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*Research article*

# Global strong solution for the compressible Poisson–Nernst–Planck–Navier–Stokes system in 1D

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**Abstract:** This paper investigates a multi-species Poisson–Nernst–Planck–Navier–Stokes (PNP-NS) system coupled with a self-consistent electrostatic potential. On a one-dimensional bounded domain, we establish the global existence and uniqueness of strong solutions that allow for vacuum in the initial fluid density. This work extends the two-ion global well-posedness results of Chen [S. Chen, J. Math. Anal. Appl. 554 (2026) 129943] to arbitrary multi-ion configurations with distinct valences and diffusion coefficients.

**Keywords:** Poisson-Nernst-Planck-Navier-Stokes global existence; multi-ion; a priori estimates
**Mathematics Subject Classification:** 35Q30, 76N10

## 1. Introduction

The compressible Poisson–Nernst–Planck–Navier–Stokes (PNP-NS) system models the transport of charged particles in a compressible fluid under a self-consistent electrostatic potential, and in the three-dimensional space it takes the following form:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p(\rho) - \mu\Delta\mathbf{u} - (\mu+\lambda)\nabla\operatorname{div}\mathbf{u} = -\sum_{k=1}^m \nabla\varphi_k(v_k) + \zeta\nabla\phi\Delta\phi, \\ \partial_t v_k + \operatorname{div}(v_k\mathbf{u}) = \operatorname{div}\left(D_k\nabla\varphi_k(v_k) + D_k z_k e v_k\nabla\phi\right), \quad k = 1, 2, \cdots, m, \\ -\zeta\Delta\phi = \sum_{k=1}^m z_k e v_k, \end{cases} \tag{1.1}$$

where $\rho$, $u$, and $p(\rho)$ denote the fluid density, velocity, and pressure, respectively. Regarding the pressure, we assume the following constitutive relation:

$$p(\rho) = \alpha\rho^\gamma,$$

where $\gamma > \frac{3}{2}$ is the adiabatic exponent and $\alpha > 0$ is a constant. The function $\varphi_k(v_k)$ represents the entropy densities of the corresponding ionic species, with functions $v_k(k = 1, 2, \cdots, m, \text{here } m \in \mathbb{N}^+)$ being the ion concentrations. The electrostatic potential is denoted by $\phi$. The physical parameters are as follows: $\mu$ and $\lambda$ are the viscosity coefficients of the fluid, and $\zeta$ is the dielectric permittivity. Additionally, $D_k$ are the positive diffusion coefficients for the ions, while $z_k$ is the (constant) charge of the $k$th ion species.

If the macroscopic fluid effects are disregarded ($\mathbf{u} = 0$), system (1.1) simplifies to the following Poisson–Nernst–Planck system:

$$\begin{cases} \partial_t v_k = \operatorname{div}(D_k \nabla \varphi_k(v_k) + D_k z_k v_k \nabla \phi), \quad k = 1, 2, \cdots, m, \\ -\zeta \Delta \phi = \sum_{k=1}^m z_k v_k. \end{cases} \tag{1.2}$$

This system serves as a model for the motion of a conductive fluid containing dilute charged species [3]. Existing literature on this system is classified based on spatial settings. For the Cauchy problem on the whole space, the local and global well-posedness of (1.2) has been extensively studied under various assumptions, including small initial data, different function spaces, and multi-ion configurations [10, 25, 26]. For bounded domains, a wider range of results is available owing to the diversity of boundary conditions, Jerome and Sacco [8] obtained the global existence of the weak solution of the initial boundary value/mixed boundary value problem of the system (1.2).Considering the ion distribution with mixed boundary and the electric potential energy with Neumann boundary, Schmuck [17] proved the global existence of the weak solution of the system (1.2) in two-dimensional and three-dimensional space. For related studies, please refer to [2, 7], while periodic cases are addressed in [1, 9, 20].

For the full system (1.1), Wang et al. [22] established the existence of local classical solutions and global solutions for small initial data, as well as the corresponding decay rates, in the two-ion case. Further generalizations and pointwise estimates were provided in [19, 21, 23], while Besov regularity was studied in [24]. For large initial data, global existence results include a 3D weak solution in [13] and 1D strong and classical solutions in [6]. It is worth noting, however, that the analysis in [6] is restricted to the two-ion case and does not extend to multi-ion systems, owing to the presence of distinct ions valences and nonuniform diffusion coefficients (see [6, Remark 1.11]). In this paper, we aim to overcome this limitation and extend Chen's results to the general multi-ion framework.

In parallel with theoretical analysis, structure-preserving numerical schemes for the PNP and PNP–NS systems have been extensively studied, with positivity and energy stability established in Liu et al. [11, 12], Qian et al. [14, 15], and Qin et al. [16]. The convergence of such schemes often relies on the well-posedness of the underlying PDE model, which further justifies the theoretical results obtained in this work.

Inspired by [6], we assume that

$$\varphi_k(v_k) = v_k.$$

For simplicity of presentation, we normalize the physical parameters by setting

$$\mu = \zeta = e = \alpha = 1, \quad \lambda = 0.$$

Under the above assumptions, the one-dimensional multi-ion system corresponding to (1.1) takes the following form:

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + (\rho u^2)_x + (\rho^\gamma)_x = u_{xx} - \sum_{k=1}^m v_{kx} + \left(\frac{1}{2}\phi_x^2\right)_x,\\ v_{kt} + (v_k u)_x = \left(D_k v_{kx} + D_k z_k v_k \phi_x\right)_x, \quad k = 1, 2, \cdots m,\\ -\phi_{xx} = \sum_{k=1}^m z_k v_k,\end{cases} \tag{1.3}$$

with initial data

$$(\rho, u, v_k)\big|_{t=0} = (\rho_0, u_0, v_{k0}) \quad \text{in } I = [0, 1] \tag{1.4}$$

and boundary conditions

$$(u, v_{kx}, \phi_x)\big|_{x=0,1} = (0, 0, 0), \quad t > 0. \tag{1.5}$$

Before presenting our main results, we introduce some notation.

**Notation 1.1.** *Let $I = [0, 1]$, $\partial I = \{0, 1\}$, $Q_T = I \times (0, T)(T > 0)$ be defined as above. Throughout this paper, $C$ denotes a generic positive constant whose value may vary from line to line. For $p \geq 1$, we denote $L^p$ as the $L^p$ space endowed with the norm $\|\cdot\|_p$. For $k \geq 1$ and $p \geq 1$, let $W^{k,p}$ denote the Sobolev space with norm $\|\cdot\|_{W^{k,p}}$, and write $H^k = W^{k,2}$.*

We now state the definition of strong solutions to the system under consideration.

**Definition 1.1.** *For $T > 0$, a tuple $(\rho, u, v_k, \phi_x)$ is called a strong solution of the PNP-NS system* (1.3) *in $Q_T$ provided that it satisfies* (1.3) *almost everywhere in $Q_T$.*

**Theorem 1.1.** *Assume that the initial data satisfy $\rho_0, v_{k0} \geq 0$, with $\rho_0 \in H^1$, $u_0 \in H_0^1$, and $v_{k0} \in H^2$. Then, for any $T > 0$, there exists a unique global strong solution $(\rho, u, v_k, \phi_x)(k = 1, 2, \cdots m)$ to the system* (1.3)–(1.5) *satisfying*

$$\begin{cases}\rho \geq 0, \quad (\rho, \rho^\gamma) \in L^\infty([0, T], H^1), \quad (\rho_t, (\rho^\gamma)_t) \in C([0, T], L^2), \quad (\rho u)_t \in C([0, T], L^2),\\ u \in C([0, T], H_0^1 \cap H^2), \quad u_t \in L^\infty(0, T; L^2) \cap L^2((0, T), H^1),\\ v_k \geq 0, \quad v_k \in C([0, T], H^2), \quad v_{kt} \in L^\infty(0, T; L^2) \cap L^2((0, T), H^1),\\ \phi_x \in C([0, T], H^2).\end{cases} \tag{1.6}$$

**Remark 1.1.** *We define the strong solution in terms of $(\rho, u, v_k, \phi_x)(k = 1, 2, \cdots, m)$ rather than $(\rho, u, v_k, \phi)(k = 1, 2, \cdots, m)$, since only the derivative $\phi_x$ appears explicitly in* (1.3)–(1.5). *Moreover, although the solution $(\rho, u, v_k, \phi_x)(k = 1, 2, \cdots, m)$ will be proved to be unique, the potential function $\phi$ itself is not uniquely determined.*

**Remark 1.2.** *The initial value of $\phi_{0x}$ is determined by* $(1.3)_4$*, together with the boundary condition for $\phi$ and the initial data $v_{k0}$.*

**Remark 1.3.** *The boundary conditions* (1.5) *imply that the boundary is impermeable in the sense that the ionic flux vanishes there.*

**Remark 1.4.** *In contrast to the work of Chen [6], which is restricted to two ionic species (denoted by v and w)with valences* $(z_v, z_w) = (-1, 1)$ *and equal diffusion coefficients* $D_w = D_v = 1$*, the present paper generalizes the model to accommodate multiple ionic species* $v_k$*, where* $k = 1, 2, \cdots m, m \geq 2$ *with distinct valences and non-uniform diffusion coefficients. This extension overcomes the key limitations in [6] and significantly broadens the applicability of the theoretical framework.*

**Remark 1.5.** *Chen's results are confined to the two-ion case* $(v, w)$*. This is primarily because the proof of [6, Lemma 3.2] relies critically on the square-difference term generated by* $(v, w)$*, which yields a nonnegative perfect-square term* $(v - w)^2$*. Together with the property* $v, w \geq 0$*, this enables the desired energy estimates. Specifically, the derivation proceeds as follows:*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I (v^2 + w^2)\,\mathrm{d}x + \int_I (v_x^2 + w_x^2)\,\mathrm{d}x \\
&= \cdots = \int_I u(vv_x + ww_x)\,\mathrm{d}x - \frac{1}{2}\int_I (v^2 - w^2)(v - w)\,\mathrm{d}x \\
&= \int_I u(vv_x + ww_x)\,\mathrm{d}x - \frac{1}{2}\int_I (v + w)(v - w)^2\,\mathrm{d}x \\
&\leq \|u\|_{L^\infty}(\|v\|_{L^2}\|v_x\|_{L^2} + \|w\|_{L^2}\|w_x\|_{L^2}) \leq \cdots .
\end{aligned}$$

*In the multi-ion case, the proof of the aforementioned lemma is no longer applicable. To overcome this difficulty, we observe that the basic energy admits a crucial perfect-square energy term of the form (see Lemma 3.1)*

$$\int_0^t \int_I \sum_{k=1}^m D_k v_k \left(\frac{v_{kx}}{v_k} + z_k\phi_x\right)^2 \mathrm{d}x\mathrm{d}t \leq C.$$

*By estimating the* $L^\infty$*-norm of* $\sqrt{v_k}$*,we are able to cleverly reconstruct this perfect-square term, which in turn enables us to close the energy estimates. For further details, we refer to the proof of Lemma 3.3.*

**Remark 1.6.** *We allow the initial fluid density* $\rho_0$ *to vanish (i.e., vacuum), which is physically relevant in applications such as electrowetting-driven droplet transport (liquid–gas interfaces), electroosmotic flow in porous media (near-wall density depletion or cavitation), and lithium-ion battery slurry coating processes (local porosity variations). Mathematically, this assumption significantly complicates the analysis, as standard parabolic regularization fails near vacuum regions. This motivates the use of the characteristic-line method based on* $\ln\rho$ *in Section 3.*

The remainder of this paper is organized as follows. In Section 2, we establish the local existence of solutions of the system (1.3)–(1.5) under the assumption that the initial density is bounded away from vacuum. We then state the Aubin–Lions lemma, which will be used later to extend the result to the vacuum case. In Section 3, we derive a series of global-in-time a priori estimates for strong solutions of (1.3)–(1.5). The proof of our main theorems is then presented in Section 4.

