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Research article

A monodromy operator-based interpretation for repetitive control of robot manipulators with controller synthesis and stability analysis

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Abstract: A monodromy operator-based interpretation for repetitive control of robot manipulators subject to periodic reference and disturbance signals is established in this paper. An inverse dynamics strategy is first employed to convert the coupled nonlinear input–output relation of a robot manipulator into a decoupled linear time-invariant (LTI) representation, for which the conventional repetitive control approach can be applied. Such an application leads to a unified expression, i.e., a delay-feedback system, suitable for both controller synthesis and stability analysis. Two linear matrix inequality (LMI) formulations are established to obtain repetitive controllers that minimize the H_∞ and generalized H_2 norms from the disturbance to the tracking error, respectively. The resulting closed-loop systems are shown to be exponentially stable if and only if the spectrum radii of their monodromy operators are less than 1. We further show that the exponential stability also implies the input-to-state stability (ISS) under bounded disturbances. Finally, simulation comparisons are presented to demonstrate the effectiveness of the overall arguments.

Keywords: robot manipulators; controller synthesis; stability analysis
Mathematics Subject Classification: 93C85

1. Introduction

As robot manipulators have become fundamental components in modern industrial automation, a wide variety of control approaches has been developed in the literature, as summarized in [1, 2]. These approaches are typically derived from the Lagrangian formulation describing dynamics of robot manipulators, which inherently yields coupled nonlinear differential equations. To circumvent the direct treatment of such coupled nonlinear models, the inverse dynamics (ID) approach has been employed in [3–6] as a sort of feedback linearizations that convert the coupled nonlinear differential equations into equivalent decoupled LTI differential equations. The capability of the ID based control approach with respect to achieving high precision trajectory tracking has been validated through numerous simulations and experimental studies in [7,8], while the influence of disturbances and model uncertainty on the relevant stability and performance has been rigorously investigated in [9, 10].

To cope with manipulator systems subjected to periodic reference and disturbance signals, on the other hand, one could raise the repetitive control framework [11], in which a delay element $\exp(-sL)$ with the relevant period L is embedded into the feedback loop based on the internal model principle [12]. More precisely, such a delay feedback component is for generating harmonic signals at frequencies $\omega = 2n\pi/L$ ($n = 0, \pm1, \pm2, \dots$), by which the asymptotic stability of the tracking error in terms of reference and disturbance signals with the period L can be attained. Application studies of repetitive control to robot manipulators are deeply discussed for flexible manipulators [13] and passive finite dimensional controller realizations [14].

Despite the aforementioned results, it is not clarified whether the relevant assertions could attain disturbance attenuation performances in some quantitative senses. In contrast, performance optimizations with respect to the repetitive control approach are introduced in [15, 16], but they are confined to partial costs of the overall systems. Although a refined low-pass filter is designed for the repetitive control loop in [17] and a stability assertion for the relevant delay term is provided in [18, 19], no systematic controller synthesis procedure is discussed in those studies. On the other hand, there also exist control approaches to dealing with uncertainties and disturbances through time-delay mechanisms, e.g., the adaptive sliding mode control [20] and the robust data-driven predictive control [21]. However, these control approaches often involve heavy computational burdens.

To address the above limitations, we are concerned with designing controllers for suppressing the reference tracking error occurring from applying repetitive control to robot manipulators in various senses. More precisely, we adopt the L_2 norm or the L_∞ norm as a quantitative measure for the reference tracking error, depending on required control objectives. The rationale behind taking these two norms is that the former norm can be effectively used for reducing the energy of the tracking error [22–27] while the latter norm can be taken for dealing with the peak value of the tracking error [27–32]. By noting this together with the fact that disturbances in robot systems have been often regarded as elements in the L_2 space [33–35], we consider the input–output relation from L_2 to L_2 or L_∞, i.e., the H_∞ performance or the generalized H_2 performance, in terms of repetitive control for robot manipulators. They are in a complementary relationship in the following aspect: Reducing the H_∞ norm promotes rapid disturbance attenuation and favorable transient responses, whereas suppressing the generalized H_2 norm could lead to making the steady state error small. To formulate such controller synthesis problems under the repetitive control approach to robot manipulators in an efficient fashion, the relevant delay-feedback representation is first introduced. On the basis of this formulation, some LMI-based conditions are derived to ensure that the H_∞ norm and the generalized H_2 norm from the disturbance to the tracking error are less than a prespecified level γ (> 0), without considering the effect of the delay element $\exp(-sL)$. We also show that the resulting closed-loop representations with taking into account the delay $\exp(-sL)$ are exponentially stable if and only if the spectral radii of the corresponding monodromy operators [36] are less than 1. It is further shown that such an exponential stability condition implies ISS in the presence of bounded disturbances. Comparative simulation results are also presented to verify both the theoretical and practical effectiveness of the proposed approach.

To summarize, the main contributions of this paper can be described as follows.

- LMI-based repetitive controller synthesis procedures with respect to the H_∞ norm and the

generalized H_2 norm are developed.
- The exponential stability and the ISS for the resulting closed-loop systems are characterized in terms of the associated monodromy operators.
- Comparative simulations are carried out to demonstrate the effectiveness of the overall arguments.

Here, it would be worthwhile to discuss the distinct novelties of this paper in comparison with the authors' previous studies [18, 19, 37]. Although the arguments in [18, 19] characterize the internal stability of repetitive control systems, no theoretical proof is provided in [19], and the proof in [18] is limited to the L_∞ case. Furthermore, the results in [18, 19] are limited to the resulting closed-loop systems after a controller synthesis is carried out. In other words, a controller synthesis and a low-pass filter synthesis procedures are not deeply discussed in those studies. Even though a controller synthesis procedure is introduced in [37], no discussions on the rigorous stability proof, the generalized H_2 controller synthesis and the frequency-domain cut-off selection for the low-pass filter are given in that study. In contrast to the existing studies [18, 19, 37], this paper deals with the controller synthesis procedures with respect to both the H_∞ and generalized H_2 norms together with an explicit guideline on taking the cut-off frequency in the low-pass filter. Furthermore, the stability proof is rigorously established for the L_2 case in this paper.

The remainder of this paper is organized as follows. The problem definition with respect to robot manipulator control subject to periodic signals is introduced in Section 2. The delay-feedback expression in terms of the repetitive control framework is derived in Section 3. The LMI-based controller synthesis procedures associated with the H_∞ norm and the generalized H_2 norm are presented in Section 4. The stability of the delay-feedback system is investigated in Section 5. Simulation validations are given in Section 6. Finally, concluding remarks are given in Section 7.

We use the following notations throughout this paper. The sets of ν dimensional real vectors and nonnegative real scalars are denoted by $\mathbb{R}^\nu$ and $\mathbb{R}_+$, respectively. The set of nonnegative integers is denoted by $\mathbb{Z}_0$. The notation $|\cdot|_p$ for $p \in \{1, 2, \infty\}$ denotes the p norm of a finite dimensional real vector, i.e.,

$$|x|_p := \begin{cases} \left(\displaystyle\sum_{i=1}^{\nu} |x_i|^p \right)^{1/p} & (p \in \{1,2\}), \\ \displaystyle\max_{1\le i\le \nu} |x_i| & (p = \infty) \end{cases}$$

for an $x \in \mathbb{R}^\nu$. The notation $|\cdot|_\infty$ is also used for implying the matrix ∞-norm defined as

$$|A|_\infty := \sup_{|x|_\infty \le 1} \frac{|Ax|_\infty}{|x|_\infty} = \max_i \sum_j |A_{ij}|,$$

where A_{ij} denotes the (i, j)th element of A.

