to p, and $\Gamma(p)$ is the gamma function.

Lemma 2.1. If $D_{0^+}^{p} x \in L(0,+\infty)$, then

$$I_{0^+}^{p} D_{0^+}^{p} x(t) = x(t) + c_1 t^{p-1} + c_2 t^{p-2} + \cdots + c_n t^{p-n},$$

where $c_i \in R, i = 1,2,\cdots,n$ and n is the smallest integer greater than or equal to p.

Lemma 2.2. Let $p > 0$, then $D^{p}_{0^+}x(t) = 0$ if, and only if,

$$x(t) = c_1 t^{p-1} + c_2 t^{p-2} + \cdots + c_n t^{p-n},$$

where $c_i (i = 1, 2, \cdots, n)$ are arbitrary constants, and n is the smallest integer greater than or equal to p.
Lemma 2.3. For $y(t) \in L^1[0, +\infty)$, the following fractional differential equation

$$\begin{cases} D^{\beta}_{0^+}(p(t)\phi_p(D^{\alpha}_{0^+}u(t))) + y(t) = 0, t \in [0, +\infty), \\ u(0) = 0,\ D^{\alpha-1}_{0^+}u(0) = \sum\limits_{i=1}^{m} a_i \int_0^{\xi_i} u(t)dt,\ D^{\alpha}_{0^+}u(0) = 0, \\ \lim\limits_{t\to+\infty} D^{\beta-1}_{0^+}(p(t)\phi_p(D^{\alpha}_{0^+}u(t))) = \lambda I^{\nu}(p(\eta)\phi_p(D^{\alpha}_{0^+}u(\eta))), \end{cases} \tag{2.1}$$

has a unique solution

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\phi_q(\frac{1}{p(s)} \int_0^{+\infty} G(s,\tau)y(\tau)d\tau)ds \\ & + \frac{t^{\alpha-1}}{\Gamma(\alpha)(\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i\xi_i^{\alpha})} \\ & \times \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha}\phi_q(\frac{1}{p(s)} \int_0^{+\infty} G(s,\tau)y(\tau)d\tau)ds, \end{aligned} \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Delta} \begin{cases} [\Gamma(\beta+\nu) - \lambda(\eta-s)^{\beta+\nu-1}]t^{\beta-1} & \\ -[\Gamma(\beta+\nu) - \lambda\eta^{\beta+\nu-1}](t-s)^{\beta-1}, & 0 \le s \le t,\ 0 \le s \le \eta, \\ [\Gamma(\beta+\nu) - \lambda(\eta-s)^{\beta+\nu-1}]t^{\beta-1}, & 0 \le t \le s \le \eta, \\ \Gamma(\beta+\nu)[t^{\beta-1} - (t-s)^{\beta-1}] & \\ +\lambda\eta^{\beta+\nu-1}(t-s)^{\beta-1}, & 0 \le \eta \le s \le t, \\ \Gamma(\beta+\nu)t^{\beta-1}, & t \le s,\ \eta \le s, \end{cases} \tag{2.3}$$

where, $\Delta = \Gamma(\beta)(\Gamma(\beta+\nu) - \lambda\eta^{\beta+\nu-1})$.
Proof: From Lemmas 2.1, 2.2, and Eq (2.1), we obtain

$$\begin{aligned} p(t)\phi_p(D^{\alpha}_{0^+}u(t)) &= -I^{\beta}_{0^+}y(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} \\ &= -\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)ds + c_1 t^{\beta-1} + c_2 t^{\beta-2}, \end{aligned}$$

where $c_1, c_2 \in R$. From $D^{\alpha}_{0^+}u(0) = 0$, we get $c_2 = 0$. Thus,

$$D^{\beta-1}_{0^+}(p(t)\phi_p(D^{\alpha}_{0^+}u(t))) = -\int_0^t y(s)ds + c_1\Gamma(\beta),$$

and

$$I^{\nu}(p(t)\phi_p(D^{\alpha}_{0^+}u(\eta))) = -\frac{1}{\Gamma(\beta+\nu)} \int_0^{\eta} (\eta-s)^{\beta+\nu-1} y(s)ds + c_1 \frac{\Gamma(\beta)}{\Gamma(\beta+\nu)}\eta^{\beta+\nu-1}.$$

According to the condition $\lim\limits_{t\to+\infty} D_{0^+}^{\beta-1}(p(t)\phi_p(D_{0^+}^{\alpha}u(t))) = \lambda I^{\nu}(p(t)\phi_p(D_{0^+}^{\alpha}u(\eta)))$, we have

$$-\int_0^{+\infty} y(s)ds + c_1\Gamma(\beta) = -\frac{\lambda}{\Gamma(\beta+\nu)}\int_0^{\eta}(\eta-s)^{\beta+\nu-1}y(s)ds + c_1\frac{\lambda\Gamma(\beta)}{\Gamma(\beta+\nu)}\eta^{\beta+\nu-1}.$$

i.e.,

$$c_1\Gamma(\beta)(1-\frac{\lambda}{\Gamma(\beta+\nu)}\eta^{\beta+\nu-1}) = \int_0^{+\infty} y(s)ds - \frac{\lambda}{\Gamma(\beta+\nu)}\int_0^{\eta}(\eta-s)^{\beta+\nu-1}y(s)ds.$$

By simple computation, we have

$$c_1 = \frac{\Gamma(\beta+\nu)}{\Gamma(\beta)(\Gamma(\beta+\nu)-\lambda\eta^{\beta+\nu-1})}\Big(\int_0^{+\infty} y(s)ds - \frac{\lambda}{\Gamma(\beta+\nu)}\int_0^{\eta}(\eta-s)^{\beta+\nu-1}y(s)ds\Big).$$

Hence,

$$\begin{aligned}
&p(t)\phi_p(D_{0^+}^{\alpha}u(t))\\
&= -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}y(s)ds+\\
&\frac{\Gamma(\beta+\nu)}{\Gamma(\beta)(\Gamma(\beta+\nu)-\lambda\eta^{\beta+\nu-1})}t^{\beta-1}\Big(\int_0^{+\infty} y(s)ds - \frac{\lambda}{\Gamma(\beta+\nu)}\int_0^{\eta}(\eta-s)^{\beta+\nu-1}y(s)ds\Big)\\
&\triangleq \int_0^{+\infty} G(t,s)y(s)ds.
\end{aligned}$$

So,

$$D_{0^+}^{\alpha}u(t) = \phi_q(\frac{1}{p(t)}\int_0^{+\infty} G(t,s)y(s)ds).$$

From Lemmas 2.1 and 2.2 again, we derive

$$\begin{aligned}
u(t) &= I_{0^+}^{\alpha}\phi_q\Big(\frac{1}{p(t)}\int_0^{+\infty} G(t,s)y(s)ds\Big) + c_3t^{\alpha-1} + c_4t^{\alpha-2}\\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\Big(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\Big)ds + c_3t^{\alpha-1} + c_4t^{\alpha-2},
\end{aligned}$$

where $c_3, c_4 \in R$. By $u(0) = 0$, we have $c_4 = 0$. Therefore,

$$D_{0^+}^{\alpha-1}u(t) = \int_0^t \phi_q\Big(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\Big)ds + c_3\Gamma(\alpha).$$

From $D_{0^+}^{\alpha-1}u(0) = \sum\limits_{i=1}^{m} a_i\int_0^{\xi_i} u(t)dt$, we conclude

$$\begin{aligned}
c_3\Gamma(\alpha) &= \sum_{i=1}^{m} a_i\int_0^{\xi_i}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\Big(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\Big)dsdt + \sum_{i=1}^{m} a_i\int_0^{\xi_i} c_3t^{\alpha-1}dt\\
&= \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} a_i\int_0^{\xi_i}\int_s^{\xi_i}(t-s)^{\alpha-1}\phi_q\Big(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)y(\tau)d\tau\Big)dtds + c_3\sum_{i=1}^{m} a_i\frac{\xi_i^{\alpha}}{\alpha}
\end{aligned}$$

$$= \frac{1}{\alpha\Gamma(\alpha)} \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha} \phi_q\Big(\frac{1}{p(s)} \int_0^{+\infty} G(s,\tau)y(\tau)d\tau\Big)ds + \frac{c_3}{\alpha} \sum_{i=1}^{m} a_i \xi_i^{\alpha}.$$

By computation, we get $c_3 = \frac{1}{\Gamma(\alpha)\big(\Gamma(\alpha+1)-\sum_{i=1}^{m} a_i\xi_i^{\alpha}\big)} \times \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha} \phi_q(\frac{1}{p(s)} \int_0^{+\infty} G(s,\tau)y(\tau)d\tau)ds$. Therefore, (2.2) and (2.3) hold. This completes the proof.

