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Research article

Applications of Mittag-Leffler functions in computing fractional integral inequalities

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Abstract: This paper develops fractional integral inequalities generated by integral operators whose kernels contain an extended generalized Mittag-Leffler function. Under two-sided derivative bounds, we establish Ostrowski-type, Ostrowski-Grüss-type, and Hermite-Hadamard-type estimates, together with midpoint consequences and error bounds. The contribution is not limited to a formal replacement of the classical power kernel: The kernel parameters enter the normalization, the fractional means, and the derivative-controlled remainder terms simultaneously, thereby producing a unified family of estimates. A concrete application to a fractional relaxation equation illustrates how the results provide computable bounds for the discrepancy between a pointwise state and its Mittag-Leffler-weighted fractional mean. Classical Riemann-Liouville inequalities are recovered through suitable parameter choices.

Keywords: Ostrowski inequality; Hadamard inequality; Grüss inequality; bounds
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1. Introduction

Fractional calculus provides a natural framework for systems with memory, hereditary effects, anomalous transport, and nonlocal interactions. Fractional integral inequalities complement this framework by giving quantitative bounds for approximation errors, weighted averages, and stability-related functionals. Such estimates are useful in control, signal processing, viscoelasticity, and numerical analysis, where the Riemann-Liouville operator and its generalizations frequently arise [1–3]. The Mittag-Leffler function is especially important because it replaces the exponential response in many fractional evolution models and captures the slower relaxation associated with memory-dependent dynamics [4, 5].

A concrete example is the fractional relaxation equation ${}^{C}D^{\alpha}_{0+}y(t) = -\kappa y(t)$, whose solution is expressed through a Mittag-Leffler function. In computations, one often compares the state at a selected time with a local or nonlocal average. An Ostrowski-type inequality supplies a rigorous upper bound for this discrepancy, whereas an Ostrowski-Grüss estimate controls the interaction between the state and its derivative bounds. Section 2 includes an explicit version of this application and shows how the derived inequalities can serve as consistency and error indicators for numerical approximations of fractional differential equations.

Integral inequalities are continuous counterparts of many classical inequalities for finite sums and sequences. They are used in optimization, statistics, economics, physics, and engineering; standard references include [6, 7]. Convexity is central to this theory because it converts qualitative shape information into commutable upper and lower bounds. A real-valued function Ψ defined on a convex set C is convex if

$$\Psi(\lambda u + (1-\lambda)v) \leq \lambda\Psi(u) + (1-\lambda)\Psi(v), \qquad u, v \in C, \quad \lambda \in [0,1].$$

For a convex function on $[u, v]$, the classical Hermite-Hadamard inequality is

$$\Psi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \Psi(x)dx \leq \frac{\Psi(u)+\Psi(v)}{2}. \tag{1.1}$$

Inequality (1.1) is the Hermite-Hadamard inequality. Its two gaps admit the following estimates [8]:

$$\frac{B(v-u)^2}{24} \leq \frac{1}{v-u}\int_u^v \Psi(\lambda)d\lambda - \Psi\left(\frac{u+v}{2}\right) \leq \frac{A(v-u)^2}{24}, \tag{1.2}$$

$$\frac{B(v-u)^2}{12} \leq \frac{\Psi(u)+\Psi(v)}{2} - \frac{1}{v-u}\int_u^v \Psi(\lambda)d\lambda \leq \frac{A(v-u)^2}{12}, \tag{1.3}$$

provided that Ψ is twice differentiable and that the second derivative is bounded above by A and bounded below by B. Under the same conditions, Ujević obtained the following refinements [9]:

$$\frac{(3\mathscr{S}-2A)(v-u)^2}{24} \leq \frac{1}{v-u}\int_u^v \Psi(x)dx - \Psi\left(\frac{u+v}{2}\right) \leq \frac{(3\mathscr{S}-2B)(v-u)^2}{24}, \tag{1.4}$$

$$\frac{(3\mathscr{S}-A)(v-u)^2}{24} \leq \frac{\Psi(u)+\Psi(v)}{2} - \frac{1}{v-u}\int_u^v \Psi(x)dx \leq \frac{(3\mathscr{S}-B)(v-u)^2}{24}, \tag{1.5}$$

where $\mathscr{S} = (\Psi'(v) - \Psi'(u))/(v-u)$. Another integral inequality was established by Grüss in [10], stated as follows:

$$\left|\frac{1}{v-u}\int_u^v \Psi(\lambda)\eta(\lambda)d\lambda - \left(\frac{1}{v-u}\int_u^v \Psi(\lambda)d\lambda\right)\left(\frac{1}{v-u}\int_u^v \eta(\lambda)d\lambda\right)\right| \leq \frac{1}{4}(A-B)(\mathscr{N}_2 - \mathscr{N}_1), \tag{1.6}$$

provided that $B \leq \Psi(\lambda) \leq A$ and that $\mathscr{N}_1 \leq \eta(\lambda) \leq \mathscr{N}_2$ for all $\lambda \in [u,v]$. Ostrowski proved the following inequality [11]:

$$\left|\Psi(\sigma) - \frac{1}{v-u}\int_u^v \Psi(\lambda)d\lambda\right| \leq \left[\frac{1}{4} + \frac{(\sigma - \frac{u+v}{2})^2}{(v-u)^2}\right](v-u)D, \tag{1.7}$$

provided that $|\Psi'(t)| \leq D$ for all $t \in [u,v]$. Farid later established the following Ostrowski-Grüss-type inequality [12]:

$$\begin{aligned}&\left|\frac{1}{2}\Psi(\sigma) - \frac{(\sigma-v)\Psi(v) - (\sigma-u)\Psi(u)}{2(v-u)} - \frac{1}{v-u}\int_u^v \Psi(\lambda)d\lambda\right| \\ \leq&\frac{(\sigma-u)^2 + (v-\sigma)^2}{4(v-u)}(A-B),\end{aligned} \tag{1.8}$$

provided that $B \leq \Psi'(\lambda) \leq A$ for all $\lambda \in [u,v]$.