For a hybrid continuous/discrete time signal $\xi(t) = \begin{bmatrix} x^T & v^T(t+\cdot) \end{bmatrix}^T$ with $x \in \mathbb{R}^{\nu_1}$ and $v(t+\cdot) \in (L_p[t, t+L))^{\nu_2}$, its L_p norm for $p \in \{1, 2, \infty\}$ is denoted by $\|\xi(t)\|_p$, i.e.,

$$\|\xi(t)\|_p := \begin{cases} \left(|x(t)|_p^p + \displaystyle\int_t^{t+L} |v(\theta)|_p^p d\theta \right)^{1/p} & (p \in \{1,2\}), \\ |x(t)|_\infty + \displaystyle\operatorname*{ess\,sup}_{t\le\theta<t+L} |v(\theta)|_\infty & (p = \infty). \end{cases}$$

When a symmetric matrix P is positive definite, we denote it by $P > 0$. A function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is strictly increasing and $\zeta(0) = 0$, and a function $\beta : \mathbb{R}_+ \times \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t and $\beta(r, \cdot)$ is strictly decreasing for each fixed r. The spectral radius of an operator is denoted by $\rho(\cdot)$.

2. Problem definition on inverse dynamics control of robot manipulators

The dynamic behavior of a robot manipulator is generally described by the Lagrangian equation

$$M(q)\ddot{q} + V(q, \dot{q}) + G(q) = \tau + \tau_d, \tag{2.1}$$

where $M(q) \in \mathbb{R}^{n\times n}$ denotes the inertia matrix, $V(q, \dot{q}) \in \mathbb{R}^n$ represents the Coriolis and centrifugal effects, and $G(q) \in \mathbb{R}^n$ is the gravitational torque vector. The generalized coordinate vector, control input, and disturbance torque are denoted by $q(t) \in \mathbb{R}^n$, $\tau(t) \in \mathbb{R}^n$ and $\tau_d(t) \in \mathbb{R}^n$, respectively.

On the basis of this Lagrange equation, this paper is concerned with achieving the asymptotic tracking performance described by

$$q(t) \to q_d(t) \quad (t \to \infty), \tag{2.2}$$

for a given reference trajectory $q_d(t) \in \mathbb{R}^n$. Here, without loss of generality, the reference trajectory $q_d(t)$ is assumed to be twice continuously differentiable with bounded time derivatives, i.e., $q_d, \dot{q}_d, \ddot{q}_d \in L_\infty$, as discussed in [2]. To address this control objective, we take the inverse dynamics approach [3–6] given by

$$M(q)(\ddot{q}_d - u) + V(q, \dot{q}) + G(q) = \tau, \tag{2.3}$$

where u is an auxiliary control input to be designed. Substituting (2.3) into (2.1) leads to

$$\begin{aligned} &M(q)\ddot{q} + V(q, \dot{q}) + G(q) = M(q)(\ddot{q}_d - u) + V(q, \dot{q}) + G(q) + \tau_d \\ \Rightarrow &M(q)u - \tau_d = M(q)(\ddot{q}_d - \ddot{q}). \end{aligned} \tag{2.4}$$

This admits the LTI differential equation given by

$$\ddot{e}(t) = u(t) - M^{-1}(q)\tau_d(t) =: u(t) + w(t), \tag{2.5}$$

with the tracking error defined as

$$e(t) := q_d(t) - q(t) \in \mathbb{R}^n, \tag{2.6}$$

where we employ the fact in [1] that $M(q)$ is positive definite for all $q \in \mathbb{R}^n$ and thus it is also invertible for any $q \in \mathbb{R}^n$.

With defining the output as $y(t) := e(t)$, (2.5) admits the representation

$$P : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Bw(t), \\ y(t) = Cx(t), \end{cases} \tag{2.7}$$

where $x(t) = \begin{bmatrix} e^T(t) & \dot{e}^T(t) \end{bmatrix}^T \in \mathbb{R}^{2n}$ and

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{2.8}$$

In practical applications of robot manipulators, the reference trajectory q_d is commonly periodic in industrial tasks. Regarding the treatment of w, both $M(q)$ and $M^{-1}(q)$ can be approximated as periodic matrices with the same period as q_d, provided that the tracking error e is sufficiently small. This premise does not affect the essentials of the overall arguments, as the asymptotic stability of e is readily ensured by employing a full state-feedback controller, such as the one introduced in [1]. Furthermore, given that τ_d contains bias and periodic components and its period is an integer multiple of that of q_d or vice versa, it is valid to regard w in (2.5) as a periodic signal. This observation motivates us to employ the repetitive control framework [11], in which a delay element is embedded into the feedback loop to establish the zero steady-state error for periodic signals based on the internal model principle [12]. Such a time-delay feedback with the length L is involved in the repetitive controller, where L is the least common multiple of the periods of external signals. In other words, the denominator of the ideal transfer function for the repetitive controller is given by $1 - e^{-sL}$.

Although the repetitive control framework is deeply considered for robot manipulators in [13–19], no systematic procedure for controller synthesis is discussed in those studies. Motivated by this fact, we are devoted to establishing controller synthesis procedures together with the associated stability analysis.

3. Delay-feedback expression for repetitive control approach

This section considers applying the repetitive control architecture [11] to the ID-based representation (2.7) and derives the relevant delay-feedback expression.

We first suppose that the control input consists of two components as

$$u(t) = u_{\mathrm{tc}}(t) + u_{\mathrm{rc}}(t), \tag{3.1}$$

where the reference tracking component is selected as

$$u_{\mathrm{tc}}(t) = Kx(t) = -K_{\mathrm{p}}e(t) - K_{\mathrm{d}}\dot{e}(t), \tag{3.2}$$

and $u_{\mathrm{rc}}(t)$ is determined by the repetitive controller.

We next assume that the reference signal $q_d(t)$ and the disturbance torque $\tau_d(t)$ are periodic functions with the common period L. With respect to this, the repetitive controller [11] is designed for generating periodic signals at the frequencies $\omega = 2n\pi/L$ ($n = 0, \pm 1, \pm 2, \dots$), by which the steady-state of the reference tracking error becomes zero in terms of the internal model principle [12]. In this sense, the ideal repetitive controller has $1 - e^{-sL}$ at its denominator of the corresponding transfer function. However, this structure leads to infinitely many poles on the imaginary axis. Hence, a direct application of the ideal repetitive controller could cause instability of the overall closed-loop systems in practice. To take into account this issue, a low-pass filter is often employed, and thus the practical repetitive controller is described by

$$H_{\mathrm{rc}}(s) = \frac{U_{\mathrm{rc}}(s)}{Y(s)} = \frac{1}{1 - \dfrac{\omega_{\mathrm{c}}}{s + \omega_{\mathrm{c}}} e^{-sL}} K_{\mathrm{rc}}, \tag{3.3}$$

as shown in Figure 1, where $H_{\mathrm{rc}}(s)$ is the transfer function of the repetitive controller, ω_{c} is the cut-off frequency of the low-pass filter, K_{rc} is the corresponding controller gain, and $U_{\mathrm{rc}}(s)$ and $Y(s)$ denote the Laplace transforms of the associated control input $u_{\mathrm{rc}}(t)$ and the output $y(t)$, respectively.

Here, it should be noted that incorporating the low-pass filter $\frac{\omega_{\mathrm{c}}}{s+\omega_{\mathrm{c}}}$ as in (3.3) does not theoretically ensure the zero steady-state error in response to external signals with the frequencies $\omega = 2n\pi/L$ ($n = 0, \pm 1, \pm 2, \dots$) since the poles of $H_{\mathrm{rc}}(s)$ are no longer located at $j\omega$. However, it does not compromise the effectiveness of the practical repetitive controller. This is because the behavior of $H_{\mathrm{rc}}(s)$ converges to that of the ideal repetitive controller as the cut-off frequency ω_{c} increases, by which the residual steady-state error can be reduced to an arbitrary level. This observation will be verified through the simulation results in Section 6.