Lemma 2.4. If (H_1) holds, then the Green function defined by (2.3) satisfies

$$G(t,s) \geq 0,$$

and

$$G(t,s) \leq \frac{\Gamma(\beta+\nu)}{\Gamma(\beta)(\Gamma(\beta+\nu) - \lambda\eta^{\beta+\nu-1})} t^{\beta-1} = \frac{\Gamma(\beta+\nu)}{\Delta} t^{\beta-1}. \tag{2.4}$$

Proof: From the positivity of $\Gamma(\alpha)$ and $\Gamma(\beta)$, combined with (H_1), we can get Lemma 2.4.

Define the function spaces

$$X = \{u(t) \in C([0,+\infty), R) | \sup_{t\in[0,+\infty)} e^{-t} \mid u(t) \mid < +\infty\}$$

with the norm $\|u\|_X = \sup_{t\in[0,+\infty)} e^{-t} \mid u(t) \mid$.

$$Y = \{u(t) \in X | D_{0+}^{\alpha} u(t) \in C([0,+\infty), R), \sup_{t\in[0,+\infty)} \mid D_{0+}^{\alpha-1} u(t) \mid < +\infty, \sup_{t\in[0,+\infty)} \mid D_{0+}^{\alpha} u(t) \mid < +\infty\}$$

with the norm

$$\|u\|_{\infty} = \max\{\sup_{t\in[0,+\infty)} e^{-t} \mid u(t) \mid, \sup_{t\in[0,+\infty)} \mid D_{0+}^{\alpha-1} u(t) \mid, \sup_{t\in[0,+\infty)} \mid D_{0+}^{\alpha} u(t) \mid\}.$$

Lemma 2.5. $(X, \|\cdot\|_X)$ is a Banach space.

Proof: Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(X, \|\cdot\|_X)$. Then, $\forall \epsilon > 0, \exists N > 0$, such that

$$|e^{-t}||u_n(t) - u_m(t)| < \epsilon,$$

for any $t \in [0,+\infty)$, and $n, m > N$. There exists $u(t) \in X$ such that $\{e^{-t}u_n\}_{n=1}^{\infty}$ converges uniformly to a function $e^{-t}u(t) \in X$. Then, $(X, \|\cdot\|_X)$ is a Banach space.

Lemma 2.6. $(Y, \|\cdot\|_{\infty})$ is a Banach space.

Proof: We divide the proof into two steps.

Step 1. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(Y, \|\cdot\|_{\infty})$. Then, $\{u_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $(X, \|\cdot\|_X)$. Thus, there exist $v(t),\ w(t) \in C([0,+\infty), R)$ such that

$$\lim_{n\to+\infty} D_{0+}^{\alpha-1} u(t) = v(t),\ \lim_{n\to\infty} D_{0+}^{\alpha} u(t) = w(t),$$

and

$$\sup_{t\in[0,+\infty)} |v(t)| < +\infty, \quad \sup_{t\in[0,+\infty)} |w(t)| < +\infty.$$

Step 2. We prove that $v(t) = D_{0+}^{\alpha-1}u(t)$ and $w(t) = D_{0+}^{\alpha}u(t)$.

In fact, from $\{e^{-s}u_n(s)\}_{n=1}^{\infty}$ converges uniformly to $e^{-s}u(s)$ and $e^{-s}u(s) \in X$, there exists $M > 0$, for $\forall s \in [0, +\infty)$, such that $|e^{-s}u_n(s)| \le M, n = 1, 2, \cdots$. Thus, for any $t \in [0, +\infty)$ and $1 < \alpha \le 2$, we can get

$$\begin{aligned}\left|\int_0^t (t-s)^{1-\alpha}u_n(s)ds\right| &= \left|\int_0^t (t-s)^{1-\alpha}\frac{1}{e^{-s}}e^{-s}u_n(s)ds\right| \\ &\le M\left|\int_0^t (t-s)^{1-\alpha}e^{s}ds\right| \\ &\le Me^t\left|\int_0^t (t-s)^{1-\alpha}ds\right| = Me^t\frac{t^{2-\alpha}}{2-\alpha}.\end{aligned}$$

By Lebesgue's dominated convergence theorem and the uniform convergence of $D_{0+}^{\alpha-1}u_n(t)$ and $D_{0+}^{\alpha}u_n(t)$, we can obtain that

$$\begin{aligned}v(t) &= \lim_{n\to\infty} D_{0+}^{\alpha-1}u_n(t) = \lim_{n\to\infty}\frac{d}{dt}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u_n(s)ds \\ &= \frac{d}{dt}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u(s)ds \\ &= D_{0+}^{\alpha-1}u(t),\end{aligned}$$

and

$$\begin{aligned}w(t) &= \lim_{n\to\infty} D_{0+}^{\alpha}u_n(t) = \lim_{n\to\infty}\frac{d^2}{dt^2}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u_n(s)ds \\ &= \frac{d^2}{dt^2}\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u(s)ds \\ &= D_{0+}^{\alpha}u(t).\end{aligned}$$

Hence, we conclude that $(Y, \|\cdot\|_\infty)$ is a Banach space. The proof is complete.

Lemma 2.7. Assume that V is bounded in Y, then V is relatively compact in Y if the continuous following conditions hold:

(i) $\{e^{-t}u|u \in V\}$, $\{D_{0+}^{\alpha-1}u|u \in V\}$, $\{D_{0+}^{\alpha}u|u \in V\}$ are equicontinuous on any compact interval of $[0, +\infty)$;

(ii) For any $\epsilon > 0$, there exists a constant $T = T(\epsilon) > 0$, such that

$$|e^{-t_1}Tu(t_1) - e^{-t_2}Tu(t_2)| < \epsilon;$$

$$|D_{0+}^{\alpha-1}Tu(t_1) - D_{0+}^{\alpha-1}Tu(t_2)| < \epsilon;$$

and

$$|D_{0+}^{\alpha}Tu(t_1) - D_{0+}^{\alpha}Tu(t_2)| < \epsilon$$

for any $u(t) \in V$ and $t_1, t_2 \ge T$.

Proof: It is sufficient to prove that V is totally bounded. By condition(ii), we divide $[0, +\infty)$ into $[0, T]$ and $[T, +\infty)$. Set $V_{[0,T]} = \{u(t)|u(t) \in V, t \in [0, T]\}$ with the norm $\|u\|_T = \sup_{t\in[0,T]} e^{-t}|u(t)|$, then $V_{[0,T]}$

is a Banach space. From condition (i) and Arzel$\grave{a}$-Ascoli theorem, we can get that $V_{[0,T]}$ is relatively compact. Thus, $V_{[0,T]}$ is totally bounded, i.e., for any $\epsilon > 0$, there exist finitely many balls $B_\epsilon(u_i)$ such that

$$V_{[0,T]} \subset \cup_{i=1}^{n} B_\epsilon(u_i),$$

where $B_\epsilon(u_i) = \left\{u(t) \in V_{[0,T]} : \|u - u_i\|_T = \sup\limits_{t\in[0,T]} e^{-t}|u(t) - u_i(t)| < \epsilon\right\}$.