## 2. Preliminaries

This section is devoted to recalling two ingredients that will be used later: the local existence theory for (1.3)–(1.5), and the Aubin–Lions lemma, which provides the compactness argument for the limiting procedure in Section 4.

**Lemma 2.1.** *Suppose that $\rho_0 \geq 1/\bar{k}$ ($\bar{k} > 0$), $(\rho_0, \rho_0^\gamma) \in H^1$, $u_0 \in H_0^1$, $v_{k0} \in H^2(k = 1, 2, \cdots m)$. Then there exists a positive time $T_0 > 0$ and a unique strong solution $(\rho, u, v_k, \phi_x)(k = 1, 2, \cdots m)$ to the system* (1.3)–(1.5) *satisfying*

$$\begin{cases} \rho \geq 0, \quad (\rho, \rho^\gamma) \in C([0, T_0], H^1), \quad (\rho_t, (\rho^\gamma)_t) \in C([0, T_0], L^2), \quad (\rho u)_t \in C([0, T_0], L^2), \\ u \in C([0, T_0], H_0^1 \cap H^2), \quad u_t \in L^\infty([0, T_0], L^2) \cap L^2([0, T_0], H^1), \\ v_k \geq 0, \quad v_k \in C([0, T_0], H^2), \quad v_{kt} \in L^\infty([0, T_0], L^2) \cap L^2([0, T_0], H^1), \\ \phi_x \in C([0, T_0]; H^2). \end{cases}$$

*Proof.* The proof of Lemma 2.1 is based on Galerkin's approximation method and the fixed point theorem similar to that in [4]. We do not include the full proof here for brevity; instead, we only present the non-negativity of each $v_k$, which follows from a standard argument as in [5].

Define

$$F(s) = \begin{cases} s^2, & s < 0, \\ 0, & s \geq 0, \end{cases}$$

which satisfies $F(s) \geq 0$. Multiplying $(1.3)_3$ by $F'(v_k)$ and integrating over $I$, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_I F(v_k)\,\mathrm{d}x &= -\int_I D_k F''(v_k) v_{kx}^2\,\mathrm{d}x + \int_I F''(v_k) v_{kx} v_k [u - D_k z_k \phi_x]\,\mathrm{d}x \\ &\leq -\frac{1}{2} \int_I F''(v_k) v_{kx}^2\,\mathrm{d}x + C(\,\|u\|_{L^\infty}^2 + \|\phi_x\|_{L^\infty}^2\,) \int_I F(v)\,\mathrm{d}x \\ &\leq C(\,\|u_x\|_{L^2}^2 + \|\phi_{xx}\|_{L^2}^2\,) \int_I F(v_k)\,\mathrm{d}x, \end{aligned}$$

where we have used Young's inequality and the Sobolev embedding $H^1(I) \hookrightarrow L^\infty(I)$, together with the boundary conditions $u|_{\partial I = \{0,1\}} = \phi_x|_{\partial I = \{0,1\}} = 0$. Since $v_{k0} \geq 0$, we have $\int_I F(v_{0k})\,\mathrm{d}x = 0$. Gronwall's inequality then implies $F(v_k) \equiv 0$ for all $t \in [0, T_0]$ and therefore $v_k(t, x) \geq 0$. This completes the proof of Lemma 2.1. □

Next, we recall a version of Aubin–Lions compactness lemma, which will be employed in our later arguments.

**Lemma 2.2.** *(Simon-Aubin-Lions [18]) Assume that $X \subset Y \subset Z$ are Banach spaces and $X \hookrightarrow\hookrightarrow Y$ (compact embedding). Then, the following embeddings are compact:*

$$(i)\ \{\varphi : \varphi \in L^q(0, T; X),\ \partial_t \varphi \in L^1(0, T; Z)\} \hookrightarrow\hookrightarrow L^q(0, T; Y), \quad 1 \leq q \leq \infty,$$

$$(ii)\ \{\varphi : \varphi \in L^\infty(0, T; X),\ \partial_t \varphi \in L^r(0, T; Z)\} \hookrightarrow\hookrightarrow C([0, T], Y), \quad 1 < r \leq \infty.$$

**Remark 2.1.** *This compactness lemma will be employed in Section 4 in the limiting procedure to pass from the approximate solutions to the strong solution, where strong convergence in $C(Q_T)$ is required.*

## 3. A priori estimates

This section is devoted to extending the local solution from Lemma 2.1 to a global one. The main strategy consists of establishing energy inequalities and improving the regularity of $v_k$ and $u$, from which we obtain several key estimates.

For notational convenience, throughout this section we use $C$ or $C(T)$ to denote positive constants that may depend on

$$\gamma, T, \|\rho_0\|_{H^1}, \|u_0\|_{H^1}, \|v_{k0}\|_{H^2}, (k = 1, 2, \cdots, m),$$

where $T > 0$ denotes the final time. Suppose that $(\rho, u, \{v_k\}_{k=1}^m, \phi_x) : I \times (0, T) \to \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$ is the classical solution obtained by Lemma 2.1. It should be noted that $v_k \geq 0$ for each $k$. This property, proven in Lemma 2.1, will be essential for the subsequent analysis.

**Lemma 3.1.** *Under the assumptions of Theorem 1.1 , for any given $T > 0$ and $t \in [0, T]$, it holds that*

$$\int_I \left[ \frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + \sum_{k=1}^m (v_k \ln v_k - v_k + 1) + \frac{\phi_x^2}{2} \right] \mathrm{d}x + \int_0^t \int_I \left[ u_x^2 + \sum_{k=1}^m D_k v_k \left( \frac{v_{kx}}{v_k} + z_k \phi_x \right)^2 \right] \mathrm{d}x\mathrm{d}t = E_0,$$

*where the initial total energy*

$$E_0 = \int_I \left[ \frac{\rho_0 u_0^2}{2} + \frac{\rho_0^\gamma}{\gamma - 1} + \sum_{k=1}^m (v_{k0} \ln v_{k0} - v_{k0} + 1) + \frac{\phi_{0x}^2}{2} \right] \mathrm{d}x.$$

*Proof.* Multiplying $(1.3)_2$ by $u$ and integrating over $I$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I \frac{\rho u^2}{2} \mathrm{d}x + \int_I (\rho^\gamma)_x u \, \mathrm{d}x + \int_I u_x^2 \, \mathrm{d}x = \int_I \Big[ - \sum_{k=1}^m v_{kx} + \phi_x \phi_{xx} \Big] u \, \mathrm{d}x. \tag{3.1}$$

Applying $(1.3)_{1,4}$, we obtain

$$\int_I (\rho^\gamma)_x u \, \mathrm{d}x = \int_I \gamma \rho^{\gamma-1} (-\rho_t - \rho u_x) \, \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} \int_I \rho^\gamma \, \mathrm{d}x + \int_I \gamma (\rho^\gamma)_x u \, \mathrm{d}x$$

and

$$\int_I \phi_x \phi_{xx} u \, \mathrm{d}x = - \int_I \sum_{k=1}^m z_k v_k \phi_x u \, \mathrm{d}x.$$

Then (3.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_I \left( \frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} \right) \mathrm{d}x + \int_I u_x^2 \, \mathrm{d}x = \int_I \Big[ - \sum_{k=1}^m v_{kx} u - \sum_{k=1}^m z_k v_k \phi_x u \Big] \mathrm{d}x. \tag{3.2}$$

Next, multiplying $(1.3)_3$ by $\ln v_k - z_k \phi$ for $(k = 1, 2, \cdots m)$ gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_I (v_k \ln v_k - v_k + 1) \, \mathrm{d}x + \int_I v_{kt} z_k \phi \, \mathrm{d}x + \int_I (v_k u)_x (\ln v_k + z_k \phi) \, \mathrm{d}x \\ &= \int_I (D_k v_{kx} + D_k z_k v_k \phi_x)_x (\ln v_k + z_k \phi) \, \mathrm{d}x, \end{aligned} \tag{3.3}$$

integrating by parts, we get

$$\int_I (v_k u)_x (\ln v_k + z_k\phi)\, \mathrm{d}x = -\int_I v_k u\left(\frac{1}{v_k} v_{kx} + z_k\phi_x\right) \mathrm{d}x \\ = -\int_I (u v_{kx} + z_k v_k u\phi_x)\, \mathrm{d}x \tag{3.4}$$

and

$$\begin{aligned} &\int_I (D_k v_{kx} + D_k z_k v_k \phi_x)_x (\ln v_k + z_k\phi)\, \mathrm{d}x \\ =& -\int_I (D_k v_{kx} + D_k z_k v_k \phi_x)\left(\frac{1}{v_k} v_{kx} + z_k\phi_x\right) \mathrm{d}x \\ =& -\int_I \left(D_k \frac{v_{kx}^2}{v_k} + D_k z_k v_{kx}\phi_x + D_k z_k \phi_x v_{kx} + D_k z_k^2 v_k \phi_x^2\right) \mathrm{d}x \\ =& -\int_I D_k v_k \left(\frac{v_{kx}}{v_k} + z_k\phi_x\right)^2 \mathrm{d}x. \end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), and summing the resulting identities for $k = 1, 2, \cdots, m$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_I \sum_{k=1}^m (v_k \ln v_k - v_k + 1)\, \mathrm{d}x + \int_I \sum_{k=1}^m z_k v_{kt}\phi\, \mathrm{d}x + \int_I \sum_{k-1}^m D_k v_k \left(\frac{v_{kx}}{v_k} + z_k\phi_x\right)^2 \mathrm{d}x \\ =& \int_I (\sum_{k-1}^m u v_{kx} + \sum_{k-1}^m z_k v_k u \phi_x)\, \mathrm{d}x. \end{aligned} \tag{3.6}$$