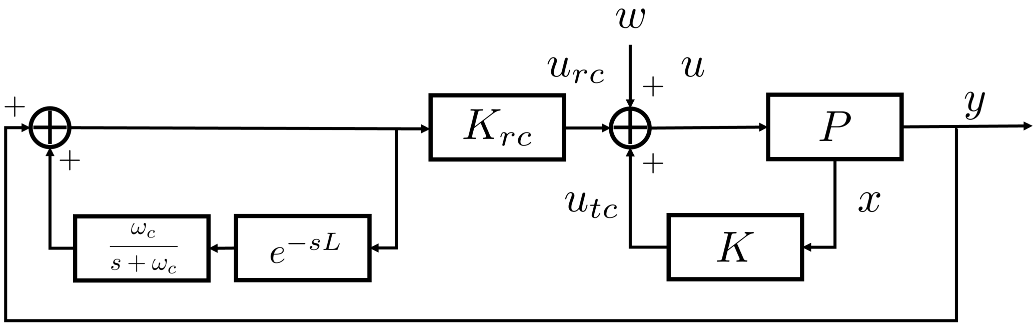


Figure 1. The controller formulation using ID with RC for robot manipulators.

To make the analysis for the resulting closed-loop system clearer, we introduce the state-space realization of (3.3) described by

$$H_{\mathrm{rc}} : \begin{cases} \dot{x}_{\mathrm{rc}}(t) = \omega_{\mathrm{c}}\big(-x_{\mathrm{rc}}(t) + x_{\mathrm{rc}}(t-L) + e(t-L)\big), \\ u_{\mathrm{rc}}(t) = K_{\mathrm{rc}}\big(x_{\mathrm{rc}}(t) + e(t)\big). \end{cases} \tag{3.4}$$

In other words, $x_{\mathrm{rc}}(t) \in \mathbb{R}^n$ means the state variable vector of the repetitive controller, whose physical meaning corresponds to the filtered and delayed memory of the tracking error $e(t) \in \mathbb{R}^n$. As shown in (3.4) and Figure 1, the dimension and the relative degree of $x_{\mathrm{rc}}(t)$ coincide with those of the position error $e(t)$, while the compensation input $u_{\mathrm{rc}}(t)$ is generated through the repetitive control gain K_{rc}. Substituting (3.2) and the second equation of (3.4) into (3.1) yields

$$u(t) = u_{\mathrm{tc}}(t) + u_{\mathrm{rc}}(t) = (K_{\mathrm{rc}} - K_{\mathrm{p}})e(t) - K_{\mathrm{d}}\dot{e}(t) + K_{\mathrm{rc}}x_{\mathrm{rc}}(t). \tag{3.5}$$

We also define the augmented state vector $\bar{x}(t)$ as

$$\bar{x}(t) := \begin{bmatrix} x^T(t) & x_{\mathrm{rc}}^T(t) \end{bmatrix}^T. \tag{3.6}$$

Then, in view of $\bar{x}(t)$ defined as (3.6), combining (2.7), (3.4), and (3.5) establishes the augmented closed-loop system given by

$$\begin{cases} \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{A}_1\bar{x}(t-L) + \bar{B}w(t) + \bar{B}u(t), \\ u(t) = \bar{K}\bar{x}(t), \end{cases} \tag{3.7}$$

with

$$\bar{A} = \begin{bmatrix} A & 0_{2n\times n} \\ 0_{n\times 2n} & -\omega_c I_n \end{bmatrix}, \quad \bar{A}_l = \begin{bmatrix} & 0_{2n\times 3n} & \\ \omega_c I_n & 0 & \omega_c I_n \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} K_{rc} - K_p & -K_d & K_{rc} \end{bmatrix}. \tag{3.8}$$

The expression (3.7) corresponds to a class of delay-differential equations (DDEs) [38], whose dynamic behavior has been investigated by using Lyapunov–Krasovskii functionals (LKFs) [39]. However, it is often difficult to develop a necessary and sufficient condition for the asymptotic or exponential stability and associated controller synthesis procedures in the LKF-based approaches. As a preliminary step to establish less conservative stability assertions and effective controller synthesis procedures, we reformulate the closed-loop DDE (3.7) into the delay-feedback system Σ as shown in Figure 2, whose dynamic behavior is described by

$$\Sigma : \begin{cases} \dot{\bar{x}}(t) = (\bar{A} + \bar{B}\bar{K})\bar{x}(t) + \bar{B}w(t) + v(t), \\ z(t) = \bar{A}_l\bar{x}(t), \\ v(t) = z(t-L), \end{cases} \tag{3.9}$$

where the pure delay operator is represented as $H(L) : v(t) = z(t-L)$. This reformulation enables the LMI-based controller synthesis and the operator-based stability analysis, which will be discussed in the subsequent sections.

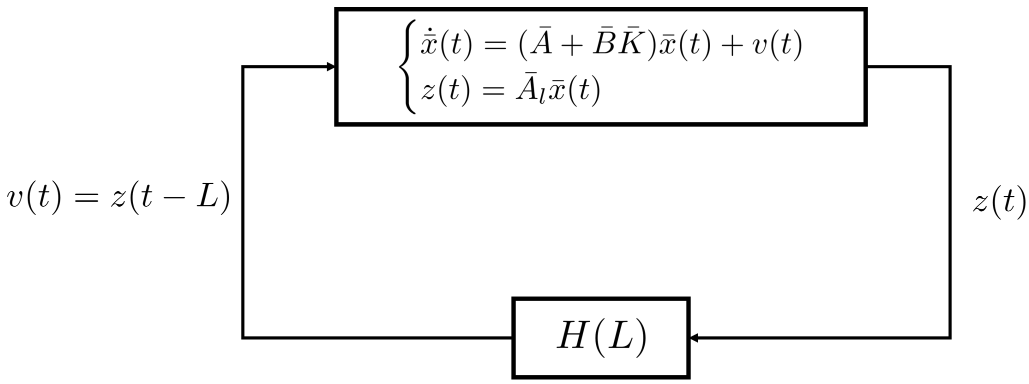


Figure 2. The delay-feedback system Σ.

4. LMI-based controller synthesis procedures

On the basis of the delay-feedback expression (3.9), this section introduces control synthesis procedures with respect to the cut-off frequency ω_c in (3.3) and the control gain $\bar{K}$ in (3.8).

4.1. Selection of the cut-off frequency ω_c

The selection of the cut-off frequency ω_c in the low-pass filter should be carefully considered in view of hardware limitations. In this regard, it should be noted as a fundamental trade-off between taking higher and lower cut-off frequencies. Selecting a smaller ω_c may result in performance degradation in tracking high-frequency components of the reference signal, whereas employing a larger ω_c can increase the sensitivity of robot manipulators to resonant dynamics. With this in mind, it might be often desirable to select ω_c such that

$$\omega_r < \omega_c < \omega_a, \tag{4.1}$$

where ω_r denotes the highest frequency contained in the reference trajectory q_d (and/or the disturbance w) and ω_a means the actuator bandwidth.

In many practical applications [40], the actuator bandwidth ω_a is generally regarded as considerably larger than the reference frequency ω_r. Under such circumstances, it is straightforward to choose the cut-off frequency ω_c as in (4.1). Furthermore, a reasonable heuristic method for taking ω_c is to initially set $\omega_c = \omega_r$ and then gradually increase it until a satisfactory tracking performance is achieved, while maintaining the constraint $\omega_c < \omega_a$.

On the other hand, when the reference signal q_d contains high-frequency components such that $\omega_r > \omega_a$, achieving both accurate tracking performance and effective vibration suppression through a filter synthesis alone becomes challenging. In such cases, it would be desired that an additional low-pass filter is implemented directly at the actuator level.

4.2. LMI-based controller synthesis

Regarding controller synthesis, the linear quadratic regulator (LQR) [41] is widely recognized as an effective strategy. However, the LQR-based approach primarily focuses on minimizing a quadratic performance index without explicitly addressing the influence of external disturbances. In contrast, this paper is also concerned with tackling the effect of the disturbance input w on the delay-feedback system Σ, as shown in (3.9). With respect to this issue, we propose controller synthesis procedures through the LMI-based approach [42].

As a preliminary step, we consider a simplified representation of (3.9) given by

$$\begin{cases} \dot{\bar{x}}(t) &= \bar{A}\,\bar{x}(t) + \bar{B}u(t) + \bar{B}w(t) \\ z_r(t) &= E\bar{x}(t) \end{cases} \tag{4.2}$$

where z_r denotes the regulated output to be minimized, and the weighting matrix E is selected according to the desired control objectives. On the basis of (4.2), we deal with synthesis procedures for two types of state-feedback controller in terms of the H_∞ norm and the generalized H_2 norm.