Similarly, the space $V_{[0,T]}^{\alpha-1} = \{D_{0+}^{\alpha-1}u(t) : u(t) \in V_{[0,T]}, t \in [0,T]\}$ with the norm $\|D_{0+}^{\alpha-1}u\|_T = \sup\limits_{t\in[0,T]} |D_{0+}^{\alpha-1}u(t)|$ is a Banach space and can be covered by finitely many balls $B_\epsilon(D_{0+}^{\alpha-1}v_j)$, i.e.,

$$V_{[0,T]}^{\alpha-1} \subset \cup_{j=1}^{m} B_\epsilon(D_{0+}^{\alpha-1}v_j),$$

where $B_\epsilon(D_{0+}^{\alpha-1}v_j) = \left\{D_{0+}^{\alpha-1}u(t) \in V_{[0,T]}^{\alpha-1} : \|u - v_j\|_T = \sup\limits_{t\in[0,T]} |D_{0+}^{\alpha-1}u(t) - D_{0+}^{\alpha-1}v_j(t)| < \epsilon\right\}$.

Again, the space $V_{[0,T]}^{\alpha} = \{D_{0+}^{\alpha}u(t) : u(t) \in V_{[0,T]}, D_{0+}^{\alpha-1}u(t) \in V_{[0,T]}^{\alpha-1}, t \in [0,T]\}$ with the norm $\|D_{0+}^{\alpha}u\|_T = \sup\limits_{t\in[0,T]} |D_{0+}^{\alpha}u(t)|$ is a Banach space and can be covered by finitely many balls $B_\epsilon(D_{0+}^{\alpha}w_k)$, i.e.,

$$V_{[0,T]}^{\alpha} \subset \cup_{k=1}^{l} B_\epsilon(D_{0+}^{\alpha}w_k),$$

where $B_\epsilon(D_{0+}^{\alpha}w_k) = \left\{D_{0+}^{\alpha}u(t) \in V_{[0,T]}^{\alpha} : \|u - w_k\|_T = \sup\limits_{t\in[0,T]} |D_{0+}^{\alpha}u(t) - D_{0+}^{\alpha}w_k(t)| < \epsilon\right\}$.

Denote

$$V_{ijk} = \{u(t) \in V : u_{[0,T]} \in B_\epsilon(u_i),\ D_{0+}^{\alpha-1}u_{[0,T]} \in B_\epsilon(D_{0+}^{\alpha-1}v_j),\ D_{0+}^{\alpha}u_{[0,T]} \in B_\epsilon(D_{0+}^{\alpha}w_k)\}.$$

Clearly, $V_{[0,T]} \subset \cup_{1\le i\le n, 1\le j\le m, 1\le k\le l} V_{ijk}$. Now take $u_{ijk} \in V_{ijk}$, then V can be covered by balls $B_{4\epsilon}(u_{ijk}),\ 1 \le i \le n,\ 1 \le j \le m,\ 1 \le k \le l$.

In fact, for $u(t) \in V$, the above discussion implies that there exist $i,\ j,\ k$ such that $u_{[0,T]} \in B_\epsilon(u_i),\ D_{0}^{\alpha-1}u_{[0,T]} \in B_\epsilon(D_{0+}^{\alpha-1}v_j),\ D_{0+}^{\alpha}u_{[0,T]} \in B_\epsilon(D_{0+}^{\alpha}w_k)$. Therefore, for $t \in [0,T]$, we can get

$$e^{-t}|u(t) - u_{ijk}(t)| \le e^{-t}|u(t) - u_i(t)| + e^{-t}|u_i(t) - u_{ijk}(t)| < 2\epsilon, \tag{2.5}$$

$$|D_{0+}^{\alpha-1}u(t) - D_{0+}^{\alpha-1}u_{ijk}(t)| \le |D_{0+}^{\alpha-1}u(t) - D_{0+}^{\alpha-1}v_j(t)| + |D_{0+}^{\alpha-1}v_j(t) - D_{0+}^{\alpha-1}u_{ijk}(t)| < 2\epsilon, \tag{2.6}$$

$$|D_{0+}^{\alpha}u(t) - D_{0+}^{\alpha}u_{ijk}(t)| \le |D_{0+}^{\alpha}u(t) - D_{0+}^{\alpha}w_k(t)| + |D_{0+}^{\alpha}w_k(t) - D_{0+}^{\alpha}w_{ijk}(t)| < 2\epsilon. \tag{2.7}$$

For $t \in [T, +\infty)$, by condition (ii) and (2.5), we have

$$\begin{aligned} &e^{-t}|u(t) - u_{ijk}(t)| \\ &\le e^{-t}|u(t) - u(T)| + e^{-t}|u(T) - u_{ijk}(T)| \\ &+ e^{-t}|u_{ijk}(T) - u_{ijk}(t)| < \epsilon + 2\epsilon + \epsilon = 4\epsilon. \end{aligned} \tag{2.8}$$

In the same way,

$$|D_{0+}^{\alpha-1}u(t) - D_{0+}^{\alpha-1}u_{ijk}(t)| < 4\epsilon, \tag{2.9}$$

$$|D_{0+}^{\alpha}u(t)-D_{0+}^{\alpha}u_{ijk}(t)|<4\epsilon. \tag{2.10}$$

From (2.8)–(2.10), we prove that $\|u-u_{ijk}\|_{\infty}<4\epsilon$. Thus, V is totally bounded in Y. The proof is complete.

Lemma 2.8. [7] Let $x,\ y\geq 0$, then

$$\phi_p(x+y)\leq\phi_p(x)+\phi_p(y),\ \text{if}\ p<2;$$

$$\phi_p(x+y)\leq 2^{p-2}(\phi_p(x)+\phi_p(y)),\ \text{if}\ p\geq 2.$$

Lemma 2.9. [16] (Schauder's fixed point theorem) Let K be a bounded closed convex set in Banach space E and $T:K\to K$ be a completely continuous operator. Then, T has at least one fixed point K.

3. Existence of the solution

In this section, we will use the Schauder's fixed point theorem to prove the existence of solution to BVP (1.2).

Define

$$B=\int_0^{+\infty}(b(t)+c(t)+d(t))dt,\ \int_0^{+\infty}a(t)dt=\|a\|_1,$$

and

$$C=\max\Big\{\int_0^{+\infty}\phi_q\Big(\frac{s^{\beta-1}}{p(s)}\Big)ds,\ \sup_{s\in[0,+\infty)}\phi_q\Big(\frac{s^{\beta-1}}{p(s)}\Big)\Big\}.$$

In addition, B and C should satisfy the following conditions:

If $q<2$, $B<\min\Big\{\frac{\Delta\phi_p(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})}{\phi_p(C\alpha)\Gamma(\beta+\nu)},\ \frac{\Delta\phi_p(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})}{\phi_p(C\Gamma(\alpha+1))\Gamma(\beta+\nu)},\ \frac{\Delta}{\phi_p(C)\Gamma(\beta+\nu)}\Big\}$.

If $q\geq 2$, $B<\min\Big\{\frac{\Delta\phi_p(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})}{\phi_p(2^{q-2}C\alpha)\Gamma(\beta+\nu)},\ \frac{\Delta\phi_p(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})}{\phi_p(2^{q-2}C\Gamma(\alpha+1))\Gamma(\beta+\nu)},\ \frac{\Delta}{\phi_p(2^{q-2}C)\Gamma(\beta+\nu)}\Big\}$.

For $u\in Y$, we define the operator A by

$$Au(t)=f(t,u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)),$$

and

$$\begin{aligned}Tu(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds\\&+\frac{t^{\alpha-1}}{\Gamma(\alpha)(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})}\times\sum_{i=1}^{m}a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds,\end{aligned}$$

for all $t\in[0,+\infty)$.

Remark 3.1. By condition (i) of (H_3), we have

$$\int_0^{+\infty}|Au(t)|dt=\int_0^{+\infty}|f(t,u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t))|dt$$

$$\leq \int_0^{+\infty}[a(t)+b(t)\mid e^{-t}u(t)\mid^{p-1}+c(t)\mid D_{0+}^{\alpha-1}u(t)\mid^{p-1}+d(t)\mid D_{0+}^{\alpha}u(t)\mid^{p-1}]dt$$

$$\leq \|a\|_1+\|u\|_\infty^{p-1}\int_0^{+\infty}(b(t)+c(t)+d(t))dt=\|a\|_1+\|u\|_\infty^{p-1}B<+\infty. \tag{3.1}$$

Therefore, the operator T is well defined.