Using $(1.3)_4$, we have

$$\int_I \sum_{k=1}^m z_k v_{kt}\phi\, \mathrm{d}x = \int_I -\phi_{xxt}\phi\, \mathrm{d}x = \int_I \phi_{xt}\phi_x\, \mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \phi_x^2\, \mathrm{d}x,$$

then (3.6) becomes

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_I \Big[\sum_{k=1}^m (v_k \ln v_k - v_k + 1) + \frac{1}{2}\phi_x^2\Big] \mathrm{d}x + \int_I \sum_{k-1}^m D_k v_k \left(\frac{v_{kx}}{v_k} + z_k\phi_x\right)^2 \mathrm{d}x \\ =& \int_I (\sum_{k-1}^m u v_{kx} + \sum_{k-1}^m z_k v_k u \phi_x)\, \mathrm{d}x. \end{aligned} \tag{3.7}$$

Combining (3.2) and (3.7), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \Big[\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + \sum_{k=1}^m (v_k \ln v_k - v_k + 1) + \frac{\phi_x^2}{2}\Big] \mathrm{d}x + \int_I \Big[u_x^2 + \sum_{k=1}^m D_k v_k \left(\frac{v_{kx}}{v_k} + z_k\phi_x\right)^2\Big] \mathrm{d}x = 0.$$

Integrating the above inequality over $[0, t]$ completes the proof. □

**Remark 3.1.** *(Physical interpretation of the dissipation functional) We pause here to comment on the physical interpretation of the dissipation functional appearing in Lemma 3.1. The integrand*

$$D_k v_k \left( \frac{v_{kx}}{v_k} + z_k \phi_x \right)^2,$$

*represents the entropy production rate associated with the kth ionic species. The quantity inside the parentheses,*

$$\frac{v_{kx}}{v_k} + z_k \phi_x$$

*is precisely the gradient of the electrochemical potential (up to a constant factor), which drives the diffusion–migration process of the kth ionic species. The quadratic form reflects the linear flux–force relation in irreversible thermodynamics, consistent with the Onsager framework. Thus, the energy identity in Lemma 3.1 describes the total free energy dissipation through fluid viscosity and ionic entropy production.*

**Remark 3.2.** *(Energy-dissipation structure) Lemma 3.1 reveals that the compressible multi-species PNP-NS system* (1.3) *admits a natural energy-dissipation law of the form*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) + D(t) = 0,$$

*where*

$$E(t) = \int_I \left[ \frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + \sum_{k=1}^m (v_k \ln v_k - v_k + 1) + \frac{\phi_x^2}{2} \right] \mathrm{d}x$$

*is the total free energy, and*

$$D(t) = \int_I \left[ u_x^2 + \sum_{k=1}^m D_k v_k \left( \frac{v_{kx}}{v_k} + z_k \phi_x \right)^2 \right] \mathrm{d}x = E_0$$

*is the dissipation functional. In contrast to the incompressible case where the fluid kinetic energy is decoupled from the ion transport, the compressible setting introduces the internal energy $\rho^\gamma/(\gamma - 1)$ and the coupling through the convective terms in the ion equations.*

**Lemma 3.2.** *Under the assumptions of Theorem 1.1, for any given $T > 0$ and $t \in [0, T]$, it holds that*

$$\sup_{0 \le t \le T} \int_I \Big( \sum_{k=1}^m v_k^2 + \phi_{xx}^2 \Big) \mathrm{d}x + \int_0^T \int_I \Big( \sum_{k=1}^m D_k v_{kx}^2 + \phi_{xxx}^2 \Big) \mathrm{d}x \mathrm{d}t \le C(T)$$

*and*

$$\|\phi_x\|_{L^\infty(0,T;L^\infty(I))} \le C(T),$$

*where $C(T)$ only depends on $E_0, \|v_{k0}\|_2, \|w_0\|_2, T$.*

*Proof.* Multiplying $(1.3)_3$ by $v_k$, integrating over $I$, applying integration by parts, and summing the resulting identities for $k = 1, 2, \cdots, m$, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_I \sum_{k=1}^m v_k^2 \,\mathrm{d}x + \int_I \sum_{k=1}^m D_k v_{kx}^2 \,\mathrm{d}x = - \int_I \sum_{k=1}^m \Big[ (v_k u)_x v_k + (D_k z_k v_k \phi_x) v_{kx} \Big] \mathrm{d}x$$

$$
\begin{aligned}
&= \int_I \sum_{k=1}^{m} \left[ v_k u v_{kx} - \frac{1}{2} D_k z_k (v_k^2)_x \phi_x \right] \mathrm{d}x = \int_I \sum_{k=1}^{m} \left[ v_k u v_{kx} + \frac{1}{2} D_k z_k v_k^2 \phi_{xx} \right] \mathrm{d}x \\
&= \int_I \sum_{k=1}^{m} v_k u v_{kx} \, \mathrm{d}x - \frac{1}{2} \int_I \sum_{k=1}^{m} D_k z_k v_k^2 \sum_{k=1}^{m} z_k v_k \, \mathrm{d}x =: A + B,
\end{aligned} \tag{3.8}
$$

where we have used Eq $(1.3)_4$.

For $A$, we can estimate it as follows:

$$
\begin{aligned}
A &\leq \left| \int_I \sum_{k=1}^{m} v_k u v_{kx} \, \mathrm{d}x \right| \leq \frac{1}{4} \int_I \sum_{k=1}^{m} D_k v_{kx}^2 \, \mathrm{d}x + C \|u\|_{L^\infty}^2 \int \sum_{k=1}^{m} v_k^2 \, \mathrm{d}x \\
&\leq \frac{1}{4} \int \sum_{k=1}^{m} D_k v_{kx}^2 \, \mathrm{d}x + C \|u_x\|_2^2 \int \sum_{k=1}^{m} v_k^2 \, \mathrm{d}x,
\end{aligned} \tag{3.9}
$$

where we have used the Sobolev embedding $\|u\|_{L^\infty} \leq c \, \|u_x\|_{L^2}$ and Young's inequality.

For $B$, we have

$$
\begin{aligned}
B &\leq \left| \frac{1}{2} \int_I \sum_{k=1}^{m} D_k z_k v_k^2 \sum_{k=1}^{m} z_k v_k \, \mathrm{d}x \right| \leq C \left\| \sum_{k=1}^{m} z_k v_k \right\|_{L^\infty} \int_I \sum_{k=1}^{m} v_k^2 \, \mathrm{d}x \\
&\leq C \sum_{k=1}^{m} \|v_k\|_{L^\infty} \int_I \sum_{k=1}^{m} v_k^2 \, \mathrm{d}x,
\end{aligned} \tag{3.10}
$$

to complete the proof, we need to control $\|v_k\|_{L^\infty}$. We can derive from Lemma 3.1 that

$$
\int_I (v_k \ln v_k - v_k + 1) \, \mathrm{d}x \leq C, \quad k = 1, 2, \cdots, m,
$$

which implies that

$$
\int_I v_k \, \mathrm{d}x \leq C, \quad k = 1, 2, \cdots, m. \tag{3.11}
$$

By virtue of the Newton–Leibniz formula, we have

$$
\left\| \sqrt{v_k} \right\|_{L^\infty} \leq C \left| \int_0^x (\sqrt{v_k})_x \, \mathrm{d}x + \sqrt{v_k}|_{x=0} \right| \leq C + \left\| (\sqrt{v_k})_x \right\|_1, \tag{3.12}
$$

here we note that the boundary term $\sqrt{v_k}|_{x=0}$ is uniformly bounded. Indeed, for the strong solutions considered in this section, Lemma 2.1 ensures $v_k \in C([0, T]; H^2(I))$, and the one-dimensional Sobolev embedding $H^2(I) \hookrightarrow C^1(I)$ yields $\|v_k\|_{C^1(I)} \leq C(T)$.

For $\left\| (\sqrt{v_k})_x \right\|_1$, we estimate as follows:

$$
\begin{aligned}
\left\| (\sqrt{v_k})_x \right\|_1^2 &= \left\| (\sqrt{v_k})_x + \frac{z_k \sqrt{v_k} \phi_x}{2} - \frac{z_k \sqrt{v_k} \phi_x}{2} \right\|_1^2 \\
&\leq C \left\| (\sqrt{v_k})_x + \frac{z_k \sqrt{v_k} \phi_x}{2} \right\|_1^2 + C \left\| z_k \sqrt{v_k} \phi_x \right\|_1^2
\end{aligned}
$$

$$\leq \int_I \left( \frac{v_{kx}}{2\sqrt{v_k}} + \frac{z_k \sqrt{v_k}\phi_x}{2} \right)^2 \mathrm{d}x + \left\| \sqrt{v_k} \right\|_{L^2}^2 \|\phi_x\|_2^2 \tag{3.13}$$
$$\leq \frac{1}{4} \int_I v_k \left( \frac{v_{kx}}{v_k} + z_k \phi_x \right)^2 \mathrm{d}x + \|v_k\|_1 \|\phi_x\|_2^2$$
$$\leq \frac{1}{4} \int_I v_k \left( \frac{v_{kx}}{v_k} + z_k \phi_x \right)^2 \mathrm{d}x + C,$$

where we used (3.11) and Lemma 3.1. Then, from (3.12) and (3.13), we derive

$$\|v_k\|_{L^\infty} \leq \|\sqrt{v_k}\|_{L^\infty}^2 \leq \frac{1}{4} \int_I v_k \left( \frac{v_{kx}}{v_k} + z_k \phi_x \right)^2 \mathrm{d}x + C. \tag{3.14}$$

Substituting (3.14) into (3.10), we obtain

$$B \leq C \Big[ \int_I \sum_{k=1}^m v_k \left( \frac{v_{kx}}{v_k} + z_k \phi_x \right)^2 \mathrm{d}x + C \Big] \int_I \sum_{k=1}^m v_k^2 \, \mathrm{d}x. \tag{3.15}$$

Combining (3.8), (3.9), and (3.15), we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_I \sum_{k=1}^m v_k^2 \, \mathrm{d}x + \int_I \sum_{k=1}^m D_k v_{kx}^2 \, \mathrm{d}x \tag{3.16}$$
$$\leq C \left[ \|u_x\|_2^2 + \int_I \sum_{k=1}^m v_k \left( \frac{v_{kx}}{v_k} + z_k \phi_x \right)^2 + 1 \right] \int_I \sum_{k=1}^m v_k^2 \, \mathrm{d}x.$$

Then Gronwall's inequality together with Lemma 3.1 yields

$$\sup_{0 \leq t \leq T} \int_I \sum_{k=1}^m v_k^2 + \int_0^T \int_I \sum_{k=1}^m D_k v_{kx}^2 \leq C(T).$$

By applying elliptic regularity theory to $(1.3)_4$, we obtain

$$\int_I \phi_{xx}^2 \, \mathrm{d}x + \int_0^T \int_I \phi_{xxx}^2 \, \mathrm{d}x \leq C(T).$$

Finally, using Sobolev interpolation together with Lemma 3.1, we get

$$\|\phi_x\|_{L^\infty(I)} \leq \|\phi_x\|_{L^2}^{1/2} \|\phi_{xx}\|_{L^2}^{1/2} \leq C(T).$$

Thus, we complete the proof. □

**Lemma 3.3.** *Under the assumptions of Theorem 1.1, for any given $T > 0$ and $t \in [0, T]$, it holds that*

$$\|\rho\|_{L^\infty(Q_T)} \leq C(T).$$

*Proof.* For any $t \in [0, T]$, define

$$\xi(x,t) = \int_0^x \rho_0 u_0(y)\,\mathrm{d}y + \int_0^t \left[u_x - \rho u^2 - \rho^\gamma - \sum_{k=1}^m v_k + \frac{1}{2}\phi_x^2\right]\mathrm{d}t.$$

We recall that the initial value $\phi_{0x}$ in the definition of $\xi$ determined by $(1.3)_4$ together with the boundary condition $\phi_x|_{\partial I} = 0$ and the initial data $v_{k0}$ (see Remark 1.2). This ensures that the expression for $\xi(x,0) = \int_0^x \rho_0 u_0\,\mathrm{d}y$ is well-defined.