The relationships among these inequalities are as follows. Inequalities (1.2) and (1.3) estimate the two gaps in the Hermite-Hadamard inequality, and (1.4) and (1.5) sharpen those estimates by incorporating derivative information [9]. The Grüss inequality (1.6) and the Ostrowski inequality (1.7) lead naturally to mixed Ostrowski-Grüss estimates such as (1.8) [13]. Fractional analogs have subsequently been developed for Caputo derivative [14], k-Riemann-Liouville integral [15, 16], Katugampola fractional integral [17], and Mittag-Leffler-based operators [18].

Several closely related contributions clarify the position of the present work. Rashid et al. derived Hermite-Hadamard-type bounds for higher-order strongly h-preinvex functions through Riemann-Liouville fractional integrals [19]. Zhou et al. studied weighted Čebyšev and Pólya-Szegő functionals for Hilfer generalized proportional fractional integral operators [20]. Rashid et al. introduced weighted nonsingular operators associated with the Atangana-Baleanu framework [21]. These studies emphasize generalized convexity, weighted functionals, or nonsingular proportional operators. By contrast, the present manuscript uses two-sided derivative envelopes and an extended generalized Mittag-Leffler kernel to obtain, within one framework, pointwise Ostrowski bounds, Ostrowski-Grüss estimates, and Hermite-Hadamard error bounds. Thus, the novelty lies in the simultaneous interaction of the generalized kernel, its normalization factors, and lower/upper derivative controls, rather than in a purely formal substitution of one kernel for another.

Recent developments also include multi-index Mittag-Leffler operators that unify several fractional constructions [22], four-parameter kernels used in geometric-function applications [23], and incomplete matrix-valued Mittag-Leffler operators [24]. These advances demonstrate that the choice of kernel affects both analytical structure and computational behavior. Motivated by this observation, we study the extended generalized Mittag-Leffler function introduced in [25] and the associated integral operators.

In [25], the Mittag-Leffler function $E_{\tau,v,\psi}^{\delta,\theta,\rho,\phi}(z;p)$ is defined as follows.

Definition 1.1. *[25] Let $\tau, v, \psi, \delta, \phi \in \mathbb{C}$, $\Re(\tau), \Re(v), \Re(\psi) > 0$, $\Re(\phi) > \Re(\delta) > 0$ with $p \geq 0$, $\theta > 0$ and $0 < \rho \leq \theta + \Re(\rho)$. Then, the extended generalized Mittag-Leffler function $E_{\tau,v,\psi}^{\delta,\theta,\rho,\phi}(z;p)$ is defined by*

$$E_{\tau,v,\psi}^{\delta,\theta,\rho,\phi}(z;p) = \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\delta + n\rho, \phi - \delta)}{\mathrm{B}(\delta, \phi - \delta)} \frac{(\phi)_{n\rho}}{\Gamma(\tau n + v)} \frac{z^n}{(\psi)_{n\theta}}. \tag{1.9}$$

Definition 1.2. *[25] Under the same parameter assumptions as in Definition 1.1, let $f \in L_1[u,v]$. The integral operators $\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi,\omega,\sigma^+}\Psi\right)(v;p)$ and $\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi,\omega,\sigma^-}\Psi\right)(v;p)$ containing the Mittag-Leffler function $E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(z;p)$ are given by*

$$\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi,\omega,u^+}f\right)(x;p) = \int_u^x (x-t)^{\nu-1} E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(x-t)^{\tau};p)f(t)dt, \tag{1.10}$$

$$\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi,\omega,v^-}f\right)(x;p) = \int_x^v (t-x)^{\nu-1} E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(t-x)^{\tau};p)f(t)dt. \tag{1.11}$$

Table 1 summarizes the principal notations that are frequently used throughout this paper.

Table 1. Principal notation and standing assumptions.

Symbol	Meaning and assumptions
u, v	Endpoints of the interval, with $u < v$.
σ	Evaluation point in $[u, v]$.
Ψ	Real-valued differentiable function to which the inequalities are applied.
ζ, η	Integrable lower and upper controls for the derivative $\zeta(t) \leq \Psi'(t) \leq \eta(t)$.
τ, ν, ψ	Complex parameters with positive real parts.
δ, ϕ	Complex parameters satisfying $\Re(\phi) > \Re(\delta) > 0$.
θ, ρ, p	Kernel parameters with $\theta > 0$, $p \geq 0$, and the restrictions stated in Definition 1.1.
ω	Scale parameter in the Mittag-Leffler kernel.
μ	Positive order parameter; the main derivative-based estimates use $\mu > 1$.
$E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}$	Extended generalized Mittag-Leffler function defined in (1.9).
$\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi,\omega,\cdot}$	Left- or right-sided integral operator defined in (1.10) and (1.11).

The following derivative identity is used repeatedly in the proofs.

Lemma 1.1. *[25] If $m \in \mathbb{N}$, $\omega, \rho, \sigma, \tau, \delta, c \in \mathbb{C}$, $\Re(\rho), \Re(\sigma), \Re(\tau) > 0$, $\Re(c) > \Re(\delta) > 0$ with $p \geq 0$, $r > 0$ and $0 < q < r + \Re(\rho)$, then*

$$\left(\frac{d}{dz}\right)^m \left[z^{\sigma-1} E^{\delta,r,q,c}_{\rho,\sigma,\tau}(\omega z^{\rho};p)\right] = z^{\sigma-m-1} E^{\delta,r,q,c}_{\rho,\sigma-m,\tau}(\omega z^{\rho};p), \quad \Re(\sigma) > m. \tag{1.12}$$

The proofs are motivated by derivative identities used in related Caputo-type estimates [14], but the extended generalized Mittag-Leffler kernel requires different normalization factors and produces a wider parameter family. Section 2 first establishes a two-sided Ostrowski estimate under the envelope condition $\zeta \leq \Psi' \leq \eta$. It then derives absolute-value Ostrowski bounds, Ostrowski-Grüss refinements, and midpoint/Hermite-Hadamard consequences. Intermediate integration-by-parts identities are displayed explicitly in the longer proofs. The section concludes with a fractional relaxation example and a numerical comparison of exponential and Mittag-Leffler responses.

2. Main results

The following result gives the fractional version of the Ostrowski inequality with general conditions.