Remark 3.2. From condition (ii) of (H_3), we obtain that for any $\epsilon>0$, there exists $N>0$, whenever $t_1,\ t_2>N$,

$$\int_{t_1}^{t_2}|Au(s)|ds\leq\int_{t_1}^{t_2}a(t)dt+\|u\|_\infty^{p-1}\int_{t_1}^{t_2}(b(t)+c(t)+d(t))dt<\epsilon.$$

Lemma 2.3 indicates that the solution of BVP (1.2) is the fixed point of the operator T.

Next, we will use the Schauder's fixed point theorem to prove Theorem 3.1.

Theorem 3.1. Assume that $f:[0,+\infty)\times R^3\to R$ is continuous and (H_1)–(H_3) hold. Then the BVP (1.2) has at least one solution.

Proof: From the definition of T, we have

$$\begin{aligned}D_{0+}^{\alpha-1}Tu(t)=&\int_0^t\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds\\&+\frac{1}{(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha)}\times\sum_{i=1}^m a_i\int_0^{\xi_i}(\xi_i-s)^\alpha\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds,\end{aligned}$$

$$D_{0+}^{\alpha}Tu(t)=\phi_q(\frac{1}{p(t)}\int_0^{+\infty}G(t,s)Au(s)ds).$$

By (H_2) and (H_3), we have that $Tu(t)$, $D_{0+}^{\alpha-1}Tu(t)$, and $D_{0+}^{\alpha}Tu(t)$ are continuous on $[0,+\infty)$.

Let $P=\{u(t)\in Y|\ \|u\|_\infty\leq r\}$. The proof can be divided into three steps.

Step 1. We prove that $T:P\to P$.

Case I. If $q<2$, set

$$\begin{aligned}r\geq\max\Bigg\{&\frac{C\alpha\phi_q(\Gamma(\beta+\nu))\phi_q(\|a\|_1)}{\phi_q(\Delta)(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha)-C\alpha\phi_q(\Gamma(\beta+\nu))\phi_q(B)},\\&\frac{C\Gamma(\alpha+1)\phi_q(\Gamma(\beta+\nu))\phi_q(\|a\|_1)}{\phi_q(\Delta)(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha)-C\Gamma(\alpha+1)\phi_q(\Gamma(\beta+\nu))\phi_q(B)},\\&\frac{C\phi_q(\Gamma(\beta+\nu))\phi_q(\|a\|_1)}{\phi_q(\Delta)-C\phi_q(\Gamma(\beta+\nu))\phi_q(B)}\Bigg\}.\end{aligned}$$

For any $u(t)\in P$, combined with (2.4), (3.1), and Lemma 2.8, we can get

$$\begin{aligned}e^{-t}\mid Tu(t)\mid\leq&\frac{1}{\Gamma(\alpha)}\int_0^t e^{-t}(t-s)^{\alpha-1}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)\mid Au(\tau)\mid d\tau)ds\\&+e^{-t}\frac{t^{\alpha-1}}{\Gamma(\alpha)(\Gamma(\alpha+1)-\sum\limits_{i=1}^m a_i\xi_i^\alpha)}\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha} \phi_q(\frac{1}{p(s)} \int_0^{+\infty} G(s,\tau) \mid Au(\tau) \mid d\tau) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) \int_0^{+\infty} \phi_q(\frac{s^{\beta-1}}{p(s)} \int_0^{+\infty} \mid Au(\tau) \mid d\tau) ds \\
&+ \frac{\sum_{i=1}^{m} a_i \xi_i^{\alpha}}{\Gamma(\alpha)(\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i \xi_i^{\alpha})} \phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) \int_0^{+\infty} \phi_q(\frac{s^{\beta-1}}{p(s)} \int_0^{+\infty} \mid Au(\tau) \mid d\tau) ds \\
&\leq \frac{C}{\Gamma(\alpha)} \phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) \phi_q(\parallel a \parallel_1 + \parallel u \parallel_{\infty}^{p-1} B) \\
&+ \frac{C \sum_{i=1}^{m} a_i \xi_i^{\alpha}}{\Gamma(\alpha)(\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i \xi_i^{\alpha})} \phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) \phi_q(\parallel a \parallel_1 + \parallel u \parallel_{\infty}^{p-1} B) \\
&\leq \frac{C\alpha}{\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i \xi_i^{\alpha}} \phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) (\phi_q(\parallel a \parallel_1) + r\phi_q(B)) \leq r;
\end{aligned}
$$

$$
\begin{aligned}
|D_{0+}^{\alpha-1} Tu(t)| &\leq \int_0^t \phi_q(\frac{1}{p(s)} \int_0^{+\infty} G(s,\tau) \mid Au(\tau) \mid d\tau) ds \\
&+ \frac{1}{(\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i \xi_i^{\alpha})} \\
&\times \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha} \phi_q(\frac{1}{p(s)} \int_0^{+\infty} G(s,\tau) \mid Au(\tau) \mid d\tau) ds \\
&\leq C\phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) \phi_q(\parallel a \parallel_1 + \parallel u \parallel_{\infty}^{p-1} B) \\
&+ \frac{C \sum_{i=1}^{m} a_i \xi_i^{\alpha}}{\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i \xi_i^{\alpha}} \phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) \phi_q(\parallel a \parallel_1 + \parallel u \parallel_{\infty}^{p-1} B) \\
&\leq \frac{C\Gamma(\alpha+1)}{\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i \xi_i^{\alpha}} \phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) (\phi_q(\parallel a \parallel_1) + r\phi_q(B)) \leq r;
\end{aligned}
$$

$$
\begin{aligned}
|D_{0+}^{\alpha} Tu(t)| &\leq \phi_q(\frac{1}{p(t)} \int_0^{+\infty} G(t,s) \mid Au(s) \mid ds) \\
&\leq C\phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) (\phi_q(\parallel a \parallel_1) + r\phi_q(B)) \leq r.
\end{aligned}
$$

Thus, with the choices of r and B, we can get $\|Tu\|_{\infty} \leq r$. This shows that $T : P \to P$.
Case II. If $q \geq 2$, set

$$
r \geq \max\Big\{ \frac{2^{q-2} C\alpha \phi_q(\Gamma(\beta+\nu)) \phi_q(\|a\|_1)}{\phi_q(\Delta)(\Gamma(\alpha+1) - \sum\limits_{i=1}^{m} a_i \xi_i^{\alpha}) - 2^{q-2} C\alpha \phi_q(\Gamma(\beta+\nu)) \phi_q(B)},
$$

$$\frac{2^{q-2}C\Gamma(\alpha+1)\phi_q(\Gamma(\beta+\nu))\phi_q(\|a\|_1)}{\phi_q(\Delta)(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha})-2^{q-2}C\Gamma(\alpha+1)\phi_q(\Gamma(\beta+\nu))\phi_q(B)},$$
$$\left.\frac{2^{q-2}C\phi_q(\Gamma(\beta+\nu))\phi_q(\|a\|_1)}{\phi_q(\Delta)-2^{q-2}C\phi_q(\Gamma(\beta+\nu))\phi_q(B)}\right\}.$$

For any $u(t)\in P$, combined with (2.4), (3.1), and Lemma 2.8 again , we can obtain that

$$\begin{aligned}&e^{-t}\mid Tu(t)\mid\\&\le\frac{C\alpha}{\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}}\phi_q(\frac{\Gamma(\beta+\nu)}{\Delta})(2^{q-2}\phi_q(\|a\|_1)+r2^{q-2}\phi_q(B))\le r;\end{aligned}\tag{3.2}$$

Notice that $B<\frac{\Delta\phi_p(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})}{\phi_p(2^{q-2}C\alpha)\Gamma(\beta+\nu)}$ and $r\ge\frac{2^{q-2}C\alpha\phi_q(\Gamma(\beta+\nu))\phi_q(\|a\|_1)}{\phi_q(\Delta)(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})-2^{q-2}C\alpha\phi_q(\Gamma(\beta+\nu))\phi_q(B)}$. By solving (3.2), we have

$$\sup_{t\in[0,+\infty)}|e^{-t}Tu(t)|\le r.\tag{3.3}$$

$$\begin{aligned}&|D_{0+}^{\alpha-1}Tu(t)|\\&\le\frac{C\Gamma(\alpha+1)}{\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha}}\phi_q(\frac{\Gamma(\beta+\nu)}{\Delta})(2^{q-2}\phi_q(\|a\|_1)+r2^{q-2}\phi_q(B))\le r.\end{aligned}\tag{3.4}$$