Applying $(1.3)_2$, we obtain

$$\xi_x = \rho u, \quad \xi_t = u_x - \rho u^2 - \rho^\gamma - \sum_{k=1}^m v_k + \frac{1}{2}\phi_x^2.$$

Using Lemma 3.1, we have

$$\begin{aligned}\int_I (|\xi| + |\xi_x|)\,\mathrm{d}x &\le \int_0^t \int_I \left(u_x + \rho u^2 + \rho^\gamma + \sum_{k=1}^m v_k + \frac{1}{2}\phi_x^2\right)\mathrm{d}x\,\mathrm{d}t + \int_I |\rho u|\,\mathrm{d}x \\ &\le C(T) + \int_I \rho u^2\,\mathrm{d}x + \int_I \rho\,\mathrm{d}x \le C(T).\end{aligned}$$

By Sobolev interpolation, we have

$$\|\xi\|_{L^\infty(Q_T)} \le C \int_I (|\xi| + |\xi_x|) \le C(T). \tag{3.17}$$

In view of the non-negativity of $\rho$, it suffices to prove the upper bound $\rho(x,t) \le C$. For any $(x,t) \in Q_T$, let $X(x,t)$ denote the solution to the following system:

$$\begin{cases} \dfrac{dX(x,t)}{dt} = u(X(x,t),t), & 0 \le t \le T, \\ X(x,0) = x, & 0 \le x \le 1. \end{cases}$$

Define

$$\eta(\rho) = \ln\rho - \frac{C(T)^2}{2}t. \tag{3.18}$$

Then, using $v_k \ge 0$ and Lemma 3.2, we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\xi + \eta)(X(x,t),t) &= (u_x - \rho u^2 - \rho^\gamma - \sum_{k=1}^m v_k + \frac{\phi_x^2}{2} + \rho u^2) - u_x - \frac{C(T)^2}{2} \\ &= -\rho^\gamma - \sum_{k=1}^m v_k + \frac{\phi_x^2}{2} - \frac{C(T)^2}{2} \le 0.\end{aligned} \tag{3.19}$$

Since

$$\xi(X(x,0),0) = \int_0^{X(x,0)} \rho_0 u_0(y)\,\mathrm{d}y = \int_0^x \rho_0 u_0(y)\,\mathrm{d}y \le \int_I |\rho_0 u_0|\,\mathrm{d}y \le \|\sqrt{\rho_0}u_0\|_2 \|\rho_0\|_1^{\frac{1}{2}} \le C,$$
$$\eta(X(x,0),0) = \ln\rho_0 \le \rho_0,$$

it follows that

$$\xi(X(x,0),0) + \eta(X(x,0),0) \le C. \tag{3.20}$$

Combining (3.17)–(3.19) and (3.21), we obtain

$$\ln\rho \le C + \|\xi\|_{L^\infty} + \frac{C(T)^2}{2}T \le C(T).$$

Hence

$$\rho(x,t) \le e^{C(T)} \le C(T).$$

□

The proofs of the following lemmas follow from a straightforward adaptation of the arguments in Chen [6] (specifically Lemmas 3.4–3.8 therein). Therefore, we only present the main steps of the proofs.

**Lemma 3.4.** *Under the hypotheses of Theorem 1.1, for any given $T > 0$ and $t \in [0,T]$, we have the following estimate:*

$$\sup_{0\le t\le T}\int_I\Big(\sum_{k=1}^m v_{kx}^2 + \phi_{xxx}^2\Big)\mathrm{d}x + \int_0^T\int_I\Big(\sum_{k=1}^m v_{kt}^2 + \phi_{xxt}^2\Big)\mathrm{d}x\,\mathrm{d}t \le C(T).$$

*Proof.* Multiplying $(1.3)_3$ by $v_{kt}$ and integrating over $I$ yields

$$\begin{aligned}
&\frac{D_k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I v_{kx}^2\,\mathrm{d}x + \int_I v_{kt}^2\,\mathrm{d}x\\
=&\int_I -(v_{kx}u + v_k u_x)v_{kt} + D_k z_k(v_{kx}\phi_x + v_k\phi_{xx})v_{kt}\,\mathrm{d}x\\
\le&(\|u\|_\infty\|v_{kx}\|_2 + \|v_k\|_\infty\|u_x\|_2)\|v_{kt}\|_2\\
&+ D_k z_k(\|\phi_x\|_\infty\|v_{kx}\|_2 + \|v_k\|_\infty\|\phi_{xx}\|_2)\|v_{kt}\|_2\\
\le&C(\|u_x\|_2\|v_{kx}\|_2 + +\|v_k\|_2^{\frac12}\|v_{kx}\|_2^{\frac12}\|u_x\|_2)\|v_{kt}\|_2\\
&+ C(\|\phi_{xx}\|_2\|v_{kx}\|_2 + \|v_k\|_2^{\frac12}\|v_{kx}\|_2^{\frac12}\|\phi_{xx}\|_2)\|v_{kt}\|_2\\
\le&\frac12\|v_{kt}\|_2^2 + (\|u_x\|_2^2 + \|\phi_{xx}\|_2^2)\|v_{kx}\|_2^2 + \|v_k\|_2^2(\|u_x\|_2^2 + \|\phi_{xx}\|_2^2),
\end{aligned}$$

where the boundary condtion $u|_{\partial I=\{0,1\}} = \phi_x|_{\partial I=\{0,1\}} = 0$ have been applied, along with the following Sobolev embeddings:

$$\|v_k\|_\infty \le C(\|v_k\|_2 + \|v_{kx}\|_2), \quad \|u\|_\infty \le C\,\|u_x\|_2\,, \quad \|\phi_x\|_\infty \le C\,\|\phi_{xx}\|_2\,.$$

Applying Gronwall's inequality to the above estimate for each $v_k, k = 1,2,\cdots,m$ and Lemmas 3.1 and 3.2, we get

$$\sup_{0\le t\le T}\int_I v_{kx}^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_I v_{kt}^2\,\mathrm{d}x \le C(T).$$

Summing these inequalities over $k = 1, 2, \cdots, m$, gives the uniform bound

$$\sup_{0\le t\le T}\int_I \sum_{k=1}^m v_{kx}^2 \,\mathrm{d}x + \int_0^T \int_I \sum_{k=1}^m v_{kt}^2 \,\mathrm{d}x\,\mathrm{d}t \le C(T).$$

From the relation $-\phi_{xx} = \sum_{k=1}^m z_k v_k$, we immediately obtain

$$-\phi_{xxx} = \sum_{k=1}^m z_k v_{kx} \quad \text{and} \quad -\phi_{xxt} = \sum_{k=1}^m z_k v_{kt}.$$

Consequently, applying the previously established uniform bounds yields

$$\sup_{0\le t\le T}\int_I \phi_{xxx}^2 \,\mathrm{d}x \le C \sup_{0\le t\le T}\int_I \sum_{k=1}^m v_{kx}^2 \,\mathrm{d}x \le C(T),$$

$$\int_0^T \int_I \phi_{xxt}^2 \,\mathrm{d}x\,\mathrm{d}t \le C \int_0^T \int_I \sum_{k=1}^m v_{kt}^2 \,\mathrm{d}x\,\mathrm{d}t \le C(T).$$

□

**Lemma 3.5.** *Under the hypotheses of Theorem 1.1, for any $T > 0$ and $t \in [0, T]$, we have the following estimate:*

$$\sup_{0\le t\le T}\int_I u_x^2 \,\mathrm{d}x + \int_0^T \int_I \rho u_t^2 \,\mathrm{d}x\,\mathrm{d}t \le C(T).$$

*Proof.* It follows from $(1.3)_{1,2}$ that

$$\rho u_t + \rho u u_x + (\rho^\gamma)_x = u_{xx} - \sum_{k=1}^m v_{kx} + (\frac{1}{2}\phi_x^2)_x. \tag{3.21}$$

Multiplying (3.21) by $u_t$ and integrating by parts over $I$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I u_x^2 \,\mathrm{d}x + \int_I \rho u_t^2 \,\mathrm{d}x \\
&= -\int_I \rho u\, u_x u_t \,\mathrm{d}x - \int_I (\rho^\gamma)_x u_t \,\mathrm{d}x - \int_I \sum_{k=1}^m v_{kx} u_t \,\mathrm{d}x + \frac{1}{2}\int_I (\phi^2)_x u_t \,\mathrm{d}x \\
&= \sum_{i=1}^4 I_i.
\end{aligned} \tag{3.22}$$

We now estimate each term on the right-hand side of (3.22).
**Estimate of $I_1$.** Using Cauchy's inequality and the Sobolev embedding, we have

$$I_1 \le \int_I |\rho u\, u_x u_t| \,\mathrm{d}x \le \frac{1}{4}\int_I \rho u_t^2 \,\mathrm{d}x + \|\rho\|_\infty \|u\|_\infty^2 \int u_x^2 \,\mathrm{d}x \le \frac{1}{4}\int_I \rho u_t^2 \,\mathrm{d}x + \|u_x\|_2^4. \tag{3.23}$$