Theorem 2.1. *Let $\Psi : [u_1, u_2] \longrightarrow \mathbb{R}$ be the function having first order derivative Ψ' that satisfies $\zeta(\lambda) \leq \Psi'(\lambda) \leq \eta(\lambda)$ for all $\lambda \in [u_1, u_2]$, where $\zeta, \eta \in L[u_1, u_2]$. Then, for $\sigma \in [u_1, u_2]$, we have the inequality given below:*

$$\begin{aligned}
&\frac{\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\zeta(\lambda)d\lambda-\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\eta(\lambda)d\lambda}{(\sigma-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u_1)^{\tau};p)+(u_2-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\sigma)^{\tau};p)} \\
&\leq \Psi(\sigma)-\frac{\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}\Psi\right)(u_1;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^+}\Psi\right)(u_2;p)\right)}{(\sigma-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u_1)^{\tau};p)+(u_2-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\sigma)^{\tau};p)} \\
&\leq \frac{\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\eta(\lambda)d\lambda-\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\zeta(\lambda)d\lambda}{(\sigma-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u_1)^{\tau};p)+(u_2-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\sigma)^{\tau};p)}.
\end{aligned} \tag{2.1}$$

Proof. Under the stated assumptions, the following four weighted inequalities hold for $\sigma \in [u_1, u_2]$:

$$\begin{aligned}
&\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\Psi'(\lambda)d\lambda \\
\geq &\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\zeta(\lambda)d\lambda \\
&\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\Psi'(\lambda)d\lambda \\
\leq &\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\eta(\lambda)d\lambda \\
&\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\Psi'(\lambda)d\lambda \\
\geq &\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\zeta(\lambda)d\lambda \\
&\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\Psi'(\lambda)d\lambda \\
\leq &\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\eta(\lambda)d\lambda.
\end{aligned}$$

Combining these four inequalities gives

$$\begin{aligned}
&\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\zeta(\lambda)d\lambda \\
&-\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\eta(\lambda)d\lambda \\
\leq &\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\Psi'(\lambda)d\lambda \\
&-\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\Psi'(\lambda)d\lambda \\
\leq &\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u_1)^{\tau};p)\eta(\lambda)d\lambda \\
&-\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(u_2-\lambda)^{\tau};p)\zeta(\lambda)d\lambda.
\end{aligned} \tag{2.2}$$

To evaluate the middle expression, apply integration by parts separately on $[u_1,\sigma]$ and $[\sigma,u_2]$. By (1.12),

$$\begin{aligned}
&\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\lambda-u_1)^{\tau};p)\Psi'(\lambda)\,d\lambda\\
&=(\sigma-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\sigma-u_1)^{\tau};p)\Psi(\sigma)-\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\sigma^-}^{\delta,\theta,\rho,\phi}\Psi\right)(u_1;p),\\
&\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-\lambda)^{\tau};p)\Psi'(\lambda)\,d\lambda\\
&=-(u_2-\sigma)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-\sigma)^{\tau};p)\Psi(\sigma)+\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\sigma^+}^{\delta,\theta,\rho,\phi}\Psi\right)(u_2;p).
\end{aligned}$$

Substitution of these identities into (2.2) yields

$$\begin{aligned}
&\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\lambda-u_1)^{\tau};p)\zeta(\lambda)d\lambda\\
&-\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-\lambda)^{\tau};p)\eta(\lambda)d\lambda\\
&\le\Big((\sigma-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\sigma-u_1)^{\tau};p)+(u_2-\sigma)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-\sigma)^{\tau};p)\Big)\Psi(\sigma)\\
&-\left(\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\sigma^-}^{\delta,\theta,\rho,\phi}\Psi\right)(u_1;p)+\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\sigma^+}^{\delta,\theta,\rho,\phi}\Psi\right)(u_2;p)\right)\\
&\le\int_{u_1}^{\sigma}(\lambda-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\lambda-u_1)^{\tau};p)\eta(\lambda)d\lambda\\
&-\int_{\sigma}^{u_2}(u_2-\lambda)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-\lambda)^{\tau};p)\zeta(\lambda)d\lambda.
\end{aligned}\tag{2.3}$$

Dividing by the positive normalization factor in (2.3) proves (2.1).

Corollary 2.1. *Setting $\sigma=(u_1+u_2)/2$ in (2.1), the following error estimation of the fractional Hadamard-type inequality is obtained:*

$$\begin{aligned}
&\frac{2^{\mu-2}\left(\int_{u_1}^{\frac{u_1+u_2}{2}}P(u_1,\lambda)\zeta(\lambda)d\lambda-\int_{\frac{u_1+u_2}{2}}^{u_2}Q(u_2,\lambda)\eta(\lambda)d\lambda\right)}{(u_2-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\frac{u_2-u_1}{2})^{\tau};p)}\\
&\le\Psi\left(\frac{u_1+u_2}{2}\right)-\frac{2^{\mu-2}\left(\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\frac{u_1+u_2}{2}^-}^{\delta,\theta,\rho,\phi}\Psi\right)(u_1;p)+\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\frac{u_1+u_2}{2}^+}^{\delta,\theta,\rho,\phi}\Psi\right)(u_2;p)\right)}{(u_2-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\frac{u_2-u_1}{2})^{\tau};p)}\\
&\le\frac{2^{\mu-2}\left(\int_{u_1}^{\frac{u_1+u_2}{2}}P(u_1,\lambda)\eta(\lambda)d\lambda-\int_{\frac{u_1+u_2}{2}}^{u_2}Q(u_2,\lambda)\zeta(\lambda)d\lambda\right)}{(u_2-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\frac{u_2-u_1}{2})^{\tau};p)},
\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}
P(u_1,\lambda)&=(\lambda-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\lambda-u_1)^{\tau};p),\\
Q(u_2,\lambda)&=(u_2-\lambda)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-\lambda)^{\tau};p).
\end{aligned}$$

The following Ostrowski-type inequality provides estimates for the operators $\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}.\right)(u;p)$ and $\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,\sigma^+}.\right)(v;p)$ for bounded differentiable functions ψ to the Mittag-Leffler function $E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(.;p)$.