Combined with $B<\frac{\Delta\phi_p(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})}{\phi_p(2^{q-2}C\Gamma(\alpha+1))\Gamma(\beta+\nu)}$ and $r\ge\frac{2^{q-2}C\Gamma(\alpha+1)\phi_q(\Gamma(\beta+\nu))\phi_q(\|a\|_1)}{\phi_q(\Delta)(\Gamma(\alpha+1)-\sum_{i=1}^{m}a_i\xi_i^{\alpha})-2^{q-2}C\Gamma(\alpha+1)\phi_q(\Gamma(\beta+\nu))\phi_q(B)}$, by solving (3.4), we have

$$\sup_{t\in[0,+\infty)}|D_{0+}^{\alpha-1}Tu(t)|\le r.\tag{3.5}$$

Again, we have

$$\begin{aligned}|D_{0+}^{\alpha}Tu(t)|&\le\phi_q(\frac{1}{p(t)}\int_0^{+\infty}G(t,s)\mid Au(s)\mid ds)\\&\le C\phi_q(\frac{\Gamma(\beta+\nu)}{\Delta})(2^{q-2}\phi_q(\|a\|_1)+r2^{q-2}\phi_q(B))\le r.\end{aligned}\tag{3.6}$$

Notice that $B<\frac{\Delta}{\phi_p(2^{q-2}C)\Gamma(\beta+\nu)}$ and $r\ge\frac{2^{q-2}C\phi_q(\Gamma(\beta+\nu))\phi_q(\|a\|_1)}{\phi_q(\Delta)-2^{q-2}C\phi_q(\Gamma(\beta+\nu))\phi_q(B)}$. By solving (3.6), we have

$$\sup_{t\in[0,+\infty)}|D_{0+}^{\alpha}Tu(t)|\le r.\tag{3.7}$$

From (3.3), (3.5), and (3.7), we get $\|Tu\|_\infty\le r$. This shows that $T:\ P\to P$. The proof is complete.
Step 2. We use Lemma 2.7 to show that TV is relatively compact; here, V is a subset of P .

Let $I \subset [0, +\infty)$ be a compact interval, $t_1, t_2 \in I$, and $t_1 < t_2$. Then, for any $u(t) \in V$, we obtain

$$\begin{aligned}
&| e^{-t_2}Tu(t_2) - e^{-t_1}Tu(t_1) | \\
&\le \frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{e^{t_2}}\phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au(\tau)d\tau)ds \\
&- \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{e^{t_2}}\phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au(\tau)d\tau)ds\Big| \\
&+ \frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}}{e^{t_2}}\phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au(\tau)d\tau)ds \\
&- \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{e^{t_1}}\phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au(\tau)d\tau)ds\Big| \\
&+ \Big|\frac{t_2^{\alpha-1}}{e^{t_2}} - \frac{t_1^{\alpha-1}}{e^{t_1}}\Big| \frac{1}{\Gamma(\alpha)(\Gamma(\alpha+1) - \sum_{i=1}^m a_i\xi_i^\alpha)} \\
&\times \sum_{i=1}^m a_i\Big|\int_0^{\xi_i} (\xi_i - s)^\alpha \phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au(\tau)d\tau)ds\Big|, \\
&\le \frac{1}{\Gamma(\alpha)}\phi_q(\frac{\Gamma(\beta+\nu)}{\Delta})\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{e^{t_2}}\phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty} |Au(\tau)|d\tau)ds \\
&+ \frac{1}{\Gamma(\alpha)}\phi_q(\frac{\Gamma(\beta+\nu)}{\Delta})\int_0^{t_1} \Big|\frac{(t_2-s)^{\alpha-1}}{e^{t_2}} - \frac{(t_1-s)^{\alpha-1}}{e^{t_1}}\Big|\phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty} |Au(\tau)|d\tau)ds \\
&+ \Big|\frac{t_2^{\alpha-1}}{e^{t_2}} - \frac{t_1^{\alpha-1}}{e^{t_1}}\Big|\phi_q(\frac{\Gamma(\beta+\nu)}{\Delta}) \frac{1}{\Gamma(\alpha)(\Gamma(\alpha+1) - \sum_{i=1}^m a_i\xi_i^\alpha)} \\
&\times \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^\alpha \phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty} |Au(\tau)|d\tau)ds;
\end{aligned}$$

$$\begin{aligned}
&|D_{0+}^{\alpha-1}Tu(t_2) - |D_{0+}^{\alpha-1}Tu(t_1)| \\
&\le \phi_q(\frac{\Gamma(\beta+\nu)}{\Delta})\int_{t_1}^{t_2} \phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty} | Au(\tau) | d\tau)ds.
\end{aligned}$$

Next, we consider $D_{0+}^\alpha Tu$.

$$\begin{aligned}
&|\phi_p(D_{0+}^\alpha Tu(t_2)) - \phi_p(D_{0+}^\alpha Tu(t_1))| \\
&= \Big|\frac{1}{p(t_2)}\int_0^{+\infty} G(t_2,s)Au(s)ds - \frac{1}{p(t_1)}\int_0^{+\infty} G(t_1,s)Au(s)ds\Big|.
\end{aligned} \tag{3.8}$$

According to the interval of integration depending on s, there are three cases.
Case I. $0 < t_1 < t_2 < \eta < +\infty$.

$$(3.8) \le \frac{1}{\Delta}\int_0^{t_1} [\Gamma(\beta+\nu) - \lambda(\eta-s)^{\beta+\nu-1}]|Au(s)|ds\Big|\frac{t_2^{\beta-1}}{p(t_2)} - \frac{t_1^{\beta-1}}{p(t_1)}\Big|$$

$$
\begin{aligned}
&+\frac{\Gamma(\beta+\nu)-\lambda\eta^{\beta+\nu-1}}{\Delta}\int_0^{t_1}\left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right||Au(s)|ds\\
&+\frac{1}{\Delta}\int_{t_1}^{t_2}[\Gamma(\beta+\nu)-\lambda(\eta-s)^{\beta+\nu-1}]|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\\
&+\frac{1}{\Delta}\frac{\Gamma(\beta+\nu)-\lambda\eta^{\beta+\nu-1}}{p(t_2)}\int_{t_1}^{t_2}[\Gamma(\beta+\nu)-\lambda\eta^{\beta+\nu-1}](t_2-s)^{\beta-1}|Au(s)|ds\\
&+\frac{1}{\Delta}\int_{t_2}^{\eta}[\Gamma(\beta+\nu)-\lambda(\eta-s)^{\beta+\nu-1}]|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\\
&+\frac{\Gamma(\beta+\nu)}{\Delta}\int_{\eta}^{+\infty}|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|.
\end{aligned}
$$

Case II. $0<\eta<t_1<t_2<+\infty$.

$$
\begin{aligned}
(3.8)\le&\frac{1}{\Delta}\int_0^{\eta}[\Gamma(\beta+\nu)-\lambda(\eta-s)^{\beta+\nu-1}]|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\\
&+\frac{1}{\Delta}\int_0^{\eta}[\Gamma(\beta+\nu)-\lambda\eta^{\beta+\nu-1}]|Au(s)|\left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right|ds\\
&+\frac{\Gamma(\beta+\nu)}{\Delta}\int_{\eta}^{t_1}|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\\
&+\frac{\Gamma(\beta+\nu)}{\Delta}\int_{\eta}^{t_1}\left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right||Au(s)|ds\\
&+\frac{\lambda\eta^{\beta+\nu-1}}{\Delta}\int_{\eta}^{t_1}\left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right||Au(s)|ds\\
&+\frac{\Gamma(\beta+\nu)}{\Delta}\int_{t_1}^{t_2}\left|\frac{[t_2^{\beta-1}-(t_2-s)^{\beta-1}]+\lambda\eta^{\beta+\nu-1}(t_2-s)^{\beta-1}}{p(t_2)}\right||Au(s)|ds\\
&+\frac{\Gamma(\beta+\nu)}{\Delta}\int_{t_1}^{t_2}|Au(s)|ds\frac{t_1^{\beta-1}}{p(t_1)}+\frac{\Gamma(\beta+\nu)}{\Delta}\int_{t_2}^{+\infty}|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|.
\end{aligned}
$$