- Method 1: H_∞ controller synthesis

An H_∞ controller is for suppressing the L_2-induced norm from the disturbance w to the regulated output z_r within a prespecified level γ (> 0), i.e.,

$$\sup_{w\neq 0} \frac{\left(\int_0^\infty z_r^T(t) z_r(t)dt\right)^{1/2}}{\left(\int_0^\infty w^T(t)w(t)dt\right)^{1/2}} < \gamma. \tag{4.3}$$

Here, γ corresponds to an upper bound on the disturbance attenuation performance in terms of the H_∞ norm. As a result, such a controller can provide fast transient responses in the presence of external disturbances. In particular, reducing this induced norm contributes to suppressing the energy of the tracking error, by which we might achieve rapid convergence of the tracking error to zero. Based on these considerations, the synthesis procedure for an H_∞ controller can be summarized as follows. Given a prespecified scalar $\gamma > 0$, there exists a state feedback controller of the form

$$u(t) = \bar{K}\bar{x}(t) \tag{4.4}$$

that achieves the H_∞ performance given by (4.3) for the closed-loop system of (4.2) if and only if there exist decision variables X and L such that

$$\mathcal{X} > 0, \quad \begin{bmatrix} \mathcal{A}^T + \mathcal{A} & \mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T & -\gamma I & 0 \\ \mathcal{C} & 0 & -\gamma I \end{bmatrix} < 0, \tag{4.5}$$

where the matrices $\mathcal{X}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are defined respectively as

$$\mathcal{X} := X, \quad \mathcal{A} := \bar{A}X + \bar{B}L, \quad \mathcal{B} := \bar{B}, \quad \mathcal{C} := EX. \tag{4.6}$$

If the LMIs in (4.5) are feasible, the corresponding state-feedback gain is obtained as

$$\bar{K} = LX^{-1}. \tag{4.7}$$

To put it another way, the H_∞ performance given by (4.3) is established if and only if the LMI-based condition (4.5) is feasible. Hence, the value of γ is often determined as its minimum in the constraint that there exist decision variables X and L for the LMIs (4.5).

- Method 2: Generalized H_2 controller synthesis

The generalized H_2 control problem seeks to reduce the induced norm from L_2 to L_∞ in terms of the mapping from the disturbance w to the regulated output z_r within a prespecified level γ (> 0), i.e.,

$$\sup_{w \neq 0} \frac{\operatorname*{ess\,sup}_{0 \le t < \infty} \left(z_r^T(t) z_r(t) dt \right)^{1/2}}{\left(\int_0^\infty w^T(t) w(t) dt \right)^{1/2}} < \gamma. \tag{4.8}$$

This γ also corresponds to an upper bound on the disturbance attenuation performance in terms of the generalized H_2 norm. Through the formulation of (4.8), we aim at suppressing the peak magnitude of z_r under external disturbances, and thus the steady-state tracking error is ensured to be bounded within a small range. In connection with this, the synthesis procedure for a generalized H_2 controller is described as follows. For a prescribed scalar $\gamma > 0$, there exists a controller of the form (4.4) rendering the generalized H_2 performance given by (4.8) for the closed-loop system of (4.2) if and only if there exist decision matrices X and L such that

$$\begin{bmatrix} \mathcal{A}^T + \mathcal{A} & \mathcal{B} \\ \mathcal{B}^T & -\gamma I \end{bmatrix} < 0, \quad \begin{bmatrix} X & \mathcal{C}^T \\ \mathcal{C} & \gamma I \end{bmatrix} > 0, \tag{4.9}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are defined as (4.6). If the LMIs (4.9) are feasible, the corresponding state-feedback gain is also given by (4.7). To summarize, in an equivalent sense to the H_∞ control, the generalized H_2 performance of (4.8) is satisfied if and only if the decision variables X and L exist in the LMIs of (4.9), and thus the corresponding controller is determined by (4.7) with the smallest feasible γ under the constraint that the LMI-based condition (4.9) is feasible.

5. Stability analysis

Although the assertions presented in the preceding section provide the H_∞ controller and generalized H_2 controller synthesis procedures in terms of the LMI-based conditions, the resulting design does not explicitly address the stability issue occurring from the time-delay feedback structure. To take into account this issue, we investigate the stability properties of the delay-feedback system Σ obtained through the proposed synthesis procedure by deriving operator-theoretic conditions. More precisely, we first characterize the exponential stability of the delay-feedback system Σ, in which no external disturbance is assumed. We then show that the exponential stability of Σ further implies the ISS [45] in the presence of bounded disturbances.

5.1. Exponential stability condition

We first define the hybrid continuous/discrete-time signal $\xi(t)$ as

$$\xi(t) := \begin{bmatrix} \bar{x}(t) \\ v(t+\cdot) \end{bmatrix}, \tag{5.1}$$

where $v(\tau+\cdot)$ denotes the history segment of $v(t)$ over the interval $\tau \leq t < \tau + L$. As a preliminary step to consider a condition ensuring that $\xi(t) \to 0$ as $t \to \infty$ with an exponential decay rate for the disturbance-free case $w = 0$, we revisit the following definition [36].

Definition 1. *For a given $p \in \{1, 2, \infty\}$, the delay-feedback system Σ is said to be p-type exponentially stable if there exist constants $c > 0$ and $\alpha > 0$ such that*

$$\|\xi(t)\|_p \leq ce^{-\alpha t}\|\xi(0)\|_p, \quad t \geq 0, \qquad \forall \xi(0) \in \mathbb{R}^{3n} \oplus (L_p[0,L))^{3n}, \tag{5.2}$$

where $\|\xi(t)\|_p$ ($p \in \{1, 2, \infty\}$) is defined at the end of Introduction.

The p-type exponential stability in Definition 1 implies that $\xi(t)$ converges to the origin exponentially as $t \to \infty$. In particular, this obviously means that both $\bar{x}(t)$ and $v(t)$ tend to 0 in an exponential order as t becomes larger. To determine whether or not the delay-feedback system Σ is p-type exponentially stable, we introduce the monodromy operator associated with the delay-feedback system Σ, which is defined as

$$\mathbf{H} := \begin{bmatrix} A_{\rm d} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \tag{5.3}$$

where $\bar{x}_k := \bar{x}(kL)$, and

$$A_{\rm d} = e^{(\bar{A}+\bar{B}\bar{K})L},\ \mathbf{B}v = \int_0^L e^{(\bar{A}+\bar{B}\bar{K})(L-\tau)}v(\tau)d\tau,\ (\mathbf{C}\bar{x}_k)(\tau) = \bar{A}_1 e^{(\bar{A}+\bar{B}\bar{K})\tau}\bar{x}_k,\ (\mathbf{D}v)(\tau) = \int_0^\tau \bar{A}_1 e^{(\bar{A}+\bar{B}\bar{K})(\tau-\theta)}v(\theta)d\theta$$

by which

$$\begin{bmatrix} \bar{x}_{k+1} \\ v((k+1)L+\cdot) \end{bmatrix} = \mathbf{H}\begin{bmatrix} \bar{x}_k \\ v(kL+\cdot) \end{bmatrix}. \tag{5.4}$$

Note that $\mathbf{H} : \mathbb{R}^{3n} \oplus (L_p[0, L))^{3n} \to \mathbb{R}^{3n} \oplus (L_p[0, L))^{3n}$ since $A_\mathrm{d} : \mathbb{R}^{3n} \to \mathbb{R}^{3n}$, $\mathbf{B} : (L_p[0, L))^{3n} \to \mathbb{R}^{3n}$, $\mathbf{C} : \mathbb{R}^{3n} \to (L_p[0, L))^{3n}$ and $\mathbf{D} : (L_p[0, L))^{3n} \to (L_p[0, L))^{3n}$. This means that $\mathbf{H}$ in (5.3) describes the hybrid continuous/discrete-time behavior of Σ. By employing this monodromy operator $\mathbf{H}$, we are in a position to establish a necessary and sufficient condition for the exponential stability of the time-delay feedback system Σ. To this end, we first introduce the following lemma.