**Estimate of $I_2$.** Integrating by parts and applying Eq $(1.3)_{1,2}$, we obtain

$$
\begin{aligned}
I_2 &= \int \rho^\gamma u_{xt}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int \rho^\gamma u_x\,\mathrm{d}x - \int \gamma\rho^{\gamma-1}\rho_t u_x\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \rho^\gamma u_x\,\mathrm{d}x + \int \gamma\rho^{\gamma-1}(\rho u)_x u_x\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \rho^\gamma u_x\,\mathrm{d}x + \int \gamma\rho^\gamma u_x^2 + \int (\rho^\gamma)_x u\,u_x\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \rho^\gamma u_x\,\mathrm{d}x + (\gamma-1)\int \gamma\rho^\gamma u_x^2\,\mathrm{d}x - \int \rho^\gamma u\,u_{xx}\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \rho^\gamma u_x\,\mathrm{d}x + (\gamma-1)\int \gamma\rho^\gamma u_x^2\,\mathrm{d}x \\
&\quad - \int \rho^\gamma u\Big[(\rho u)_t + (\rho u^2)_x + (\rho^\gamma)_x + \sum_{k=1}^m v_{kx} - \phi_x\phi_{xx}\Big]\,\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int \rho^\gamma u_x\,\mathrm{d}x + C\|u_x\|^2 + I_2^3,
\end{aligned}
\tag{3.24}
$$

where $I_2^3$ denotes the last integral term. For $I_2^3$, we have

$$
\begin{aligned}
I_2^3 &= -\int \rho^\gamma u\Big[\rho u_t + \rho u u_x + (\rho^\gamma)_x + \sum_{k=1}^m v_{kx} - \phi_x\phi_{xx}\Big]\,\mathrm{d}x \\
&\le \|\rho^\gamma\|_\infty \int \Big[u\rho u_t + \rho u^2 u_x + (\rho^\gamma)_x u + \sum_{k=1}^m v_{kx} u + \phi_x\phi_{xx} u\Big]\,\mathrm{d}x \\
&\lesssim \frac{1}{4}\|\sqrt{\rho}\,u_t\|_2^2 + \|\sqrt{\rho}\,u\|_2^2 + \|\sqrt{\rho}\,u\|_2^2 + \|\rho\|_\infty\|u\|_\infty^2\|u_x\|_2^2 \\
&\quad - \int \rho^\gamma u_x\,\mathrm{d}x + \int_I \sum_{k=1}^m v_{kx}^2\,\mathrm{d}x + \|u_x\|_2^2 + \|u\|_\infty\|\phi_x\|_2\|\phi_{xx}\|_2 \\
&\lesssim \frac{1}{4}\|\sqrt{\rho}\,u_t\|_2^2 + \|\sqrt{\rho}\,u\|_2^2 + \|u_x\|_2^4 + \|\rho^\gamma\|_\infty\,\|\rho^\gamma\|_1 + \|u_x\|_2^2 \\
&\quad + \sum_{k=1}^m \|v_{kx}\|_2^2 + \|\phi_{xx}\|_2^2 + \|\phi_x\|_2^2\|u_x\|_2^2 \\
&\le \frac{1}{4}\|\sqrt{\rho}\,u_t\|_2^2 + C(1 + \|u_x\|_2^2)^2,
\end{aligned}
\tag{3.25}
$$

where we have used Lemmas 3.1–3.4, the Poincaré inequality, and Hölder's inequality.

Substituting (3.25) into (3.24) yields

$$
I_2 \le \frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho^\gamma u_x\,\mathrm{d}x + \frac{1}{4}\|\sqrt{\rho}\,u_t\|_2^2 + C(1 + \|u_x\|_2^2)^2. \tag{3.26}
$$

**Estimate of $I_3 + I_4$.** Integrating by parts and using Lemma 3.1 together with the Sobolev embedding, we obtain

$$
\begin{aligned}
I_3 + I_4 &= \int_I \left( \sum_{k=1}^{m} v_k - \frac{1}{2}\phi_x^2 \right) u_{xt}\, \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_I \left( \sum_{k=1}^{m} v_k - \frac{1}{2}\phi_x^2 \right) u_x\, \mathrm{d}x - \int_I \left( \sum_{k=1}^{m} v_{kt} - \phi_x \phi_{xt} \right) u_x\, \mathrm{d}x \\
&\leq \frac{\mathrm{d}}{\mathrm{d}t} \int_I \left( \sum_{k=1}^{m} v_k - \frac{1}{2}\phi_x^2 \right) u_x\, \mathrm{d}x + \big( \sum_{k=1}^{m} \|v_{kt}\|_2 + \|\phi_{xxt}\|_2 \|\phi_x\|_2 \big) \|u_x\|_2 \\
&\leq \frac{\mathrm{d}}{\mathrm{d}t} \int_I \left( \sum_{k=1}^{m} v_k - \frac{1}{2}\phi_x^2 \right) u_x\, \mathrm{d}x + \sum_{k=1}^{m} \|v_{kt}\|_2^2 + \|\phi_{xxt}\|_2^2 + C\|u_x\|_2^2.
\end{aligned}
\tag{3.27}
$$

Substituting (3.23), (3.26), and (3.27) into (3.22), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_I u_x^2\, \mathrm{d}x + \int_I \rho u_t^2\, \mathrm{d}x \\
\leq & C(1 + \|u_x\|^2)^2 + \frac{\mathrm{d}}{\mathrm{d}t} \int_I \rho^\gamma u_x + \frac{\mathrm{d}}{\mathrm{d}t} \int_I \left( \sum_{k=1}^{m} v_k - \frac{1}{2}\phi_x^2 \right) u_x\, \mathrm{d}x + \sum_{k=1}^{m} \|v_{kt}\|_2^2 + \|\phi_{xxt}\|^2.
\end{aligned}
$$

Integrating the above inequality over $(0, T)$ gives

$$
\begin{aligned}
&\int_I u_x^2\, \mathrm{d}x + \int_0^t \int_I \rho u_t^2\, \mathrm{d}x\, \mathrm{d}t \\
\leq & C(\rho_0, u_{0x}, v_{k0}, \phi_{0x}) + C \int_0^t (1 + \|u_x\|_2^2)^2\, \mathrm{d}t + \int_I \rho^\gamma u_x\, \mathrm{d}x \\
&+ \int_I \left( \sum_{k=1}^{m} v_k - \frac{1}{2}\phi_x^2 \right) u_x\, \mathrm{d}x + \int_0^t \sum_{k=1}^{m} \|v_{kt}\|_2^2 + \|\phi_{xxt}\|_2^2\, \mathrm{d}t \\
\leq & C(T) + \int_0^t \|u_x\|_2^4\, \mathrm{d}t + \|\rho^\gamma\|_2 \|u_x\|_2 + \left( \sum_{k=1}^{m} \|v_k\|_2 + \|\phi_x\|_\infty \|\phi_x\|_2 \right) \|u_x\|_2 \\
\leq & C(T) + \int_0^t \|u_x\|_2^4\, \mathrm{d}t + \frac{1}{4}\|u_x\|_2^2 + \sum_{k=1}^{m} \|v_k\|_2^2 + \|\phi_x\|_2^2 \|\phi_{xx}\|_2^2 \\
\leq & C(T) + \int_0^t \|u_x\|_2^4\, \mathrm{d}t + \frac{1}{4}\|u_x\|_2^2,
\end{aligned}
\tag{3.28}
$$

where we have again used Lemmas 3.1–3.4. Combining this with the already established bound $\int_0^t \int_I u_x^2\, \mathrm{d}x\, \mathrm{d}t \leq C(T)$ (in Lemma 3.1) and applying Gronwall's inequality yields the desired estimate. This completes the proof. □

**Lemma 3.6.** *Under the hypotheses of Theorem 1.1, for any given $T > 0$ and $t \in [0, T]$, we have the following estimate:*

$$
\sup_{0 \leq t \leq T} \int_I (\rho_x^2 + \rho_t^2)\, \mathrm{d}x + \int_0^T (\|u_x\|_{L^\infty}^2 + \|u_{xx}\|_{L^2}^2)\, \mathrm{d}t \leq C(T).
$$

*Proof.* Applying (3.21), we obtain the following estimate for $\|u_x\|_\infty$ :

$$
\begin{aligned}
\|u_x\|_\infty^2 &\le \Big\|u_x - \rho^\gamma - \sum_{k=1}^m v_k + \frac{1}{2}\phi_x^2\Big\|_\infty^2 + \Big\|\rho^\gamma + \sum_{k=1}^m v_k - \frac{1}{2}\phi_x^2\Big\|_\infty^2 \\
&\lesssim \Big\|u_x - \rho^\gamma - \sum_{k=1}^m v_k + \frac{1}{2}\phi_x^2\Big\|_2^2 + \Big\|u_x - (\rho^\gamma)_x - \sum_{k=1}^m v_{kx} + \frac{1}{2}(\phi_x^2)_x\Big\|_2^2 \\
&\quad + \|\rho^\gamma\|_\infty^2 + \sum_{k=1}^m \|v_k\|_{H^1}^2 + \|\phi_{xx}\|_2^4 \\
&\lesssim \|u_x\|_2^2 + \|\rho^\gamma\|_\infty \|\rho\|_1^2 + \sum_{k=1}^m \|v_k\|_2^2 + \|\phi_x\|_\infty^2 \|\phi_x\|_2^2 + \|\rho u_t + \rho u u_x\|_2^2 + C(T) \\
&\le C(T) + \|\rho\|_\infty \|\sqrt{\rho} u_t\|_2^2 + \|\rho\|_\infty^2 \|u\|_\infty^2 \|u_x\|_2^2.
\end{aligned}
$$

Integrating the above inequality over $(0, T)$ yields

$$
\int_0^T \|u_x\|_\infty^2 \,\mathrm{d}t \le C(T),
$$

where we have used the previously established bounds for $u, \rho, v_k, \phi_x$.

Next, differentiating equation $\rho_t + (\rho u)_x = 0$ in $(1.3)_1$ with respect to $x$, multiplying the resulting equation by $\rho_x$, and integrating by parts over $I$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho_x^2 \,\mathrm{d}x &= -\frac{3}{2}\int_I \rho_x^2 u_x \,\mathrm{d}x - \int_I \rho\rho_x u_{xx} \,\mathrm{d}x \\
&\lesssim \|u_x\|_\infty \|\rho_x\|_2^2 + \int_I \rho\rho_x \Big[\rho u_t + \rho u u_x + (\rho^\gamma)_x + \sum_{k=1}^m v_{kx} - (\tfrac{1}{2}\phi_x^2)_x\Big] \mathrm{d}x \\
&\lesssim \|u_x\|_\infty \|\rho_x\|_2^2 + \|\rho\|_\infty^{\frac{3}{2}} \|\rho_x\|_2 \|\sqrt{\rho} u_t\|_2 + \|\rho\|_\infty^2 \|u\|_\infty \|\rho_x\|_2 \|u_x\|_2 \\
&\quad + \|\rho\|_\infty^\gamma \|\rho_x\|_2^2 + \|\rho\|_\infty \|\rho_x\|_2 \sum_{k=1}^m \|v_{kx}\|_2 + \|\rho\|_\infty \|\rho_x\|_2 \|\phi_x\|_\infty \|\phi_{xx}\|_2 \\
&\le C(T)(1 + \|u_x\|_\infty)\|\rho_x\|_2^2 + C(T),
\end{aligned}
$$

where we have used (3.21) to substitute $u_{xx}$ in the second step. Applying Gronwall's inequality then gives

$$
\sup_{0\le t\le T} \|\rho_x\|_2^2 \le C(T).
$$

Furthermore, applying Lemmas 3.1 and 3.5 together with $(1.3)_1$ yields

$$
\sup_{0\le t\le T} \|\rho_t\|_2^2 \le C(T).
$$

Finally, from (3.21) we derive the $L^2(0, T; H^2)$ bound for $u$:

$$
\int_0^T \|u_{xx}\|_2^2 \,\mathrm{d}t \le C(T) \int_0^T \Big[\|\sqrt{\rho}\, u_t\|_2^2 + \|u_x\|_2^4 + \|\rho_x\|_2^2 + \sum_{k=1}^m \|v_{kx}\|_2^2 + \|\phi_x\|_\infty^2 \|\phi_{xx}\|_2^2\Big] \mathrm{d}t
$$

$$\leq C(T).$$

where all terms on the right-hand side are bounded by the estimates previously established. This completes the proof.