Case III. $0<t_1<\eta<t_2<+\infty$.

$$
\begin{aligned}
(3.8)\le&\frac{1}{\Delta}\int_0^{t_1}[\Gamma(\beta+\nu)-\lambda(\eta-s)^{\beta+\nu-1}]|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\\
&+\frac{1}{\Delta}\int_0^{t_1}[\Gamma(\beta+\nu)-\lambda\eta^{\beta+\nu-1}]\left|[\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}]Au(s)\right|ds\\
&+\frac{1}{\Delta}\int_{t_1}^{\eta}[\Gamma(\beta+\nu)-\lambda(\eta-s)^{\beta+\nu-1}]|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|\\
&+\frac{1}{\Delta}\int_{t_1}^{\eta}[\Gamma(\beta+\nu)-\lambda(\eta-s)^{\beta+\nu-1}]\frac{(t_2-s)^{\beta-1}}{p(t_2)}|Au(s)|ds\\
&+\frac{\Gamma(\beta+\nu)}{\Delta}\int_{\eta}^{t_2}|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|
\end{aligned}
$$

$$+\frac{\Gamma(\beta+\nu)+\lambda\eta^{\beta+\nu-1}}{\Delta}\int_{\eta}^{t_2}\frac{(t_2-s)^{\beta-1}}{p(t_2)}|Au(s)|ds$$
$$+\frac{\Gamma(\beta+\nu)}{\Delta}\int_{t_2}^{+\infty}|Au(s)|ds\left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|.$$

Combining (3.1) and (H_3), condition (i) of Lemma 2.7 holds.

It remains to prove that for any $u(t)\in V$, $e^{-t}Tu(t)$, $D_{0+}^{\alpha-1}Tu(t)$, $D_{0+}^{\alpha}Tu(t)$ satisfy the condition (ii) of Lemma 2.7.

From (2.4) and (3.1), we have

$$\begin{aligned}
&\left|\int_0^{+\infty}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds\right|\\
&\le\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)}\int_0^{+\infty}\phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty}|Au(\tau)|d\tau)ds\\
&\le\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)}\int_0^{+\infty}\phi_q(\frac{s^{\beta-1}}{p(s)})ds\phi_q(\|a\|_1+\|u\|_\infty^{p-1}B)\\
&\le\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)}\int_0^{+\infty}\phi_q(\frac{s^{\beta-1}}{p(s)})ds\phi_q(\|a\|_1+\|u\|_\infty^{p-1}B)\\
&\le\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)}C\phi_q(\|a\|_1+\|u\|_\infty^{p-1}B).
\end{aligned}\tag{3.9}$$

By (3.1), we know that for any $u(t)\in V$, $\int_0^{+\infty}|Au(t)|dt$ is bounded. From (H_2) and Remark 3.2, for any given $\epsilon>0$, there exists a constant $N>\eta>0$ such that $t_1,t_2>N$,

$$\int_{t_1}^{t_2}\phi_q(\frac{t^{\beta-1}}{p(t)})dt<\epsilon,\ \int_{t_1}^{t_2}|Au(t)|dt<\epsilon.\tag{3.10}$$

In addition, $\lim\limits_{t\to+\infty}\frac{t^{\beta-1}}{e^t}=0$, $\lim\limits_{t\to+\infty}\frac{t^{\beta-1}}{p(t)}=0$, and there exists a constant $T_1>N$, such that for any $t_1,\ t_2>T_1$ we have

$$\left|\frac{t_1^{\beta-1}}{e^{t_1}}\right|<\epsilon,\ \left|\frac{t_1^{\beta-1}}{p(t_1)}\right|<\epsilon,\ \left|\frac{t_2^{\beta-1}}{p(t_2)}-\frac{t_1^{\beta-1}}{p(t_1)}\right|<\epsilon,\ \left|\frac{t_2^{\beta-1}}{e^{t_2}}-\frac{t_1^{\beta-1}}{e^{t_1}}\right|<\epsilon.\tag{3.11}$$

Similarly, $\lim\limits_{t\to+\infty}\frac{(t-s)^{\beta-1}}{p(t)}=0$, and there exists a constant $T_2>N$, such that for any $t_1,t_2>T_2, 0\le s\le N$, we have

$$\left|\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right|<\epsilon,\ \left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}\right|<\epsilon,\ \left|\frac{(t_2-s)^{\beta-1}}{p(t_2)}-\frac{(t_1-s)^{\beta-1}}{p(t_1)}\right|<\epsilon.\tag{3.12}$$

Set $T=\max\{T_1,T_2\}$, then for $t_1,t_2>T$, combined with (3.9)–(3.12), the following statements hold.

$$\begin{aligned}
&|\ e^{-t_2}Tu(t_2)-e^{-t_1}Tu(t_1)\ |\\
&\le\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{e^{t_2}}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{e^{t_1}}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds\Big|\\
&+\Big|\frac{t_2^{\alpha-1}}{e^{t_2}}-\frac{t_1^{\alpha-1}}{e^{t_1}}\Big|\frac{1}{\Gamma(\alpha)(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha})}\\
&\times\sum_{i=1}^{m}a_i\Big|\int_0^{\xi_i}(\xi_i-s)^{\alpha}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds\Big|\\
&\leq\frac{1}{\Gamma(\alpha)}\int_0^{N}\Big|\frac{(t_2-s)^{\alpha-1}}{e^{t_2}}-\frac{(t_1-s)^{\alpha-1}}{e^{t_1}}\Big|\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)\mid Au(\tau)\mid d\tau)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_N^{t_1}\frac{(t_1-s)^{\alpha-1}}{e^{t_1}}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)|Au(\tau)|d\tau)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_N^{t_2}\frac{(t_2-s)^{\alpha-1}}{e^{t_2}}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)\mid Au(\tau)\mid d\tau)ds\\
&+\Big|\frac{t_2^{\alpha-1}}{e^{t_2}}-\frac{t_1^{\alpha-1}}{e^{t_1}}\Big|\frac{1}{\Gamma(\alpha)(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha})}\\
&\times\sum_{i=1}^{m}a_i\Big|\int_0^{\xi_i}(\xi_i-s)^{\alpha}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds\Big|\\
&\leq\frac{\phi_q(\Gamma(\beta+\nu))}{\Gamma(\alpha)\phi_q(\Delta)}3C\phi_q(\|a\|_1+\|u\|_\infty^{p-1}B)\epsilon+\frac{\epsilon}{\Gamma(\alpha)(\Gamma(\alpha+1)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha})}\\
&\times\sum_{i=1}^{m}a_i\Big|\int_0^{\xi_i}(\xi_i-s)^{\alpha}\phi_q(\frac{1}{p(s)}\int_0^{+\infty}G(s,\tau)Au(\tau)d\tau)ds\Big|.
\end{aligned}
$$

$$
\begin{aligned}
&|D_{0+}^{\alpha-1}Tu(t_1)-D_{0+}^{\alpha-1}Tu(t_2)|\\
&\leq\frac{\phi_q(\Gamma(\beta+\nu)}{\phi_q(\Gamma(\Delta))}\int_{t_1}^{t_2}\phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty}\mid Au(\tau)\mid d\tau)ds\\
&\leq\frac{\phi_q(\Gamma(\beta+\nu)}{\phi_q(\Gamma(\Delta))}\phi_q(\int_0^{+\infty}\mid Au(\tau)\mid d\tau)\epsilon.
\end{aligned}
$$

Next, we consider $D_{0+}^{\alpha}Tu$.

$$
\begin{aligned}
&|\phi_p(D_{0+}^{\alpha}Tu(t_2))-\phi_p(D_{0+}^{\alpha}Tu(t_1))|\\
&\leq\Big|\frac{1}{p(t_2)}\int_0^{+\infty}G(t_2,s)Au(s)ds\Big|+\Big|\frac{1}{p(t_1)}\int_0^{+\infty}G(t_1,s)Au(s)ds\Big|\\
&\leq\frac{\Gamma(\beta+\nu)}{\Delta}\frac{t_2^{\beta-1}}{p(t_2)}\int_0^{+\infty}|Au(s)|ds+\frac{\Gamma(\beta+\nu)}{\Delta}\frac{t_1^{\beta-1}}{p(t_1)}\int_0^{+\infty}|Au(s)|ds\\
&\leq\frac{2\Gamma(\beta+\nu)}{\Delta}\epsilon\int_0^{+\infty}|Au(s)|ds.
\end{aligned}
$$

Using Lemma 2.7, TV is relatively compact.
Step 3. We prove the operator $T : P \to P$ is continuous.