Theorem 2.2. *Let ψ be differentiable on $[u,v]$ and $|\Psi'(t)| \leq \eta(t)$ for all $t \in [u,v]$, where $\eta \in L[u,v]$. Then, for $\sigma \in [u,v]$ and $\mu,\nu > 1$, we obtain*

$$\begin{aligned}
&\Bigg|\left((\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-\sigma)^{\tau};p)\right)\Psi(\sigma) \\
&\quad-\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}\Psi\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,\sigma^+}\Psi\right)(v;p)\right)\Bigg| \\
&\leq \int_u^{\sigma}(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\eta(\lambda)d\lambda \\
&\quad+\int_{\sigma}^{v}(v-\sigma)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-\sigma)^{\tau};p)\eta(\lambda)d\lambda.
\end{aligned} \tag{2.5}$$

Proof. Under given conditions, for $\lambda \in [u,\sigma]$, we can have the following integral inequality:

$$\begin{aligned}
&\int_u^{\sigma}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\Psi'(\lambda)d\lambda \\
&+\int_u^{\sigma}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\eta(\lambda)d\lambda \geq 0.
\end{aligned} \tag{2.6}$$

The first term of (2.6) is calculated as follows:

$$\int_u^{\sigma}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\Psi'(\lambda)d\lambda.$$

Putting $\lambda-u=\mathscr{Z}$, that is, $\lambda=u+\mathscr{Z}$ and using the formula (1.12), we have

$$\begin{aligned}
&\int_0^{\sigma-u}\mathscr{Z}^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega\mathscr{Z}^{\tau};p)\Psi'(u+\mathscr{Z})d\mathscr{Z} \\
&=(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\Psi(\sigma)-\int_0^{\sigma-u}\mathscr{Z}^{\mu-2}E^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi}(\omega\mathscr{Z}^{\tau};p)\Psi(u+\mathscr{Z})d\mathscr{Z}.
\end{aligned}$$

Now, put $u+\mathscr{Z}=\lambda$ in the second term of the above equation and use Definition 1.1 to get

$$(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\Psi(u)-\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}\Psi\right)(u;p). \tag{2.7}$$

By using (2.7) and $(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p) \geq (\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)$ in (2.6), we have

$$\begin{aligned}
&(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\Psi(\sigma)-\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}\Psi\right)(u;p) \\
&\geq -\int_u^{\sigma}(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\eta(\lambda)d\lambda.
\end{aligned} \tag{2.8}$$

Similarly, for $\lambda \in [\sigma, v]$, we have

$$\begin{aligned}
&\int_{\sigma}^{v}(v-\lambda)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\lambda)^{\tau};p)\eta(\lambda)d\lambda \\
&-\int_{\sigma}^{v}(v-\lambda)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\lambda)^{\tau};p)\Psi'(\lambda)d\lambda \geq 0.
\end{aligned} \tag{2.9}$$

By using $(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p) \geq (v-\lambda)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\lambda)^{\tau};p)$ in the first term of (2.9) and for the second term of (2.9), following the same procedure as we did for the first term of (2.6), we have

$$\begin{aligned}
&(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p)\Psi(\sigma)-\left(\varepsilon_{\tau,\nu-1,\psi,\omega,\sigma^{+}}^{\delta,\theta,\rho,\phi}\Psi\right)(v;p) \\
\geq &-\int_{\sigma}^{v}(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p)\eta(\lambda)d\lambda.
\end{aligned} \tag{2.10}$$

Adding (2.8) and (2.10) gives

$$\begin{aligned}
&\left((\sigma-u)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p)\right)\Psi(\sigma) \\
&-\left(\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\sigma^{-}}^{\delta,\theta,\rho,\phi}\Psi\right)(u;p)+\left(\varepsilon_{\tau,\nu-1,\psi,\omega,\sigma^{+}}^{\delta,\theta,\rho,\phi}\Psi\right)(v;p)\right) \\
\geq &-\left(\int_{u}^{\sigma}(\sigma-u)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\sigma-u)^{\tau};p)\eta(\lambda)d\lambda\right. \\
&\left.+\int_{\sigma}^{v}(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p)\eta(\lambda)d\lambda\right).
\end{aligned} \tag{2.11}$$

Because the expressions

$$\begin{aligned}
&(\lambda-u)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\lambda-u)^{\sigma};p)(\eta(\lambda)-\Psi'(\lambda)), \\
&(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p)(\eta(\lambda)+\Psi'(\lambda)),
\end{aligned}$$

are non-negative, one can get

$$\int_{u}^{\sigma}(\lambda-u)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\lambda-u)^{\sigma};p)(\eta(\lambda)-\Psi'(\lambda))d\lambda \geq 0, \tag{2.12}$$

$$\int_{\sigma}^{v}(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p)(\eta(\lambda)+\Psi'(\lambda))d\lambda \geq 0. \tag{2.13}$$

Integrating by parts and then adding the above inequalities, the following inequality is obtained:

$$\begin{aligned}
&\left((\sigma-u)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p)\right)\Psi(\sigma) \\
&-\left(\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\sigma^{-}}^{\delta,\theta,\rho,\phi}\Psi\right)(u;p)+\left(\varepsilon_{\tau,\nu-1,\psi,\omega,\sigma^{+}}^{\delta,\theta,\rho,\phi}\Psi\right)(v;p)\right) \\
\leq &\int_{u}^{\sigma}(\sigma-u)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\sigma-u)^{\tau};p)\eta(\lambda)d\lambda+\int_{\sigma}^{v}(v-\sigma)^{\nu-1}E_{\tau,\nu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\sigma)^{\tau};p)\eta(\lambda)d\lambda.
\end{aligned} \tag{2.14}$$

Combining (2.11) and (2.14) proves (2.5).

Corollary 2.2. *If η is increasing, then from Theorem 2.2, one can get*

$$\begin{aligned}&\Big|\Big((\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-\sigma)^{\tau};p)\Big)\Psi(\sigma)\\&-\Big(\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^{-}}\Psi\Big)(u;p)+\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,\sigma^{+}}\Psi\Big)(v;p)\Big)\Big|\\\leq&\Big((\sigma-u)^{\mu}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\eta(\sigma)+(v-\sigma)^{\nu}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-\sigma)^{\tau};p)\eta(v)\Big).\end{aligned}\tag{2.15}$$

Corollary 2.3. *If η is decreasing, then from Theorem 2.2, one can get*

$$\begin{aligned}&\Big|\Big((\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-\sigma)^{\tau};p)\Big)\Psi(\sigma)\\&-\Big(\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^{-}}\Psi\Big)(u;p)+\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,\sigma^{+}}\Psi\Big)(v;p)\Big)\Big|\\\leq&\Big((\sigma-u)^{\mu}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\eta(u)+(v-\sigma)^{\nu}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-\sigma)^{\tau};p)\eta(\sigma)\Big).\end{aligned}\tag{2.16}$$

The Hadamard-type inequality is proved in the next result.