□

**Lemma 3.7.** *Under the conditions of Theorem 1.1, for any given $T > 0$ and $t \in [0, T]$, it holds that*

$$\sup_{0\leq t\leq T} \int_I \sum_{k=1}^{m} v_{kxx}^2 \,\mathrm{d}x + \int_0^T \int_I \sum_{k=1}^{m} v_{kxt}^2 \,\mathrm{d}x\,\mathrm{d}t \leq C(T).$$

*Proof.* Differentiating $(1.3)_3$ with respect to $x$, multiplying the resulting equation by $v_{kxt}$ and integrating over $I$, we obtain

$$\begin{aligned}
&\frac{D_k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I v_{kxx}^2 \,\mathrm{d}x + \int_I v_{kxt}^2 \,\mathrm{d}x \\
=&\int_I \big( - v_{kxx}u - 2v_{kx}u_x - v_k u_{xx} + D_k z_k (v_{kxx}\phi_x + 2v_{kx}\phi_{xx} + v_k\phi_{xxx})\big) v_{kxt} \,\mathrm{d}x \\
\leq&\frac{1}{2}\int_I v_{kxt}^2 \,\mathrm{d}x + C(\|u\|_\infty^2 + \|\phi_x\|_\infty^2)\|v_{kxx}\|_2^2 + C(\|u_x\|_\infty^2 + \|\phi_{xx}\|_\infty^2)\|v_{kx}\|_2^2 \\
&+ C\|v_k\|_\infty^2(\|u_{xx}\|_2^2 + \|\phi_{xxx}\|_2^2) \\
\leq&\frac{1}{2}\int_I v_{kxt}^2 \,\mathrm{d}x + C(\|u_x\|_{H^1}^2 + \|\phi_{xx}\|_{H^1}^2)(\|v_{kxx}\|_2^2 + \|v_{kx}\|_2^2) + C\|v_k\|_{H^1}^2(\|u_{xx}\|_2^2 + \|\phi_{xxx}\|_2^2).
\end{aligned}$$

where we have used the Sobolev embeddings $\|(u, u_x)\|_\infty \leq C\,\|(u, u_x)\|_{H^1}$ and $\|(\phi_x, \phi_{xx})\|_\infty \leq C\,\|(\phi_x, \phi_{xx})\|_{H^1}$, as well as $\|v_k\|_\infty \leq C\|v_k\|_{H^1}$.

Consequently, absorbing the first term on the right-hand side into the left-hand side yields

$$\frac{D_k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I v_{kxx}^2 \,\mathrm{d}x + \int_I v_{kxt}^2 \,\mathrm{d}x \leq C(\|u_x\|_{H^1}^2 + \|\phi_x\|_{H^1}^2)\|v_{kxx}\|_2^2 + C(T)(\|u_x\|_{H^1}^2 + \|\phi_{xx}\|_{H^1}^2).$$

where we have used the previously established bounds for $v_k$.

Applying Gronwall's inequality to the above estimate for each $k = 1, 2, \cdots, m$, we obtain

$$\int_I v_{kxx}^2 \,\mathrm{d}x + \int_0^T \int_I v_{kxt}^2 \,\mathrm{d}x\,\mathrm{d}t \leq C(T),$$

where the constant $C(T) > 0$ is independent of $k$. Summing these inequalities over $k = 1, 2, \cdots, m$ then gives the desired uniform bound

$$\sup_{0\leq t\leq T} \sum_{k=1}^{m} \|v_{kxx}\|_2^2 + \int_0^T \sum_{k=1}^{m} \|v_{kxt}\|_2^2 \,\mathrm{d}t \leq C(T).$$

This completes the proof. □

**Lemma 3.8.** *Under the conditions of Theorem 1.1, for any given $T > 0$ and $t \in [0, T]$, it holds that*

$$\int_0^T t \int_I u_{xt}^2 \,\mathrm{d}x\,\mathrm{d}t \leq C(T).$$

*Proof.* Differentiating $(1.3)_2$ with respect to $t$, multiplying the resulting equation by $u_t$, integrating by parts over $I$, and using Eq $(1.3)_1$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho u_t^2 dx + \int_I u_{xt}^2\,\mathrm{d}x \\
\leq & C\sum_{k=1}^m \|v_{kt}\|_2^2 + \|\phi_{xt}\|_\infty^2\|\phi_x\|_2^2 + C(1+\|u_x\|_\infty + \|\rho\|_\infty\|u\|_\infty^2)\|\sqrt{\rho}u_t\|_2^2 \\
&+ C\Big(\gamma^2\|\rho\|_\infty^{2\gamma} + \|\rho\|_\infty\|u\|_\infty^2\|u_x\|_\infty^2 + \|\rho\|_\infty^2\|u\|_\infty^4\Big)\|u_x\|_2^2 + C\|\rho\|_\infty\|u\|_\infty^4\|u_{xx}\|_2^2 \\
&+ C\Big(\|u\|_\infty^4 + \|u\|_\infty^6 + \gamma^2\|\rho\|_\infty^{2\gamma-2}\|u\|_\infty^2\Big)\|\rho_x\|_2^2.
\end{aligned}$$

Applying Lemmas 3.1, 3.3, 3.5 and 3.6, and the Sobolev embedding, together with the boundary condition $\phi_x|_{\partial I} = 0$, we obtain the simplified estimate

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho u_t^2\,\mathrm{d}x + \int_I u_{xt}^2\,\mathrm{d}x \\
\leq & C(T)\left[\sum_{k=1}^m \|v_{kt}\|_2^2 + \|\phi_{xxt}\|_2^2 + (1+\|u_x\|_\infty)\|\sqrt{\rho}u_t\|_2^2 + \|u_{xx}\|_2^2 + 1\right].
\end{aligned}$$

Multiplying the above inequality by $t$ gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(t\int_I \rho u_t^2\,\mathrm{d}x\Big) + t\int_I u_{xt}^2\,\mathrm{d}x \\
\leq & \int_I \rho u_t^2\,\mathrm{d}x + C(T)t\left[\sum_{k=1}^m \|v_{kt}\|_2^2 + \|\phi_{xxt}\|_2^2 + C(1+\|u_x\|_\infty)\|\sqrt{\rho}u_t\|_2^2 + \|u_{xx}\|_2^2 + 1\right].
\end{aligned} \tag{3.29}$$

From Lemma 3.5, we have

$$\lim_{t_l\to 0} t_l\int_I \rho u_t^2(t_l)\,\mathrm{d}x \to 0. \tag{3.30}$$

Integrating (3) over $(t_l, t)$, and using (3.30) together with Lemmas 3.1, 3.4–3.6, and Gronwall's inequality, we complete the proof. □

## 4. Proof of main results

*Proof of Theorem 1.1.* The proof of the main theorem follows by combining the local existence result (Lemma 2.1), the Aubin–Lions compactness lemma (Lemma 2.2), and the a priori estimates established in Section 3.

To prove our result which allows for vacuum, we construct an approximate solution sequence for system (1.3) by introducing a positive lower bound to the initial density. More precisely, for any large $i > 0$ and each $k = 1, 2, \cdots, m$, we define

$$\rho_0^i = \eta_i * \rho_0 + \frac{1}{i}, \quad u_0^i = \eta_i * u_0, \quad v_{0k}^i = \eta_{ik} * v_{0k},$$

where $\eta_i$ denotes the standard mollifier used to improve the regularity of the initial data. Then $\rho_0^i \geq i^{-1}$ and

$$\lim_{i\to+\infty}\left\{\|\rho_0^i-\rho_0\|_{H^1}+\|u_0^i-u_0\|_{H^1}+\sum_{k=1}^m\|v_{k0}^i-v_{k0}\|_{H^2}\right\}=0.$$

Thus, for some small $T_0 > 0$, there exists a unique local classical solution $(\rho, u, v_k)$ to (1.3)–(1.5). From the a priori estimates in Section 3, we have

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\|\rho^i\|_{H^1}+\|\rho_t^i\|_2+\|u^i\|_{H^1}+\|(v_k^i,\phi_x^i)\|_{H^2}+\|v_{kt}^i\|_2\right)\\&\quad+\int_0^T\left(t\|u_{xt}^i\|_2^2+\|(\rho^iu^i)_t\|_2^2+\|u^i\|_{H^2}^2+\|v_{kxx}^i\|_2^2+\|v_{kt}^i\|_{H^1}^2+\|\phi_{xxt}\|_2^2\right)\mathrm{d}t\leq C(T),\end{aligned}$$

where the constant $C(T) > 0$ is independent of $i$.

Passing to the limit $i \to +\infty$ (taking the subsequence if necessary), we may assume the following weak and weak$^*$ convergences:

$$\begin{aligned}(\rho^i,\rho_x^i,\rho_t^i)&\rightharpoonup(\rho,\rho_x,\rho_t)\text{ weak}^*\text{ in }L^\infty(0,T;L^2(I)),\\(u^i,u_x^i)&\rightharpoonup(u,u_x)\text{ weak}^*\text{ in }L^\infty(0,T;L^2(I)),\\(u_{xx}^i,\ \sqrt{t}\,u_{xt}^i)&\rightharpoonup(u_{xx},\ \sqrt{t}\,u_{xt})\text{ weak in }L^2(0,T;L^2(I)),\\(\rho^iu^i)_t&\rightharpoonup M\text{ weakly in }L^2(0,T;L^2(I)),\\(v_k^i,v_{kx}^i,v_{kxx}^i,v_{kt}^i)&\rightharpoonup(v_k,v_{kx},v_{kxx},v_{kt})\text{ weakly in }L^\infty(0,T;L^2(I)),\\v_{kxt}^i&\rightharpoonup v_{kxt}\text{ weakly in }L^2(0,T;L^2(I)),\\(\phi_x^i,\phi_{xx}^i)&\rightharpoonup(\phi_x,\phi_{xx})\text{ weak}^*\text{ in }L^\infty(0,T;L^2(I)),\\\phi_{xxt}^i&\rightharpoonup\phi_{xxt}\text{ weakly in }L^2(0,T;L^2(I)).\end{aligned}$$