Lemma 1 ([47])**.** *There exist $m > 0$ and $0 < r < 1$ such that*

$$\|\mathbf{H}^n\|_p \le mr^n, \quad \forall n \in \mathbb{N} \tag{5.5}$$

if and only if $\rho(\mathbf{H}) < 1$.

On the basis of Lemma 1, we are led to the following theorem.

Theorem 1. *For any $p \in \{1, 2, \infty\}$, the delay-feedback system Σ is p-type exponentially stable if and only if $\rho(\mathbf{H}) < 1$.*

The proof of Theorem 1 is based on the spectral analysis of the monodromy operator $\mathbf{H}$ and the equivalence between exponential decay of its powers and the spectral radius condition [36]. The proof of this theorem is provided in [18] for the case of $p = \infty$ and in [36] for the case of $p = 1$. In particular, by following the arguments in [18, 36], one can verify that the delay-feedback system Σ is p-type exponential stable for any $p \in \{1, 2\}$ if and only if $\rho(\mathbf{H}) < 1$. Accordingly, we construct the following proof for the case of $p = 2$, by which the assertions of Theorem 1 can be completed uniformly for $p \in \{1, 2, \infty\}$.

Proof. It should first be noted that the essential spectrum [46] of $\mathbf{H}$ is $\{0\}$ because no feedthrough term exists in $\mathbf{D}$, and thus $\mathbf{D}$ is a compact operator. To put it another way, the essential spectrum of $\mathbf{H}$ does not affect our target assertion of $\rho(\mathbf{H}) < 1$. Hence, it is only required to take into account the point spectrum of $\mathbf{H}$. In connection with this issue, we also note from [36] that $\gamma \in \mathbb{R}$ is a point spectrum and $s := [\bar{x}_0^T \; v^T(\cdot)]^T$ with $\bar{x}_0 \in \mathbb{R}^{3n}$ and $v(t) \in (L_p[0, L))^{3n}$ is the corresponding eigenvector if and only if

$$v(t) = e^{(\bar{A}+\bar{A}_1/\gamma)t} v_0, \quad \bar{x}_0 = \gamma v_0, \tag{5.6}$$

where

$$e^{(\bar{A}+\bar{A}_1/\gamma)L} v_0 = \gamma v_0. \tag{5.7}$$

Since this s is an element of $\mathbb{R}^{3n} \oplus (L_p[0, L))^{3n}$ regardless of $p \in \{1, 2, \infty\}$, the spectrum of $\mathbf{H}$ is invariant for the choice of p.

(*Necessary condition*) Let us suppose that (5.2) holds for some $c > 0$ and $\alpha > 0$. By substituting $t = kL$ into (5.2), we can see that

$$\left\| \begin{bmatrix} \bar{x}_k \\ v(kL + \cdot) \end{bmatrix} \right\|_p \le c e^{-\alpha k L} \left\| \begin{bmatrix} \bar{x}_0 \\ v(0 + \cdot) \end{bmatrix} \right\|_p = (e^{-\alpha L})^k c \left\| \begin{bmatrix} \bar{x}_0 \\ v(0 + \cdot) \end{bmatrix} \right\|_p. \tag{5.8}$$

Letting $m = c\|[\bar{x}_0^T \; v^T(0 + \cdot)]^T\|_p$ (> 0) and $r = e^{-\alpha L}$ $(\in [0, 1))$ obviously establishes the necessary condition (i.e., $\rho(\mathbf{H}) < 1$) by Lemma 1.

(*Sufficient condition*) Here, we confine ourselves to the case of $p = 2$ since the other cases are essentially equivalent to this specific case. Let us assume that $\rho(\mathbf{H}) < 1$. Then, by Lemma 1, there exist $m > 0$ and $0 < r < 1$ such that

$$\left\|\begin{bmatrix} \bar{x}_k \\ v(kL+\cdot) \end{bmatrix}\right\|_2 \le mr^k \left\|\begin{bmatrix} \bar{x}_0 \\ v(0+\cdot) \end{bmatrix}\right\|_2, \quad \forall k \in \mathbb{N}, \quad \forall \bar{x}(0) \in \mathbb{R}^{3n}, \quad \forall v(0+\cdot) \in (L_2[0,L))^{3n}. \tag{5.9}$$

On the other hand, we note from (3.9) that

$$\bar{x}(kL+\theta) = e^{(\bar{A}+\bar{B}\bar{K})\theta}\bar{x}(kL) + \int_0^\theta e^{(\bar{A}+\bar{B}\bar{K})(\theta-\tau)} v(kL+\tau)d\tau, \quad \forall \theta \in [0,L). \tag{5.10}$$

Combining (5.10) with (5.9) clearly leads to

$$\begin{aligned} |\bar{x}(kL+\theta)|_2 &\le e^{|\bar{A}+\bar{B}\bar{K}|_2\theta}|\bar{x}(kL)|_2 + \left(\int_0^\theta \left|e^{(\bar{A}+\bar{B}\bar{K})(\theta-\tau)}\right|_2^2 d\tau\right)^{1/2} \left(\int_0^\theta |v(kL+\tau)|_2^2 d\tau\right)^{1/2} \\ &\le e^{|\bar{A}+\bar{B}\bar{K}|_2 L}|\bar{x}(kL)|_2 + \sqrt{L} e^{|(\bar{A}+\bar{B}\bar{K})|_2 L} \left(\int_0^L |v(kL+\tau)|_2^2 d\tau\right)^{1/2} \\ &\le (1+\sqrt{L}) e^{|\bar{A}+\bar{B}\bar{K}|_2 L} mr^k s_0 =: m_1 r^k s_0, \quad \forall \theta \in [0,L), \end{aligned} \tag{5.11}$$

where

$$s_0 := \left\|\begin{bmatrix} \bar{x}_0 \\ v(0+\cdot) \end{bmatrix}\right\|_p. \tag{5.12}$$

From the definition of the L_2 norm of a vector function, it readily follows that

$$\begin{aligned} &\left(\int_0^L v^T(kL+\theta+\tau)v^T(kL+\theta+\tau)d\tau\right)^{1/2} \le \left(\int_0^{2L} v^T(kL+\tau)v^T(kL+\tau)d\tau\right)^{1/2} \\ &= \left(\int_0^L v^T(kL+\tau)v^T(kL+\tau)d\tau + \int_0^L v^T((k+1)L+\tau)v^T((k+1)L+\tau)d\tau\right)^{1/2} \\ &\le \sqrt{2} mr^k s_0, \quad \forall \theta \in [0,L). \end{aligned} \tag{5.13}$$

Then, it immediately follows from (5.11) and (5.13) that

$$\left\|\begin{bmatrix} \bar{x}(kL+\theta) \\ v(kL+\theta+\cdot) \end{bmatrix}\right\|_2 \le ce^{-\alpha(kL+\theta)} s_0, \tag{5.14}$$

where

$$c := \frac{\sqrt{2m^2+m_1^2}}{r}, \quad \alpha := -\frac{\log r}{L}. \tag{5.15}$$

This completes the proof. □

Remark 1. *The decay constants $c > 0$ and $\alpha > 0$ defined in* (5.15) *are independent of the initial state $\xi(0)$, while they do depend on the constants m and r in Lemma 1. This dependence does not affect the validity of the proof of the sufficient condition in Theorem 1; we show that there exist the decay constants $c > 0$ and $\alpha > 0$ which are uniform with respect to $\xi(0)$. On the other hand, the constant m in the proof of the necessary condition does depend on the initial state $\xi(0)$. However, the uniformity of m with respect to $\xi(0)$ is not required because the exponential stability is assumed a priori in the necessary condition, and thus it is sufficient to show an existence of the constants m and r in Lemma 1 (since this lemma establishes a necessary and sufficient condition for $\rho(\mathbf{H}) < 1$). Hence, allowing m to depend on the initial state $\xi(0)$ in the proof of the necessary condition causes no issue.*

It is obviously clarified from Theorem 1 that $\xi(t) \to 0$ as $t \to \infty$ if and only if $\rho(\mathbf{H}) < 1$. Here, it should be required to compute the spectrum of the monodromy operator $\mathbf{H}$ for employing the assertions in this theorem, but it is nontrivial since $\mathbf{H}$ is an infinite-rank operator. However, this does not affect the feasibility of Theorem 1 because $\rho(\mathbf{H})$ can be obtained to any degree of accuracy by using the arguments in [43].