Let $u_n, u \in P, n = 1, 2, \cdots$, and $u_n \to u(n \to +\infty)$. Then, from (3.1), we have

$$
\begin{aligned}
&| e^{-t}Tu_n(t) - e^{-t}Tu(t) | \\
&\leq \frac{1}{\Gamma(\alpha)}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{e^t}[\phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au_n(\tau)d\tau)ds - \phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au(\tau)d\tau)]ds\right| \\
&+ \frac{t^{\alpha-1}}{e^t}\frac{1}{\Gamma(\alpha)(\Gamma(\alpha+1) - \sum_{i=1}^m a_i\xi_i^\alpha)} \\
&\times \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^\alpha \left|[\phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au_n(\tau)d\tau) - \phi_q(\frac{1}{p(s)}\int_0^{+\infty} G(s,\tau)Au(\tau)d\tau)]ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)}\left|\int_0^{+\infty}\phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty} Au_n(\tau)d\tau)ds\right| \\
&+ \frac{1}{\Gamma(\alpha)}\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)}\left|\int_0^{+\infty}\phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty} Au(\tau)d\tau)ds\right| \\
&+ \frac{1}{\Gamma(\alpha)(\Gamma(\alpha+1) - \sum_{i=1}^m a_i\xi_i^\alpha)}\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)} \\
&\times \sum_{i=1}^m a_i\xi_i^\alpha\left[\left|\int_0^{+\infty}\phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty} Au_n(\tau)d\tau)ds\right| + \left|\int_0^{+\infty}\phi_q(\frac{s^{\beta-1}}{p(s)}\int_0^{+\infty} Au(\tau)d\tau)ds\right|\right] \\
&\leq 2C\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)}\frac{\alpha}{\Gamma(\alpha+1) - \sum_{i=1}^m a_i\xi_i^\alpha}\phi_q(\|a\|_1 + r^{p-1}B).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&| D_{0+}^{\alpha-1}Tu_n(t) - D_{0+}^{\alpha-1}Tu(t) | \\
&\leq 2C\frac{\phi_q(\Gamma(\beta+\nu))}{\phi_q(\Delta)}\frac{\alpha}{\Gamma(\alpha+1) - \sum_{i=1}^m a_i\xi_i^\alpha}\phi_q(\|a\|_1 + r^{p-1}B).
\end{aligned}
$$

$$
\begin{aligned}
&| D_{0+}^{\alpha}Tu_n(t) - D_{0+}^{\alpha}Tu(t) | \\
&= \left|\phi_q(\frac{1}{p(t)}\int_0^{+\infty} G(t,s)Au_n(s)ds) - \phi_q(\frac{1}{p(t)}\int_0^{+\infty} G(t,s)Au(s)ds)\right| \\
&\leq \frac{1}{\phi_q(\Gamma(\Delta))}\left(\phi_q(\frac{t^{\beta-1}}{p(t)}\int_0^{+\infty} |Au_n(s)|ds) + \phi_q(\frac{t^{\beta-1}}{p(t)}\int_0^{+\infty} |Au(s)|ds)\right) \\
&\leq \frac{\phi_q(\Gamma(\beta+\nu)}{\phi_q(\Gamma(\Delta))}2C\phi_q(\|a\|_1 + r^{p-1}B).
\end{aligned}
$$

From Lebesgue's dominated convergence theorem, the operator T is continuous. So, the operator $T : P \to P$ is completely continuous. Therefore, by Lemma 2.9, BVP (1.2) has at least one solution in P. The proof is complete.

4. Examples

In this section, some examples are given to illustrate our result.

Example 4.1. Consider the following BVP

$$\begin{cases} D_{0^+}^{\frac{3}{2}}((t+1)^6\phi_4(D_{0^+}^{\frac{3}{2}}u(t))) + \frac{cost}{(t+3)^4} + e^{-t}\frac{u^3(t)}{(t+4)^4} + \frac{(D_{0^+}^{\frac{1}{2}}u(t))^3}{(t+5)^4} + \frac{(D_{0^+}^{\frac{3}{2}}u(t))^3}{(t+6)^4} = 0, t \in (0,+\infty), \\ u(0) = 0, \ \ D_{0^+}^{\frac{1}{2}}u(0) = \frac{1}{8}\int_0^{\sqrt[3]{4}} u(t)dt + \frac{1}{12}\int_0^{\sqrt[3]{9}} u(t)dt + \frac{1}{16}\int_0^{\sqrt[3]{16}} u(t)dt, \ \ D_{0^+}^{\alpha}u(0) = 0, \\ \lim\limits_{t\to+\infty} D_{0^+}^{\frac{1}{2}}(t+1)^6\phi_4(D_{0^+}^{\frac{3}{2}}u(t))) = \frac{1}{2}I^{\frac{1}{2}}((\frac{1}{2}+1)^6\phi_4(D_{0^+}^{\frac{3}{2}}u(\frac{1}{2}))), \end{cases} \tag{4.1}$$

where $p = 4$, $p(t) = (t+1)^6$, $\alpha = \beta = \frac{3}{2}$, $\nu = \frac{1}{2}$, $a_1 = \frac{1}{8}$, $a_2 = \frac{1}{12}$, $a_3 = \frac{1}{16}$, $\xi_1 = \sqrt[3]{4}$, $\xi_2 = \sqrt[3]{9}$, $\xi_3 = \sqrt[3]{16}$, $\lambda = \eta = \frac{1}{2}$,

$$f(t,u,v,w) = \frac{cost}{(t+3)^4} + e^{-t}\frac{u^3(t)}{(t+3)^4} + \frac{v^3(t)}{(t+4)^4} + \frac{w^3(t)}{(t+5)^4}.$$

From $p = 4$, we have $q = \frac{4}{3} < 2$.

First,

$$\Gamma(\alpha+1) = \Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi} \doteq 1.3293, \sum_{i=1}^{3} a_i\xi_i^3 = 0.75 < \Gamma(\alpha+1).$$

$$\Gamma(\beta+\nu) = \Gamma(2) = 1, \lambda\eta^{\beta+\nu-1} = 0.25 < \Gamma(\beta+\nu).$$

This shows condition (H_1) holds.

Second,

$$p(t) = (t+1)^6 \in C[0,+\infty), \ p(t) > 0, \ \phi_q(\frac{t^{\beta-1}}{p(t)}) = \frac{\sqrt[6]{t}}{(t+1)^2} \in L^1[0,+\infty), \ \lim_{t\to+\infty}\frac{t^{\beta-1}}{p(t)} = 0,$$

which show that (H_2) is satisfied and

$$\int_0^{+\infty}\phi_q(\frac{t^{\beta-1}}{p(t)})dt = \int_0^{+\infty}\frac{\sqrt[6]{t}}{(t+1)^2} = \frac{\pi}{3} \doteq 1.0472, \sup_{t\in[0,+\infty)}\phi_q(\frac{t^{\beta-1}}{p(t)}) = \frac{\sqrt{\frac{1}{3}}}{(\frac{1}{3}+1)^2} \doteq 0.3248.$$

So, $C = \max\{\int_0^{+\infty}\phi_q(\frac{t^{\beta-1}}{p(t)})dt, \sup\limits_{t\in[0,+\infty)}\phi_q(\frac{t^{\beta-1}}{p(t)})\} = \frac{\pi}{3}$.