Since $\rho^i$ is bounded in $L^\infty(0,T;H^1(I))$ and $\rho_t^i$ bounded in $L^\infty(0,T;L^2(I))$, the Simon-Aubin-Lions Lemma 2.2(ii) implies that as $i \to +\infty$,

$$\rho^i\to\rho\text{ strongly in }C(Q_T).$$

Consequently, since $\rho^i \geq 0$ is uniformly bounded in $L^\infty(Q_T)$ and the map $s \mapsto s^\gamma$ is continuous for $\gamma > 1$, from Lebesgue's dominated convergence theorem, we obtain

$$(\rho^i)^\gamma\to\rho^\gamma\quad\text{strongly in }C(\overline{Q_T}),$$

which justifies the passage to the limit in the pressure term:

$$[(\rho^i)^\gamma]_x=\gamma(\rho^i)^{\gamma-1}\rho_x^i\rightharpoonup[\rho^\gamma]_x\text{ weakly in }L^2(0,T;L^2).$$

Moreover, we have

$$\rho^iu^i\rightharpoonup\rho u\text{ weak}^*\text{ in }L^\infty(0,T;L^2(I)),$$

thus $M = (\rho u)_t$. Then Lemma 2.2(ii), yields

$$\rho^iu^i\to\rho u\text{ strongly in }C(Q_T),$$

due to $\rho u$ being bounded in $L^\infty(0,T;H^1(I))$ and $(\rho^i u^i)_t$ being bounded in $L^2(0,T;L^2(I))$. Combining the above convergences, we obtain

$$\rho^i(u^i)^2 \rightharpoonup \rho u^2 \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2(I)).$$

Furthermore, since $(\rho^i(u^i)^2)_x$ is bounded in $L^\infty(0,T;L^2(I))$, we have

$$(\rho^i(u^i)^2)_x \rightharpoonup (\rho u^2)_x \text{ weak}^* \text{ in } L^\infty(0,T;L^2(I)).$$

Since $(v_k^i, v_{kx}^i)$ is bounded in $L^\infty(0,T;L^2(I))$ and $v_{kt}^i, v_{kxt}^i$ are bounded in $L^\infty(0,T;L^2(I))$ and $L^2(0,T;L^2(I))$, respectively, we can derive from Lemma 2.2(ii) that

$$(v_k^i, (v^i)_{kx}) \to (v_k, v_{kx}) \text{ strongly in } C(Q_T). \tag{4.1}$$

Then, from $(1.3)_4$, we can derive

$$\phi_{xx}^i \to \phi_{xx} \text{ strongly in } C(Q_T).$$

Then, by the boundary condition $\phi_x^i(0) = 0$, integration gives

$$\phi_x^i \to \phi_x \text{ strongly in } C(Q_T),$$

and together with (4.1) we obtain

$$v_k^i \phi_x^i \to v_k \phi_x \text{ strongly in } C(Q_T).$$

By (4.1), we can also derive that

$$(v_k^i u^i)_x \rightharpoonup (v_k u)_x \text{ weakly in } L^2(0,T;L^2(I)).$$

It should be noted that we do not need the strong convergence of $v_k^i u^i$; it is sufficient to establish the convergence of the product in the weak formulation. Indeed, for any test function $\psi \in C_c^\infty(Q_T)$, we have

$$\int_{Q_T} (v_k^i u^i - v_k u)\psi \, dxdt = \int_{Q_T} (v_k^i - v_k)u^i \psi \, dxdt + \int_{Q_T} v_k(u^i - u)\psi \, dxdt.$$

The first integral tends to 0 because $v_k^i \to v_k$ uniformly and $u^i$ is bounded in $L^2(0,T;L^2)$. The second integral tends to 0 because $u^i \rightharpoonup u$ weakly in $L^2(0,T;L^2)$ and $v_k\psi \in L^2(0,T;L^2)$. Hence

$$v_k^i u^i \;\rightharpoonup\; v_k u \quad \text{weakly in } L^2(0,T;L^2).$$

This weak convergence is precisely what is required to pass to the limit in the weak form of the momentum equation. Moreover, the limit functions satisfy $v_k \in C([0,T];H^2)$ and $u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2)$, which ensures that $v_k u$ is well defined and the resulting distributional identity is equivalent to the almost-everywhere equation. Hence the convergence is sufficient to obtain the strong solution in the sense of Theorem 1.1

Based on these convergences, we conclude that $(\rho, u, v_k, \phi_x)$ is a strong solution to the system (1.3)–(1.5).

To establish uniqueness, we need the estimate $\sqrt{t}u_t \in L^2(0,T;H_0^1)$ obtained in Lemma 3.8. Let us denote $\widetilde{\rho} = \rho_1 - \rho_2$, $\tilde{u} = u_1 - u_2$, $\tilde{v_k} = v_k^1 - v_k^2$, $\tilde{w} = w_1 - w_2$, $\tilde{\phi}_x = \phi_{1x} - \phi_{2x}$, where $(\rho_j, u_j, v_k^j, \phi_{jx})$ $(j = 1,2)$ are two strong solutions to (1.3)–(1.5). Then the difference system reads

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}u_1)_x + (\rho_2\tilde{u})_x = 0, \\ \rho_1\tilde{u}_t - \tilde{u}_{xx} = -\tilde{\rho}u_{2t} - \tilde{\rho}u_2u_{2x} - \rho_1\tilde{u}u_{2x} - \rho_1u_1\tilde{u}_x \\ \qquad\qquad - (\rho_1^\gamma - \rho_2^\gamma)_x - \displaystyle\sum_{k=1}^m \tilde{v}_{kx} + \tilde{\phi}_x\phi_{1xx} + \phi_{2x}\tilde{\phi}_{xx}, \\ \tilde{v}_{kt} - D_k\tilde{v}_{kxx} = -\tilde{v}_{kx}(u_1 + D_kz_k\phi_{1x}) - v_{kx}^2(\tilde{u} + D_kz_k\tilde{\phi}_x) \\ \qquad\qquad - \tilde{v}_k(u_{1x} + D_kz_k\phi_{1xx}) - v_k^2(\tilde{u}_x + D_kz_k\tilde{\phi}_{xx}), \\ -\tilde{\phi}_{xx} = \sum_{k=1}^m z_k\tilde{v}_k, \end{cases} \tag{4.2}$$

for $(x,t) \in (0,1) \times (0,+\infty)$, with initial and boundary conditions

$$(\widetilde{\rho}, \widetilde{u}, \tilde{v_k})|_{t=0} = 0 \text{ in } [0,1], \;\; (\widetilde{u}, \widetilde{v}_{kx}, \tilde{\phi}_x)\big|_{\partial I} = 0, \;\; t > 0.$$

**Estimate for $\tilde{\rho}$:** Since the first equation is identical to that in Chen's work [6], we directly cite the result here for brevity:

$$\|\tilde{\rho}\|_2^2 \le Ct\int_0^t \|\tilde{u}_x\|_2^2 \mathrm{d}s, \quad t \in [0,T]. \tag{4.3}$$

**Estimate for $\tilde{u}$:** Multiplying $(4.2)_2$ by $\tilde{u}$ and integrating by parts over $I$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho_1|\tilde{u}|^2\,\mathrm{d}x + \int_I |\tilde{u}_x|^2\,\mathrm{d}x \\ =&\frac{1}{2}\int_I \rho_{1t}|\tilde{u}|^2\,\mathrm{d}x - \int_I \tilde{\rho}u_{2t}\tilde{u}\,\mathrm{d}x - \int_I \tilde{\rho}u_2u_{2x}\tilde{u}\,\mathrm{d}x - \int_I \rho_1|\tilde{u}^2u_{2x} - \int_I \rho_1u_1\tilde{u}_x\tilde{u}\,\mathrm{d}x \\ &+ \int_I(\rho_1^\gamma - \rho_2^\gamma)\tilde{u}_x - \int_I\sum_{k=1}^m \tilde{v}_{kx}\tilde{u} - \int_I \tilde{\phi}_x\phi_{1xx}\tilde{u} - \int_I \phi_{2x}\tilde{\phi}_{xx}\tilde{u}, \end{aligned}$$

where we have used integration by parts on the pressure term. Noting that

$$\frac{1}{2}\int_I \rho_{1t}|\tilde{u}|^2 - \int_I \rho_1u_1\tilde{u}_x\tilde{u} = \frac{1}{2}\int_I \rho_{1t}|\tilde{u}|^2 + \frac{1}{2}\int_I (\rho_1u_1)_x|\tilde{u}|^2 = 0,$$

we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho_1|\tilde{u}|^2\,\mathrm{d}x + \int_I |\tilde{u}_x|^2\,\mathrm{d}x \\ \le&\|\tilde{u}\|_\infty\|\tilde{\rho}\|_2\|u_{2t}\|_2 + \|\tilde{\rho}\|_2\|\tilde{u}\|_\infty\|u_2\|_\infty\|u_{2x}\|_2 + \|u_{2x}\|_\infty\int_I \rho_1|\tilde{u}|^2\,\mathrm{d}x + C\|\tilde{\rho}\|_2\|\tilde{u}_x\|_2 \\ &+ \sum_{k=1}^m \|\tilde{v}_k\|_2\|\tilde{u}_x\|_2 + \|1\|_2\|\tilde{\phi}_x\|_\infty\|\phi_{1xx}\|_2\|\tilde{u}\|_\infty + \|1\|_{L^2}\|\phi_{2x}\|_\infty\|\tilde{\phi}_{xx}\|_2\|\tilde{u}\|_\infty \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\|\tilde{u}_x\|_2^2 + C\|u_{2t}\|_2^2\|\tilde{\rho}\|_2^2 + C(\|u_{2x}\|_2^4 + 1)\|\tilde{\rho}\|_2^2 + \|u_{2x}\|_{H^1} \int_I \rho_1|\tilde{u}|^2 \, \mathrm{d}x \\
&\quad + C\sum_{k=1}^{m} \|\tilde{v}_k\|_2^2 + C(\|\phi_{1x}\|_2^2 + \|\phi_{2x}\|_2^2)\|\tilde{\phi}_{xx}\|_2^2.
\end{aligned}
$$

Using the regularity of $\rho_j, u_j$, and $\phi_j$ and the estimate (4.3), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_I \rho_1|\tilde{u}|^2 \, \mathrm{d}x + \int_0^t \int_I |\tilde{u}_x|^2\right) \mathrm{d}x \\
\leq & Ct\|u_{2t}\|_2^2 \int_0^t \int_I |\tilde{u}_x|^2 \, \mathrm{d}x \, \mathrm{d}s + Ct \int_0^t \int_I |\tilde{u}_x|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&+ C\|u_2\|_{H^1} \int_I \rho_1|\tilde{u}|^2 \, \mathrm{d}x + C(\sum_{k=1}^{m} \|\tilde{v}_k\|_2^2 + \|\tilde{\phi}_{xx}\|_{L^2}^2).
\end{aligned} \tag{4.5}
$$