5.2. Input-to-state stability condition

This subsection is devoted to showing that the exponential stability also implies the ISS for the delay-feedback system Σ. With respect to this, we first introduce the definition of the ISS as follows [44].

Definition 2. *The delay-feedback system Σ is said to satisfy the ISS if there exist a class $\mathcal{K}$ function ζ and a class $\mathcal{KL}$ function β such that*

$$\|\xi(t)\|_\infty \leq \beta(\|\xi(0)\|_\infty, t) + \zeta(\|w\|_{L_\infty}) \tag{5.16}$$

where $\|w\|_\infty$ is defined as

$$\|w\|_{L_\infty} := \operatorname*{ess\,sup}_{0\leq t<\infty} |w(t)|_\infty. \tag{5.17}$$

With respect to Definition 2, we can obtain the following theorem.

Theorem 2. *The delay-feedback system Σ satisfies the ISS if $\rho(\mathbf{H}) < 1$.*

Even though a simplified version of the proof is presented in [18], a detailed derivation is provided as follows.

Proof. By noting (5.4), we can obtain another expression of (3.9) by

$$\begin{cases} \bar{x}_{k+1} = A_d \bar{x}_k + \mathbf{B}v(kL + \cdot) + \displaystyle\int_0^L e^{(\bar{A}+\bar{B}\bar{K})(L-\tau)} \bar{B}w(\tau)d\tau, \\ v((k+1)L + \theta) = (\mathbf{C}\bar{x}_k)(\theta) + (\mathbf{D}v(kL + \cdot))(\theta) + \displaystyle\int_0^\theta \bar{A}_l e^{(\bar{A}+\bar{B}\bar{K})(\theta-\tau)} \bar{B}w(kL + \tau)d\tau, \end{cases} \tag{5.18}$$

with defining

$$\mathbf{B}_w w := \mathbf{B}(\bar{B}w), \quad (\mathbf{D}_w w)(\theta) := (\mathbf{D}(\bar{B}w))(\theta). \tag{5.19}$$

(5.18) also admits the representation

$$\xi((k+1)L) = \mathbf{H}\xi(kL) + \begin{bmatrix}\mathbf{B}_w \\ \mathbf{D}_w\end{bmatrix} w(kL+\cdot). \tag{5.20}$$

This leads to a closed-form expression of $\xi(kL)$ as

$$\xi(kL) = \mathbf{H}^k\xi(0) + \sum_{j=0}^{k-1} \mathbf{H}^j \begin{bmatrix}\mathbf{B}_w \\ \mathbf{D}_w\end{bmatrix} w((k-1-j)L+\cdot). \tag{5.21}$$

Then, it readily follows from (5.21) that

$$\|\xi(kL)\|_\infty \le \|\mathbf{H}^k\xi(0)\|_\infty + \sum_{j=0}^{k-1} \left\| \mathbf{H}^j \begin{bmatrix}\mathbf{B}_w \\ \mathbf{D}_w\end{bmatrix} w((k-1-j)L+\cdot) \right\|_\infty. \tag{5.22}$$

By Theorem 1, there exist $c > 0$ and $0 < \alpha < 1$ such that

$$\|\xi(kL)\|_\infty \le ce^{-\alpha kL}\|\xi(0)\|_\infty + \left\|\begin{bmatrix}\mathbf{B}_w \\ \mathbf{D}_w\end{bmatrix}\right\|_\infty \sum_{j=0}^{k-1}\{ce^{-\alpha jL}\|w((k-1-j)L+\cdot)\|_{L_\infty}\}, \tag{5.23}$$

where $\left\|\begin{bmatrix}\mathbf{B}_w \\ \mathbf{D}_w\end{bmatrix}\right\|_\infty$ denotes the induced norm from w to ξ. Because $\rho(\mathbf{H}) < 1$ implies $\bar{A} + \bar{B}\bar{K}$ is Hurwitz stable, $\left\|\begin{bmatrix}\mathbf{B}_w \\ \mathbf{D}_w\end{bmatrix}\right\|_\infty$ is bounded. This together with the fact that $\sum_{j=0}^{\infty} ce^{-\alpha jL}$ is bounded allows us to obtain an upper bound on (5.23) as

$$\|\xi(kL)\|_\infty \le ce^{-\alpha kL}\|\xi(0)\|_\infty + \left\|\begin{bmatrix}\mathbf{B}_w \\ \mathbf{D}_w\end{bmatrix}\right\|_\infty \frac{c}{1-e^{-\alpha L}}\|w\|_{L_\infty} =: ce^{-\alpha kL}\|\xi(0)\|_\infty + c_w\|w\|_{L_\infty}. \tag{5.24}$$

Since this inequality holds regardless of the choice of $k \in \mathbb{Z}_0$ and $v(kL+\cdot)$ in $\xi(kL)$ is a continuous-time function defined on the interval $[kL, (k+1)L)$ from (5.1), the remaining task is to show the boundness of $\bar{x}(t)$ on the same interval $t \in [kL, (k+1)L)$. With respect to this, we can see from the operator-based arguments on sampled-data and time-delay systems [48–55] that for any $\theta \in [0, L)$, the following relation holds.

$$\bar{x}(kL+\theta) = e^{(\bar{A}+\bar{B}\bar{K})\theta}\bar{x}(kL) + \int_0^\theta e^{(\bar{A}+\bar{B}\bar{K})(\theta-\tau)}v(kL+\tau)d\tau + \int_0^\theta e^{(\bar{A}+\bar{B}\bar{K})(\theta-\tau)}\bar{B}w(kL+\tau)d\tau. \tag{5.25}$$

We also note from (5.24) that

$$\operatorname*{ess\,sup}_{0\le\tau<L} |v(kL+\tau)|_\infty \le \|\xi(kL)\|_\infty. \tag{5.26}$$

Combining (5.24)–(5.26) leads to

$$\begin{aligned}|\bar{x}(kL+\theta)|_\infty &\le e^{|\bar{A}+\bar{B}\bar{K}|_\infty\theta}|\bar{x}(kL)|_\infty + \theta e^{|\bar{A}+\bar{B}\bar{K}|_\infty L}\|\xi(kL)\|_\infty + \theta e^{|\bar{A}+\bar{B}\bar{K}|_\infty L}|\bar{B}|_\infty\|w\|_{L_\infty} \\ &\le e^{|\bar{A}+\bar{B}\bar{K}|_\infty L}|\bar{x}(kL)|_\infty + Le^{|\bar{A}+\bar{B}\bar{K}|_\infty L}\|\xi(kL)\|_\infty + Le^{|\bar{A}+\bar{B}\bar{K}|_\infty L}|\bar{B}|_\infty\|w\|_{L_\infty}\end{aligned}$$

$$
\begin{aligned}
&\leq (1+L)e^{|\bar{A}+\bar{B}\bar{K}|_\infty L}\|\xi(kL)\|_\infty + Le^{|\bar{A}+\bar{B}\bar{K}|_\infty L}|\bar{B}|_\infty\|w\|_{L_\infty} \\
&\leq ce^{-\alpha kL}(1+L)e^{|\bar{A}+\bar{B}\bar{K}|_\infty L}\|\xi(0)\|_\infty + e^{|\bar{A}+\bar{B}\bar{K}|_\infty L}\left((1+L)\left\|\begin{bmatrix}\mathbf{B}_w\\ \mathbf{D}_w\end{bmatrix}\right\|_\infty \frac{c}{1-e^{-\alpha L}} + L|\bar{B}|_\infty\right)\|w\|_{L_\infty} \\
&=: c_0 e^{-\alpha kL}\|\xi(0)\|_\infty + \bar{c}_w\|w\|_{L_\infty}, \quad \forall\theta\in[0,L).
\end{aligned} \tag{5.27}
$$

On the basis of (5.24) and (5.27), we can take a class $\mathcal{K}$ function ζ and a class $\mathcal{KL}$ function β by

$$
\zeta(\|w\|_{L_\infty}) := (c_w + \bar{c}_w)\|w\|_{L_\infty}, \quad \beta(\|\xi(0)\|_\infty, t) := (c+c_0)e^{-\frac{\alpha}{2}t}\|\xi(0)\|_\infty. \tag{5.28}
$$

This completes the proof.