Theorem 2.3. *Let ψ be differentiable on $[u,v]$, and $|\Psi'(t)|\leq\eta(t)$ for all $t\in[u,v]$, where $\eta\in L[u,v]$. Then, for $\sigma\in[u,v]$, one can have*

$$\left|\frac{\Psi(u)+\Psi(v)}{2}-\frac{\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^{-}}\Psi\Big)(u;p)+\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,u^{+}}\Psi\Big)(v;p)}{2(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-u)^{\tau};p)}\right|\leq\int_u^v\eta(\lambda)d\lambda.\tag{2.17}$$

Proof. By putting $\sigma=v$ in (2.8) and $\sigma=u$ in (2.10), respectively, the following inequalities are obtained:

$$\Psi(v)-\frac{\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^{-}}\Psi\Big)(u;p)}{(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-u)^{\tau};p)}\geq-\int_u^v\eta(\lambda)d\lambda,$$

$$\Psi(u)-\frac{\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,u^{+}}\Psi\Big)(v;p)}{(v-u)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-u)^{\tau};p)}\geq-\int_u^v\eta(\lambda)d\lambda.$$

By adding the above two inequalities, we have

$$\begin{aligned}&\frac{\Psi(u)+\Psi(v)}{2}-\Bigg(\frac{\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^{-}}\Psi\Big)(u;p)}{2(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-u)^{\tau};p)}+\frac{\Big(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,u^{+}}\Psi\Big)(v;p)}{2(v-u)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-u)^{\tau};p)}\Bigg)\\&\geq-\int_u^v\eta(\lambda)d\lambda.\end{aligned}\tag{2.18}$$

Similarly, by putting $\sigma = v$ in the resulting inequality of (2.12) and $\sigma = u$ in the resulting inequality of (2.13), respectively, then after adding, we get the following inequality:

$$\frac{\Psi(u)+\Psi(v)}{2} - \left(\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^-}\Psi\right)(u;p)}{2(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-u)^{\tau};p)} + \frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,u^+}\Psi\right)(v;p)}{2(v-u)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-u)^{\tau};p)} \right) \tag{2.19}$$

$$\leq \int_u^v \eta(\lambda)d\lambda.$$

From (2.18) and (2.19), we can write

$$\left| \frac{\Psi(u)+\Psi(v)}{2} - \left(\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^-}\Psi\right)(u;p)}{2(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-u)^{\tau};p)} + \frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,u^+}\Psi\right)(v;p)}{2(v-u)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}(\omega(v-u)^{\tau};p)} \right) \right| \tag{2.20}$$

$$\leq \int_u^v \eta(\lambda)d\lambda.$$

By taking $\mu = \nu$ in (2.20), the required inequality (2.17) is obtained.

Theorem 2.4. *Under the assumptions of Theorem 2.3, the following inequality also holds:*

$$\left| \frac{2^{\mu-2}\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^-}\Psi\right)(u;p) + \left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,\frac{u+v}{2}^+}\Psi\right)(v;p) \right)}{(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)} - \Psi\left(\frac{u+v}{2}\right) \right| \tag{2.21}$$

$$\leq \frac{1}{2}\int_u^v \eta(\lambda)d\lambda.$$

Proof. Putting $\sigma = (u+v)/2$ in (2.5), we get

$$\left| \left(\left(\frac{v-u}{2}\right)^{\mu-1} E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right) + \left(\frac{v-u}{2}\right)^{\nu-1} E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right) \right) \right. \tag{2.22}$$

$$\left. \Psi\left(\frac{u+v}{2}\right) - \left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^-}\Psi\right)(u;p) + \left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\nu-1,\psi,\omega,\frac{u+v}{2}^+}\Psi\right)(v;p) \right) \right|$$

$$\leq \int_u^{\frac{u+v}{2}} 2^{1-\mu}(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)\eta(\lambda)d\lambda$$

$$+ \int_{\frac{u+v}{2}}^v 2^{1-\nu}(v-u)^{\nu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\nu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)\eta(\lambda)d\lambda.$$

By taking $\mu = \nu$ in (2.22), the required inequality (2.21) is obtained.

Theorem 2.5. *Under the assumptions of Theorem 2.3, the following inequality also holds:*

$$\left| \frac{2^{\mu-2}\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^-}\Psi\right)(u;p) + \left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^+}\Psi\right)(v;p) \right)}{(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)} - \Psi\left(\frac{u+v}{2}\right) \right| \tag{2.23}$$

$$\leq \frac{2^{\mu-2}\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi,\omega,\frac{u+v}{2}^-}\zeta\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi,\omega,\frac{u+v}{2}^+}\zeta\right)(v;p)\right)}{(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)}.$$

Proof. Let $\lambda \in [u,(u+v)/2]$, and by using the given condition on Ψ', we have

$$\begin{aligned}&\int_u^{\frac{u+v}{2}}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\Psi'(\lambda)d\lambda \\ &+\int_u^{\frac{u+v}{2}}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\zeta(\lambda)d\lambda \geq 0.\end{aligned} \tag{2.24}$$

Let $m=(u+v)/2$. Applying (1.12) and integrating by parts gives

$$\begin{aligned}&\int_u^m(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\Psi'(\lambda)\,d\lambda \\ =&(m-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(m-u)^{\tau};p)\Psi(m)-\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,m^-}\Psi\right)(u;p).\end{aligned}$$

Substituting this identity into (2.24) and dividing by the positive boundary kernel yields

$$\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^-}\Psi\right)(u;p)}{\left(\frac{v-u}{2}\right)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)}-\Psi\left(\frac{u+v}{2}\right)\leq\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^-}\zeta\right)(u;p)}{\left(\frac{v-u}{2}\right)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)}. \tag{2.25}$$

Similarly, for $\lambda \in [(u+v)/2, v]$, we have

$$\begin{aligned}&\int_{\frac{u+v}{2}}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)\zeta(\lambda)d\lambda \\ &-\int_{\frac{u+v}{2}}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)\Psi'(\lambda)d\lambda \geq 0.\end{aligned} \tag{2.26}$$