**Estimate for $\tilde{v}_k$:** Multiplying $(4.2)_3$ by $\tilde{v}_k(k = 1, 2, \cdots, m)$, integrating over $I$, and integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I |\tilde{v}_k|^2 \, \mathrm{d}x + \int_I |\tilde{v}_{kx}|^2 \, \mathrm{d}x \\
\leq & \int_I -\Big[\tilde{v}_{kx}(u_1 + \phi_{1x})\tilde{v}_k + v_{kx}^2(\tilde{u} + \tilde{\phi}_x)\tilde{v}_k + (u_{1x} + \phi_{1xx})\tilde{v}_k^2 + v_k^2(\tilde{u}_x + \tilde{\phi}_{xx})\tilde{v}_k\Big] \, \mathrm{d}x \\
\leq & \|\tilde{v}_{kx}\|_2\|u_1 + \phi_{1x}\|_\infty\|\tilde{v}_k\|_2 + \|v_{kx}^2\|_2\|\tilde{u} + \tilde{\phi}_x\|_\infty\|\tilde{v}_k\|_2 \\
&+ \|u_{1x} + \phi_{1xx}\|_\infty\|\tilde{v}_k\|_2^2 + \|v_k^2\|_\infty\|\tilde{u}_x + \tilde{\phi}_{xx}\|_2\|\tilde{v}_k\|_2 \\
\leq & \frac{1}{2}\|\tilde{v}_{kx}\|_2^2 + \epsilon\|\tilde{u}_x\|_2^2 + \|\tilde{\phi}_{xx}\|_2^2 + C_\epsilon\|\tilde{v}\|_2^2.
\end{aligned} \tag{4.6}
$$

From $(4.2)_4$, we have

$$
\|\tilde{\phi}_{xx}\|_2^2 \leq C\sum_{k=1}^{m} \|\tilde{v}_k\|_2^2. \tag{4.7}
$$

Substituting (4.7) into (4.6) and summing over $k = 1, 2, \cdots, m$, we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \sum_{k=1}^{m} |\tilde{v}_k|^2 \, \mathrm{d}x + \int_I \sum_{k=1}^{m} |\tilde{v}_{kx}|^2 \, \mathrm{d}x \leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \int_I \tilde{u}_x^2 \, \mathrm{d}x \, \mathrm{d}s + C\sum_{k=1}^{m} \|\tilde{v}_k\|_2^2. \tag{4.8}
$$

Combining (4.5), (4.7), and (4.8), we deduce that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_I \rho_1|\tilde{u}|^2 \, \mathrm{d}x + \int_I \sum_{k=1}^{m} |\tilde{v}_k|^2 \, \mathrm{d}x + \frac{1}{2}\int_0^t \int_I |\tilde{u}_x|^2\right) \mathrm{d}x \, \mathrm{d}s + C\int_I \sum_{k=1}^{m} |\tilde{v}_{kx}|^2 \, \mathrm{d}x \\
&\leq Ct\|u_{2t}\|_2^2 \int_0^t \int_I |\tilde{u}_x|^2 \, \mathrm{d}x \, \mathrm{d}s + Ct\int_0^t \int_I |\tilde{u}_x|^2 \, \mathrm{d}x \, \mathrm{d}s + C\int_I \rho_1|\tilde{u}|^2 \, \mathrm{d}x + C\int_I \sum_{k=1}^{m} |\tilde{v}_k|^2 \, \mathrm{d}x.
\end{aligned}
$$

Since $(\tilde{\rho}, \tilde{u}, \tilde{v}_k)|_{t=0} = 0$, applying Lemma 3.8 and Gronwall's inequality yields

$$
\tilde{\rho} \equiv, \quad \tilde{u} \equiv, \quad \tilde{v}_k) \equiv 0(k = 1, 2, \cdots, m).
$$

Consequently, from

$$-\tilde{\phi}_{xx} = \sum_{k=1}^{m} z_k \tilde{v}_k = 0,$$

and the boundary condition $\tilde{\phi}_x \equiv 0$ we deduce that $-\tilde{\phi}_x =\equiv 0$ as well.

Thus, the uniqueness of the strong solution is established, and the proof of Theorem 1.1 is complete.

## 5. Conclusions

We have established the global existence and uniqueness of strong solutions for a multi-species PNP-NS system coupled with a self-consistent electrostatic potential on a bounded domain, allowing for vacuum in the initial fluid density. This work generalizes the two-ion results of Chen [6] to arbitrary multi-ion configurations with distinct valences and diffusion coefficients. The results provide a rigorous mathematical framework for multi-species electrochemical systems. Future research will be directed toward exploring the long-time dynamics and extending these results to more complex physical settings

## Use of Generative-AI tools declaration

The author declares that she has not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

## References

1. E. Abdo, M. Ignatova, Long time finite dimensionality in charged fluids, *Nonlinearity*, **34** (2021), 6173. https://doi.org/10.1088/1361-6544/ac13bf
2. D. Bothe, A. Fischer, J. Saal, Global well-posedness and stability of electrokinetic flows, *SIAM J. Math. Anal.*, **46** (2014), 1263–1316. https://doi.org/10.1137/120880926
3. C. Cercignani, R. Illner, M. Pulvirenti, *The mathematical theory of Dilute Gases*, 3 Eds., New York: Springer, 1994. https://doi.org/10.1007/978-1-4419-8524-8

4. H. Choe, H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, *Manuscripta Math.*, **120** (2006), 91–129. https://doi.org/10.1007/s00229-006-0637-y

5. P. Constantin, M. Ignatova, On the Nernst-Planck-Navier-Stokes system, *Arch. Ration. Mech. An.*, **232** (2019), 1379–1428. https://doi.org/10.1007/s00205-018-01345-y

6. S. Chen, Global solution for the compressible Poisson–Nernst–Planck–Navier–Stokes system in 1D, *J. Math. Anal. Appl.*, **554** (2026), 129943. https://doi.org/10.1016/j.jmaa.2025.129943

7. J. Fan, H. Gao, Uniqueness of weak solutions to a model of electro-kinetic fluid, *Commun. Math. Sci.*, **7** (2009), 411–421. https://doi.org/10.4310/cms.2009.v7.n2.a7

8. J. Jerome, R. Sacco, Global weak solutions for an incompressible charged fluid with multi-scale couplings: Initial-boundary-value problem, *Nonlinear Anal.*, **71** (2009), e2487–e2497. https://doi.org/10.1016/j.na.2009.05.047

9. F. Li, Quasineutral limit of the electro-diffusion model arising in electrohydrodynamics, *J. Differ. Equations*, **246** (2009), 3620–3641. https://doi.org/10.1016/j.jde.2009.01.027

10. J. Liu, J. Wang, Global existence for Nernst–Planck–Navier–Stokes system in $\mathbb{R}^N$, *Commun. Math. Sci.*, **18** (2020), 1743–1754. https://doi.org/10.4310/CMS.2020.v18.n6.a9

11. C. Liu, C. Wang, S. Wise, X. Yue, S. Zhou, A positivity-preserving, energy stable and convergent numerical scheme for the Poisson–Nernst–Planck system, *Math. Comput.*, **90** (2021), 2071–2106. https://doi.org/10.1090/mcom/3642

12. C. Liu, C. Wang, S. Wise, X. Yue, S. Zhou, A second order accurate, positivity preserving numerical method for the Poisson-Nernst-Planck system and its convergence analysis, *J. Sci. Comput.*, **97** (2023), 23. https://doi.org/10.1007/s10915-023-02345-9

13. D. Marroquin, D. Wang, Global weak solutions for the compressible Poisson–Nernst–Planck–Navier–Stokes system, *J. Lond. Math. Soc.*, **113** (2026), e70552. https://doi.org/10.1112/jlms.70552

14. Y. Qian, C. Wang, S. Zhou, A positive and energy stable numerical scheme for the Poisson-Nernst-Planck-Cahn-Hilliard equations with steric interactions, *J. Comput. Phys.*, **426** (2021), 109908. https://doi.org/10.1016/j.jcp.2020.109908

15. Y. Qian, C. Wang, S. Zhou, Convergence analysis on a structure-preserving numerical scheme for the Poisson–Nernst–Planck–Cahn–Hilliard system, *CSIAM T. Appl. Math.*, **4** (2023), 345–380. https://doi.org/10.4208/csiam-am.so-2021-0022

16. Y. Qin, C. Wang, A second-order accurate, positivity-preserving numerical scheme for the Poisson–Nernst–Planck–Naiver–Stokes system, *IMA J. Numer. Anal.*, **46** (2026), 1383–1430. https://doi.org/10.1093/imanum/draf027

17. M. Schmuck, Analysis of the Navier–Stokes–Nernst–Planck–Poisson system, *Math. Mod. Meth. Appl. S.*, **19** (2009), 993–1014. https://doi.org/10.1142/S0218202509003693

18. J. Simon, Nonhomogeneous viscous incompressible fluids: Existence of velocity, density, and pressure, *SIAM J. Math. Anal.*, **21** (1990), 1093–1117. https://doi.org/10.1137/0521061

19. L. Tong, Z. Tan, Optimal decay rates of the solution for generalized Poisson–Nernst–Planck–Navier–Stokes equations in $\mathbb{R}^3$, *Z. Angew. Math. Phys.*, **72** (2021), 200. https://doi.org/10.1007/s00033-021-01627-2

20. S. Wang, L. Jiang, C. Liu, Quasi-neutral limit and the boundary layer problem of Planck–Nernst–Poisson–Navier–Stokes equations for electro-hydrodynamics, *J. Differ. Equations*, **267** (2019), 3475–3523. https://doi.org/10.1016/j.jde.2019.04.011

21. Y. Wang, C. Liu, Z. Tan, A generalized Poisson–Nernst–Planck–Navier–Stokes model on the fluid with the crowded charged particles: Derivation and its well-posedness, *SIAM J. Math. Anal.*, **48** (2016), 3191–3235. https://doi.org/10.1137/16M1055104

22. Y. Wang, C. Liu, Z. Tan, Well-posedness on a new hydrodynamic model of the fluid with the dilute charged particles, *J. Differ. Equations*, **262** (2017), 68–115. https://doi.org/10.1016/j.jde.2016.09.026

23. Z. Wu, W. Wang, Green's function and pointwise estimate for a generalized Poisson–Nernst–Planck–Navier–Stokes model in dimension three, *Z. Angew. Math. Me.*, **98** (2018), 1066–1085. https://doi.org/10.1002/zamm.201700109

24. X. Zhai, Z. Wu, On the well-posedness and decay rates of solutions to the Poisson–Nernst–Planck–Navier–Stokes system, *J. Math. Fluid Mech.*, **26** (2024), 37. https://doi.org/10.1007/s00021-024-00867-2

25. Z. Zhang, Z. Yin, Global well-posedness for the Navier–Stokes–Nernst–Planck–Poisson system in dimension two, *Appl. Math. Lett.*, **40** (2015), 102–106. https://doi.org/10.1016/j.aml.2014.10.002

26. J. Zhao, C. Deng, S. Cui, Global existence and asymptotic behavior of self-similar solutions for the Navier–Stokes–Nernst–Planck–Poisson system in $\mathbb{R}^3$, *Int. J. Differ. Equat.*, **2011** (2011), 329014. https://doi.org/10.1155/2011/329014