□

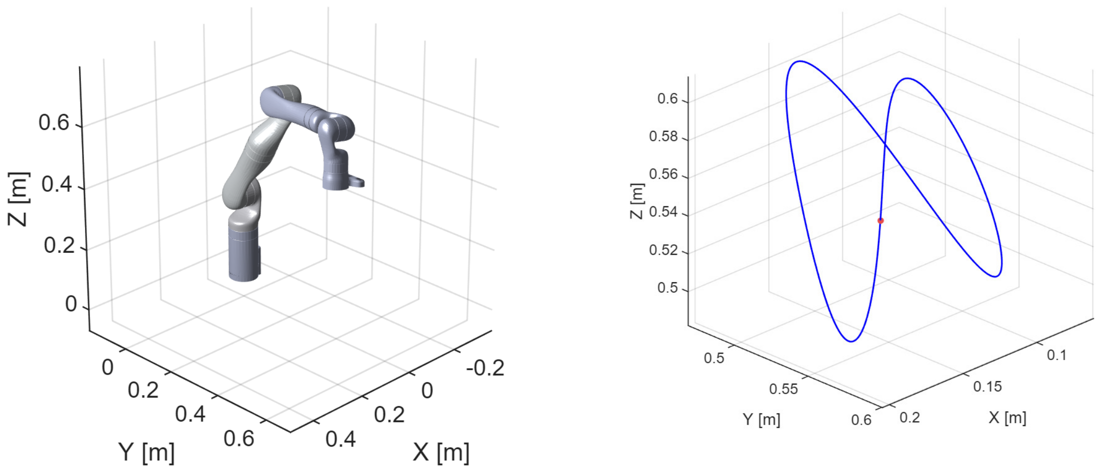
We can obviously see from Theorem 2 that ξ is bounded for the case of $w \neq 0$ if $\rho(\mathbf{H}) < 1$. To put it another way, combining Theorems 1 and 2 implies that the delay-feedback system Σ satisfies the ISS if it is p-type exponentially stable for any $p \in \{1, 2, \infty\}$.

Finally, we would like to summarize the arguments developed in Sections 4 and 5 as follows. In Section 4, controller synthesis methods are developed by using some LMI-based conditions to minimize the H_∞ norm and the generalized H_2 norm of the nominal system, without considering the time-delay associated with the repetitive controller. Subsequently, in Section 5, after the controller synthesis, the stability of the overall system is formulated in terms of the operator-theoretic approach by explicitly taking into account the time-delay involved in the repetitive controller. To put it another way, such operator-based conditions ensure the exponential stability in the absence of disturbances and the ISS in the presence of disturbances.

6. Simulation results

This section presents simulation results to validate the theoretical soundness and practical relevance of the overall arguments developed in this paper. The simulations are conducted by employing the robot manipulator model and the periodic reference trajectory illustrated in Figure 3(a),(b), respectively. To emulate external disturbances acting on the robot manipulator, a random signal is arbitrarily applied in the simulation environment. For controller synthesis, the common weighting matrix is adopted as

$$
E = \begin{bmatrix} 5I_7 & I_7 & I_7 \end{bmatrix}. \tag{6.1}
$$



(a) Robot manipulator. (b) Reference trajectories.

Figure 3. Simulation environments.

The model parameters of the robot manipulator utilized in simulations are shown in Table 1, while the reference joint trajectory $q_d(t) \in \mathbb{R}^7$ is set to

$$q_d(t) = \begin{bmatrix} -0.1\sin(\pi t) \\ 0.1\sin(\pi t) + 0.05 \\ -0.05\sin(\pi t) \\ -0.05\sin(\pi t) - 0.1 \\ 0.025\sin(\pi t) \\ 0.075\sin(\pi t) - 0.025 \\ 0.15\sin(\pi t) \end{bmatrix}. \tag{6.2}$$

Table 1. Model information of the robot manipulator used in simulations.

Joint	1	2	3	4	5	6	7
Mass [kg]	1.377	1.164	1.164	0.930	0.678	0.678	0.500
Length [m]	0.285	0.210	0.210	0.208	0.106	0.106	0.062

For a verification of the assertion in Theorem 2 on the ISS, a random disturbance signal with the period of 2 s and the magnitude bounded by 0.1 is applied to the system. The cut-off frequency of the low-pass filter and the associated control parameters are determined according to the synthesis procedures proposed in the previous section, by which we can design an H_∞ controller and a generalized H_2 controller. In the controller synthesis, the delay period is set to $L = 2$ s to synchronize the periods of q_d and τ_d, and the cut-off frequency of the low-pass filter is chosen as $\omega_c = 20$ rad/s. The MOSEK LMI solver in MATLAB is employed to compute the optimal controllers. For the H_∞ control synthesis, the corresponding control parameters are given by

$$K_{\mathrm{p}} = \begin{bmatrix} 35.42 & -0.32 & 0.24 & -0.09 & 0.05 & -0.03 & 0.01 \\ -0.32 & 36.81 & -0.34 & 0.14 & -0.04 & 0.02 & -0.01 \\ 0.22 & -0.33 & 34.15 & -0.29 & 0.11 & -0.04 & 0.01 \\ -0.11 & 0.13 & -0.31 & 33.90 & -0.31 & 0.09 & -0.02 \\ 0.04 & -0.05 & 0.12 & -0.30 & 37.05 & -0.26 & 0.07 \\ -0.02 & 0.02 & -0.05 & 0.08 & -0.26 & 36.20 & -0.23 \\ 0.01 & -0.01 & 0.02 & -0.03 & 0.07 & -0.23 & 38.10 \end{bmatrix},$$

$$K_{\mathrm{d}} = \begin{bmatrix} 11.50 & 0.42 & -0.18 & 0.08 & -0.04 & 0.02 & -0.01 \\ 0.39 & 12.10 & 0.46 & -0.19 & 0.07 & -0.03 & 0.01 \\ -0.17 & 0.44 & 11.20 & 0.38 & -0.14 & 0.05 & -0.02 \\ 0.07 & -0.18 & 0.40 & 10.85 & 0.41 & -0.13 & 0.04 \\ -0.03 & 0.06 & -0.15 & 0.39 & 12.30 & 0.36 & -0.09 \\ 0.01 & -0.03 & 0.06 & -0.12 & 0.34 & 11.90 & 0.32 \\ -0.01 & 0.01 & -0.02 & 0.04 & -0.09 & 0.31 & 12.45 \end{bmatrix},$$

$$K_{\mathrm{rc}} = \begin{bmatrix} 14.80 & 0.52 & -0.21 & 0.09 & -0.04 & 0.02 & -0.01 \\ 0.49 & 15.20 & 0.58 & -0.24 & 0.11 & -0.05 & 0.02 \\ -0.20 & 0.55 & 14.30 & 0.47 & -0.18 & 0.07 & -0.03 \\ 0.08 & -0.22 & 0.48 & 13.95 & 0.51 & -0.17 & 0.06 \\ -0.04 & 0.09 & -0.19 & 0.50 & 15.40 & 0.45 & -0.12 \\ 0.02 & -0.04 & 0.07 & -0.16 & 0.43 & 15.10 & 0.41 \\ -0.01 & 0.02 & -0.03 & 0.06 & -0.11 & 0.39 & 15.60 \end{bmatrix}.$$