Similarly,

$$\begin{aligned}&\int_m^v(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)\Psi'(\lambda)\,d\lambda \\ =&-(v-m)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-m)^{\tau};p)\Psi(m)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,m^+}\Psi\right)(v;p).\end{aligned}$$

Using this identity in (2.26) gives

$$\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^+}\Psi\right)(v;p)}{\left(\frac{v-u}{2}\right)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)}-\Psi\left(\frac{u+v}{2}\right)\leq\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^+}\zeta\right)(v;p)}{\left(\frac{v-u}{2}\right)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)}. \tag{2.27}$$

By adding inequalities (2.25) and (2.27), we have

$$\begin{aligned}
&\frac{2^{\mu-2}\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^-}\Psi\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^+}\Psi\right)(v;p)\right)-\Psi\left(\frac{u+v}{2}\right)}{(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)} \\
\leq&\frac{2^{\mu-2}\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi,\omega,\frac{u+v}{2}^-}\zeta\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi,\omega,\frac{u+v}{2}^+}\zeta\right)(v;p)\right)}{(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)}.
\end{aligned} \tag{2.28}$$

Next, interchange the non-negative kernel terms $(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)$ and $(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)$ in inequalities (2.24) and (2.26). Then, by adding the achieved inequalities after integrating the same as in (2.25) and (2.27), we get the following expression:

$$\begin{aligned}
&\frac{2^{\mu-2}\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^-}\Psi\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\frac{u+v}{2}^+}\Psi\right)(v;p)\right)-\Psi\left(\frac{u+v}{2}\right)}{(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)} \\
\geq&-\frac{2^{\mu-2}\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi,\omega,\frac{u+v}{2}^-}\zeta\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi,\omega,\frac{u+v}{2}^+}\zeta\right)(v;p)\right)}{(v-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}\left(\omega\left(\frac{v-u}{2}\right)^{\tau};p\right)}.
\end{aligned} \tag{2.29}$$

From inequalities (2.28) and (2.29), the required inequality (2.23) is obtained.

Theorem 2.6. *Let $\Psi:[u,v]\longrightarrow\mathbb{R}$ be differentiable, and suppose that $\zeta(\lambda)\leq\Psi'(\lambda)\leq\eta(\lambda)$ for all $\lambda\in[u,v]$, where $\zeta,\eta\in L[u,v]$. Then, for $\sigma\in[u,v]$, the following inequality holds:*

$$\begin{aligned}
&\Bigg|\Psi(\sigma)+\frac{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\Psi(u)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)\Psi(v)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)} \\
&-\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}\Psi\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,u^+}\Psi\right)(\sigma;p)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)} \\
&-\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^-}\Psi\right)(\sigma;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^+}\Psi\right)(v;p)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\Bigg| \\
\leq&\frac{1}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)} \\
&\times\Bigg[(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\int_u^{\sigma}\eta(\lambda)d\lambda+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)\int_{\sigma}^{v}\eta(\lambda)d\lambda \\
&-\int_u^{\sigma}(\sigma-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-\lambda)^{\tau};p)\zeta(\lambda)d\lambda-\int_{\sigma}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)\zeta(\lambda)d\lambda\Bigg].
\end{aligned} \tag{2.30}$$

Proof. Let $\lambda \in [u,\sigma]$. By using the positivity of $(\sigma-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)(\Psi'(\lambda)-\zeta(\lambda))$ and $(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\sigma};p)(\eta(\lambda)-\Psi'(\lambda))$, we have

$$\begin{aligned}&\int_u^{\sigma}(\sigma-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)(\Psi'(\lambda)-\zeta(\lambda))d\lambda \\ &+\int_u^{\sigma}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)(\eta(\lambda)-\Psi'(\lambda))d\lambda\geq 0.\end{aligned} \tag{2.31}$$

That gives

$$\begin{aligned}&\int_u^{\sigma}(\sigma-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-\lambda)^{\tau};p)\Psi'(\lambda)d\lambda-\int_u^{\sigma}(\sigma-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-\lambda)^{\tau};p)\zeta(\lambda)d\lambda \\ &+\int_u^{\sigma}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\eta(\lambda)d\lambda-\int_u^{\sigma}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\Psi'(\lambda)d\lambda\geq 0.\end{aligned}$$

By using $(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)\leq(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)$ for $\lambda\in[u,\sigma]$ and doing integration, we have

$$\begin{aligned}&(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\Big(\Psi(u)+\Psi(\sigma)\Big) \\ &-\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}\Psi\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,u^+}\Psi\right)(\sigma;p)\right) \\ \leq&(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\int_u^{\sigma}\eta(\lambda)d\lambda \\ &-\int_u^{\sigma}(\sigma-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-\lambda)^{\tau};p)\zeta(\lambda)d\lambda.\end{aligned} \tag{2.32}$$

Similarly, for $\lambda\in[\sigma,v]$, we have

$$\begin{aligned}&\int_{\sigma}^{v}(\lambda-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-\sigma)^{\tau};p)(\eta(\lambda)-\Psi'(\lambda))d\lambda \\ &+\int_{\sigma}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)(\Psi'(\lambda)-\zeta(\lambda))d\lambda\geq 0.\end{aligned} \tag{2.33}$$

That gives

$$\begin{aligned}&\int_{\sigma}^{v}(\lambda-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-\sigma)^{\tau};p)\eta(\lambda)d\lambda-\int_{\sigma}^{v}(\lambda-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-\sigma)^{\tau};p)\Psi'(\lambda)d\lambda \\ &+\int_{\sigma}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)\Psi'(\lambda)d\lambda-\int_{\sigma}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)\zeta(\lambda)d\lambda\geq 0.\end{aligned}$$

By using $(\lambda-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-\sigma)^{\tau};p)\leq(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)$ for $\lambda\in[\sigma,v]$ and doing integration, we have

$$\begin{aligned}&(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)\Big(\Psi(\sigma)+\Psi(v)\Big) \\ &-\left(\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^-}\Psi\right)(\sigma;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^+}\Psi\right)(v;p)\right)\end{aligned} \tag{2.34}$$

$$\leq (v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)\int_{\sigma}^{v}\eta(\lambda)d\lambda - \int_{\sigma}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)\zeta(\lambda)d\lambda.$$