Let $a(t) = \frac{1}{(t+3)^4}$, $b(t) = \frac{1}{(t+4)^4}$, $c(t) = \frac{1}{(t+5)^4}$, $d(t) = \frac{1}{(t+6)^4}$. Clearly, $a(t), b(t), c(t), d(t)$ satisfy (H_3). Then,

$$B = \int_0^{+\infty}(b(t)+c(t)+d(t))dt = \frac{1}{3}(\frac{1}{64}+\frac{1}{125}+\frac{1}{219}) \doteq 0.0094.$$

By computation, we can get

$$\min\left\{\frac{\Delta\phi_p(\Gamma(\alpha+\nu)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha})}{\phi_p(C\alpha)\Gamma(\beta+\nu)}, \frac{\Delta\phi_p(\Gamma(\alpha+\nu)-\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha})}{\phi_p(C\Gamma(\alpha+1))\Gamma(\beta+\nu)}, \frac{\Delta}{\phi_p(C)\Gamma(\beta+\nu)}\right\} = 0.0189.$$

Obviously, $B < \min\left\{\frac{\Delta\phi_p(\Gamma(\alpha+\nu)-\sum_{i=1}^{m} a_i\xi_i^{\alpha})}{\phi_p(C\alpha)\Gamma(\beta+\nu)}, \frac{\Delta\phi_p(\Gamma(\alpha+\nu)-\sum_{i=1}^{m} a_i\xi_i^{\alpha})}{\phi_p(C\Gamma(\alpha+1))\Gamma(\beta+\nu)}, \frac{\Delta}{\phi_p(C)\Gamma(\beta+\nu)}\right\}$. Therefore, we can conclude that bvp (4.1) has at least one solution by Theorem 3.1.

Example 4.2. Consider the following BVP

$$\begin{cases} D_{0^+}^{\frac{3}{2}}((t+1)^6\phi_{\frac{4}{3}}(D_{0^+}^{\frac{3}{2}}u(t))) + \frac{cost}{(t+3)^4} + e^{-t}\frac{\sqrt[3]{u(t)}}{(t+4)^4} + \frac{\sqrt[3]{D_{0^+}^{\frac{1}{2}}u(t)}}{(t+5)^4} + \frac{\sqrt[3]{D_{0^+}^{\frac{3}{2}}u(t)}}{(t+6)^4} = 0, t \in (0,+\infty), \\ u(0) = 0, \quad D_{0^+}^{\frac{1}{2}}u(0) = \frac{1}{8}\int_0^{\sqrt[3]{4}} u(t)dt + \frac{1}{12}\int_0^{\sqrt[3]{9}} u(t)dt + \frac{1}{16}\int_0^{\sqrt[3]{16}} u(t)dt, \quad D_{0^+}^{\alpha}u(0) = 0, \\ \lim\limits_{t\to+\infty} D_{0^+}^{\frac{1}{2}}(t+1)^6\phi_{\frac{4}{3}}(D_{0^+}^{\frac{3}{2}}u(t)) = \frac{1}{2}I^{\frac{1}{2}}((\frac{1}{2}+1)^6\phi_{\frac{4}{3}}(D_{0^+}^{\frac{3}{2}}u(\frac{1}{2}))), \end{cases} \tag{4.2}$$

where $p = \frac{4}{3},\ p(t) = (t+1)^6,\ \alpha = \beta = \frac{3}{2}, \nu = \frac{1}{2}, a_1 = \frac{1}{8},\ a_2 = \frac{1}{12},\ a_3 = \frac{1}{16}\ , \xi_1 = \sqrt[3]{4},\ \xi_2 = \sqrt[3]{9},\ \xi_3 = \sqrt[3]{16},\ \lambda = \eta = \frac{1}{2}$,

$$f(t,u,v,w) = \frac{cost}{(t+3)^4} + e^{-t}\frac{\sqrt[3]{u(t)}}{(t+3)^4} + \frac{\sqrt[3]{v(t)}}{(t+4)^4} + \frac{\sqrt[3]{w(t)}}{(t+5)^4}.$$

From $p = \frac{4}{3}$, we have $q = 4 > 2$.

Note that in Example 4.2, the only difference in Example 4.1 is the value of p. So, (H_1) holds. By Example 4.1, we can get

$$p(t) = (t+1)^6 \in C[0,+\infty), p(t) > 0, \phi_q(\frac{t^{\beta-1}}{p(t)}) = \frac{\sqrt{t^3}}{(t+1)^{18}} \in L^1[0,+\infty), \lim_{t\to+\infty}\frac{t^{\beta-1}}{p(t)} = 0,$$

which shows that (H_2) is satisfied. By simple computation, we obtain

$$\int_0^{+\infty}\phi_q(\frac{t^{\beta-1}}{p(t)})dt = \int_0^{+\infty}\frac{\sqrt{t^3}}{(t+1)^{18}} = B(\frac{31}{2},\frac{5}{2}) \doteq 0.0013, \sup_{t\in[0,+\infty)}\phi_q(\frac{t^{\beta-1}}{p(t)}) = \frac{\sqrt{11^{33}}}{12^{18}} \doteq 0.0057.$$

Hence, $C = \max\{\int_0^{+\infty}\phi_q(\frac{t^{\beta-1}}{p(t)})dt, \sup\limits_{t\in[0,+\infty)}\phi_q(\frac{t^{\beta-1}}{p(t)})\} = 0.0057$.

By computation, we have

$$\min\left\{\frac{\Delta\phi_p(\Gamma(\alpha+\nu)-\sum_{i=1}^{m} a_i\xi_i^{\alpha})}{\phi_p(2^{q-2}C\alpha)\Gamma(\beta+\nu)}, \frac{\Delta\phi_p(\Gamma(\alpha+\nu)-\sum_{i=1}^{m} a_i\xi_i^{\alpha})}{\phi_p(2^{q-2}C\Gamma(\alpha+1))\Gamma(\beta+\nu)}, \frac{\Delta}{\phi_p(2^{q-2}C)\Gamma(\beta+\nu)}\right\} = 13.1232.$$

Obviously, $B < \min\left\{\frac{\Delta\phi_p(\Gamma(\alpha+\nu)-\sum_{i=1}^{m} a_i\xi_i^{\alpha})}{\phi_p(2^{q-2}C\alpha)\Gamma(\beta+\nu)}, \frac{\Delta\phi_p(\Gamma(\alpha+\nu)-\sum_{i=1}^{m} a_i\xi_i^{\alpha})}{\phi_p(2^{q-2}C\Gamma(\alpha+1))\Gamma(\beta+\nu)}, \frac{\Delta}{\phi_p(2^{q-2}C)\Gamma(\beta+\nu)}\right\}$ holds. Therefore, we can conclude that BVP (4.2) has at least one solution by Theorem 3.1.

Remark 4.1. The distinction between Example 4.1 and Example 4.2 arises from the different values of the parameter p, which consequently yields different admissible values for the conjugate exponent q. When $q < 2$ in Example 4.1, $B < 0.0189$, while $q \geq 2$ in Example 4.2, $B < 13.1232$. Obviously, when $q \geq 2$, B is obtained more easily than $q < 2$. This means that in BVP (1.2), the requirements of nonlinear function $f(t,x,y,z)$ are satisfied more easily.

5. Conclusions

This paper investigates a class of Riemann-Liouville fractional boundary-value problems involving the p-Laplacian operator on the semi-infinite interval. Owing to the loss of compactness on unbounded domains and the intrinsic nonlinearity arising from the p-Laplacian, theoretical analysis for such systems is nontrivial. By applying the fixed point theorem, the existence result for the solution is established. The obtained result complements the theoretical framework for p-Laplacian fractional differential equations and furnishes theoretical foundations for subsequent numerical investigations. Possible further research directions include the study of positive solutions and the construction of corresponding numerical algorithms.

Author contributions

Yanping Zheng: Conceptualization, methodology, formal analysis, writing–original draft; Wenxia Wang: Validation, writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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