whereas for the generalized H_2 control synthesis, the corresponding control parameters are obtained as

$$K_{\mathrm{p}} = \begin{bmatrix} 30.75 & -0.30 & 0.24 & -0.10 & 0.06 & -0.03 & 0.02 \\ -0.30 & 32.20 & -0.32 & 0.14 & -0.04 & 0.02 & -0.01 \\ 0.22 & -0.32 & 29.90 & -0.27 & 0.11 & -0.05 & 0.01 \\ -0.10 & 0.13 & -0.27 & 29.00 & -0.30 & 0.09 & -0.02 \\ 0.05 & -0.05 & 0.12 & -0.30 & 32.60 & -0.27 & 0.07 \\ -0.03 & 0.02 & -0.04 & 0.08 & -0.27 & 31.80 & -0.25 \\ 0.01 & -0.02 & 0.02 & -0.04 & 0.06 & -0.24 & 33.50 \end{bmatrix},$$

$$K_{\mathrm{d}} = \begin{bmatrix} 8.25 & 0.35 & -0.15 & 0.07 & -0.03 & 0.01 & -0.01 \\ 0.32 & 8.90 & 0.39 & -0.16 & 0.06 & -0.02 & 0.01 \\ -0.14 & 0.37 & 8.10 & 0.32 & -0.12 & 0.04 & -0.01 \\ 0.06 & -0.15 & 0.34 & 7.80 & 0.35 & -0.11 & 0.03 \\ -0.02 & 0.05 & -0.12 & 0.33 & 9.15 & 0.30 & -0.08 \\ 0.01 & -0.02 & 0.05 & -0.10 & 0.28 & 8.70 & 0.27 \\ -0.01 & 0.01 & -0.02 & 0.03 & -0.07 & 0.26 & 9.20 \end{bmatrix},$$

$$K_{\mathrm{rc}} = \begin{bmatrix} 17.50 & 0.65 & -0.28 & 0.12 & -0.05 & 0.03 & -0.01 \\ 0.62 & 18.10 & 0.72 & -0.31 & 0.14 & -0.06 & 0.03 \\ -0.26 & 0.69 & 17.10 & 0.61 & -0.23 & 0.09 & -0.04 \\ 0.11 & -0.29 & 0.62 & 16.80 & 0.66 & -0.22 & 0.08 \\ -0.05 & 0.12 & -0.24 & 0.64 & 18.25 & 0.58 & -0.16 \\ 0.02 & -0.05 & 0.09 & -0.21 & 0.55 & 17.90 & 0.53 \\ -0.01 & 0.02 & -0.04 & 0.07 & -0.15 & 0.51 & 18.45 \end{bmatrix}.$$

For computing the spectral radius of the associated monodromy operator, we employ the quasi-finite-rank approximation technique [43], in which the associated discretization resolution is chosen to be sufficiently fine to ensure numerical convergence and accuracy.

To carry out comparative analysis, two conventional control strategies are also considered. The first controller of the form (4.4) is designed by using the standard linear quadratic regulator [41] for the LTI system (4.2) under the assumption $w = 0$. The second controller of the form (4.4) is based on the advanced low-pass filter design proposed in [17], in which no optimization procedure is incorporated. For notational convenience, the proposed H_∞ and generalized H_2 controllers are referred to as 'Method 1: H_∞' and 'Method 2: GH_2', respectively, whereas the two conventional approaches are denoted by 'LQR [41]' and the complex-coefficient filter ('CC-filter') [17].

For Method 1: H_∞, Method 2: GH_2, LQR [41], and CC-filter [17], the spectral radii of the corresponding monodromy operators are obtained by 0.936, 0.931, 0.958, and 0.972, respectively, through the computation method introduced in [43]. Since all these values are strictly less than 1, it follows from Theorems 1 and 2 that each control scheme stabilizes the delay-feedback system Σ.

The simulation results for the root mean square (RMS) values of the tracking error e are shown in Table 2. As observed in Table 2, the RMS values obtained with Method 1: H_∞ and Method 2: GH_2 are significantly smaller than those with the conventional LQR [41] and CC-filter [17] for all joints. In particular, Method 1: H_∞ consistently yields the smallest RMS values across all the joints. These results clearly indicate the superior capability of Method 1: H_∞ over other methods in reducing the L_2 norm, or equivalently the RMS, of the tracking error.

On the other hand, the simulation results for the peak magnitudes of the tracking error e are shown in Table 3. In contrast to the results observed from Table 2, we can see from Table 3 that the peak values of the tracking error e achieved by Method 2: GH_2 are consistently smaller than those obtained with the other control methods for all the joints. This observation suggests that Method 2: GH_2 is particularly effective in suppressing the maximum magnitude of the tracking error, i.e., the L_∞ norm of the tracking error.

For comparisons in a more practical fashion, the simulation results for the RMS values of the control torque inputs are also shown in Table 4. It could be observed from this table that both Method 1: H_∞ and Method 2: GH_2 require less control effort, in terms of RMS torque, than the two conventional methods. This observation demonstrates that the proposed methods achieve better energy efficiency than the conventional methods [17, 41].

To summarize, the simulation results validate the stability conditions developed in this paper. Furthermore, we can confirm that Method 1: H_∞ and Method 2: GH_2 significantly enhance tracking accuracy while improving energy efficiency within the repetitive control framework, compared with the conventional LQR [41] and CC-filter [17] methods. Finally, the distinct theoretical roles of the two proposed controllers in suppressing the L_2 norm and the L_∞ norm of the tracking error are clearly verified through the simulation results.

Table 2. RMS values of the position error e in simulations 10^{-3}[rad].

Joint	1	2	3	4	5	6	7
Method1: H_∞	0.32	0.53	0.42	0.64	1.12	1.05	1.36
Method2: GH_2	0.65	1.02	0.65	0.82	2.25	2.14	2.74
LQR [41]	1.10	1.88	2.15	3.12	3.06	3.62	3.81
CC-filter [17]	1.31	1.98	2.31	2.89	3.27	3.35	3.64

Table 3. Peak values of the position error e in simulations 10^{-3}[rad].

Joint	1	2	3	4	5	6	7
Method1: H_∞	3.83	4.72	5.65	7.12	7.98	8.42	9.86
Method2: GH_2	1.62	2.13	2.43	2.96	3.22	3.85	4.24
LQR [41]	1.74	2.37	3.15	3.68	4.41	6.05	8.25
CC-filter [17]	2.55	3.12	3.94	4.50	6.42	7.20	8.57

Table 4. RMS values of the torque input τ in simulations [Nm].

Method	Method1: H_∞	Method2: GH_2	LQR [41]	CC-filter [17]
RMS	2.531	3.426	4.788	5.431

7. Conclusions

This paper introduced controller synthesis procedures and operator-based stability conditions for robot manipulators subject to periodic reference and disturbance signals, by integrating the repetitive control approach [11] and the inverse dynamics formulation [3–6]. To solve the difficulties occurring from coupled nonlinearities and infinite-dimensional delay dynamics in repetitive control of robot manipulators, a delay-feedback representation was first introduced. Because it is non-trivial to obtain an adequate controller with considering the time-delay, we next designed nominal controllers via LMI-based approach to attenuate disturbances in terms of the H_∞ norm and the generalized H_2 norm for the delay-free part. After such controllers were designed, the p-type exponential stability and the ISS for the delay-feedback representation were characterized by the monodromy operator-based arguments [36]. Due to the infinite-rank property of the monodromy operator, its spectrum is quite difficult to be explicitly obtained, and thus an approximate but an asymptotically exact computation method [43] was employed. Comparative simulation results demonstrated that the proposed schemes achieve superior tracking accuracy while consuming less control energy compared with conventional control approaches [17, 41]. Finally, extending the developed arguments to reducing the associated L_1-induced norm [56] is also left as a promising future study.

Author contributions

Geun Il Song: Original draft preparation, methodology, investigation, and formal analysis; Jung Hoon Kim: Supervision, validation, manuscript-review, project administration, funding acquisition, and revisions. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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