Adding (2.32) and (2.34) gives

$$\begin{aligned}
&\Psi(\sigma)+\frac{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\Psi(u)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)\Psi(v)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\\
&-\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}\Psi\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,u^+}\Psi\right)(\sigma;p)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\\
&-\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^-}\Psi\right)(\sigma;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^+}\Psi\right)(v;p)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\Bigg|\\
\leq&\frac{1}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\\
&\times\Bigg[(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\int_{u}^{\sigma}\eta(\lambda)d\lambda+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)\int_{\sigma}^{v}\eta(\lambda)d\lambda\\
&-\int_{u}^{\sigma}(\sigma-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-\lambda)^{\tau};p)\zeta(\lambda)d\lambda-\int_{\sigma}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)\zeta(\lambda)d\lambda\Bigg].
\end{aligned}\tag{2.35}$$

We also have the following inequalities:

$$\begin{aligned}
&\int_{u}^{\sigma}(\lambda-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)(\Psi'(\lambda)-\zeta(\lambda))d\lambda\\
&+\int_{u}^{\sigma}(\sigma-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-u)^{\tau};p)(\eta(\lambda)-\Psi'(\lambda))d\lambda\geq 0,
\end{aligned}\tag{2.36}$$

$$\begin{aligned}
&\int_{\sigma}^{v}(v-\lambda)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\lambda)^{\tau};p)(\eta(\lambda)-\Psi'(\lambda))d\lambda\\
&+\int_{\sigma}^{v}(\lambda-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\lambda-\sigma)^{\tau};p)(\Psi'(\lambda)-\zeta(\lambda))d\lambda\geq 0.
\end{aligned}\tag{2.37}$$

Applying integration by parts to (2.36) and (2.37) and then adding the resulting inequalities gives

$$\begin{aligned}
&\Psi(\sigma)+\frac{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\Psi(u)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)\Psi(v)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\\
&-\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^-}\Psi\right)(u;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,u^+}\Psi\right)(\sigma;p)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\\
&-\frac{\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,v^-}\Psi\right)(\sigma;p)+\left(\varepsilon^{\delta,\theta,\rho,\phi}_{\tau,\mu-1,\psi,\omega,\sigma^+}\Psi\right)(v;p)}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\Bigg|\\
\geq&\frac{-1}{(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)}\\
&\times\Bigg[(\sigma-u)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(\sigma-u)^{\tau};p)\int_{u}^{\sigma}\eta(\lambda)d\lambda+(v-\sigma)^{\mu-1}E^{\delta,\theta,\rho,\phi}_{\tau,\mu,\psi}(\omega(v-\sigma)^{\tau};p)\int_{\sigma}^{v}\eta(\lambda)d\lambda
\end{aligned}\tag{2.38}$$

$$-\int_{u}^{\sigma}(\sigma-\lambda)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(\sigma-\lambda)^{\tau};p)\zeta(\lambda)d\lambda-\int_{\sigma}^{v}(v-\lambda)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(v-\lambda)^{\tau};p)\zeta(\lambda)d\lambda\Bigg].$$

The upper and lower estimates (2.35) and (2.38) together prove (2.30).

Corollary 2.4. *Setting $\sigma=(u_1+u_2)/2$ in (2.30), the following error estimation of the fractional Hadamard inequality is obtained:*

$$\begin{aligned}&\left|\frac{\Psi(u_1)+\Psi(u_2)}{2}-\frac{\left(\varepsilon_{\tau,\mu-1,\psi,\omega,u_2^-}^{\delta,\theta,\rho,\phi}\Psi\right)(u_1;p)+\left(\varepsilon_{\tau,\mu-1,\psi,\omega,u_1^+}^{\delta,\theta,\rho,\phi}\Psi\right)(u_2;p)}{2(u_2-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-u_1)^{\tau};p)}\right| \\ \leq&\frac{1}{2(u_2-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-u_1)^{\tau};p)}\Bigg[(u_2-u_1)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-u_1)^{\tau};p) \\ &\times\int_{u_1}^{u_2}\eta(\lambda)d\lambda-\int_{u_1}^{u_2}(u_2-\lambda)^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega(u_2-\lambda)^{\tau};p)\zeta(\lambda)d\lambda\Bigg].\end{aligned} \tag{2.39}$$

Illustrative application to a fractional relaxation model

Consider the Caputo fractional relaxation problem

$$^{C}D_{0+}^{\alpha}y(t)=-\kappa y(t),\qquad y(0)=y_0,\qquad 0<\alpha\leq 1,\quad \kappa>0, \tag{2.40}$$

whose exact response is $y(t)=y_0E_{\alpha}(-\kappa t^{\alpha})$ [4]. Fix $0<a<b\leq T$ so that y is differentiable on $[a,b]$, and define

$$M_{a,b}=\sup_{t\in[a,b]}|y'(t)|.$$

Set $\mu=\nu>1$ in Theorem 2.2, and choose $\eta(t)=M_{a,b}$. With

$$K_{\mu}(r)=r^{\mu-1}E_{\tau,\mu,\psi}^{\delta,\theta,\rho,\phi}(\omega r^{\tau};p),$$

we obtain, for every $\sigma\in[a,b]$,

$$\begin{aligned}&\left|\big(K_{\mu}(\sigma-a)+K_{\mu}(b-\sigma)\big)y(\sigma)-\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\sigma^-}^{\delta,\theta,\rho,\phi}y\right)(a;p)-\left(\varepsilon_{\tau,\mu-1,\psi,\omega,\sigma^+}^{\delta,\theta,\rho,\phi}y\right)(b;p)\right| \\ \leq& M_{a,b}\big((\sigma-a)K_{\mu}(\sigma-a)+(b-\sigma)K_{\mu}(b-\sigma)\big).\end{aligned} \tag{2.41}$$

At the midpoint $m=(a+b)/2$, division by $2K_{\mu}((b-a)/2)$ yields the transparent estimate

$$\left|y(m)-\frac{\left(\varepsilon_{\tau,\mu-1,\psi,\omega,m^-}^{\delta,\theta,\rho,\phi}y\right)(a;p)+\left(\varepsilon_{\tau,\mu-1,\psi,\omega,m^+}^{\delta,\theta,\rho,\phi}y\right)(b;p)}{2K_{\mu}((b-a)/2)}\right|\leq\frac{M_{a,b}(b-a)}{2}. \tag{2.42}$$

Thus, the difference between the fractional state at the midpoint and its normalized Mittag-Leffler-weighted mean is controlled by the interval length and a derivative bound. For a numerical approximation y_h, the same expression can be evaluated with a computable estimate of $M_{a,b}$; an observed violation indicates either insufficient temporal resolution or excessive quadrature error. The estimate can therefore be used as a local consistency indicator in simulations of relaxation, diffusion, and other memory-dependent models.

Figure 1 summarizes the analytical progression used in the manuscript. The first transition replaces the classical integral mean by a fractional mean, and the second introduces the extended generalized Mittag-Leffler kernel and its parameter-dependent normalization. The results of this paper retain the classical inequalities as limiting or special cases, but also allow two-sided derivative envelopes and a broader family of memory kernels.

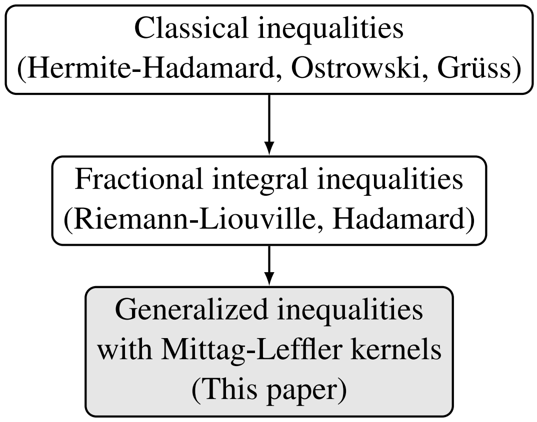

Figure 1. Hierarchy of inequalities: From classical convexity-based results to fractional integrals, further extended in this paper by incorporating Mittag-Leffler kernels.

Figure 2 illustrates how the fractional order changes the effective growth of the kernel. Over the plotted interval, the $\alpha = 0.9$ truncation remains closer to the exponential response than the $\alpha = 0.5$ truncation, whereas the latter exhibits a stronger departure from exponential behavior. The plot is intended as a qualitative comparison of kernel shapes; because only terms $k = 0, \ldots, 4$ are retained, it should not be interpreted as a high-accuracy evaluation of $E_\alpha(t)$ near the right endpoint. Increasing the truncation order provides a direct numerical refinement.

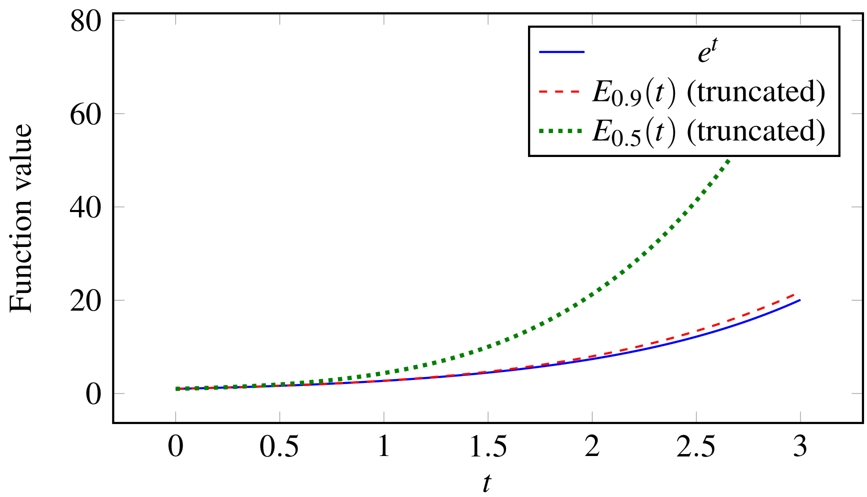

Figure 2. Comparison of e^t with the four-term truncations $\sum_{k=0}^{4} t^k/\Gamma(1+\alpha k)$ of $E_\alpha(t)$ for $\alpha = 0.9$ and $\alpha = 0.5$ on $0 \le t \le 3$. The curves were generated directly in `PGFPlots` (TeX Live, compatibility level 1.18) using the numerical coefficients displayed in the source code.

3. Conclusions

This paper established a unified collection of Ostrowski-type, Ostrowski-Grüss-type, and Hermite-Hadamard-type inequalities for integral operators containing an extended generalized Mittag-Leffler kernel. The principal distinction from closely related Riemann-Liouville, Hilfer proportional, and weighted nonsingular formulations is the simultaneous use of a highly-parameterized Mittag-Leffler kernel, its associated normalization factors, and two-sided derivative envelopes. Consequently, the results control both pointwise-to-mean deviations and mixed endpoint/midpoint errors within the same operator framework. Suitable parameter choices recover classical Riemann-Liouville inequalities, which provides a direct consistency check and clarifies the relation between the new estimates and established results. The fractional relaxation example demonstrates a concrete use of the theory: The discrepancy between a state value and a normalized Mittag-Leffler-weighted mean is bounded by the interval length and a derivative estimate. This interpretation connects the abstract inequalities with error estimation and local consistency testing for numerical solutions of fractional differential equations. The two figures further clarify the hierarchy of the results and the effect of fractional order on the kernel response. Several extensions are natural. In numerical analysis, the bounds can be incorporated into adaptive quadrature and time-stepping algorithms. For fractional differential equations, they may be developed into residual-based estimates, stability criteria, and a posteriori error controls. In optimization and control theory, analogous inequalities could bound nonlocal cost functionals and memory-dependent feedback terms. Further work may also replace ordinary convexity assumptions by generalized convexity classes, including preinvex, h-convex, and strongly convex functions. Finally, the same proof strategy can be investigated for Hilfer, Caputo-Fabrizio, Atangana–Baleanu, Prabhakar, tempered, variable-order, and matrix-valued fractional operators, as well as for multivariable and stochastic settings.

Author contributions

H. H. Taha, J. Pecaric, J. Ro and G. Farid: Conceptualization, Validation, Formal analysis, Writing manuscript, Editing and review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest.

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